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Abstract

The characteristic distribution of a polytope is the functional taking a test function to its integral over the polytope. The characteristic distributions of the cells in a polyhedral cell complex K generate a module M_K over the ring D of differential operators with polynomial coefficients. We describe this module and compute its de Rham cohomology and various direct images.
A B-spline is a distribution given by integration over the fibers of a projected polytope. We show that the D-module generated by the B-splines associated to the cells of K is isomorphic to the D-module direct image of M_K under the given projection, given a certain mild condition on K.

Sammanfatning

Den karakteristiska distributionen δ_K till ett polyhedralt cellcomplex K genererar en D-modul M_K. Vi ger en beskrivning av modulen, och visar ett sammanhang mellan direkta bilder av modulen under projektionsavbildningar och vissa såkallade B-splines.

Acknowledgements

Oh brave new world, that has such people in it!

First and foremost, I would like to thank my advisor Rikard Bøgvad for doing his job and doing it well; without him, this work would contain a whole lot more handwaving. My gratitude also extends to my other colleagues at Stockholm University and in particular to my fellow graduate students, for welcoming me into this strange and foreign country, and for being excellent human beings; my various friends and family, for getting me where I am today; and last but not least, *you the reader*. If you've bothered to read the acknowledgements, you surely are interested enough to read the rest; it warms my heart to know that my toil was not in vain.

Contents

1 Introduction

Multivariate B-splines are as functions given by projecting a (usually convex) polytope in $\mathbb{R}^m$ to a lower-dimensional linear subspace, and integrating over the fibers of this projection. In terms of distributions, this is a direct image (under the projection) of the *characteristic distribution* of the polytope, given by integration over the polytope. In this work, we replace the polytope by the polyhedral cell complex K it defines, and describe the D-module M_K generated by the characteristic distributions of all the cells. The main result is that under fairly nice conditions, the D-module direct image of this module under a projection $\mathbb{R}^m \to \mathbb{R}^s$ is isomorphic to the D-module S_K generated by the associated B-splines.

This provides a concrete worked-out example of a D-module direct image, of which the existing corpus of D-module theory is somewhat short. Some generalisations to more complicated settings than polyhedral cell complexes are within easy reach, but are not pursued in this work.

Some good sources for the basic properties of multivariate B-splines are [DCP06], [DBH82] and [CLR87]. A good introductory source for modules over the Weyl algebra is [Cou95], more comprehensive treatments of $\mathcal{D}$-modules in the full context of derived categories of sheaves are given in [B$^+$87], [Mal93], [Dim04] and [Bj3]. We tangentially touch the subject of currents in Section 4, the best source here is [Dem09].

1.1 Notation and terminology

The *Weyl algebra* is the ring of differential operators with polynomial coefficients on the space $\mathbb{C}^m$. It is freely generated as a $\mathbb{C}$-algebra by variables x_i, ∂_i, subject to the relations that $[\partial_i, \partial_j] = [x_i, x_j] = [\partial_i, x_j] = 0$ for $i \neq j$; and $[\partial_i, x_i] = 1$ (where $[-,-]$ denotes the commutator as usual). In particular, one has $[\partial_i, f] = \frac{\partial f}{\partial x_i}$ for any polynomial f. One uses the same name for the corresponding ring on $\mathbb{C}^s$, and one says *the Weyl algebra in m variables* (say) if one wishes to specify. As this ring is not commutative, one must distinguish between left and right modules; as is traditional, we will consider only left modules in this text (although a few bimodules do appear).

A *distribution* (on a subset of a manifold) is an ($\mathbb{R}$- or $\mathbb{C}$- valued) functional that operates on the space of (compactly supported) test functions by $\delta : \phi \mapsto \delta(\phi)$. A ring of differential operators (e.g. the Weyl algebra) acts on a distribution δ by

$$(p(x)\partial^\alpha \cdot \delta)(\phi) = \delta\left((-1)^{|\alpha|}\partial^\alpha(p(x) \cdot \phi)\right).$$

We will use the following conventions in this article: The Weyl algebra will be denoted W, or if it is important which space it is associated to, D_X, D_Y etc. We let $\mathcal{O}_X, \mathcal{D}_X$

and $\mathbb{C}_X$, respectively, denote the sheaves of regular functions, differential operators and locally constant $\mathbb{C}$-valued functions on X. The differential operator $\frac{\partial}{\partial x_i}$ will be denoted ∂_i, or ∂_{x_i} if necessary. For a vector $v \in \mathbb{R}^m$, we denote by ∂_v the directional derivative $\sum_i \langle v|e_i\rangle \partial_i$. Standard multiindex notation will be employed.

$\mathbb{R}^m$ is as usual, with $\langle \cdot|\cdot \rangle$ denoting the standard inner product. We will consider various (linear) maps $\pi : \mathbb{R}^m \to \mathbb{R}^s$, and their complexification $\pi : \mathbb{C}^m \to \mathbb{C}^s$, and denote $X := \mathbb{C}^m$, $Y := \mathbb{C}^s$ etc., and write $\pi : X \to Y$. In the simple case that π is the projection on the first s coordinates, we implicitly choose a splitting and write $\pi : X = Y \times Z \to Y$. Correspondingly, we use the shorthand $\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_m]$ etc., for the polynomial ring in m variables, and in general $\mathbb{C}[H] := \Gamma(H, \mathcal{O}_H^{alg})$ for an affine subspace $H \subset X$. Throughout the text, m, s are reserved to mean $\dim(X), \dim(Y)$, respectively.

A 'polyhedral complex' or 'polyhedral complex in $\mathbb{R}^m$' means a 'finite polyhedral cell complex embedded in $\mathbb{R}^m$', that is, a collection of polyhedral cells in $\mathbb{R}^m$ such that the intersection of two is a face of either. When we write e.g. K, we mean $K \subset \mathbb{R}^m$, considered as a subspace. We will thus consider two such complexes to be isomorphic if there is some affine coordinate change that takes one to the other. In particular, the 'standard m-simplex' is the convex hull in $\mathbb{R}^m$ of the coordinate vectors $\{e_i\}_{1 \le i \le m}$ and the origin. Note that we do not in general require anything to be convex.

For a subset $S \subset \mathbb{R}^m$ or in $\mathbb{C}^m$, we let $I(S)$ denote the ideal in $\mathbb{C}[x_1, \ldots, x_m]$ associated to its Zariski closure. We also let H_S denote its affine hull, which in the case that $S \subset \mathbb{C}^m$ is defined by linear equations is the same as the Zariski closure of S, that is, here $I(H_S) = I(S)$.

2 The characteristic distribution

Definition 2.1. Given a measurable subset $\sigma \subset \mathbb{R}^m$, the *characteristic distribution* on σ, denoted by δ_σ, is given by

$$f \mapsto \int_\sigma f$$

for any suitable (i.e. compactly supported or swiftly decreasing) test function f, and where $\int_\sigma$ denotes the $(\dim \sigma)$-dimensional integral with respect to the standard measure. It has a left action by the Weyl algebra W, given by

$$p(x)\partial^\alpha \cdot \int_\sigma f = \int_\sigma (-1)^{|\alpha|} \partial^\alpha (p(x) f)$$

where $p(x)$ is some polynomial, and $\alpha \in \mathbb{Z}^m$ is a multiindex, with $|\alpha| = \sum_i \alpha_i$ as usual.

An interesting case is when $\sigma \subset \mathbb{R}^m$ is a polyhedral body, with facets σ_i and outward unit normal vectors n_i (where by *outward* we mean pointing outward when considering σ as lying in its affine hull H_σ). We will more generally consider a finite polyhedral cell complex $K = \bigcup_{\sigma \subset K} \sigma$, where the σ are the cells of K; and the intersection of two cells is a face of either. Throughout the document, whenever we write "$\sigma \subset K$", we are implicitly saying that σ is a cell in K. For technical reasons we will require that K is closed in $\mathbb{R}^m$, but we do not in general require it to be convex or compact.

Here, and onward, we adopt the notational convention that when σ is some face or cell (or union of such) of K, δ_σ is denoted by $\delta_{\text{generators for } I(\sigma)}$, e.g. in the standard 3-simplex, for the facet that lies in the $y-z$-plane we write δ_x, for the diagonal facet $\delta_{x+y+z-1}$, for

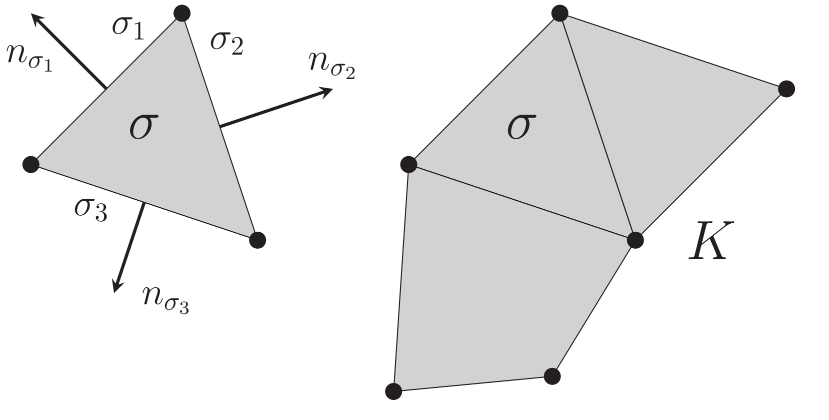

Figure 2.1: A cell σ with facets and normal vectors; sitting in the cell complex K.

the face on the z-axis $\delta_{x,y}$ etc. We will also be sloppy and refer to the distribution of a face or cell of K as a 'face/cell of δ_K' or even a 'face/cell of K' when no confusion is likely to arise.

For any cell $\sigma \subset K$, one has the following relations between δ_σ and its facets δ_{σ_j}, which we call the *standard relations.*

Proposition 2.2 (Standard relations). *(i) If* $\dim(\sigma) > 0$*, then for any directional derivative* ∂_v *where* v *is a vector tangent to* H_σ *(the affine flat spanned by* σ*), we have*

$$\partial_v \cdot \delta_\sigma = -\sum_i \langle v | n_i \rangle \delta_{\sigma_i},$$

and

(ii) for any $p \in I(\sigma)$*, we have*

$$p \cdot \delta_\sigma = 0.$$

As σ *is a polyhedral body,* H_σ *is an affine space defined by* $m - dim(\sigma)$ *equations of degree 1, and the corresponding polynomials generate* $I(\sigma)$*.*

Proof. (i) is Stokes' theorem, applied to σ.

(ii) is obvious, and it follows immediately by definition. To find the action of x_i on δ_σ, simply evaluate it in $k[X]/I(\sigma)$, e.g. if $I(\sigma) = \langle x_i - p(x) \rangle$ (where $p(x)$ will be some linear polynomial in the other variables), we have $x_i \cdot \delta_\sigma = p(x)\delta_\sigma$. □

Lemma 2.3. *For any cell* σ*,* m *of the standard relations suffice to generate all of them.*

Proof. The relations of the type 2.2(i) are generated by a spanning set of the tangent space of H_σ, which is $\dim(\sigma)$-dimensional. The relations of type 2.2(ii) are generated by a generating set for $I(H_\sigma)$, and as H_σ is defined by $m - \dim(\sigma)$ equations, it suffices to have $dim(\sigma) + m - \dim(\sigma) = m$ standard relations to generate all of them. □

Example 2.4. Let K be a single point $p = (p_1, \ldots, p_m)$. Then $\delta_K = \delta_p$, the Dirac distribution on that point. It is annihilated by any form $(x_i - p_i)$. Indeed, $I(p) = \langle x_1 - p_1, \ldots, x_m - p_m \rangle$. In this case, there are no standard relations given by differentials, as H_p is zero-dimensional and has no tangents.

Example 2.5. Let $K = [0,1]^n \subset \mathbb{R}^n$. Then $\partial_i \delta_K = \delta_{x_i} - \delta_{x_i - 1}$, so δ_K is annihilated by any element on the form $x_i(x_i - 1)\partial_i$.

Example 2.6. Let $K = \Delta_2 \subset \mathbb{R}^2$ be the standard 2-simplex. In this case, $\partial_i \delta_K = \delta_{x_i} - \frac{1}{\sqrt{2}} \delta_{x_1 + x_2 - 1}$, and one sample annihilator of δ_{Δ_2} is $x_i(x_1 + x_2 - 1)\partial_i$.

Remark 2.7 (Topology). Above, we made the requirement that K be closed. This is because we wish to connect the relations between the δ_σ to the geometry of K, and in order for this to work, we must require that K be closed. An example is that $\delta_{[0,1]}, \delta_{[0,1)}$ and $\delta_{(0,1)}$ are all the same distribution (on $\mathbb{R}$), even though $[0,1], [0,1)$ and $(0,1)$ are very different topologically.

We can note from the standard relations 2.2 that the ∂_i's act on δ_σ like a topological boundary map, in the sense that the result of applying ∂_v to δ_K is, informally, 'what K looks like when viewed in the direction v', or 'the boundary in the v direction'. We will see later, in 4.6, that the ∂_i's in a very precise sense act like a simplicial boundary map.

3 The module M_K

We can consider all of this taking place in the W-module

$$M_K := W \cdot \{\delta_\sigma | \sigma \subset K\},$$

the module generated by the characteristic distributions of all the cells of K. In the sequel, this is the module we will work with.

Let us note right away that M_K does *not* uniquely determine K. In general, the annihilator ideal in W of a distribution μ annihilates many other distributions. This is of no greater concern for us, however, as we are mainly concerned with what features of K are applicable to M_K.

Proposition 3.1. *M_K is a quotient of the free module generated by the cells of K, by the ideal generated by the standard relations given in Proposition 2.2. Letting c be the number of cells in K, there is by Lemma 2.3 a total of $m \cdot c$ relations generating this ideal, and hence there is a canonical presentation*

$$W^{m \cdot c} \to W^c \twoheadrightarrow M_K,$$

where the last map is given by $\sum_{\sigma \subset K} p_\sigma \cdot (\sigma) \mapsto \sum_{\sigma \subset K} p_\sigma \delta_\sigma$.

We postpone the proof of this until section 3.2.

Example 3.2 (The 1-simplex)**.** Consider the standard 1-simplex in $\mathbb{R}$, i.e. the closed unit interval $I = [0, 1]$. M_I is generated by three simplices, δ_I, δ_0 and δ_1, with the relations $\partial_x \delta_I = \delta_0 - \delta_1$, $x\delta_0 = 0$ and $(x-1)\delta_1 = 0$. The canonical presentation is therefore

$$W^3 \xrightarrow{\cdot \begin{pmatrix} \partial_x & -1 & 1 \\ 0 & x & 0 \\ 0 & 0 & (x-1) \end{pmatrix}} W^3 \twoheadrightarrow M_I.$$

We note that the multiplication is from the right, so as to make the quotient a left W-module.

Given that K is a polyhedral cell complex, we would like to describe the module M_K in terms of the modules of the cells of K and so preserve some of its cell complex structure. Some such results come easily:

Proposition 3.3. *If $K \subset L$ is a subcomplex, closed in L, then $M_K \subset M_L$ is a submodule.*

Proof. Considering that the cells of K are also cells of L, and that because K is closed in L the relations between the generators δ_σ of M_K are also relations between generators in L, it is clear that we have a diagram with exact rows:

$$\begin{tikzcd}
W^{r_K} \arrow[r] \arrow[d, hook] & W^{s_K} \arrow[r, two heads] \arrow[d, hook] & M_K \arrow[d, dotted] \\
W^{r_L} \arrow[r] & W^{s_L} \arrow[r, two heads] & M_L
\end{tikzcd}$$

where r_K, s_K are the number of cells and standard relations (resp.) of M_K, and *vice versa* for M_L; it is clear that the map $M_K \to M_L$ is an injection. $\square$

Proposition 3.4. *If K' is a subdivision of K, there exists a canonical injective map $M_K \hookrightarrow M_{K'}$.*

Proof. Characteristic distributions have the property that if $K, L \subset \mathbb{R}^m$ are of equal dimension and $K \cap L$ has dimension strictly less than $dim(K)$, then $\delta_{K\cup L} = \delta_K + \delta_L$. Using this, we can just take the map that sends a simplex δ_L to the sum of the simplices that subdivide it. $\square$

Example 3.5. To subdivide $[0, 2]$ into $[0, 1] \cup_1 [1, 2]$, we send $\delta_{[0,2]}$ to $\delta_{[0,1]} + \delta_{[1,2]}$ and the relation $\langle \partial_x \delta_{[0,2]} - \delta_0 + \delta_2 = 0\rangle$ to $\langle \partial_x \delta_{[0,1]} - \delta_0 + \delta_1 = 0\rangle + \langle \partial_x \delta_{[1,2]} - \delta_1 + \delta_2 = 0\rangle$. The points δ_0, δ_2 and their relations $\langle x\delta_0 = 0\rangle$,$\langle (x - 2)\delta_2 = 0\rangle$ (in $M_{[0,2]}$) are sent to δ_0, δ_2 and $\langle x\delta_0 = 0\rangle$,$\langle (x - 2)\delta_2 = 0\rangle$ (in $M_{[0,1]\cup_1[1,2]}$).

Proposition 3.6. *If K and L are glued along a subcomplex F, we have $M_{K\cup_F L} \simeq M_K \oplus_{M_F} M_L$.*

Proof. By considering the generators we see that M_F is a submodule of both M_K and M_L, so we can take the pushout of the two inclusions and be done. The crucial part is observing that both the generators and their relations are matched correctly, which is obvious if one writes out the presentations $W^r \to W^s \twoheadrightarrow M_K$ for all the involved modules, along with their inclusions into eachother. $\square$

Example 3.7. Consider the gluing $[0, 1] \cup_1 [1, 2]$. We know from Example 2.5 that in general $M_{[a,b]}$ has generators $\delta_{[a,b]}, \delta_a, \delta_b$, and relations $\partial_x \delta_{[a,b]} = \delta_a - \delta_b$, $(x-a)\delta_a = 0$ and $(x - b)\delta_b = 0$. The module $M_{[0,1]\cup_1[1,2]}$ has generators $\delta_{[0,1]}, \delta_{[1,2]}, \delta_0, \delta_1, \delta_2$ and the same relations, as appropriate. Gluing $M_{[0,1]}$ and $M_{[1,2]}$ along δ_1 simply amounts to saying that the δ_1's appearing in each module is in fact the same.

3.1 The support of M_K

Given that M_K is a module over the complex Weyl algebra, it is worth taking a moment to think about what the support of M_K in $\mathbb{C}^m$ looks like, and to take advantage of any structure it has.

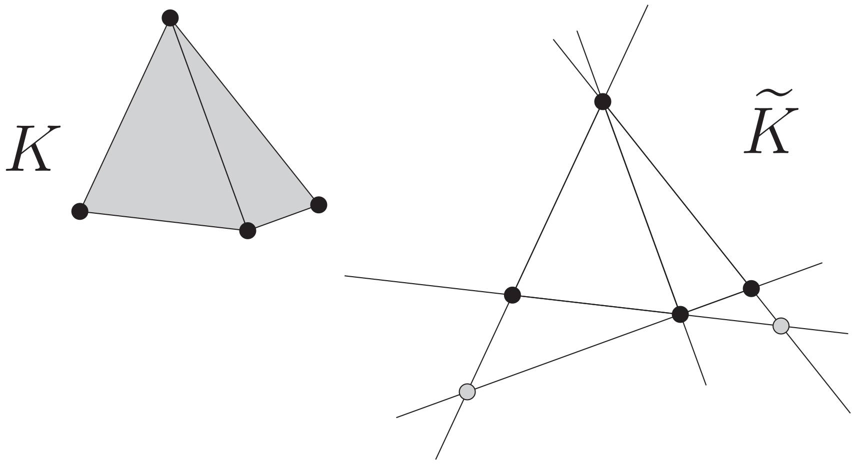

Figure 3.1: The cell complex K and the associated affine space arrangement $\widetilde{K}$.

Definition 3.8. Recall from 1.1 that for any cell $\sigma \subset \mathbb{R}^m$, we let H_σ denote its affine hull $Z(I(\sigma))$ in $\mathbb{C}^m$. This is a complex variety of degree one and dimension $\dim(\sigma)$.

Now, define

$$\widetilde{K} = \bigcup_{\sigma \subset K} H_\sigma$$

with the union taken over the cells of K. This is an arrangement of affine spaces in $\mathbb{C}^m$. Note that if K is pure dimensional of dimension m, this is in fact a hyperplane arrangement.

We can stratify this by dimension of σ:

Definition 3.9 (The skeleton stratification)**.** Consider the arrangement $\widetilde{K}$ in $\mathbb{C}^m$. The *skeleton stratification*, denoted $\mathcal{S}$, is defined by

$$\widetilde{K}_{\leq i} = \bigcup_{dim(\sigma) \leq i} H_\sigma,$$

and the strata are

$$\mathcal{S}_i = \widetilde{K}_{\leq i} \setminus \widetilde{K}_{\leq i-1}$$

One notable feature of the arrangement $\widetilde{K}$ is that not all the components of the strata correspond to a cell of K. That is, we may have cells σ, τ with $\sigma \cap \tau = \emptyset$, but $H_\sigma \cap H_\tau \neq \emptyset$. These 'excess' intersections do not cause any problems, as we will see.

Definition 3.10. Let $\mathcal{M}_K$ be the sheaf associated to M_K (considered as a $\mathbb{C}[X]$-module) on $\mathbb{C}^m$. This has a natural structure as $\mathcal{D}_X$-module.

The sections of $\mathcal{M}_K$ are determined by the arrangement $\widetilde{K}$:

Proposition 3.11. *Over an open set $U \subset \mathbb{C}^m$, the sections $\mathcal{M}_K(U)$ are generated by those δ_σ such that $H_\sigma \cap U \neq \emptyset$.*

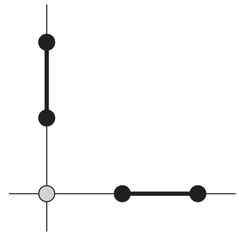

Figure 3.2: Thick lines/points are K, thin lines are $\widetilde{K}$, extra intersection point shown in gray.

Proof. Localizing M_K at some point $p \in \mathbb{C}^m$ has the effect of killing those δ_σ that are not supported at p, explicitly if $p \notin H_\sigma$, and $q(x) \cdot \delta_\sigma = 0$, then in the localization, $\delta_\sigma = \frac{q(x)}{q(x)}\delta_\sigma = \frac{1}{q(x)}q(x) \cdot \delta_\sigma = 0$. Also, the relations are modified accordingly.

By definition (see e.g. [Har77, II.5]), the sections over an open U are those functions $s : U \to \coprod_{p \in U}(M_K)_p$ such that $s(p) \in (M_K)_p$ and that are locally a fraction m/f with $m \in M_K$ and $f \in \mathbb{C}[X]$. Here, $m = \sum_\sigma p_\sigma(x, \partial)\delta_\sigma$, and those δ_σ that appear can be only those that are not killed by localization at some point of U. These are exactly those such that $H_\sigma \cap U \neq \emptyset$. □

We now see that the extra intersections we get when passing from K to $\widetilde{K}$ cause no real trouble, as there are no generators that correspond to any extra intersection.

Example 3.12. Let $K \subset \mathbb{R}^2$ be the union of the line segments $\sigma_1 = [(1,0),(2,0)]$ and $\sigma_2 = [(0,1),(0,2)]$. Then $\widetilde{K}$ is the zero-set of x_1x_2, and the origin appears as an extra point in the arrangement (see 3.2).

By the standard relations and 3.1, M_K is generated by δ_{σ_1}, δ_{σ_1}, and $\delta_{(p,q)}$ for the points $(1,0),(2,0),(0,1)$ and $(0,2)$. The standard relations give that $\partial_{x_1}\delta_{\sigma_1} = \delta_{(1,0)} - \delta_{(2,0)}$ and $x_2\delta_{\sigma_1} = 0$ (and similarly for δ_{σ_2}), and $(x_1 - p)\delta_{(p,q)} = (x_2 - q)\delta_{(p,q)} = 0$ for the given points.

Localizing this at the origin, we see that the vertices $(1,0),(2,0),(0,1)$ and $(0,2)$ disappear, indeed $\delta_{(p,q)} = \frac{x_1-p}{x_1-p}\delta_{(p,q)} = \frac{1}{x_1-p}(x_1-p)\delta_{(p,q)} = 0$ etc., as $x_1 - p$ is invertible in the localisation. What remains is the generators $\delta_{\sigma_1}, \delta_{\sigma_2}$, with the relations $\partial_{x_i}\delta_{\sigma_i} = 0, x_j\delta_{\sigma_i} = 0$ for $i \neq j$, so the module does not see the extra intersection point. There are no relations between δ_{σ_1} and δ_{σ_2} in the localisation, so the localised module is a direct sum.

As $\mathbb{C}^m$ is affine, the categories of W-modules and of $\mathcal{D}_{\mathbb{C}^m}$-modules are equivalent; all the results for M_K will also hold for $\mathcal{M}_K$. We will not have any further use for $\mathcal{M}_K$ and will work only with M_K for the remainder.

3.2 The skeleton filtration

Associated to the skeleton stratification is a natural filtration on M_K:

Definition 3.13 (Skeleton filtration)**.** Let F^iM_K be the submodule of M_K generated by those δ_σ with $\dim(\sigma) \leq i$. These submodules form a filtration

$$F^0M_K \subset F^1M_K \subset \cdots \subset F^{m-1}M_K \subset F^mM_K = M_K$$

which we call the *skeleton filtration*.

Thus, F^0M_K is generated by the vertices of M_K, F^1M_K by the vertices and 1-cells, etc.

Proposition 3.14. *Let $i : H_\sigma \hookrightarrow X$ be the inclusion map. The filtration quotients $Q_k := F^kM_K/F^{k-1}M_K$ are semisimple, with summands isomorphic to the direct image under the inclusion $i^0_+\mathbb{C}[H_\sigma]$, one for each k-cell $\sigma \subset K$.*

Remark 3.15. See Section 5 for definitions of the direct image functors i^0_+, i_+.

Proof. It is clear that Q_k is generated by the (classes of the) k-cells, namely $Q_k = \sum W \cdot \overline{\delta_\sigma}$. We must show two things: that $W \cdot \overline{\delta_K}$ is of the given form, and that the sum is direct.

We may assume by choosing coordinates appropriately that H_σ is the affine flat $x_{k+1} - p_{k+1} = \cdots = x_m - p_m = 0$. From the standard relations given in Proposition 2.2, it follows that

$$\begin{aligned} \partial_j\overline{\delta_\sigma} &= 0, \quad j \leq k \\ (x_j - p_j)\overline{\delta_\sigma} &= 0, \quad j > k. \end{aligned}$$

Indeed, we have $\partial_v \cdot \delta_\sigma = \sum_j \langle v|n_j\rangle \delta_{\sigma_j}$ for v parallel to H_σ, and in the quotient the right-hand side disappears, so we are left with $\partial_v\overline{\delta_\sigma} = 0$.

The existence of these relations implies that there is a surjective map

$$i^0_+\mathbb{C}[H_\sigma] \to W \cdot \overline{\delta_\sigma}$$

and as the first module is simple by Kashiwara's Theorem ([Mal93, IV]), this is an isomorphism unless $W \cdot \overline{\delta_\sigma}$ is the zero module. It is not, as $\overline{\delta_\sigma} = 0$ would imply that δ_σ is some linear combination of distributions with support on lower-dimensional cells, which cannot be true as their supports have different dimension.

For directness of the sum, the only obstruction is that there might be relations between cells of the same dimension. For some cells σ, τ_j of the same dimension, we could have $p(x,\partial)\overline{\delta_\sigma} = \sum p_j(x,\partial)\overline{\delta_{\tau_j}}$, with p nonzero and not all p_j zero. This is impossible, however, because the $\overline{\delta_{\sigma_j}}$ all have disjoint supports, and acting by an operator $p(x,\partial)$ can never expand the support of a distribution, only restrict it ([Hö90, 3.1.1]). Thus, for some test function f with support strictly contained in the interior of σ, the right-hand side is zero and the left-hand side is non-zero, so the equality cannot hold. □

We have actually proved a little more:

Proposition 3.16. *The modules $W \cdot \delta_{H_\sigma}$ and $W \cdot \overline{\delta_\sigma}$ are isomorphic, and also simple.*

Proof. It follows from the proof of 3.14 that the module $W \cdot \overline{\delta_\sigma}$ is simple, being isomorphic to the simple module $i_+^0 \mathbb{C}[H_\sigma]$. The generator $\overline{\delta_\sigma}$ obeys the same relations as δ_{H_σ}, as the facets δ_{σ_i} of δ_σ are killed in the quotient. We now have $W \cdot \overline{\delta_\sigma} \simeq W \cdot \delta_{H_\sigma}$ and the claim follows. □

Remark 3.17. Proposition 3.14 gives us a very useful tool for calculation: the quotient summands $W \cdot \overline{\delta_\sigma}$ are each, for a suitable choice of coordinates x_i (depending on σ), isomorphic to the module $\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m] = W / \sum_{i \le k} W \cdot \partial_i + \sum_{k < i \le m} W \cdot x_i \simeq i_+^0 \mathbb{C}[H_\sigma]$ (see [B$^+$87, V.3]). Here x_i, ∂_j have the natural actions induced in the quotient.

We are now equipped to give a proof of Proposition 3.1.

Proof of Proposition 3.1. We want to show the exactness of $W^r \to W^c \to M_K$, where c is the number of cells in K, $r = m \cdot c$, the map $W^r \to W^c$ is given by the relations from Proposition 2.2, and the map $W^c \to M_K$ makes the appropriate identifications.

Let us first define the maps properly. We label the generators of W^c by the cells of K — so that W^c is freely generated by generators g_σ — for all the $\sigma \subset K$, and let the map $W^c \to M_K$ be given by $g_\sigma \mapsto \delta_\sigma$.

We give a corresponding labelling on the generators of W^r, by the standard relations from Proposition 2.2. By Lemma 2.3, there is for each $\sigma \subset K$ a total of m generators for the standard relations. We can write each of these as some W-linear combination $P^\sigma(\delta_\sigma, \ldots, \delta_{\sigma_k}) = 0$, e.g. $\partial_v \cdot \delta_\sigma + \sum_i \langle v | n_i \rangle \delta_{\sigma_i} = 0$ for some v tangent to H_σ, or $(\sum_i a_i x_i - d)\delta_\sigma = 0$ for some generator $(\sum_i a_i x_i - d)$ of $I(H_\sigma)$. We now let W^r be freely generated by generators r_{P^σ}, one for each generating standard relation, and define the map $W^r \to W^c$ by $r_{P^\sigma} \mapsto P^\sigma(g_\sigma, \ldots, g_{\sigma_k})$.

The skeleton filtration on M_K induces filtrations on W^c and W^r, in both cases by dimension of σ: $F'^i W^c$ and $F''^i W^r$ are generated, respectively, by those g_σ and r_{P^σ} with $\dim(\sigma) \le i$. We can immediately observe that both maps respect the filtration.

The map $W^c \to M_K$ is clearly surjective, what remains is exactness in the middle. We can check this by passing to the associated graded modules of the skeleton filtration and checking exactness of the sequence

$$gr(W^r) \to gr(W^c) \twoheadrightarrow gr(M_K).$$

This reduces to checking the direct summands of the filtration quotients.

For a quotient summand $W \cdot \overline{\delta_\sigma} \simeq \mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m]$, the sequence is

$$W^m \xrightarrow{\cdot \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_k \\ x_{k+1} \\ \vdots \\ x_m \end{pmatrix}} W \twoheadrightarrow W \cdot \overline{\delta_\sigma}$$

The cokernel of the first map is $W / (\sum_{i \le k} W \cdot \partial_i + \sum_{i > k} W \cdot x_i)$, and this is clearly isomorphic to $\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m] \simeq W \cdot \overline{\delta_\sigma}$. □

Proposition 3.14 also gives us the following, almost for free:

Proposition 3.18. *M_K is holonomic.*

Proof. It suffices to check the filtration quotients in the skeleton filtration, which reduces to checking the direct summands. The summands for M_K are by 3.14 isomorphic to $i^0_+\mathbb{C}[H_\sigma]$. $\mathbb{C}[H_\sigma]$ is holonomic, and since the functor i_+ preserves holonomicity, (see [B$^+$87, VII.12]), we are done. $\square$

4 De Rham cohomology of M_K

We can also use the skeleton filtration to calculate the de Rham cohomology of M_K. We recall first the definitions, and here we follow the conventions of [Mal93]:

Definition 4.1. The *de Rham complex* $DR_X(M)$ (or $DR(M)$ for short) of a left W-module M is the complex $\Omega^\bullet \otimes_{\mathbb{C}[X]} M[m]$, that is,

$$0 \to M \to \Omega^1 \otimes_{\mathbb{C}[X]} M \to \cdots \to \Omega^m \otimes_{\mathbb{C}[X]} M,$$

graded such that $\Omega^i \otimes M$ has degree $i - m$, so e.g. M has degree $-m$ and $\Omega^m \otimes M$ has degree 0. The differential is given by the exterior derivative,

$$d(\omega \otimes m) = d\omega \otimes m + \sum_i dx_i \wedge \omega \otimes \partial_i m$$

or equivalently

$$d(dx_I \otimes m) = \sum_j dx_j \wedge dx_I \otimes \partial_j m$$

which suggests as a shorthand $d = \sum_i dx_i \otimes \partial_i$.

The *de Rham cohomology* $H^*_{dR}(M, \mathbb{C})$ of M is the cohomology of this complex.

Remark 4.2. We remark that this can be stated in terms of the derived category (of vector spaces or $\mathbb{C}_X$-modules, respectively): we can define $DR(M) = \Omega^m \otimes^L M$, and this will be equivalent to the above definition (see [Mal93, I.2] for details). DR can be viewed as a functor from the bounded derived category of W-modules to the bounded derived category of $\mathbb{C}$-vector spaces, see [Dim04, 5.3] for details.

We wish to use the skeleton filtration to compute the cohomology of M_K, and to do this we start with the filtration quotients Q_k, which reduces to their summands.

Remark 4.3. For ease of terminology, we make a minor deviation from our context of distributions and W-modules into the realm of *currents*, for which the best source is [Dem09]. Currents are (in one equivalent formulation) differential forms with distribution coefficients, and the generators that will appear in our de Rham complexes can be viewed as currents. In computing with the skeleton filtration quotients, we will encounter a generator of the form $\overline{\delta_\sigma} \otimes dx_{k+1} \wedge \ldots \wedge dx_m$ (where we have chosen coordinates such that H_σ is parallel to the flat $x_1 = \cdots = x_k = 0$). Viewed as a current, this is nothing other than the *current of integration over* H_σ (in Demailly's notation $[H_\sigma]$, see [Dem09, (2.4),(2.9)]) because $\overline{\delta_\sigma}$ is isomorphic to δ_{H_σ} (by 3.16). It should be clear (again from 3.16) that the $[H_\sigma]$ that appears in the quotient $DR(Q_k)$ comes from $[\sigma]$ (the current of integration over σ) in $DR(M_K)$.

We can act on currents with differential operators, by acting on the distribution coefficients. That is, if $p(x,\partial) \in W$, δ_i are distributions and ω_i are k-forms, we have

$$p(x,\partial) \cdot \sum_i \delta_i \omega_i = \sum_i (p(x,\partial) \cdot \delta_i)\omega_i.$$

The notation $[\sigma]$ will also be used later to refer to the singular homology class of σ, an overloading of notation we permit ourselves to commit as it should not cause any real confusion.

Lemma 4.4. *The de Rham cohomology of $W \cdot \overline{\delta_\sigma}$ is $\mathbb{C}^1$ in degree $-k$, zero otherwise. In the derived category, $DR(W \cdot \overline{\delta_\sigma})$ is isomorphic to the one-term complex that has $\mathbb{C} \cdot [H_\sigma]$ in degree $-k$.*

Proof. We can by 3.17 replace $W \cdot \overline{\delta_\sigma}$ by $i_*\mathbb{C}[H_\sigma]$. Choosing coordinates and using the decomposition $\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m] \simeq \mathbb{C}[x_1] \otimes \cdots \otimes \mathbb{C}[\partial_m]$, we can express the de Rham complex of $i_*\mathbb{C}[H_\sigma]$ as a tensor product of m complexes, k of the form

$$0 \to \mathbb{C}[x_i] \xrightarrow{\partial_i} \mathbb{C}[x_i]dx_i \to 0$$

for $1 \leq i \leq k$; and $m-k$ of the form

$$0 \to \mathbb{C}[\partial_j] \xrightarrow{\cdot\partial_j} \mathbb{C}[\partial_j]dx_j \to 0$$

for $k < j \leq m$ (see [Wei94, 4.5] and [DCP10, 10.1]). Recall that our degree convention means that these are complexes with first non-zero term in degree -1 and second term in degree 0.

The first k of these m maps are surjective, and the last $m-k$ are injective. It is obvious that respectively the inclusion of the kernel in degree -1 and the projection on the cokernel in degree 0 are quasi-isomorphisms, which means that we have k complexes quasi-isomorphic to the single-term complex with $\mathbb{C}$ in degree -1, and $m-k$ quasi-isomorphic to a complex of the form $\mathbb{C}dx_j$ in degree zero. Their tensor product is the de Rham complex of $\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m]$, which is then quasi-isomorphic to the complex $\mathbb{C}dx_{k+1} \wedge \cdots \wedge dx_m$ concentrated in degree $-k$. Recalling that we are really working with $W \cdot \overline{\delta_\sigma}$, this is $\mathbb{C} \cdot \overline{\delta_\sigma} dx_{k+1} \wedge \cdots \wedge dx_m$, or (by 4.3) $\mathbb{C} \cdot [H_\sigma]$.

We have even more: the inclusion

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C}[x] \arrow[r, "\partial"] & \mathbb{C}[x]dx \end{tikzcd}$$

is a quasi-isomorphism, but so is

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\partial"] \arrow[d, "x \mapsto 0"'] & \mathbb{C}[x]dx \arrow[d, "0"] \\ \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

and the same for $(\mathbb{C}[\partial_j] \xrightarrow{\cdot\partial_j} \mathbb{C}[\partial_j]dx_j) \leftrightarrow (0 \to \mathbb{C}dx_j)$, *mutatis mutandis.* Combining these we get two explicit quasi-isomorphisms

$$\mathbb{C}dx_{k+1} \wedge \cdots \wedge dx_m \leftrightarrow DR(\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_m]),$$

one given by the inclusion, and one by the projection $x_i, \partial_i \mapsto 0, dx_j \mapsto 0$ for $j > k$; moreover one is a section of the other (this fact is not immediately useful, but it is nice to know anyway). $\square$

Corollary 4.5. *$H^*_{dR}(Q_k, \mathbb{C})$ has dimension equal to the number of k-simplices in degree $-k$, and zero otherwise. In the derived category, $DR(Q_k)$ is isomorphic in the derived category to the one-term complex $\Omega^{m-k} \otimes \mathbb{C}^{a_k} = \bigoplus_{\dim(\sigma)=k} \mathbb{C} \cdot [H_\sigma]$, concentrated in degree $k - m$, where a_k is the number of k-cells in K.*

Theorem 4.6. *The de Rham cohomology of M_K (with coefficients in $\mathbb{C}$) is isomorphic to the Borel-Moore homology of K. In the derived category of vector spaces, $DR(M_K)$ is isomorphic to the singular homology chain complex $C^{BM}_\bullet(K, \mathbb{C})$.*

Remark 4.7. As Borel-Moore homology is isomorphic to singular homology for compact spaces, we note that for compact K, we have $DR(M_K) \simeq C^{sing}_\bullet(K, \mathbb{C})$.

Proof. We use the spectral sequence associated to the skeleton filtration. The E_0 page is then the de Rham complexes of the filtration quotients Q_k, $E_0^{p,q} = \Omega^{m+p+q} \otimes Q_{-p}$ (to match our degree convention for the de Rham complex):

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^1 \otimes Q_m \arrow[u] & \Omega^2 \otimes Q_{m-1} \arrow[u] & \Omega^3 \otimes Q_{m-2} \arrow[u] & \cdots \\
Q_m \arrow[u, "d"] & \Omega^1 \otimes Q_{m-1} \arrow[u, "d"] & \Omega^2 \otimes Q_{m-2} \arrow[u, "d"] & \cdots \\
 & Q_{m-1} \arrow[u, "d"] & \Omega^1 \otimes Q_{m-2} \arrow[u, "d"] & \cdots \\
 & & Q_{m-2} \arrow[u, "d"] & \cdots
\end{tikzcd}$$

Here, the term Q_m is in position $(-m, 0)$, the term Q_{m-1} in position $(1 - m, -1)$ etc. The E_1 page is given by

$$E_1^{p,q} = H^{m+p+q}_{dR}(Q_{-p}),$$

and from Proposition 4.5 we have that $E_1^{p,q}$ is $\Omega^{m-k} \otimes \mathbb{C}^{a_k}$ for $p = k, q = 0$ and zero otherwise, where again a_k is the number of k-cells in K. Thus, the E_1 page consists of a single complex

$$0 \to \mathbb{C}^{a_m} \to \Omega^1 \otimes_{\mathbb{C}[X]} \mathbb{C}^{a_{m-1}} \to \cdots \to \Omega^{m-1} \otimes_{\mathbb{C}[X]} \mathbb{C}^{a_1} \to \Omega^m \otimes_{\mathbb{C}[X]} \mathbb{C}^{a_0} \to 0.$$

We must now examine what the differential is. It suffices to check what happens to a single generator $[\sigma] = dx_{I_\sigma} \otimes \delta_\sigma$, assuming suitable coordinates.

$$\begin{aligned}
d(dx_{I_\sigma} \otimes \delta_\sigma) &= \sum_{i=1}^{k} dx_i \wedge dx_{I_\sigma} \otimes \partial_i \delta_\sigma \\
&= -\sum_{i=1}^{k} \sum_{j} \langle e_i | n_j \rangle dx_i \wedge dx_{I_\sigma} \otimes \overline{\delta_{\sigma_j}} \\
&= -\sum_{j} \langle \sum_{i=1}^{k} e_i | n_j \rangle dx_i \wedge dx_{I_\sigma} \otimes \overline{\delta_{\sigma_j}} \\
&= -\sum_{j} d(\sum_{i=1}^{k} e_i | n_j \rangle x_i) \wedge dx_{I_\sigma} \otimes \overline{\delta_{\sigma_j}} \\
&= -\sum_{j} d(n_j) \wedge dx_{I_\sigma} \otimes \overline{\delta_{\sigma_j}}
\end{aligned}$$

We see that the generator corresponding to δ_σ is sent to the sum of the generators corresponding to the boundary cells δ_{σ_j}. Using the language of currents, this could be seen directly from the fact that $d[\sigma] = \pm[\partial\sigma]$ ([Dem09, (2.7)]). Note that each generator $[\sigma]$ has closed support; d thus corresponds to the boundary maps for chains of *closed support*, i.e. the Borel-Moore homology boundary map, and we are done.

The claim about isomorphism in the derived category is straightforward: the inclusion $C_\bullet^{BM}(K) \hookrightarrow DR(M_K)$ given by sending the *homology class* $[\sigma]$ to the *current* $[\sigma]$ is a quasi-isomorphism (because $d[\sigma] = [\partial\sigma]$), and even more: the map $P : DR(M_K) \to C_\bullet^{BM}(K)$ given in degree $-k$ by

$$P(p(x, \partial)\omega) = \begin{cases} p(0)[\sigma] & \text{if } \omega = [\sigma] \text{ for some } \sigma \text{ with } \dim(\sigma) = k, \\ 0 & \text{otherwise} \end{cases}$$

is a chain map:

$$\partial(P(x^\alpha \partial^\beta [\sigma])) = \begin{cases} \partial[\sigma] & \text{if } \alpha = \beta = 0 \\ 0 & \text{if } \alpha, \beta \neq 0, \end{cases}$$

(letting ∂ denote the differential in $C_\bullet^{BM}(K)$) and choosing suitable coordinates as before,

$$\begin{aligned}
&P(d(dx_{I_\sigma} \otimes x^\alpha \partial^\beta \delta_\sigma)) = P(\sum_i dx_i \wedge dx_{I_\sigma} \otimes \partial_i (x^\alpha \partial^\beta \delta_\sigma)) \\
&= P\Big(\sum_{i \le k} dx_i \wedge dx_{I_\sigma} \otimes (x^\alpha \partial^\beta \partial_i \delta_\sigma + \sum_i \alpha_i x^{\alpha - 1_i} \partial^\beta \delta_\sigma) \\
&+ \sum_{i > k} dx_i \wedge dx_{I_\sigma} \otimes (x^\alpha \partial^{\beta + 1_i} \delta_\sigma) \Big) = P(\sum_{i \le k} dx_i \wedge dx_{I_\sigma} \otimes x^\alpha \partial^\beta \partial_i \delta_\sigma) \\
&= \begin{cases} P(d[\sigma]) = \partial[\sigma] & \text{if } \alpha = \beta = 0 \\ 0 & \text{if } \alpha, \beta \neq 0, \end{cases}
\end{aligned}$$

and is a quasi-isomorphism as it is the identity on homology. Moreover, the inclusion is a section of P (again, this is of no particular use to us, but we mention it for completeness). □

Example 4.8 (The standard 2-simplex)**.** Let us run through the whole calculation. We have one 2-simplex, δ_Δ, three 1-simplices δ_x, δ_y and δ_{x+y-1}, and three 0-simplices $\delta_{(0,0)}, \delta_{(1,0)}$ and $\delta_{(0,1)}$.

First, we recall what the simplicial homology complex looks like. Let us give the simplices names corresponding to the generators for M_K, e.g. σ_x is the simplex corresponding to δ_x etc. For the orientation $(0,1) < (0,0) < (1,0)$, the boundary maps are given by $\partial\Delta = \sigma_x - \sigma_{x+y-1} + \sigma_y$, $\partial\sigma_x = \sigma_{(1,0)} - \sigma_{(0,0)}$, $\partial\sigma_y = \sigma_{(0,0)} - \sigma_{(0,1)}$ and $\partial\sigma_{x+y-1} = \sigma_{(1,0)} - \sigma_{(0,1)}$. The complex then becomes

$$\mathbb{C}^1(\Delta) \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \mathbb{C}^3(\sigma_x, \sigma_y, \sigma_{x+y-1}) \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}} \mathbb{C}^3(\sigma_{(0,0)}, \sigma_{(1,0)}, \sigma_{(0,1)})$$

As in 3.17, we can write each skeleton quotient summand of M_K as follows, for suitable coordinates u, v: the 2-simplex as $\overline{\mathbb{C}[u,v]\delta_\Delta}$, the 1-simplices as $dv \otimes \overline{\mathbb{C}[u,\partial_v]\delta_{v=v_i}}$, and the 0-simplices as $du \wedge dv \otimes \overline{\mathbb{C}[\partial_u,\partial_v]\delta_{p_j}}$, with appropriate actions. Taking cohomology and keeping track of the dx's and dy's, we get the complex

$$\begin{aligned} &\mathbb{C} \cdot \overline{\delta_\Delta} \xrightarrow{d} \mathbb{C} \cdot dx \otimes \overline{\delta_x} \oplus \mathbb{C} \cdot dy \otimes \overline{\delta_y} \oplus \mathbb{C} \cdot \tfrac{dx+dy}{\sqrt{2}} \otimes \overline{\delta_{x+y-1}} \\ &\qquad \xrightarrow{d} \mathbb{C} \cdot dx \wedge dy \otimes \overline{\delta_{(0,0)}} \oplus \mathbb{C} \cdot dx \wedge dy \otimes \overline{\delta_{(1,0)}} \oplus \mathbb{C} \cdot dx \wedge dy \otimes \overline{\delta_{(0,1)}} \end{aligned}$$

Now it is simple to see what the differentials are, just apply $d = \sum_i dx_i \otimes \partial_i$ to the generators and keep track of where to go regarding dx and dy. We'll first take $\overline{\delta_\Delta}$:

$$\begin{aligned} d(\overline{\delta_\Delta}) &= (dx \otimes \partial_x + dy \otimes \partial_y)\overline{\delta_\Delta} \\ &= dx \otimes \partial_x \overline{\delta_\Delta} + dy \otimes \partial_y \overline{\delta_\Delta} \\ &= dx \otimes (\overline{\delta_x} - \frac{1}{\sqrt{2}}\overline{\delta_{x+y-1}}) + dy \otimes (\overline{\delta_y} - \tfrac{1}{\sqrt{2}}\overline{\delta_{x+y-1}}) \\ &= dx \otimes \overline{\delta_x} + dy \otimes \overline{\delta_y} - (dx+dy) \otimes \tfrac{1}{\sqrt{2}}\overline{\delta_{x+y-1}} \\ &= dx \otimes \overline{\delta_x} + dy \otimes \overline{\delta_y} - d(\tfrac{x+y}{\sqrt{2}}) \otimes \overline{\delta_{x+y-1}} \end{aligned}$$

This corresponds exactly to the topological boundary map, as we can see that $\overline{\delta_\Delta}$ is sent to the sum of its boundary simplices $d(n_\sigma) \otimes \overline{\delta_\sigma}$.

Further, we have

$$\begin{aligned} d(dx \otimes \overline{\delta_x}) &= (dx \otimes \partial_x + dy \otimes \partial_y) dx \otimes \overline{\delta_x} \\ &= dx \wedge dx \otimes \partial_x \overline{\delta_x} + dy \wedge dx \otimes \partial_y \overline{\delta_x} \\ &= dy \wedge dx \otimes (\overline{\delta_{(0,0)}} - \overline{\delta_{(0,1)}}) \\ &= dx \wedge dy \otimes (\overline{\delta_{(0,1)}} - \overline{\delta_{(0,0)}}) \end{aligned}$$

and similarly

$$d(dy \otimes \overline{\delta_y}) = dx \wedge dy \otimes (\overline{\delta_{(0,0)}} - \overline{\delta_{(1,0)}}).$$

To get $d(d(\frac{x+y}{\sqrt{2}})) \otimes \overline{\delta_{x+y-1}})$ we apply the coordinate change $u = \frac{x-y}{\sqrt{2}}, v = \frac{x+y}{\sqrt{2}}$, so $d(\frac{x+y}{\sqrt{2}}) \otimes \overline{\delta_{x+y-1}} = dv \otimes \overline{\delta_{v-\frac{1}{\sqrt{2}}}}$, and $d = du \otimes \partial_u + dv \otimes \partial_v$. We now get

$$d(dv \otimes \overline{\delta_{v-\frac{1}{\sqrt{2}}}}) = du \wedge du \otimes \partial_u \overline{\delta_{v-\frac{1}{\sqrt{2}}}}$$

which is, in the original coordinates,

$$d(\tfrac{x-y}{\sqrt{2}}) \wedge d(\tfrac{x+y}{\sqrt{2}}) \otimes (\partial_x - \partial_y)\overline{\delta_{x+y-1}} = dx \wedge dy \otimes (\overline{\delta_{(0,1)}} - \overline{\delta_{(1,0)}}).$$

Suppressing now the generators, we assemble this into the complex

$$\mathbb{C}^1 \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \mathbb{C}^3 \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}} \mathbb{C}^3$$

which we recognize as the singular homology complex of Δ_2 above.

5 Direct images under linear maps

For a polynomial map $\pi : \mathbb{C}^m \to \mathbb{C}^s$ (which we denote as $\pi : X \to Y$), thereis a direct image functor $\pi_+ : D^b(D_X) \to D^b(D_Y)$ from the (bounded) derived category of D_X-modules to the (bounded) derived category of D_Y-modules. It is defined by

$$\pi_+ M := D_{Y \leftarrow X} \otimes^L_{D_X} M$$

for a left D_X-module M, where $D_{Y \leftarrow X}$ is a so-called 'transfer module' defined in terms of D_X and D_Y, which is a left D_Y-module and a right D_X-module. The definitions are somewhat involved, so we leave the details to [B$^+$87] (chapter V.3 in particular), and restrict ourselves to quoting the parts we are interested in.

In particular, we will focus on the zeroth-level part π^0_+, which is a functor from D_X-modules to D_Y-modules; π_+ is the left derived functor of π^0_+. We can define π^0_+ by $\pi^0_+ M = D_{Y \leftarrow X} \otimes_{D_X} M$, that is, the ordinary tensor product of modules rather than the derived tensor product of complexes.

Because our K is defined by linear equations, and we wish to preserve this feature, we must restrict our treatment to *linear* maps $\pi : X \to Y$. Any linear map can be decomposed as an inclusion followed by a projection, so we need only consider these cases. Furthermore, as K is intrinsically a *real* object, we will tacitly assume whenever necessary that π is the complexification of a map $\pi : \mathbb{R}^m \to \mathbb{R}^s$. For technical reasons, we will also always assume that the fibers $\pi^{-1}(x) \cap K$ of this map are compact.

Proposition 5.1. *Let $i : X \hookrightarrow Y$ be an inclusion. Then, $i^0_+ M_K \simeq M_{i(K)}$, where $i(K)$ is simply K considered as lying in Y.*

Proof. This is really a consequence of *Kashiwara's Theorem* (see [Mal93, IV]), which states that i_+ is an equivalence of categories between the categories of D_X-modules, and D_Y-modules with support in $i(X)$.

Indeed, we may assume i is the inclusion on the first m coordinates, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$, that is, we embed X as the subspace $x_{m+1} = \cdots = x_s = 0$. In this case, the transfer module $D_{Y \leftarrow X}$ is isomorphic to $D_Y / \sum_{i=m+1}^s D_Y \cdot x_i$, with the natural left D_Y-module structure; and the right D_X-module structure given by letting x_i and ∂_{x_i} act by multiplication with x_i and $-\partial_{x_i}$, respectively (see [B$^+$87, V.3.3.3]). This gives us that $i^0_+ M_K$ is $D_Y / \sum_{i=m+1}^s D_Y \cdot x_i \otimes_{D_X} M_K$. This amounts to adding variables x_j, ∂_j for $j > m$, and because the variables x_i, x_j commute pairwise, the relations that for $j > m$, x_j kills the generators of $i^0_+ M_K$. These relations are exactly the additional information we need to describe $i(K)$ in Y, indeed $I(H_{i(K)}) = I(H_K) + (x_{m+1}, \dots, x_s)$ (considering here $\mathbb{C}[X]$ as a subring of $\mathbb{C}[Y]$). The other relations are unchanged. $\square$

Next up is the case of isomorphisms. This next result is in some sense trivial, as it is obvious that $\phi^0_+ M_K$ is isomorphic to $M_{\phi(K)}$ when ϕ is a linear isomorphism, but for the purpose of enabling explicit calculations in the case of projections, we work out what exactly happens to the generators. Later we will compute $\pi^0_+ M_K$ when π is a projection on the first coordinates of the ambient space, and precomposing with an isomorphism gives us any general projection.

Proposition 5.2. *Let $\phi : X \to Y$ be a linear isomorphism. $\phi^0_+ M_K$ is isomorphic to the module generated by the distributions $|\det(\phi|_{H_\sigma})|^{-1}\delta_{\phi(\sigma)}$. Here, we mean by $\phi|_{H_\sigma} : H_\sigma \to \phi(H_\sigma)$ the restriction to the subspaces $H_\sigma, \phi(H_\sigma)$ with the measures induced from X and Y, respectively.*

Proof. By [B+87, V.3.3.5], for a linear isomorphism ϕ, the transfer module $D_{Y\leftarrow X}$ is isomorphic to D_X with the usual right D_X-structure and the left D_Y-structure given by letting y_i act by multiplication with $\phi_i(x)$, and letting ∂_{y_i} act by multiplication with $\sum_j \frac{\partial x_j}{\partial y_i}\partial_{x_j}$.

As this is a coordinate change, it is clear that the generators $\widehat{\delta_\sigma}$ are of the form $c_\sigma \delta_{\phi(\sigma)}$ for some scaling factor $c_\sigma \in \mathbb{R}$, depending on σ. Now, by the usual change of variables formula, $\delta_{\phi(\sigma)}(f) = |\det \phi|_{H_\sigma}| \cdot \delta_\sigma(f \circ \phi)$ for some test function f supported on $\phi(\sigma)$. As $\widehat{\delta_\sigma}$ must satisfy the same relations as δ_σ, expressed in the new coordinates, we must have that

$$\widehat{\delta_\sigma}(f) = c_\sigma \delta_{\phi(\sigma)}(f) = c_\sigma |\det \phi|_{H_\sigma}| \delta_\sigma(f \circ \phi),$$

which implies $c_\sigma = |\det \phi|_{H_\sigma}|^{-1}$. □

Example 5.3. Consider the standard 2-simplex $\Delta_2 \subset \mathbb{R}^2$ and the linear isomorphism ϕ given by the matrix $\phi = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. As we already know, δ_{Δ_2} satisfies the relations $\partial_{x_1}\delta_{\Delta_2} = \delta_{x_2} - \frac{1}{\sqrt{2}}\delta_{x_1+x_2-1}$ and $\partial_{x_2}\delta_{\Delta_2} = \delta_{x_1} - \frac{1}{\sqrt{2}}\delta_{x_1+x_2-1}$. The image $\phi(\Delta_2)$ has relations $\partial_{y_1}\delta_{\phi(\Delta_2)} = \delta_{y_2} - \frac{1}{\sqrt{5}}\delta_{y_1+2y_2-1}$ and $\partial_{y_2}\delta_{\Delta_2} = \delta_{y_1} - \frac{2}{\sqrt{5}}\delta_{y_1+2y_2-1}$.

In the direct image, we have $\partial_{y_1}\widehat{\delta_{\Delta_2}} = \partial_{y_1}(\frac{1}{2}\delta_{\phi(\Delta_2)}) = \frac{1}{2}(\delta_{y_2} - \frac{1}{\sqrt{5}}\delta_{y_1+2y_2-1}) = \frac{1}{2}(\widehat{\delta_{x_1}} - \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{2}}\delta_{y_1+2y_2-1}) = \frac{1}{2}(\widehat{\delta_{x_1}} - \frac{1}{\sqrt{2}}\widehat{\delta_{x_1+x_2-1}})$ etc. Note that $\phi^0_+ M_K$ in this presentation is not equal to $M_{\phi(K)}$, but they are isomorphic by rescaling of generators.

Finally comes the case of projections. From here on, $\pi : X \to Y$ will always be a projection. If π is the projection on the subspace spanned by the first s variables, the transfer module $D_{Y\leftarrow X}$ is isomorphic to $D_X / \sum_{j>s} \partial_j D_X$, with the natural left D_Y-structure (induced by considering D_Y as a subring of D_X) and the right D_X-structure given by letting x_i, ∂_{x_i} act by multiplication with $x_i, -\partial_{x_i}$, respectively. It follows that

$$\pi^0_+ M \simeq M / \sum_{j>s} \partial_j M. \qquad (*)$$

In fact, we can say what $\pi_+ M$ is too, following [Mal93] or [B+87, V,VI] we have that $\pi_+ M \simeq DR_{X/Y} M$, the *relative de Rham complex* of M, defined below. We only give the definition in our case of a projection $Y \times Z \to Y$, the general definition of the module of relative differentials can be found in [Har77, II.8].

Definition 5.4 (The relative de Rham complex)**.** Let $\pi : X = Y \times Z \to Y$ be the projection on the first factor, and denote by ξ the projection on the last factor Z. The *module of relative differentials* $\Omega_{X/Y}$ is defined to be the $\mathbb{C}[X]$-module $\xi^*\Omega_Z = \mathbb{C}[X] \otimes_{\mathbb{C}[Z]} \Omega_Z$.

The *relative de Rham complex* of a D_X-module M with respect to π is the complex

$$DR_{X/Y}M := \Omega^\bullet_{X/Y} \otimes_{\mathbb{C}[X]} M$$

(where as in the usual de Rham complex (4.1) we write $\Omega^i_{X/Y} := \bigwedge^i \Omega_{X/Y}$) with differential d_Z given by

$$d_Z(dz_I \otimes m) = \sum_{i=1}^{m-s} dz_i \wedge dz_I \otimes \partial_{z_i} m$$

or as a shorthand we can write $d_Z = \sum_i dz_i \otimes \partial_{z_i}$. We grade this in a manner similar to the de Rham complex, so that the final term $\Omega^{m-s}_{X/Y} \otimes M$ has degree zero and the first term M has degree $-(m-s)$. This is a complex of D_Y-modules.

The derived category formulation is to let $\omega_{X/Y} := \xi^*\omega_Z$ (where $\omega_Z = \Omega_Z^{m-s}$), and define

$$DR_{X/Y}M := \omega_{X/Y} \otimes^L_{\mathbb{C}[X]} M.$$

The definitions for sheaves are completely analogous, and can be found in e.g. [B$^+$87, VI.5.3].

We will compute $\pi^0_+ M_K$ for now, and return to the case of $\pi_+ M_K$ later.

By composing with an isomorphism, it suffices to consider this case. Explicitly, if we let Y' denote the orthogonal complement to $\ker(\pi)$ in X, we have $X = Y' \times \ker(\pi)$ and $Y' \simeq Y$, so we may write

$$X \xrightarrow{\sim} Y \times Z \xrightarrow{pr_1} Y,$$

where $Z \simeq \ker(\pi)$ and the last map is the orthogonal projection on the first factor.

Definition 5.5. For a cell $\sigma \subset K$, let $v(\sigma) := \dim(\sigma) - \dim(\pi(\sigma))$ (which is the same as the dimension of a generic fiber $\pi^{-1}(x) \cap K$ for a point $x \in int(\pi(\sigma))$). We call $v(\sigma)$ the *fiber dimension* of σ.

We also extend this notation to the whole complex, and let $v(K) := \dim(K) - \dim(\pi(K))$. Do note that a complex K can contain cells σ with $v(\sigma) > v(K)$.

Proposition 5.6 (Standard relations for $\pi^0_+ M_K$)**.** *Let σ be a cell in K of top dimension, with boundary cells σ_i with outward unit normals n_i, and let $\pi : X = Y \times Z \to Y$ be the projection on the first s coordinates. Denote the class of δ_σ in the direct image $\pi^0_+ M_K$ by $\overline{\delta_\sigma}$. Then the following relations hold:*

(i) $\partial_{\pi(z)}\overline{\delta_\sigma} = -\sum_i \langle \pi(z) | n_i \rangle \overline{\delta_{\sigma_i}}$*, for any point z in H_σ (where $\partial_{\pi(z)} := \sum_i \langle e_i | \pi(z) \rangle \partial_i$),*

(ii) $\sum_i \langle v|n_i\rangle \overline{\delta_{\sigma_i}} = 0$, *for any* $v \in \ker(\pi)$, *and*

(iii) $v(\sigma)\overline{\delta_\sigma} = \sum_i (d_i - \sum_{j\leq s}(n_i)_j x_j)\cdot \overline{\delta_{\sigma_i}}$, *where* $\sum_j (n_i)_j x_j - d_i = 0$ *is the defining equation of* H_{σ_i}.

(iv) $p(x)\cdot\overline{\delta_\sigma} = 0$ *for any* $p(x) \in I(H_{\pi(\sigma)})$.

Proof. For $j \leq s$, the action of ∂_j is unchanged in the quotient $(*)$, which implies (i). For $j > s$, $\partial_j\delta_\sigma$ is zero in the quotient $(*)$: $0 = \partial_j\delta_\sigma = \sum_i \langle e_j|n_i\rangle\overline{\delta_{\sigma_i}}$ and since $\ker(\pi) = \langle e_j | j > s\rangle$, we get (ii).

The affine spans H_{σ_i} of the boundary cells σ_i are of course defined by equations $\langle x|n_i\rangle = d_i$ for some constants d_i. Now $\sum_{j>s}\partial_j x_j\overline{\delta_\sigma} = 0$, because $\sum \partial_j x_j$ is in the ideal $\sum_{j>s}\partial_j D_X$. We then get $0 = \sum_{j>s}\partial_j x_j \overline{\delta_\sigma} = \sum_{j>s}(1 + x_j\partial_j)\overline{\delta_\sigma}$, or (using $v(\sigma) = m - s$)

$$\left((m-s) + \sum_{j>s} x_j\partial_j\right)\overline{\delta_\sigma} = \left(v(\sigma) + \sum_{j>s} x_j\partial_j\right)\overline{\delta_\sigma} = 0.$$

Let us expand this:

$$\begin{aligned} v(\sigma)\overline{\delta_\sigma} &= -\sum_{j>s} x_j\partial_j\overline{\delta_\sigma} \\ &= \sum_{j>s} x_j \sum_i \langle e_j|n_i\rangle\overline{\delta_{\sigma_i}} \\ &= \sum_i (\sum_{j>s}\langle e_j|n_i\rangle x_j)\overline{\delta_{\sigma_i}} \\ &= \sum_i (d_i - \langle e_1|n_i\rangle x_1 - \cdots - \langle e_s|n_i\rangle x_s)\overline{\delta_{\sigma_i}} \\ &= \sum_i (d_i - \sum_{j\leq s}(n_i)_j x_j)\overline{\delta_{\sigma_i}}, \end{aligned}$$

where the second-to-last equality uses the standard relation $(\sum_j (n_i)_j x_j - d_i)\delta_{\sigma_i} = 0$; and we have (iii). The claim (iv) is of course obvious, it follows by definition that $\mathrm{supp}(\overline{\delta_\sigma}) = \pi(\mathrm{supp}(\delta_\sigma)) = \pi(H_\sigma) = H_{\pi(\sigma)}$. □

Remark 5.7. By an application of Kashiwara's theorem, the restriction of π to H_σ induces similar relations for cells σ of arbitrary dimension. That is, we consider the map $\pi|_{H_\sigma} : H_\sigma \to \pi(H_\sigma)$, decompose $\pi = pr \circ \phi$ (where ϕ is an isomorphism and pr is a projection on one factor), and get a description like $(*)$. The application of Kashiwara's Theorem is that the relations of δ_σ in $M_{K\cap H_\sigma}$ are the same as those of δ_σ in M_K (and subsequently their direct images).

Lemma 5.8. *For any cell* $\sigma \subset K$, $\dim(Y) + v(\sigma) + (1 - \delta_{0,v(\sigma)})$ *of the relations of 5.6 suffice to generate all of them (here,* $\delta_{0,v(\sigma)}$ *is the Kronecker delta function).*

Proof. Clearly we can choose for each cell σ, a generating set of $\dim(\pi(\sigma))$ linearly independent relations of type 5.6(i), and $\operatorname{codim}(\pi(\sigma))$ linearly independent relations of type 5.6(iv). This is exactly as in 2.3, and only depends on the support of the generators $\overline{\delta_\sigma}$. Further, as the relations of type 5.6(ii) essentially say that $\partial_v \delta_\sigma = 0$ for those v in the kernel of π parallel to σ, we can choose $v(\sigma)$ linearly independent ones of these for each σ, to generate all. Finally, there is one relation of type 5.6(iii) if $v(\sigma) \neq 0$ (when $v(\sigma) = 0$, both sides of the equation are zero).

Thus, for each cell $\sigma \subset K$, there is a total of $\dim(Y) + v(\sigma) + (1 - \delta_{0,v(\sigma)})$ generating relations. □

Let us now attempt a description of $\pi^0_+ M_K$ with the aid of the above relations. We would like to introduce a skeleton stratification on $\pi(K)$, and an associated skeleton filtration on $\pi^0_+ M_K$ as in 3.9 and 3.13, and use these for computations. This cannot be done directly, because in general $\pi(K)$ is not a polyhedral cell complex (its cells can overlap), and so the construction of 3.9 does not work. We can, however, make a stratification of K, and a filtration on M_K, that induces a filtration on $\pi^0_+ M_K$ that behaves like the skeleton filtration in all important respects.

Definition 5.9 (The π-skeleton stratification)**.** We let $\mathcal{S}^\pi$ denote the stratification of X given by

$$\mathcal{S}^\pi_{\leq i} := \bigcup_{\substack{\sigma \subset K \\ \dim(\pi(\sigma)) \leq i}} H_\sigma$$

with strata $\mathcal{S}^\pi_i = \cup_{\dim(\pi(\sigma))=i} H_\sigma$. We call this stratification the *π-skeleton stratification.*

This induces a filtration on M_K: we let $F^\pi_{\leq i}$ be the submodule generated by those δ_σ with $\dim(\pi(\sigma)) \leq i$. These form an ascending filtration $F^\pi_0 \subset F^\pi_{\leq 1} \subset \cdots \subset F^\pi_{\leq s}$, with filtration quotients Q^π_i generated by those δ_σ with $\dim(\pi(\sigma)) = i$. We call this filtration the *π-skeleton filtration.*

In a natural way, the π-skeleton filtration is passed on to the quotient $\pi^0_+ M_K$:

Definition 5.10. Let ${F^\pi_{\leq i}}'$ be the submodule of $\pi^0_+ M_K$ generated by those $\overline{\delta_\sigma}$ with $\dim(\pi(\sigma)) \leq i$. These form a filtration as above, with filtration quotients ${Q^\pi_i}'$ minimally generated by the classes (in ${F^\pi_{\leq i}}'/{F^\pi_{\leq i-1}}'$) of those $\overline{\delta_\sigma}$ with $\dim(\pi(\sigma)) = i$ (this claim follows from considering the support of each generator, only things with equal-dimensional image can be identified in the direct image). Without fear of overloading the name, we call this filtration the *skeleton filtration on $\pi^0_+ M_K$*.

Let us see how these fit together. We first introduce some notation that will be of use:

Definition 5.11. We let $K_i, K_{\leq i}$ denote the subcomplexes of K given by respectively $K_i := \bigcup_{\dim(\pi(\sigma))=i} \sigma$ and $K_{\leq i} := \bigcup_{\dim(\pi(\sigma))\leq i} \sigma$. Note that $K_{\leq i}$ is closed in K.

Lemma 5.12. $F^\pi_{\leq k} \simeq M_{K_{\leq k}}$.

Proof. This is obvious by construction. □

Remark 5.13. It follows from that Q^π_k is generated by generators $\overline{\delta_\sigma}$ corresponding to the cells of K_k.

Lemma 5.14. *The skeleton filtration quotients ${Q^\pi}'_k$ of $\pi^0_+ M_K$ are naturally isomorphic to the direct images $\pi^0_+ Q^\pi_k$ of the π-skeleton filtration quotients of M_K.*

Proof. From the definitions of the two filtrations, it follows that we have a surjection $\theta : \pi^0_+ F^\pi_{\le k} \twoheadrightarrow {F^\pi_{\le k}}'$, and it is clear that if this map is an isomorphism, we also have ${Q^\pi}'_k \simeq \pi^0_+ Q^\pi_k$. Moreover, because $F^\pi_{\le k} \simeq M_{K_{\le k}}$ by 5.13, and $K_{\le k}$ is closed in K, we get by 3.3 a canonical presentation for $F^\pi_{\le k}$ that injects into the canonical presentation for M_K; and putting this together we have the following commutative diagram:

$$\begin{tikzcd}
W^{r_k} \arrow[r, "P_k"] \arrow[d, hook, "i_r"'] & W^{s_k} \arrow[r, two heads, "\rho_k"] \arrow[d, hook, "i_s"'] & F^\pi_{\le k} \arrow[r, two heads, "q_k"] \arrow[d, hook, "i"'] & \pi^0_+ F^\pi_{\le k} \arrow[d, "\widehat{i}"'] \arrow[dr, "\theta"] & \\
W^r \arrow[r, "P"] & W^s \arrow[r, two heads, "\rho"] & M_K \arrow[r, two heads, "q"] & \pi^0_+ M_K & {F^\pi_{\le}}' \arrow[l, hook', "\iota"']
\end{tikzcd}$$

The vertical arrows i_r, i_s, i are the canonical inclusions, $\widehat{i}$ is the induced map in the direct image, and ι is the inclusion of ${F^\pi_{\le k}}'$ as a submodule in $\pi^0_+ M_K$; P, P_k, ρ, ρ_k are the maps in the canonical presentations, and $q : M_K \twoheadrightarrow M_K / \sum \partial_z M_K \simeq \pi^0_+ M_K$ (and similar for q_k) is the quotient map from $(*)$.

In particular, we can build from this the diagram

$$\begin{tikzcd}
W^{r_k} \arrow[r, "P_k"] \arrow[d, hook, "i_r"'] & W^{s_k} \arrow[r, two heads, "q_k \circ \rho_k"] \arrow[d, hook, "i_s"'] & \pi^0_+ F^\pi_{\le k} \arrow[d, "\widehat{i}"'] \\
W^r \arrow[r, "P"] & W^s \arrow[r, two heads, "q \circ \rho"] & \pi^0_+ M_K
\end{tikzcd}$$

where the horizontal maps compose to zero. We wish to show that $\ker(\widehat{i}) = 0$, i.e. that $\pi^0_+ F^\pi_{\le k}$ is a submodule of $\pi^0_+ M_K$; because $\widehat{i} = \iota \circ \theta$ this in turn implies that θ is an isomorphism and we are done.

We consider the previous diagram as a double complex, and examine the associated spectral sequence. Taking the horizontal direction first, we get E_1 to be

$$\begin{tikzcd}
\ker(P_k) \arrow[d, hook, "\overline{i_r}"'] & * \arrow[d, "\overline{i_s}"'] & 0 \\
\ker(P) & * & 0
\end{tikzcd}$$

(where we have marked by $*$ the entries we do not care about); and the sequence stops with E_2:

$$\begin{matrix}
0 & * & 0 \\
 & & \\
\ker(P)/\ker(P_k) & * & 0
\end{matrix} \tag{A}$$

Taking then the vertical direction first, E_1 is

$$\begin{tikzcd} 0 & 0 & \ker(\widehat{i}) \\ \operatorname{cok}(i_r) \arrow[r, "\overline{P}"] & \operatorname{cok}(i_s) \arrow[r, "\overline{q\circ\rho}"] & \operatorname{cok}(\widehat{i}) \end{tikzcd}$$

and finally E_2 is

$$\begin{tikzcd} 0 & 0 & \ker(\widehat{i}) \\ \ker(\overline{P}) \arrow[urr, "f"] & * & * \end{tikzcd} \tag{B}$$

We now have from diagram (B) a map $f : \ker(\overline{P}) \to \ker(\widehat{i})$, which we know from diagram (A) has to be surjective, and must have kernel isomorphic to $\ker(P)/\ker(P_k)$. In other words, we have an exact sequence

$$0 \to \ker(P)/\ker(P_k) \to \ker(\overline{P}) \to \ker(\widehat{i}) \to 0 \tag{C}$$

Now, from the Snake Lemma applied to the diagram

$$\begin{tikzcd} W^{r_k} \arrow[r, hook, "i_r"] \arrow[d, "P_k"'] & W^r \arrow[r, two heads] \arrow[d, "P"'] & \operatorname{cok}(i_r) \arrow[d, "\overline{P}"'] \\ W^{s_k} \arrow[r, hook, "i_s"] & W^s \arrow[r, two heads] & \operatorname{cok}(i_s) \end{tikzcd}$$

we get the exact sequence

$$0 \to \ker(P_k) \to \ker(P) \to \ker(\overline{P}) \to \operatorname{cok}(P_k) \to \operatorname{cok}(P) \to \operatorname{cok}(\overline{P}) \to 0$$

Recalling that by definition of P_k and P, we have $\operatorname{cok}(P_k) \simeq F^{\pi}_{\leq k}$ and $\operatorname{cok}(P) \simeq M_K$; so the map $\operatorname{cok}(P_k) \to \operatorname{cok}(P)$ is the inclusion map $F^{\pi}_{\leq k} \hookrightarrow M_K$. This means that the sequence

$$0 \to \ker(P_k) \to \ker(P) \to \ker(\overline{P}) \to 0$$

is exact, so $\ker(\overline{P}) \simeq \ker(P)/\ker(P_k)$, and from the sequence (C) we see that $\ker(\widehat{i}) = 0$. $\square$

The direct image $\pi_+ Q_k^{\pi}$ is intimately connected to the homology of the K_k's:

Theorem 5.15. *The cohomology modules $h^i(DR_{X/Y} Q_k^{\pi})$ of $DR_{X/Y} Q_k^{\pi}$ are semisimple D_Y-modules, with summands isomorphic to $D_Y \cdot \overline{\delta_{\pi(\sigma)}}$ for $\sigma \in K_k$. For $h^i(DR_{X/Y} Q_k^{\pi})$, the number of such summands is equal to $\dim H^{BM}_{i+k}(K_k, \mathbb{C})$.*

Proof. We recall our convention that $X = Y \times Z$, with π the projection on Y. We will (begin to) compute the relative de Rham complex by means of the skeleton filtration (3.13) on K_k. We can express each skeleton filtration quotient summand $D_X \cdot \overline{\delta_\sigma}$ as a module $\mathbb{C}[y_1, \dots, \partial_{y_s}, z_1, \dots, \partial_{z_{m-s}}]$ by choosing suitable coordinates, in the following manner. We choose the y_i such that

$$D_Y \cdot \overline{\delta_{\pi(\sigma)}} \simeq \mathbb{C}[y_1, \dots, y_{\dim(\pi(\sigma))}, \partial_{y_{\dim(\pi(\sigma))+1}}, \dots, \partial_{y_s}]$$

in the same way as in 3.17. Similarly, we choose the z_j such that for a generic fiber $F := \pi^{-1}(p) \cap \mathring{\sigma}$ (where $p \in int(\pi(\sigma))$ is some point), we have, also as in 3.17, that

$$D_Z \cdot \overline{\delta_F} \simeq \mathbb{C}[z_1, \dots, z_{v(\sigma)}, \partial_{z_{v(\sigma)-1}}, \dots, \partial_{z_{m-s}}].$$

This isomorphism of course depends on which point p we choose, but the coordinates do not. In particular we have $\partial_{z_j}\overline{\delta_\sigma} = 0$ for $j \leq v(\sigma)$. These coordinates are not as natural as those of 3.17, and complicate the description of the actions of y_i, ∂_{y_i} and z_j, but the action of ∂_{z_j} is easy to describe, which is all we need.

The relative de Rham complex $DR_{X/Y}(D_X \cdot \overline{\delta_\sigma})$ is now of the form

$$\Omega^\bullet_{X/Y} \otimes \mathbb{C}[y_1, \dots, y_{\dim(\pi(\sigma))}, \partial_{y_{\dim(\pi(\sigma))+1}}, \dots, \partial_{y_s}, z_1, \dots, z_{v(\sigma)}, \partial_{z_{v(\sigma)-1}}, \dots, \partial_{z_{m-s}}],$$

and since the differential $d_Z = \sum_j dz_j \otimes \partial_{z_j}$ (using our shorthand from 5.4) commutes with the Y variables, this becomes

$$\mathbb{C}[y_1, \dots, \partial_{y_s}] \otimes_{\mathbb{C}} \left(\Omega^\bullet_Z \otimes_{\mathbb{C}[Z]} \mathbb{C}[z_1, \dots, \partial_{z_{m-s}}] \right).$$

We can now follow the cohomology calculation in 4.4, applied to the complex $\Omega^\bullet_Z \otimes \mathbb{C}[z_1, \dots, z_{v(\sigma)}, \partial_{z_{v(\sigma)-1}}, \dots, \partial_{z_{m-s}}]$. By 4.4 we have that

$$\Omega^\bullet_Z \otimes \mathbb{C}[z_1, \dots, z_{v(\sigma)}, \partial_{z_{v(\sigma)-1}}, \dots, \partial_{z_{m-s}}] \simeq C^{BM}_\bullet(\pi^{-1}(p) \cap \mathring{\sigma}),$$

and the cohomology modules of $DR_{X/Y}(D_X \cdot \overline{\delta_\sigma})$ are zero except in degree $-v(\sigma)$, where we have

$$\mathbb{C}[y_1, \dots, y_{\dim(\pi(\sigma))}, \partial_{y_{\dim(\pi(\sigma))+1}}, \dots, \partial_{y_s}] dz_1 \wedge \dots \wedge dz_{v(\sigma)} \simeq D_Y \cdot \overline{\delta_{\pi(\sigma)}} dz_{J_\sigma}$$

(where $dz_{J_\sigma} := dz_1 \wedge \dots \wedge dz_{v(\sigma)}$).

The cohomology of $DR_{X/Y} Q^\pi_k$ can be calculated by a spectral sequence associated to the skeleton filtration (following the calculation for the de Rham complex $DR_X M_K$ in 4.6). The sequence begins with $E_0^{pq} = \Omega^{m-s+p+q}_{X/Y} \otimes Q_{-p}$. As for all cells in K_k we have $\dim(\sigma) = k + v(\sigma)$, each Q_{-p} has only a single cohomology module, in degree $-v(\sigma) = k + p$. This gives us that the E_1 page is a single-row complex

$$\bigoplus_{v(\sigma)=m-s-k} D_Y \cdot \overline{\delta_{\pi(\sigma)}} \to \dots \to \bigoplus_{v(\sigma)=0} D_Y \cdot \overline{\delta_{\pi(\sigma)}} dz_{J_\sigma}$$

(we let $dz_{J_\sigma} = dz_1 \wedge \dots \wedge dz_{v(\sigma)}$, in the coordinates suiting each σ as above), and the differential is d_Z as before.

We can now show that the cohomology modules must be direct sums of simple modules: as the differential d_Z commutes with D_Y, it acts only on the generators $\overline{\delta_{\pi(\sigma)}} dz_{J_\sigma}$, and so taking cohomology only involves identification of generators. This implies that the cohomology modules are of the form $\sum D_Y \cdot \overline{\delta_{\pi(\sigma)}} dz_{J_\sigma}$, and one gets a (non-canonical) direct sum decomposition by choosing some generating set. We recall from 3.16 that each summand $D_Y \cdot \overline{\delta_{\pi(\sigma)}} dz_{J_\sigma}$ is simple.

We want to relate this to the homology of K_k. We recall from 4.6 that the de Rham differential d_X corresponds to the topological boundary map, because for a generator $\delta_\sigma dx_{I_\sigma}$ we had

$$d(\delta_\sigma dx_{I_\sigma}) = \sum_{\sigma_i \subset \partial\sigma} \delta_{\sigma_i} dx_{I_{\sigma_i}}.$$

Now, in the relative de Rham complex we have the relative differential d_Z acting on generators $\delta_\sigma dz_{J_\sigma}$, and this also behaves like the topological boundary map, the same computation as in 4.6 works:

$$\begin{aligned}
d_Z(dz_{J_\sigma} \otimes \delta_\sigma) &= \sum_{j=1}^{m-s} dz_j \wedge dz_{J_\sigma} \otimes \partial_{z_j} \delta_\sigma \\
&= \sum_{j=1}^{m-s} dz_j \wedge dz_{J_\sigma} \otimes (-\sum_{\sigma_i \subset \partial\sigma} \langle e_j | n_i \rangle \delta_{\sigma_i}) \\
&= -\sum_{j=1}^{m-s} \sum_{\sigma_i \subset \partial\sigma} \langle e_j | n_i \rangle dz_j \wedge dz_{J_\sigma} \otimes \delta_{\sigma_i} \\
&= -\sum_{\sigma_i \subset \partial\sigma} d(\sum_{j=1}^{m-s} \langle e_j | n_i \rangle z_j) \wedge dz_{J_\sigma} \otimes \delta_{\sigma_i} \\
&= -\sum_{\sigma_i \subset \partial\sigma} d((n_i)_Z) \wedge dz_{J_\sigma} \otimes \delta_{\sigma_i} \\
&= -\sum_{\sigma_i \subset \partial\sigma} dz_{J_{\sigma_i}} \otimes \delta_{\sigma_i}
\end{aligned}$$

Here, $(n_i)_Z$ denotes the projection of the vector $n_i \in \mathbb{R}^m$ on the subspace $\{0\} \times \mathbb{R}^{m-s}$. We see that the correspondence of the de Rham differential to the topological boundary map holds, except for one subtlety: those cells $\sigma_i \subset \partial\sigma$ such that $(n_i)_Z = 0$ do not appear in the final sum. These are precisely those cells in the boundary of σ that have image of dimension strictly lower than $\dim(\pi(\sigma))$. Thus, if we restrict our attention to the subcomplex K_k, where these cells are removed, the correspondence to the topological boundary map remains. Just as we had $d[\sigma] = d(dx_{J_\sigma} \otimes \delta_\sigma) = \sum dx_{J_{\sigma_i}} \otimes \delta_{\sigma_i} = [\partial\sigma]$ (up to orientation), we have now $d_Z(dz_{J_\sigma}) = \sum dz_{J_{\sigma_i}} \otimes \delta_{\sigma_i}$, the only difference is instead of constant coefficients we now have D_Y-coefficients.

We recall the observation that the cells in K_k all have $\dim(\sigma) = k + v(\sigma)$, and accordingly the generators of $h^i(DR_{X/Y} Q_k^\pi)$ correspond to cells with $\dim(\sigma) = k + i$. This

H_σ	$\dim(\sigma)$	$v(\sigma)$	$\dim(\pi(\sigma))$
$x = 0$	2	2	0
$y = 0$	2	1	1
$z = 0$	2	1	1
$x + y + z - 1 = 0$	2	1	1
$x, y = 0$	1	1	0
$x, z = 0$	1	1	0
$y, z = 0$	1	0	1
$x, y + z - 1 = 0$	1	1	0
$y, x + z - 1 = 0$	1	0	1
$z, x + y - 1 = 0$	1	0	1
any vertex	0	0	0

gives us that the number of summands in $h^i(DR_{X/Y}Q_k^\pi)$ is equal to the dimension of the homology group $H_{k+i}^{BM}(K_k, \mathbb{C})$, and one can choose as generators any set of $\overline{dz_{J_\sigma} \otimes \delta_\sigma}$'s such that the associated homology classes $[\sigma]$ generate $H_{k+i}^{BM}(K_k, \mathbb{C})$. □

Remark 5.16. Note that the decomposition $h^i(DR_{X/Y}Q_k^\pi) \simeq \bigoplus D_Y \cdot \overline{\delta_{\pi(\sigma)}}$ is not canonical, as any generating set of $\pi(\sigma)$'s will do.

Corollary 5.17. *The skeleton filtration quotient $Q_k^{\pi\prime} \simeq \pi_+^0 Q_k^\pi$ of $\pi_+^0 M_K$ is a (non-canonical) direct sum of simple modules, and the number of summands is equal to the dimension of the homology group $H_k^{BM}(K_k, \mathbb{C})$.*

Proof. The isomorphism $Q_k^{\pi\prime} \simeq \pi_+^0 Q_k^\pi$ is 5.14 and 5.13. The statement for $\pi_+^0 Q_k^\pi$ follows from 5.15, because $\pi_+^0 M \simeq h^0(DR_{X/Y}M)$. □

Example 5.18. Let K be the boundary of the standard 3-simplex $\Delta_3 \subset \mathbb{R}^3$, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^1$ be the projection $(x, y, z) \mapsto x$ on the first coordinate. We apply the π-skeleton filtration to M_K, here we list the cells and their place in the filtration (the cells are listed by their affine hull) in the table.

Thus, the first quotient Q_1^π is generated by $\delta_y, \delta_z, \delta_{x+y+z-1}, \delta_{y,z}, \delta_{y,x+z-1}$ and $\delta_{z,x+y-1}$, while the zeroth quotient Q_0^π is generated by the vertices, $\delta_x, \delta_{x,y}, \delta_{x,z}$ and $\delta_{x,y+z-1}$ (see Figure 5.1).

The generators of the direct image $\pi_+^0 Q_k^\pi$ are given by choosing cells corresponding to a generating set for the homology group $H_k(K_k, \mathbb{C})$, so let us calculate what these are. We have that K_1, being the part of K that lies over the open unit interval, is homeomorphic to a product of an open interval (the image) with a circle (2-simplex with empty interior, the fiber); this has $H_1(K_1, \mathbb{C}) \simeq \mathbb{C}$ generated by (e.g.) the open unit interval. Further, K_0 is the disjoint union of point (the point $(1, 0, 0)$ and a closed 2-simplex (the part lying in the plane $x = 0$). This has $H_0(K_0, \mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}$, generated by (e.g.) the points $(0, 0, 0)$ and $(1, 0, 0)$.

This gives us the skeleton filtration quotients of $\pi_+^0 M_K$; the zeroth quotient $Q_0^{\pi\prime} = \pi_+^0 Q_0^\pi$ is isomorphic to the D_Y-module $M_0 \oplus M_1$, and the first quotient $Q_1^{\pi\prime} = \pi_+^0 Q_1^\pi$ is isomorphic to the D_Y-module $M_{(0,1)}$.

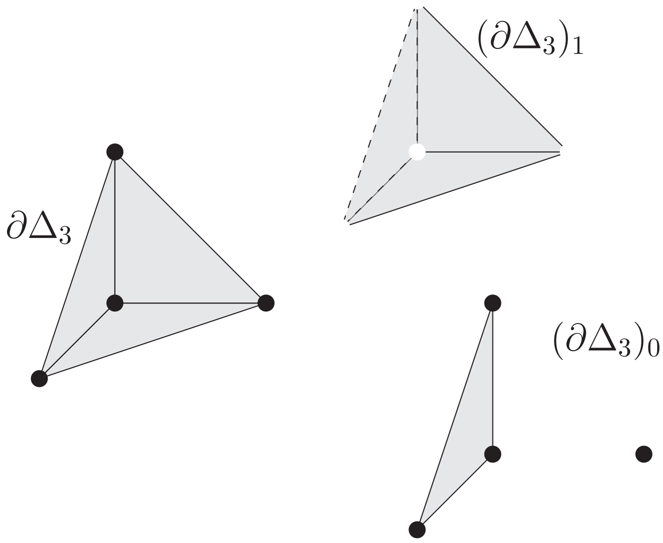

Figure 5.1: The strata in the π-skeleton filtration on $\partial\Delta_3$.

Proposition 5.19. *There is a canonical presentation*

$$(D_Y)^r \to (D_Y)^c \twoheadrightarrow \pi_+^0 M_K$$

where c is the number of cells in K, and r is equal to $(\dim(Y)+1)\cdot c+\sum_{\sigma\subset K}(v(\sigma)-\delta_{0,v(\sigma)})$ *(here, $\delta_{0,v(\sigma)}$ is the Kronecker delta function).*

Proof. We proceed as in the proof of 3.1. We label the generators of $(D_Y)^c$ by the cells of K, and label the generators of $(D_Y)^r$ by a generating set for the relations in 5.6. Recall that each standard relation induced by a cell σ can be written as a D_Y-linear combination $P^\sigma(\delta_\sigma,\ldots,\delta_{\sigma_k}) = 0$ involving σ and its boundary cells σ_i, e.g. relations of type 5.6(i) can be written as $P(\delta_\sigma,\ldots,\delta_{\sigma_n}) = \partial_{\pi(z)}\overline{\delta_\sigma} + \sum_i \langle\pi(z)|n_i\rangle\overline{\delta_{\sigma_i}} = 0$ for any point $z \in H_\sigma$, etc. So, we let $(D_Y)^c$ be freely generated by g_σ for each $\sigma \subset K$, and $(D_Y)^r$ by r_{P^σ} where the P^σ are a generating set (following Lemma 5.8) for the standard relations induced σ, as in the proof of 3.1.

The maps $(D_Y)^r \to (D_Y)^c$ and $(D_Y)^c \to \pi_+^0 M_K$ are defined, respectively, by letting

$$r_{P^\sigma} \mapsto P^\sigma(g_\sigma,\ldots,g_{\sigma_k}),$$

and

$$g_\sigma \mapsto \overline{\delta_\sigma}.$$

Surjectivity of the last map is clear, what remains is checking exactness in the middle.

The count of generating relations is given by Lemma 5.8: there is for each cell $\sigma \subset K$ a total of $\dim(Y) + v(\sigma) + (1 - \delta_{0,v(\sigma)})$ generating relations. Adding this up over all the cells, we get $(\dim(Y) + 1)\cdot c + \sum_{\sigma\subset K}(v(\sigma) - \delta_{0,v(\sigma)})$, where c is the number of cells.

We now consider the skeleton filtration on $\pi^0_+ M_K$ and the induced filtrations on $(D_Y)^c$ and $(D_Y)^r$, to reduce to the associated graded modules. The induced filtrations on $(D_Y)^c$ and $(D_Y)^r$ are given, as in 5.6, by letting

$$F'_i(D_Y)^c := D_Y \cdot \{g_\sigma | \dim(\pi(\sigma)) \leq i\}$$

and

$$F''_i(D_Y)^r := D_Y \cdot \{r_{P^\sigma} | \dim(\pi(\sigma)) \leq i\}.$$

We denote the filtration quotients by $((D_Y)^c)_i$ etc., it is clear that they are generated by generators g_σ, r_{P^σ} respectively, such that $\dim(\pi(\sigma)) = i$.

We can now pass to the associated graded modules and check exactness of the sequence

$$\bigoplus_{i=0}^{s}((D_Y)^r)_i \to \bigoplus_{i=0}^{s}((D_Y)^c)_i \to \bigoplus_{i=0}^{s} Q_i^{\pi\prime}$$

which reduces to the filtration quotients. Let us now look at the cokernel C_k of the map $((D_Y)^r)_k \to ((D_Y)^c)_k$, that is, what happens to the generators g_σ when we impose the relations of 5.6. The generators $\overline{\delta_\sigma}$ of Q_k must of course also satisfy these, the obstacle is proving that nothing else happens.

Those g_σ with $v(\sigma) > 0$ can, by 5.6(iii), be written as a sum of their boundary cells,

$$v(\sigma)\overline{\delta_\sigma} = \sum_i (d_i - \sum_{j \leq s}(n_i)_j x_j) \cdot \overline{\delta_{\sigma_i}},$$

where $\sum_j (n_i)_j x_j - d_i = 0$ is the defining equation of H_{σ_i}. The boundary cells σ_i all have $\dim(\sigma_i) = \dim(\sigma) - 1$, and either $v(\sigma_i) = v(\sigma) - 1$ or $v(\sigma_i) = v(\sigma)$ (in which case $\dim(\pi(\sigma_i)) = \dim(\pi(\sigma)) - 1$, and these are removed in the skeleton filtration quotients). Iterating the application of 5.6(iii), we can see that each C_k is generated by cells g_σ with $\dim(\sigma) = \dim(\pi(\sigma)) = k, v(\sigma) = 0$. There are some relations between these, given by 5.6(ii), so we can eliminate some of them and express C_k as a direct sum of modules generated by g_σ with $v(\sigma) = 0$. Reducing to the direct summands, we get surjective maps

$$D_Y \cdot g_\sigma \to (\text{summand of } Q_k^{\pi\prime})$$

and as the first of these is simple, the map must be an isomorphism. □

5.1 Higher direct images

The derived direct image π_+ contains more information about M_K and K than π^0_+, so let us examine it further.

Example 5.20. Recall from Example 5.18, that for K be the boundary of the standard 3-simplex in $\mathbb{R}^3$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^1$ the projection on the first coordinate, the direct image $\pi^0_+ M_K$ was isomorphic to the module $M_{[0,1]} = M_{\pi(K)}$. On the other hand, taking K to be the whole 3-simplex, *with* interior, we get the same module in the direct image, as the zeroth homology of the fibers remain the same. We see that π^0_+ fails to detect this difference.

The derived direct image π_+, however, does detect the homological information, which will come as no surprise given 5.15. Let us first remark that Proposition 4.6 has an immediate corollary:

Proposition 5.21. *The de Rham cohomology of $\pi_+ M_K$ is equal to the Borel-Moore homology of K.*

Proof. One has the standard isomorphism of functors $DR_Y \circ \pi_+ \simeq R\pi_* \circ DR_X$ (see [B+87, VIII]), which gives us $H^\bullet_{DR}(\pi_+ M_K) = H^\bullet(Y, DR_Y(\pi_+ M_K)) \simeq H^\bullet(Y, R\pi_* DR_X(M_K)) \simeq H^\bullet(X, DR_X(M_K)) \simeq H^{BM}_\bullet(K, \mathbb{C})$. □

To compute $\pi_+ M_K$, we will use the π-skeleton filtration as we did for the ordinary direct image. We consider $DR_{X/Y} M_K$ as a filtered complex with respect to this filtration, and run the spectral sequence. The E_0 page is $DR_{X/Y}$ of the Q_i^π's:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^1_{X/Y} \otimes Q^\pi_s \arrow[u] & \Omega^2_{X/Y} \otimes Q^\pi_{s-1} \arrow[u] & \Omega^3_{X/Y} \otimes Q^\pi_{s-2} \arrow[u] & \cdots \\
Q^\pi_s \arrow[u, "d_Z"'] & \Omega^1_{X/Y} \otimes Q^\pi_{s-1} \arrow[u, "d_Z"'] & \Omega^2_{X/Y} \otimes Q^\pi_{s-2} \arrow[u, "d_Z"'] & \cdots \\
 & Q^\pi_{s-1} \arrow[u, "d_Z"'] & \Omega^1_{X/Y} \otimes Q^\pi_{s-2} \arrow[u, "d_Z"'] & \cdots \\
 & & Q^\pi_{s-2} \arrow[u, "d_Z"'] & \cdots
\end{tikzcd}
$$

Each column here is the relative de Rham complex of a Q^π_k, and it follows from 5.15 that the cohomology modules are each given by generators corresponding to a generating set for the homology group $H^{BM}_{k+i}(K_k, \mathbb{C})$. So, in position $(-p, -q)$ we get the $-q$'th relative de Rham cohomology module, which is generated by $\overline{\delta_\sigma}$ corresponding to our favourite set of generators $[\sigma]$ for $H^{BM}_{p+q}(K_p, \mathbb{C})$.

We now have the E_1 page consisting of these modules, and the process stops here: the π-skeleton filtration is constructed in such a way that no interaction between the levels of the filtration takes place in the Z direction, that is one cannot get from Q^π_k to Q^π_{k-1}, say, by acting with a ∂_z differential. If this were the case, we would have a generator of Q^π_k and a generator of Q^π_{k-1} having image of the same dimension, which contradicts the definition of the π-skeleton filtration. The E_1 page now has in position $(-p, -q)$ the p'th skeleton filtration quotient of $h^q(\pi_+ M_K)$ (by the natural definition of the skeleton filtration on $h^q(\pi_+ M_K)$). We do not get the module explicitly, but knowing the skeleton filtration quotients is enough for most purposes.

Example 5.22. We extend Example 5.18 to find the higher direct images as well. Recall that the π-skeleton filtration decomposed K into K_1, the product of an open interval with a 2-simplex with empty interior; and K_0, the disjoint union of a point and a 2-simplex (*with* interior).

We already have the zeroth higher direct image $\pi_+^0 M_K = h^0(\pi_+ M_K)$, and as the fiber of π is two-dimensional, we should check for higher direct images in degree -1 and -2. The homology of K_1 is given by

$$H_i^{BM}(K_1, \mathbb{C}) = \begin{cases} 0 & \text{for i=0} \\ \mathbb{C} & \text{for i=1} \\ \mathbb{C} & \text{for i=2,} \end{cases}$$

and the homology of K_0 is

$$H_i^{BM}(K_0, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{for i=0} \\ 0 & \text{otherwise.} \end{cases}$$

This gives us the following final page of the spectral sequence:

$$\begin{array}{cc} M_{(0,1)} & 0 \\ & \\ M_{(0,1)} & M_0 \oplus M_1 \end{array}$$

which displays the skeleton filtration quotients of the direct image; in degree zero $(h^0(\pi_+ M_K))$ we have $Q_1 = M_{(0,1)}$ and $Q_0 = M_0 \oplus M_1$, in degree minus one $(h^{-1}(\pi_+ M_K))$ we have $Q_1 = M_{(0,1)}$ and $Q_0 = 0$.

6 Distributional direct images and B-splines

There is also a notion of direct image of distributions, which in our case, for δ_σ, when $\pi : \mathbb{R}^m \to \mathbb{R}^s$, is given by

$$\pi_*\delta_\sigma := [f \mapsto \int_\sigma f \circ \pi]$$

for a test function f on $\mathbb{R}^s$. In particular the case of projections is interesting here, because the distributional direct image $\pi_*\delta_\sigma$ is the distribution form of the *multivariate B-spline*

$$\sigma_\pi(x) = \frac{1}{\sqrt{\det(\pi \cdot \pi^t)}} \operatorname{vol}(\pi^{-1}(x) \cap \sigma),$$

(where we by abuse of notation write π for the matrix associated to π) which has a long history in applied mathematics. Typically, one has studied simple shapes like boxes, simplices and cones, as they are most suited for applications (see for instance [CLR87], [DBH82] or [DCP10]).

Note that we can also express $\pi_*\delta_\sigma$ as the distribution

$$\phi \mapsto \int_{\mathbb{R}^s} \phi(x)\sigma_\pi(x)dx$$

(see e.g. [DCP10, chapter 7] for details).

Before tackling the case of projections, let briefly see how the distributional direct images of δ_σ behaves under (linear) inclusions and isomorphisms:

For an inclusion, we have $\int_K f(i(x))dx = \int_{i(K)} f(x)dx$, so $i_*\delta_\sigma = \delta_{i(\sigma)}$. An isomorphism η likewise gives, by the usual change of variables formula, $\eta_*\delta_\sigma = |\det(\eta)|^{-1}\delta_{\eta(\sigma)}$.

The reader may notice that what follows below follows Section 5 almost exactly. This is no coincidence, as much of the material in this section is old stuff about B-splines, and the previous section consisted in large part of checking whether the same statements and relations applied to π_*M_K. This is true in particular for 5.6, the statement of which is almost a verbatim copy of 6.2, rephrased to suit the altered context; and the proof merely consists of checking that the same relations hold.

De Concini and Procesi in [DCP10] investigate some of the properties of the module $W \cdot \pi_*\delta_K$, when π is a projection, in the cases when K is a box or a cone. In light of what we have done so far, we might say that for general K, the module generated by all the $\pi_*\delta_\sigma$ is the more natural object, so let us investigate it closer.

Definition 6.1. We let $S_K := W \cdot \{\pi_*\delta_\sigma | \sigma \subset K\}$.

The fundamental relations for S_K are much the same as for M_K, given by this result from [DBH82], which also exists in similar versions elsewhere in the literature (follow references from [DBH82],[CLR87] or [DCP10]). We rephrase the original result to suit our context: De Boor and Höllig in [DBH82] only prove (i) and the statement of (iii) for the *functions* $\sigma_\pi, (\sigma_i)_\pi$, but the rest is implicit and follows directly.

Theorem 6.2 (De Boor - Höllig, [DBH82])**.** *Let σ be a polyhedral body in $\mathbb{R}^m$, with facets σ_i, and corresponding outward unit normals n_i, and let π be the projection on the first s coordinates. Assume also that the fibers $\pi^{-1}(x) \cap K$ are compact. Then the following hold:*

(i) $\partial_{\pi(z)}\pi_*\delta_\sigma = -\sum_i \langle \pi(z)|n_i\rangle \pi_*\delta_{\sigma_i}$*, for any* $z \in \mathbb{R}^m$*,*

(ii) $\sum_i \langle v|n_i\rangle \pi_*\delta_{\sigma_i} = 0$*, for* $v \in \mathbb{R}^m$ *orthogonal to* $\mathbb{R}^s$*, and*

(iii) $v(\sigma)\pi_*\delta_\sigma = \sum_i \langle k_i - x|n_i\rangle \pi_*\delta_{\sigma_i}$*, where* k_i *is an arbitrary point of* σ_i *and* $x \in \mathbb{R}^s$*.*

Proof, adapted from [DBH82]. To ensure convergence of the integrals, we assume that the fibers $K \cap \pi^{-1}(p)$ are compact. This is necessary because although our test functions ϕ have compact support in $\mathbb{R}^s$, $\phi \circ \pi$ does *not* have compact support in σ unless the fibers $\pi^{-1} \cap \sigma$ are compact.

(i) and (ii) are fairly straightforward: if we let $z \in \mathbb{R}^m$ be such that $\pi(z) = v$, then

$$\partial_{\pi(z)} \int_\sigma \phi \circ \pi = -\int_\sigma (\partial_{\pi(z)}\phi) \circ \pi = -\int_\sigma \partial_z(\phi \circ \pi) = -\sum_i \int_{\sigma_i} \langle z|n_i\rangle \phi \circ \pi.$$

This is almost (i), except that 'z' is substituted for '$\pi(z)$' on the right-hand side. To resolve this, write $z = \pi(z) + v$, where v is orthogonal to $\mathbb{R}^s$. We then get

$$\partial_{\pi(z)}\pi_*\delta_\sigma = -\sum_i \langle \pi(z) + v|n_i\rangle \pi_*\delta_{\sigma_i}$$

that is,

$$\partial_{\pi(z)}\pi_*\delta_\sigma = -\sum_i \langle \pi(z)|n_i\rangle \pi_*\delta_{\sigma_i} - \sum_i \langle v|n_i\rangle \pi_*\delta_{\sigma_i}$$

and for this to make sense, the last sum must be zero. This then gives us (i) and (ii).

(iii) is proved in two steps, first we prove that for $\theta := \sum_{j=1}^s x_j \partial_{x_j}$, we have $\theta \cdot \sigma_\pi(v) = (m - s)\sigma_\pi(v) - \sum_i \langle k_i|n_i\rangle (\sigma_i)_\pi(v)$, where $k_i \in H_{\sigma_i}$ is an arbitrary point. Let's write this out:

$$\begin{aligned}
\sum_{j=1}^{s} x_j \partial_{x_j} \int_\sigma \phi(\pi(z))dz &= -\int_\sigma \sum_{j=1}^{s} \partial_{x_j}(x_j \phi(\pi(z)))dz \\
&= -s \int_\sigma \phi(\pi(z))dz - \int_\sigma \sum_{j=1}^{s} x_j \partial_{x_j}(\phi(\pi(z)))dz \\
&= -s \int_\sigma \phi(\pi(z))dz - \int_\sigma \sum_{j=1}^{m} z_j \partial_{z_j}(\phi \circ \pi(z)))dz \\
&= (m-s) \int_\sigma \phi(\pi(z))dz - \int_\sigma \sum_{j=1}^{s} \partial_{z_j}(z_j(\phi \circ \pi)(z)))dz \\
&= (m-s) \int_\sigma \phi(\pi(z))dz - \sum_i \int_{\sigma_i} \langle z|n_i\rangle (\phi \circ \pi)(z)dz
\end{aligned}$$

and as $\langle z|n_i\rangle$ is constant on each σ_i, we're done. Next we observe that $m - s = v(\sigma)$ and $(\theta - \partial_{\pi(z)})\sigma_\pi(\pi(z)) = 0$, and expanding this we get

$$\begin{aligned}
0 &= (\theta - \partial_{\pi(z)})\sigma_\pi(\pi(z)) \\
&= v(\sigma)\sigma_\pi(\pi(z)) - \sum_i \langle k_i|n_i\rangle(\sigma_i)_\pi(\pi(z)) + \sum_i \langle \pi(z)|n_i\rangle(\sigma_i)_\pi(\pi(z)),
\end{aligned}$$

which rephrased in distribution form is

$$v(\sigma)\pi_*\delta_\sigma = \sum_i \langle k_i - \sum_{j \le s} x_j|n_i\rangle \pi_*\delta_{\sigma_i}$$

that is, statement (iii) above. □

Remark 6.3. By restricting π to the appropriate H_σ and composing with a coordinate change we see that the above holds for σ of any dimension, not merely the top-dimensional ones. Explicitly, if we factor $\pi = p \circ \tau$, where p is the projection on the first coordinates as above and τ is an isomorphism, we have

$$\pi_*\delta_\sigma(f) = \int_\sigma f(p(\tau(z)))dz = \int_{\tau(\sigma)} f(p(y))\frac{dy}{|\det(\tau)|} = \frac{1}{|\det(\tau)|}p_*\delta_\sigma(f).$$

Definition 6.4. Let F'_k be the submodule of S_K generated by those $\pi_*\delta_\sigma$ with $\dim(\pi(\sigma)) \le k$. These form a filtration of S_K, with filtration quotients Q'_k generated by those $\pi_*\delta_\sigma$ with $\dim(\pi(\sigma)) = k$. We call this filtration the *skeleton filtration on* S_K.

We can now, exactly parallel to 5.19 use 6.2(iii) to reduce the generators for Q'_k to those $\pi_*\delta_\sigma$ with $\dim(\sigma) = k, v(\sigma) = 0$, and then 6.2$(iii)$ to further eliminate some generators, and express Q'_k as a direct sum of $\delta_{\pi(\sigma)}$'s. Just as before, the number of summands is given by the k'th singular homology of K_k. Further, we may produce a presentation

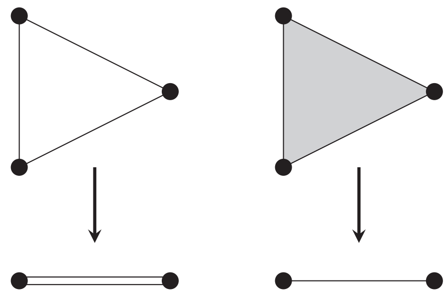

Figure 6.1: Left: empty interior, gives 'double line'; right: filled interior, gives single line. $\pi^0_+ M_K$ respects this difference, S_K always gives the filled-interior situation.

$D^r_Y \to D^c_Y \to S_K$ in the same way as in 5.19 (we omit the tedious repetition of the argument).

However, this is not entirely true: At this point we must point out a crucial difference between $\pi_*\delta_\sigma$ and the $\overline{\delta_\sigma}$: 6.2(ii) and 5.6(ii) both essentially say that $\partial_j\delta_\sigma = 0$ for $j > s$, which means a certain linear combination of the boundary cells δ_{σ_i} is zero. The important observation is that 6.2(ii) applies even if the σ in question is *not* in K, while 5.6(ii) does not. The reason is obvious: the $\pi_*\delta_{\sigma_i}$, being concrete distributions, inherit that relation from $S_{\sigma\cup\bigcup_i \sigma_i}$, while the abstract generators $\overline{\delta_\sigma}$ are not so lucky (see Figure 6.1).

This means that the number of summands of Q'_k is equal to the dimension of $H^{BM}_k(\overline{K_k})$, where $\overline{K}$ is the cell complex we get from K by 'filling in' all the relevant holes. The consequence for the presentation $D^r_Y \to D^c_Y \to S_K$ is that we must increase the count of relations, to add some of type 6.2(ii) for each hole in K we fill in.

Let us try to make this a bit more precise. Exactly what extra identifications are made in S_K? We have seen that the determining factor for the number of generators is the k'th closed support homology of the subcomplex K_k. This suggests that it is the subcomplex of K consisting of cells with $v(\sigma) \leq 1$ that determines what remains in the projection, so it is enough to add relations for the 'missing' cells with $v(\sigma) = 1$.

The extra relations are all of the type 6.2(ii), that is they are induced by a relation $\partial_z\overline{\delta_\sigma} = 0$ that exists in the complex $\sigma \cup \bigcup_{\sigma_i\subset\partial\sigma} \sigma_i$. In other words, we get such a relation if we can express the σ_i's as the boundary of a cell σ with $\dim(\sigma) = \dim(\sigma_i) + 1$, and $v(\sigma) = v(\sigma_i) + 1$.

There are two cases: first, if σ_1 and σ_2 are cells of the same dimension with $v = 0$, such that $\pi(\sigma_1) = \pi(\sigma_2)$, they are the boundary of a cell with $v = 1$ filling the space between them, and so are identified in S_K. The second, slightly more complicated, case is when $K' \subset K$ is a compact subcomplex with $\partial K' = 0$, with $\dim K' = \dim(\pi(K')) = i$ (that is, with $v(K') = 0$, per 5.5), and K' is minimal among such subcomplexes. Then K' is the boundary of a cell σ with $\dim(\sigma) = i + 1$ and $v(\sigma) = 1$, so in S_K we get a relation between the cells in K' of type 6.2(ii).

Let us summarise this:

Definition 6.5. We say that K is π-*connected* if it satisfies the following:

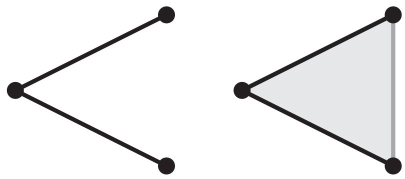

Figure 6.2: The left complex is not π-*connected*. Adding the grey cells produces a complex that is π-*connected*.

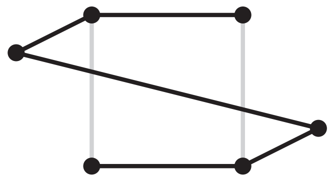

Figure 6.3: This complex is not π-*connected*, and adding the missing rectangle (marked by grey lines) cannot be done without violating the cell complex structure.

(I) For all cells $\sigma_1, \sigma_2 \subset K$ such that $v(\sigma_1) = v(\sigma_2) = 0$ and $\pi(\sigma_1) = \pi(\sigma_2)$, there exists a cell $\sigma \subset K$ with $v(\sigma) = 1$, $\pi(\sigma) = \pi(\sigma_1)$ such that $\partial(\sigma) = \sigma_1 - \sigma_2$,

(II) For all compact subcomplexes $K' \subset K$ with $\dim(K') = i$, $v(K') = 0$, $\partial K' = 0$ and having no proper compact subcomplex K'' with the same conditions, there exists a cell $\sigma \subset K$ with $\dim(\sigma) = i + 1$ and $v(\sigma) = 1$ such that $\partial\sigma = K'$.

Example 6.6. Consider the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^1$ on the first coordinate. The left complex in Figure 6.2 is not π-*connected*, the right complex is.

There are also complexes K where $\pi^0_+ M_K$ and S_K differ that can not be fixed by the simple addition of cells, figure 6.3 provides an example (with the same π as above) where no appropriate cell can be added without violating the cell complex structure of K.

We now stand ready to prove a main result:

Theorem 6.7. *There is a canonical surjection $\pi^0_+ M_K \twoheadrightarrow S_K$, given by $\overline{\delta_\sigma} \mapsto \pi_* \delta_\sigma$, which is an isomorphism when K is* π-connected.

Proof. There are two claims to prove: that the given map is surjective, and that under certain conditions it is an isomorphism.

That the map is well-defined and surjective is clear from the following diagram:

$$\begin{tikzcd}
D^r_Y \arrow[r] \arrow[d, hook] & D^c_Y \arrow[r, two heads] \arrow[d, equal] & \pi^0_+ M_K \arrow[d, dotted] \\
D^{r+r'}_Y \arrow[r] & D^c_Y \arrow[r, two heads] & S_K
\end{tikzcd}$$

When K is π-*connected*, there are no extra relations for S_K to satisfy, so the left vertical map is an isomorphism, and the two rows in the diagram above are identical. $\square$

Remark 6.8. It is of course clear that the π-*connected* condition is satisfied for nice K, e.g. convex or with connected fibers. Indeed, the name π-*connected* is due to the condition being very close to that of K having connected fibers.

7 Closing remarks

We note that almost everything in our description of the module M_K still works with only minor modifications if we replace our complex K of polyhedral cells (defined by linear equations) with a complex of semi-algebraic sets, with similar restrictions on cell complex structure. As in the standard relations of 2.2, we get relations given by tangent vector fields sending a δ_σ to its boundary cells, and defining equations annihilating. Skeleton filtrations are defined in exactly the same way, and from there everything else follows. The local point of view is essential throughout, and there is a possible issue with singularities. There are no problems if the defining varieties are smooth, but the author has not examined what may go wrong otherwise.

The constructions given here may lead to convenient descriptions of the annihilator ideals of various δ_σ. This may be of use for performing computer calculations, as current software is better adapted to the use of D-ideals than to the use of presentations like 3.1.

Perhaps also interesting is a possible connection to the large existing body of problems involving polytopes (usually convex, but not necessarily). It is not immediately clear if the D-module description presented here is of any use in attacking these problems, but to the author's admittedly limited knowledge, tools of this kind have not been used in this setting.

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