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Topics in analysis - Modulus of continuity of mappings between Euclidean spaces - Toeplitz operators on the Bergman spaces of the unit ball in $\mathbb{C}^N$

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TOPICS IN ANALYSIS:
- MODULUS OF CONTINUITY OF MAPPINGS BETWEEN EUCLIDEAN SPACES
-TOEPLITZ OPERATORS ON THE BERGMAN SPACES OF THE UNIT BALL IN $\mathbf{C}^N$.

By

Agbor Dieudonne Agbor

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Dedication

This work is dedicated to the Almighty God, to my wife Marie and my children Nema, Grace, Glenda, Dieudonne Junior and Rhema, my parents Late Mr. Akpey Samuel and Mrs Agbor Emilia.

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Table of Contents

Abstract

Let f be a function from $\mathbf{R}^p$ to $\mathbf{R}^q$ and let Λ be a finite set of pairs $(\theta, \eta) \in \mathbf{R}^p \times \mathbf{R}^q$. For each $(\theta, \eta) \in \Lambda$, the function $x \mapsto \langle \eta, f(x) \rangle$ has certain regularity properties in the direction θ, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^q$. We determine (algebraic) conditions on Λ in order that these regularity properties on $\langle \eta, f(x) \rangle$ in the direction θ imply the same regularity properties for the function f.

Finally, we determine L^1-symbols whose associated Toeplitz operators are bounded (resp. compact) on the Bergman spaces $L^p_a(\mathbf{B}_n, d\nu)$, for $1 \leq p < \infty$, where $\mathbf{B}_n$ is the unit ball of $\mathbf{C^n}$. For $p > 1$, we give a new presentation and improvements of earlier results. For $p = 1$, our results are new and are related to the reproducing kernel thesis and to earlier results.

Keywords: Modulus of continuity, Fourier transform, Toeplitz operators, Berezin transform, Carleson measure, Bergman spaces.

Part I

Modulus of Continuity of Mappings between Euclidean Spaces

Chapter 1

Measures with Finite Total Mass

1.1 Preliminaries

The space $L^1(\mathbf{R}^p)$ consist of all measurable functions, f, on $\mathbf{R}^p$ such that

$$\|f\|_1 = \int_{\mathbf{R}^p} |f(x)| dx < \infty.$$

If $f \in L^1(\mathbf{R}^p)$ then the Fourier transform $\hat{f}$ is the bounded continuous function on $\mathbf{R}^p$ defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^p} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbf{R}^p.$$

If $\hat{f}$ also happens to be integrable one can express f in terms of $\hat{f}$ by the Fourier inversion formula,

$$f(x) = (2\pi)^{-p} \int_{\mathbf{R}^p} e^{-ix\cdot\xi} \hat{f}(\xi) d\xi. \tag{1.1}$$

The space $C_0^\infty(\mathbf{R}^p)$ consist of all smooth functions with compact support. By the Schwartz class, $S = S(\mathbf{R}^p)$, we mean the set of all smooth (infinitely differentiable) functions, ψ, such that

$$\sup_x |x^\beta \partial^\alpha \psi(x)| < \infty$$

for all multi-indices α and β.

An important concept about the space S, is that the Fourier transform $F : \phi \to \hat{\phi}$ is an isomorphism of the space S with inverse given by the inversion formula (1.1), see for example Theorem 7.1.5 of [26].

Let $C_o(\mathbf{R}^p)$ be the space of continuous functions on $\mathbf{R}^p$ that tend to zero at infinity equipped with the supremum norm. The space $M(\mathbf{R}^p)$ is the dual space of $C_o(\mathbf{R}^p)$, which is the space of measures on $\mathbf{R}^p$ with finite total mass. The action of a linear form μ on test functions will be denoted by $\langle \mu, \varphi \rangle$. Thus $\mu \in M(\mathbf{R}^p)$ if and only if $|\langle \mu, \varphi \rangle| \leq C \sup |\varphi|$ for any $\varphi \in C_o(\mathbf{R}^p)$, where C is a positive constant and the norm of $\mu \in M(\mathbf{R}^p)$ will be denoted by denoted by $\|\mu\|_M$, which is given by

$$\|\mu\|_M = \sup\{|\langle \mu, \varphi \rangle|; \varphi \in C_o(\mathbf{R}^p), \|\varphi\| = \sup |\varphi| \leq 1\}.$$

The space $L^1(\mathbf{R}^p)$ is identified with a subspace of $M(\mathbf{R}^p)$ by $f \in L^1(\mathbf{R}^p)$ being identified with the linear form

$$\varphi \mapsto \int_{\mathbf{R}^p} f(x)\varphi(x)dx.$$

Any element $\mu \in M(\mathbf{R}^p)$ can be uniquely extended as a linear form to the space $C_b(\mathbf{R}^p)$ of bounded continuous functions on $\mathbf{R}^p$. Indeed, if μ is considered as a linear form on $C_o(\mathbf{R}^p)$, we take a compactly supported continuous function χ that is equal to 1 in some neighbourhood of the origin and define

$$\langle \mu, \varphi \rangle = \lim_{A \to \infty} \langle \mu, \chi(\cdot/A)\varphi \rangle, \ \varphi \in C_b(\mathbf{R}^p).$$

If $\mu \in M(\mathbf{R}^p)$ then the Fourier transform of μ is given by

$$\widehat{\mu}(\xi) = \int_{\mathbf{R}^p} e^{-ix\cdot\xi} d\mu(x) = \mu(e^{-ix\cdot\xi}).$$

Therefore if μ is a point mass at the point $a \in \mathbf{R}^p$, that is, $\langle \mu, \varphi \rangle = \varphi(a)$ for all test functions φ, then $\hat{\mu}(\xi) = e^{-ia\cdot\xi}$.

Let $f \in L^1(\mathbf{R}^p)$ and $\mu \in M(\mathbf{R}^p)$. Then the convolution of f and μ, $\mu * f$, is given by

$$(\mu * f)(x) = \int_{\mathbf{R}^p} f(x-y)\, d\mu(y)$$

and

$$\|\mu * f\|_1 \leq \|\mu\|_M \|f\|_1.$$

If $\mu \in M(\mathbf{R}^p)$ then μ is a difference measure in the direction θ if

$$\langle \mu, \varphi \rangle = \varphi(\theta) - \varphi(0) \tag{1.2}$$

and its Fourier transform is

$$e^{-i\theta\cdot\xi} - e^0 = e^{-i\theta\cdot\xi} - 1.$$

Also its convolution with a continuous function, f, gives

$$(\mu * f)(x) = f(x-\theta) - f(x).$$

We will also be very interested in the following inclusion which can easily be verified.

$$C_0^\infty(\mathbf{R}^p) \subset S(\mathbf{R}^p) \subset L^1(\mathbf{R}^p) \subset M(\mathbf{R}^p).$$

Notation.

We will denote the set of Fourier transforms of a space X by $\widehat{X}$. Also, we will always use the following identity:

$$\|f\|_{\widehat{L}^1} = \|\hat{f}\|_1.$$

The next lemma will be used always.

1.1.1 Lemma

Suppose χ is smooth function on $\mathbf{R^p}$ which is a constant outside some compact set containing the origin. Then χ is the Fourier transform of a measure in $M(\mathbf{R}^p)$.

Proof. We may assume $\chi(x) = 1$ for $|x| > 1$. Then

$$\chi(x) = 1 - \psi(x) \ \Leftrightarrow \psi(x) = 1 - \chi(x), \quad x \in \mathbf{R}^p$$

where $\psi \in C_0^\infty(\mathbf{R}^p)$ with the support of ψ been $|x| \leq 1$. $\psi \in C_0^\infty(\mathbf{R}^p)$ implies $\psi \in S(\mathbf{R}^p)$ and thus ψ is the Fourier transform of a function $\phi \in S(\mathbf{R}^p)$, that is $\hat{\phi} \in S(\mathbf{R}^p) \subset M(\mathbf{R}^p)$. Thus

$$\chi = 1 - \hat{\phi} = (\widehat{\delta_0 - \phi})$$

where δ_0 is the Dirac measure, and hence $\chi \in \widehat{M}(\mathbf{R}^p)$. $\square$

Let $M_\theta = M_\theta(\mathbf{R}^p)$ denote the subset of $M(\mathbf{R}^p)$ consisting of measures with mean equal zero on all lines with direction $\theta \in \mathbf{R}^p$.

1.1.2 Definition

$\mu \in M_\theta$, if and only if $\langle \mu, \varphi \rangle = 0$ for all test functions, φ, that are constant in direction θ , or equivalently,

$$\varphi(x + t\theta) - \varphi(x) = 0 \text{ for all } \ t \in \mathbf{R}, x \in \mathbf{R}^p. \tag{1.3}$$

Let M_θ^c denote the set of μ in M_θ with compact support.

1.1.3 Lemma

$\mu \in M_\theta$ if and only if $\widehat{\mu}(\xi) = 0$ whenever $\theta \cdot \xi = 0$.

Proof Let $\mu \in M_\theta$. If $\theta \cdot \xi = 0$, then the function $\psi(x) = e^{-ix\cdot\xi}$ satisfies (1.3) and hence $\widehat{\mu}(\xi) = \mu(e^{-ix\cdot\xi}) = 0$.

The converse follows from the fact that a continuous function $\varphi : \mathbf{R^p} \to \mathbf{R}$ such that $\varphi(x + t\theta)$ is independent of t for every x can be approximated uniformly on compact sets by linear combinations of exponential functions $e^{i\xi\cdot\theta}$ such that $\theta \cdot \xi = 0$.

For $\theta \in \mathbf{R^p}$, $\theta \neq 0$, let $N_\theta(\mathbf{R^p})$ denote the set of measures $\mu \in M(\mathbf{R^p})$ for which

$$\frac{\widehat{\mu}(\xi)}{\theta \cdot \xi} \in \widehat{M}(\mathbf{R^p}).$$

It is clear that $N_\theta(\mathbf{R^p}) \subset M_\theta(\mathbf{R^p})$, but the opposite inclusion is not true as seen by the following example, in the case $p = 1$. Let

$$h(x) = \pi^{-1}/(1 + x^2) - \psi(x),$$

where $\psi \in L^1(\mathbf{R})$ is even , has compact support and $\int_{\mathbf{R}} \psi(x)dx = 1$. Then

$$\hat{h}(\xi) = e^{-|\xi|} - \hat{\psi}(\xi).$$

Thus $\hat{h}(0) = 0$ and $\hat{h}(\xi) = -|\xi| + \bigcirc(|\xi|^2)$ as $|\xi| \to 0$, hence $\hat{h}(\xi)/\xi$ is discontinuous at the origin and therefore cannot be in $\widehat{M}(\mathbf{R})$. Our next lemma shows that this inclusion is possible for some subsets of $M_\theta(\mathbf{R^p})$.

1.1.4 Lemma

If $\mu \in M_\theta^c$ then $\mu \in N_\theta(\mathbf{R^p})$.

Proof Let $\mu \in M_\theta^c$, supp$\mu = K$, $\theta = (1, 0, \cdots, 0)$ and let φ be a test function. Define the function

$$\psi(x_1, y) = \int_{-\infty}^{x_1} \varphi(s, y)\, ds$$

where $y = (x_2, x_3, \cdots, x_p)$. Define a measure ω by

$$\langle \omega, \varphi \rangle = -\langle \mu, \psi \rangle \tag{1.4}$$

then $\omega \in M(\mathbf{R}^p)$. To prove that $\omega \in M(\mathbf{R}^p)$ we write $\varphi = \varphi_0 + \varphi_1$ where $\text{supp}\varphi_0 \subset V$, $K \subset V$ and $\varphi_1 = 0$ in some neighbourhood of K. Let

$$\psi_0(x_1, y) = \int_{-\infty}^{x_1} \varphi_0(s, y)\, ds \text{ and } \psi_1(x_1, y) = \int_{-\infty}^{x_1} \varphi_1(s, y)\, ds.$$

We claim that $\langle \omega, \varphi_1 \rangle = 0$. Indeed, since $\partial_1 \psi_1 = 0$ in a neighbourhood of K, it implies ψ_1 is a constant in the direction θ which implies $\langle \omega, \varphi_1 \rangle = -\langle \mu, \psi_1 \rangle = 0$, thus the claim is proved. Also, since

$$|\langle \omega, \varphi_0 \rangle| = |\langle \mu, \psi_0 \rangle| \leq C \sup |\psi_0| \leq C_1 \sup |\varphi_0|$$

where $C_1 = C\text{diam}(\text{supp}\varphi_0)$ we have that $\omega \in M(\mathbf{R}^p)$. Finally, (1.4) implies $\partial_1 \omega = \mu$ if and only if $i\xi_1 \widehat{\omega}(\xi) = \widehat{\mu}(\xi)$ as required. The result follows by rotation of the axis. $\square$

1.1.5 Lemma

Let χ be a smooth function on $\mathbb{R}$ that is zero in some neighborhood of the origin and is 1 outside some compact set containing the origin. Then $\frac{\chi(\xi)}{\xi} \in \widehat{M}(\mathbf{R})$.

Proof Let H be the Heaviside function, that is $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \leq 0$ and let $\psi = 1 - \chi$, that is, ψ is smooth and compactly support. Consider the function ν given by

$$\nu(x) = H(x) - \frac{1}{2\pi} \int_{-\infty}^{x} \widehat{\psi}(y)\, dy.$$

We claim that $\nu \in L^1$. To prove this claim we first observe that $\psi(0) = 1$ implies

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\psi}(y)\, dy = 1.$$

Thus

$$\begin{aligned}\nu(x) &= H(x) - \frac{1}{2\pi}\int_{-\infty}^{x} \widehat{\psi}(y)\,dy \\ &= H(x) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\psi}(y)\,dy + \frac{1}{2\pi}\int_{x}^{\infty} \widehat{\psi}(y)\,dy.\end{aligned}$$

This shows that

$$\nu(x) = \begin{cases} \frac{1}{2\pi}\int_x^\infty \widehat{\psi}(y)\,dy & \text{when } x > 0, \\ -\frac{1}{2\pi}\int_{-x}^\infty \widehat{\psi}(-y)\,dy & \text{when } x \leq 0, \end{cases}$$

since

$$-1 + \frac{1}{2\pi}\int_x^\infty \widehat{\psi}(y)\,dy = -\frac{1}{2\pi}\int_{-x}^\infty \widehat{\psi}(-y)\,dy, \quad \text{for} \quad x \leq 0.$$

That $\nu \in L^1$ follows from the fact that $\widehat{\psi}$ is in the Schwartz class, that is, $|\widehat{\psi}(y)| \leq C(1+|y|)^{-a},\ a > 2$. We now observe that $-\nu' = \delta_0 - \widehat{\psi}$, indeed, if Ψ is a test function and $\mu(x) = 2\pi\int_{-\infty}^{x} \widehat{\psi}(y)\,dy$ then

$$\begin{aligned}\langle \nu, \Psi' \rangle &= \langle H, \Psi' \rangle - \langle \mu, \Psi' \rangle \\ &= -\langle H', \Psi \rangle + \langle \mu', \Psi \rangle \\ &= -\langle \delta, \Psi \rangle + \langle \mu', \Psi \rangle.\end{aligned}$$

This implies, $\nu' = -\delta_0 + \widehat{\psi}$. Thus $-i\xi\widehat{\nu}(\xi) = 1 - \psi(\xi) = \chi(\xi)$ which proves the lemma.

□

Lemma 1.1.5 can be refined to get the following;

1.1.6 Lemma

Let $\mu \in M(\mathbf{R})$ and assume that $\hat{\mu}(\xi)$ is twice continuously differentiable in some neighborhood of the origin and that $\hat{\mu}(0) = 0$. Then $\hat{\mu}(\xi)/\xi$ is in $\widehat{M}(R)$.

Proof. Let $\chi \in C^\infty$, equal 1 for $|\xi| \geq \epsilon/2$ and vanishing in a smaller neighborhood of the origin and let $1 - \chi = \psi$. Then $\hat{\mu}(\xi) = \chi(\xi)\hat{\mu}(\xi) + \psi(\xi)\hat{\mu}(\xi)$. For $\chi(\xi)\hat{\mu}(\xi)$ the conclusion of the lemma follows by Lemma 1.1.5 since $\chi(\xi)/\xi \in \widehat{M}(\mathbf{R})$. Also since $\psi(\xi)\hat{\mu}(\xi)$ is compactly supported, twice continuously differentiable and equals zero at $\xi = 0$, we see that $\tau(\xi) = \psi(\xi)\hat{\mu}(\xi)/\xi$ is continuously differentiable and compactly supported. Thus τ' is compactly supported, belongs to L^2, and we have that $\widehat{\tau}'$ is bounded. By Plancherel, $\widehat{\tau}' \in L^2$. It is easy to see by applying Schwartz inequality that $\widehat{\tau} \in L^1$ or $\tau \in \widehat{L}^1 \subset \widehat{M}(\mathbf{R})$. This completes the proof of the lemma. $\square$

1.1.7 Lemma

Let r be a positive integer greater than 1 and let the function f be homogeneous of degree r, and smooth on $\mathbf{R}^p/\{0\}$ and let the function g be smooth and compactly supported. Then $fg \in \widehat{M}(\mathbf{R}^p)$.

Proof It is clear that the $p+1^{th}$ partial derivatives of f will be homogenous of degree at least $r-p-1$. Thus if $h = fg$ then

$$|\partial_k^{p+1} h| \leq C(|g(x)||x|^{r-p-1} + |\partial_k g(x)||x|^{r-p} + \cdots + |\partial_k^p g(x)||x|^{r-1} + |\partial_k^{p+1} g(x)||x|^r).$$

Now, since $\widehat{\partial_k^{p+1} h}(\xi) = \xi_k^{p+1}\widehat{h}(\xi)$ together with the fact that $r-p-1 > -p$ and g is compactly supported we have

$$\begin{aligned} |\xi_k^{p+1}\widehat{h}(\xi)| &\leq \int_{\mathbf{R}^p} |\partial_k^{p+1} h(x)|\, dx \\ &\leq \int_{\mathbf{R}^p} C(|g(x)||x|^{r-p-1} + |\partial_k g(x)||x|^{r-p} + \cdots + |\partial_k^{p+1} g(x)||x|^{r-p})\, dx \\ &\leq C' \end{aligned}$$

for every k. Thus

$$(1 + |\xi_k^{p+1}|)|\widehat{h}(\xi)| \leq C'$$

for every k. This implies

$$|\widehat{h}(\xi)| \leq C'(1 + |\xi_k|^{p+1})^{-1}$$

for all k. Thus there exists a constant C'' such that

$$|\widehat{h}(\xi)| \leq C''(1 + |\xi|^{p+1})^{-1}$$

which implies $\widehat{h}(\xi) \in L^1$ and hence $h \in \widehat{M}(\mathbf{R}^p)$. $\square$

1.2 Modulus of Continuity

Let σ be a function defined on the non-negative reals which tends to zero at the origin.

1.2.1 Definition

$K(\sigma)$ denotes the set of functions $f : \mathbf{R^p} \to \mathbf{R^q}$ such that to every compact subset $K \subset \mathbf{R^p}$ there exists a constant C such that for x and $x + y \in K$

$$|f(x+y) - f(x)| \leq C\sigma(\epsilon),\ if\ |y| \leq \epsilon.$$

If $0 \neq \theta \in \mathbf{R^p}$, we denote by $K(\theta, \sigma)$ the set of functions $f : \mathbf{R^p} \to \mathbf{R^q}$ such that to every compact subset $K \subset \mathbf{R^p}$ there exists a constant C such that for x and $x + t\theta \in K$, t real

$$|f(x+t\theta) - f(x)| \leq C\sigma(\epsilon),\ if\ |t| \leq \epsilon.$$

If $f \in K(\sigma)$ we say that f has modulus of continuity $\leq \sigma(\epsilon)$ or simply f has modulus of continuity $\sigma(\epsilon)$ while $f \in K(\theta, \sigma)$ will mean f has modulus of continuity $\sigma(\epsilon)$ in the direction θ.

Denote by Σ the set of all real-valued continuous subadditive and increasing functions from $\{t \in \mathbf{R}; t > 0\}$ into itself, tending to zero at the origin. We observe that any class $K(\sigma)$ is equal to some $K(\sigma_1)$ where $\sigma_1 \in \Sigma$. Indeed, we can take

$$\sigma_1(\epsilon) = \inf \left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq \epsilon, \epsilon_i \geq 0 \right\} \tag{1.5}$$

which is the largest subadditive and increasing lowerbound of σ. We first show that σ_1 is subadditive an increasing.

Subadditivity. We observe that if

$$E_1 = \left\{ z = \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq s + t, \epsilon_i \geq 0 \right\}$$

and

$$E_2 = \left\{ z = \sum_{k=1}^{p} \sigma(\epsilon_k) + \sum_{j=1}^{q} \sigma(\epsilon_j); \sum_{k=1}^{p} \epsilon_k \geq s, \sum_{j=1}^{q} \epsilon_j \geq t \quad \epsilon_j, \epsilon_k \geq 0, \ p + q = n \right\},$$

then $E_1 \supset E_2$. Thus

$$\begin{aligned} \sigma_1(s+t) &= \inf\{z; z \in E_1\} \leq \inf\{z; z \in E_2\} \\ &= \inf \left\{ \sum_{k=1}^{p} \sigma(\epsilon_k); \sum_{k=1}^{p} \epsilon_k \geq s, \quad \epsilon_k \geq 0 \right\} \\ &+ \inf \left\{ \sum_{j=1}^{q} \sigma(\epsilon_j); \sum_{j=1}^{q} \epsilon_j \geq t, \ \epsilon_j \geq 0 \right\} = \sigma_1(s) + \sigma_1(t). \end{aligned}$$

Monotone increasing. Let $s > t$. Then

$$\left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq s, \epsilon_i \geq 0 \right\} \subset \left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq t, \epsilon_i \geq 0 \right\}.$$

Thus

$$\begin{aligned} \sigma_1(s) &= \inf \left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq s, \epsilon_i \geq 0 \right\} \\ &\geq \inf \left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq t, \epsilon_i \geq 0 \right\} \\ &= \sigma_1(t). \end{aligned}$$

Now, let $f \in K(\sigma_1)$. Then $|f(x+y) - f(x)| \leq C\sigma_1(t)$ whenever $|y| \leq t$. But we know that

$$\sigma_1(t) = \inf \left\{ \sum_{i=1}^{n} \sigma(\epsilon_i); \sum_{i=1}^{n} \epsilon_i \geq t, \epsilon_i \geq 0 \right\}.$$

Thus if we take $\sum_{i=1}^n \sigma(\epsilon_i) = \epsilon_1 = t$, then $\sum_{i=1}^n \sigma(\epsilon_i) = \sigma(\epsilon_1) = \sigma(t)$. By the definition of $\sigma_1(t)$ we must have $\sigma_1(t) \leq \sigma(t)$ and thus $f \in K(\sigma)$. Conversely, suppose $f \in K(\sigma)$. Let $\omega_f(t)$ be the modulus of continuity of f, defined by

$$\omega_f(t) = \sup\{|f(x+y) - f(x)|; |y| \leq t\}.$$

Then $\omega_f(t)$ is subadditive and increasing. Indeed, if $y = y_1 + y_2$ then by the triangle inequality

$$\begin{aligned} |f(x+y) - f(x)| &= |f(x+y_1+y_2) - f(x+y_1) + f(x+y_1) - f(x)| \\ &\leq |f(x+y_1+y_2) - f(x+y_1)| + |f(x+y_1) - f(x)|. \end{aligned}$$

Thus if $|y| \leq s+t$ we can choose y_1 and y_2 such that $|y_1| \leq s$ and $|y_2| \leq t$. And hence,

$$\omega_f(t+s) \leq \omega_f(t) + \omega_f(s)$$

that is $\omega_f(t)$ is subadditive. Also if $t > s$ then

$$\{|f(x+y) - f(x)|; |y| \leq t\} \subset \{|f(x+y) - f(x)|; |y| \leq s\},$$

and thus $\omega_f(t) \leq \omega_f(s)$. Now, since σ_1 is the largest subadditive and increasing lower-bound of $\sigma(t)$, we see that if $\omega_f(t) \leq \sigma(t)$ then $\omega_f(t)$ is subadditve and increasing lowerbound of $\sigma(t)$. But $\sigma_1(t)$ is the largest of all subadditive and increasing lower-bounds of $\sigma(t)$. Hence it must in particular be larger than $\omega_f(t)$, that is, $\omega_f(t) \leq \sigma_1(t)$. This shows that if $f \in K(\sigma)$ then $f \in K(\sigma_1)$.

We will often use the following simple inequality.

1.2.2 Lemma

Let $\sigma \in \Sigma$, $a > 0$, $t > 0$, and $[a]$ denotes the integral part of a. Then

$$\sigma(at) \leq \sigma(([a]+1)t) \leq (a+1)\sigma(t).$$

Proof Since σ is subadditive, $\sigma(2t) \le 2\sigma(t)$ and thus by induction $\sigma(nt) \le n\sigma(t)$. Also since $at \le ([a]+1)t$ and σ is increasing we have

$$\sigma(at) \le \sigma(([a]+1)t) \le ([a]+1)\sigma(t) \le (a+1)\sigma(t).\Box$$

If $\sigma \in \Sigma$, we set

$$\hat{\sigma}(\epsilon) = \epsilon \left\{ \sigma(1) + \int_{\min(\epsilon,1)}^{1} t^{-2}\sigma(t)\, dt \right\},\ \epsilon > 0. \tag{1.6}$$

1.2.3 Lemma

If $\sigma \in \Sigma$, then $\hat{\sigma} \in \Sigma$ and $\sigma(t) \le \hat{\sigma}(t)$, for $0 < t < 1$.

Proof. We first show that $\sigma(t) \le \hat{\sigma}(t)$, for $0 < t < 1$. Since σ is increasing, we have

$$t\sigma(t) \int_t^1 s^{-2} ds \le t \int_t^1 s^{-2}\sigma(s) ds$$

which implies

$$(1-t)\sigma(t) \le t \int_t^1 s^{-2}\sigma(s) ds,\ 0 < t < 1.$$

Thus

$$\begin{aligned} \sigma(t) &\le t \int_t^1 s^{-2}\sigma(s) ds + t\sigma(t) \\ &\le t \left\{ \int_t^1 s^{-2}\sigma(s) ds + \sigma(1) \right\} = \hat{\sigma}(t), \quad 0 < t < 1 \end{aligned}$$

since σ is increasing.

We show that $\hat{\sigma}(t) \to 0$ as $t \to 0$. If $0 < t < \delta < 1$ then

$$t \int_t^1 s^{-2}\sigma(s) dt \le \sigma(\delta) t \int_t^\delta s^{-2} ds + t \int_\delta^1 s^{-2}\sigma(s) ds$$

since σ is increasing. Thus given $\epsilon > 0$ we can find a $\delta \in (0,1)$ such that $\sigma(\delta) < \epsilon/2$. For such a δ we can make the second term on the right hand side of the previous equation smaller then $\epsilon/2$ as $t \to 0$. We thus get

$$t \int_t^1 s^{-2} \sigma(s) ds \to 0 \text{ as } t \to 0.$$

Subadditivity of $\hat{\sigma}$ follows from the fact that if $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is decreasing, then the function $g(t) = t\psi(t)$ is subadditive.

Finally, we will show that $\hat{\sigma}$ is increasing. Now

$$\hat{\sigma}'(t) = \sigma(1) + \int_t^1 s^{-2} \sigma(s) ds - \sigma(t)/t, \ 0 < t < 1.$$

Which implies

$$\begin{aligned} t\hat{\sigma}'(t) &= \hat{\sigma}(t) - \sigma(t) \geq t\sigma(1) + \sigma(t) \left(t \int_t^1 s^{-2} ds - 1 \right) \\ &= t(\sigma(1) - \sigma(t)). \end{aligned}$$

Thus $\hat{\sigma}'(t) \geq \sigma(1) - \sigma(t) > 0$ for $0 < t < 1$ since σ is increasing. Thus $\hat{\sigma}'(t) > 0$ and hence $\hat{\sigma}$ is increasing.

Now if $\sigma(\epsilon) = \epsilon^p, \ 0 < p < 1$, then $\hat{\sigma}(\epsilon) = \epsilon\sigma(1) + (\epsilon^p - \epsilon)(1-p)$ when $\epsilon < 1$, hence $\hat{\sigma} \approx \sigma$. If $\sigma(\epsilon) = \epsilon$, $f \in K(\sigma)$ is said to be Lipschitz continuous. Furthermore $\hat{\sigma}(\epsilon) = \epsilon\sigma(1) + \epsilon \log(1/\epsilon)$, that is $\hat{\sigma}(\epsilon) \approx \epsilon \log(1/\epsilon)$ which shows that sometimes $K(\sigma)$ is a proper subset of $K(\hat{\sigma})$.

Set $\Delta_\theta \psi(x) = \psi(x+\theta) - \psi(x)$ and define $\Delta_\theta^k \psi(x)$ recursively by

$$\Delta_\theta^j \psi(x) = \Delta_\theta \Delta_\theta^{j-1} \psi(x).$$

1.2.4 Definition

Let g be a real-valued function defined on a compact subset K of $\mathbf{R^p}$ and k a natural number. Then the k^{th} order difference is

$$\Delta_h^k g(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{j+k} g(x+jh)$$

and the modulus of smoothness of order k is

$$\omega_k(g,t) = \sup\{|\Delta_h^k g(x)|; x \in K, x+kh \in K, |h| \leq t,\ t > 0\}.$$

When $k=1,\ \omega_k(g,t)$ is the modulus of continuity of g. For some basic properties of $\omega_k(g,t)$ see [39].

The generalized modulus of continuity will also be useful to us, which we now define. For $t > 0$ and $\mu \in M(\mathbf{R}^p)$ we denote by μ_t the dilation of μ defined by $\widehat{\mu}_t(\xi) = \widehat{\mu}(t\xi)$.

1.2.5 Definition

Let $\mu \in M(\mathbf{R}^p)$. Then the generalized modulus of continuity $\omega_\mu(f,t)$, for a function f on $\mathbf{R}^p$ with respect to the measure μ on $\mathbf{R}^p$ is given by

$$\omega_\mu(f,t) = \sup\{\|\mu_s * f\|; 0 < s \leq t\}$$

where $\|\cdot\|$ is some norm, e.g supremum norm.

We have the following easy consequences of the definition.

$$\omega_{\mu*\rho}(f,t) \leq \|\mu\|_M \omega_\rho(f,t)$$

and

$$\omega_{\mu+\rho}(f,t) \leq \omega_\mu(f,t) + \omega_\rho(f,t)$$

for all $\mu, \rho \in M(\mathbf{R}^p)$.

We give a special partition of unity that will be useful to us.

1.2.6 Lemma

There exist a function $\psi \in C_0^\infty(\mathbf{R}^p)$ such that

1. $\psi(x)$ is non-negative and supp$\psi \subset \{1/2 < |x| < 2\}$,

2. $\sum_{k=-\infty}^{\infty} \psi(2^k x) = 1$ if $x \neq 0$

Proof. Let $h(t)$ denote a function defined for $0 \leq t < \infty$ and equal to 1 on $[0,1]$, to 0 for $t \geq 2$, strictly decreasing on $[1,2]$, and infinitely differentiable. Then $\psi(x) = h(|x|) - h(2|x|)$ satisfies the requirements. Observe that in the series above , atmost two terms are different from zero, for each $x \neq 0$. $\square$

Finally we present a useful and interesting lemma, which is a special case of a result in [13].

1.2.7 Lemma

Let $\mu \in M(\mathbf{R})$ be a measure such that $\hat{\mu}(\xi) = 1$ outside some compact set. Then

$$\omega(f,t) \leq C(\omega_\mu(f,t) + t\int_t^\infty s^{-2}\omega_\mu(f,s)ds), \quad f \in C_0(\mathbf{R}). \tag{1.7}$$

Proof. If $\hat{\mu}(0) \neq 0$, then $\omega_\mu(f,t)$ in general does not tend to zero as $t \to 0$, so the statement is empty in this case. If $\hat{\mu}(\xi)$ vanishes at the origin of order at most 1 in the sense that $\xi/\hat{\mu}(\xi)$ is locally in $\widehat{L^1}(\mathbf{R})$ near the origin, then the stronger conclusion $\omega(f,t) \leq C\omega_\mu(f,t)$ holds. Indeed, if

$$\xi/\hat{\mu}(\xi) = \hat{\nu}(\xi), \quad \nu \in L^1(\mathbf{R}), \quad |\xi| \leq \epsilon, \quad \epsilon > 0$$

and $\Delta = \delta_1 - \delta_0$ then

$$\hat{\Delta}(\xi) = \hat{\mu}(\xi)\hat{\nu}(\xi)\frac{\hat{\Delta}(\xi)}{\xi}. \tag{1.8}$$

By Lemma 1.1.6 $\hat{\rho}(\xi) = \frac{\hat{\Delta}(\xi)}{\xi} \in \widehat{M(\mathbf{R})}$ and thus equation (1.8) shows that $\Delta = \mu * \nu * \rho$ and hence $\omega(f,t) \leq \|\rho * \nu\|_M \omega_\mu(f,t)$.

The interesting case is therefore when $\hat{\mu}(\xi)$ vanishes at the origin of higher order than 1. The statement of the lemma is a special case of Corollary 2.4 in [13], but the case considered here is much simpler because of the assumption that $\hat{\mu}(\xi) = 1$ for large $|\xi|$. We therefore give the short proof here. By scaling we easily see that we may assume that $\hat{\mu}(\xi) = 1$ for $|\xi| > 1/2$. Let χ be a smooth function on $\mathbf{R}$ such that $\chi(\xi) = 0$ for $|\xi| < 1/2$ and $\chi(\xi) = 1$ for $|\xi| > 1$, and let ρ be the measure defined by $\hat{\rho}(\xi) = \chi(\xi)$. Then $\hat{\rho}(\xi) = \hat{\rho}(\xi)\hat{\mu}(\xi)$, hence $\omega_\rho(f,t) \leq C\omega_\mu(f,t)$, so it is enough to prove the lemma with μ replaced by ρ. Set $\phi(\xi) = \chi(2\xi) - \chi(\xi)$. Then

$$1 = \chi(\xi) + \sum_{k=0}^{\infty} \phi(2^k \xi), \quad \xi \neq 0.$$

Multiplying by $\hat{\Delta}(\xi)$ and using the fact that $\hat{\Delta}(\xi) = e^{i\xi} - 1 = \xi\hat{h}(\xi)$ for some $h \in L^1(\mathbf{R})$ we obtain

$$\hat{\Delta}(\xi) = \hat{\Delta}(\xi)\chi(\xi) + \sum_{k=0}^{\infty} \xi\hat{h}(\xi)\phi(2^k\xi) = \hat{\Delta}(\xi)\chi(\xi) + \sum_{k=0}^{\infty} 2^{-k}\hat{h}(\xi)2^k\xi\phi(2^k\xi). \tag{1.9}$$

Let ρ and ν be the measures defined by $\hat{\rho}(\xi) = \chi(\xi)$ and $\hat{\nu}(\xi) = \xi\phi(\xi)$. Replacing ξ by $t\xi$ in (1.9) and taking inverse Fourier transforms we obtain

$$\Delta_t = \Delta_t * \rho_t + \sum_{k=0}^{\infty} 2^{-k} h_t * \nu_{t2^k}. \tag{1.10}$$

Since $\chi(4\xi) = 1$ on the support of ϕ we have $\xi\phi(\xi) = \xi\phi(\xi)\chi(4\xi)$, hence $\omega_\nu(f,t) \leq$

$C\omega_\rho(f, 4t)$. It now follows from (1.10) that

$$\begin{aligned} \omega(f,t) &\le 2\omega_\rho(f,t) + C\sum_{k=0}^{\infty} 2^{-k}\omega_\rho(f, 2^{k+2}t) \\ &= 2\omega_\rho(f,t) + 4C\sum_{k=2}^{\infty} 2^{-k}\omega_\rho(f, 2^{k}t). \end{aligned} \tag{1.11}$$

To estimate the last sum by integral we use the fact that for any increasing function $\omega(s)$ and $A > 0$

$$2^{-kA}\omega(s) \le (\log 2)^{-1}2^{A}\int_{2^k}^{2^{k+1}} \omega(s)s^{-A-1}ds,$$

and hence

$$\sum_{k=0}^{\infty} \omega(s)2^{-kA} \le C\int_{2}^{\infty} \omega(s)s^{-A-1}ds.$$

Applying this to the sum on the right side of (1.11), with $A = 1$, and $\omega(s) = \omega_\rho(f, st)$ gives

$$\begin{aligned} \sum_{k=2}^{\infty} 2^{-k}\omega_\rho(f, 2^k t) &\le C\int_{2}^{\infty} \omega_\rho(f, st)s^{-2}ds \\ &= Ct\int_{ct}^{\infty} \omega_\rho(f, s)s^{-2}ds \\ &\le C't\int_{t}^{\infty} \omega_\rho(f, s)s^{-2}ds \end{aligned}$$

which gives 1.7.

1.2.8 Lemma

Let χ be a smooth function that is zero in some neighborhood of the origin and is 1 outside some compact set containing the origin and let $\mu \in M(\mathbf{R})$ be the measure defined by $\widehat{\mu}(\xi) = \chi(\xi)$. Assume that $\omega_\mu(f, t) \le \sigma(t)$. Then

$$\omega(f, t) \le C\hat{\sigma}(t), \quad f \in C_0(\mathbf{R}).$$

Proof Follows immediately from Lemma 1.2.7, since

$$\begin{aligned} t\int_t^\infty \omega_\mu(f,s)s^{-2}ds &\leq t\int_t^1 \omega_\mu(f,s)s^{-2}ds + C't\sup|f| \\ &\leq Ct(\int_t^1 \sigma(s)s^{-2}ds + \sigma(1)) \end{aligned}$$

and $\sigma(t) \leq \hat{\sigma}(t)$, for $0 < t < 1$.

1.3 Wiener-Levy Theorem

In this section we will study the Wiener-Levy's theorem and some applications that will be useful to our study. Our next lemma will help in the prove the version of the Wiener-Levy Theorem needed in our work.

1.3.1 Lemma

Assume $k \in L^1(\mathbf{R^p})$, $\int_{\mathbf{R^p}} k(x)dx = 0$, and $u \in L^1(\mathbf{R^p})$. Set $k_\delta(x) = \frac{1}{\delta^p}k(\frac{x}{\delta})$. Then

$$\|k_\delta * u\|_{L^1} \to 0 \quad \text{as} \quad \delta \to 0.$$

Proof. Observe that by a suitable change of variable we have

$$\begin{aligned} (k_\delta * u)(x) &= \int_{\mathbf{R^p}} k_\delta(y)u(x-y)dy \\ &= \int \frac{1}{\delta^p}k(\frac{y}{\delta})u(x-y)dy \\ &= \int_{\mathbf{R^p}} k(y)u(x-\delta y)dy. \end{aligned}$$

Set

$$F_\delta(x) := \int_{\mathbf{R^p}} k(y)u(x-\delta y)dy.$$

Since $\int_{\mathbf{R^p}} k(x)dx = 0$, we can write

$$F_\delta(x) = \int_{\mathbf{R^p}} k(y)(u(x-\delta y) - u(x))dy.$$

Hence, using Fubini's theorem we have

$$\begin{aligned}\int_{\mathbf{R^p}} |F_\delta(x)|dx &\leq \int_{\mathbf{R^p}}\int_{\mathbf{R^p}} |k(y)(u(x-\delta y)-u(x))|dydx \\ &\leq \int_{\mathbf{R^p}}\left(\int_{\mathbf{R^p}} |u(x-\delta y)-u(x)|dx\right)|k(y)|dy. \qquad (1.12)\end{aligned}$$

Since $u \in L^1$, we know that

$$\int_{\mathbf{R^p}} |u(x-t)-u(x)|dx \to 0 \text{ as } t \to 0. \qquad (1.13)$$

Indeed, (1.13) is true for any test function ψ and by density it holds for L^1 functions.

Since $k \in L^1$, given $\epsilon > 0$ we can choose a positive constant A so that

$$\int_{|y|>A} |k(y)|dy < \epsilon.$$

Using (1.12) we have

$$\begin{aligned}\|F_\delta\|_1 &\leq \int_{|y|>A}\left(\int_{\mathbf{R^p}} |u(x-\delta y)-u(x)|dx\right)|k(y)|dy \\ &+ \int_{|y|\leq A}\left(\int_{\mathbf{R}} |u(x-\delta y)-u(x)|dx\right)|k(y)|dy. \qquad (1.14)\end{aligned}$$

Now,

$$\begin{aligned}\int_{|y|>A}\left(\int_{\mathbf{R}} |u(x-\delta y)-u(x)|dx\right)|k(y)|dy &\leq \int_{|y|>A} 2\|u\|_1|k(y)|dy \\ &\leq 2\|u\|_1\epsilon = C_1\epsilon.\end{aligned}$$

Also because of (1.13) we can make

$$\int_{\mathbf{R^p}} |u(x-\delta y)-u(x)|dx < \epsilon$$

uniformly in the interval $|y| \leq A$, by choosing $\delta < \delta_0$. Hence,

$$\int_{|y|\leq A}\left(\int_{\mathbf{R^p}} |u(x-\delta y)-u(x)|dx\right)|k(y)|dy \leq \|k\|_1\epsilon \leq C_2\epsilon.$$

Hence $\|F_\delta\|_1 \leq C\epsilon$ if $\delta < \delta_0$. $\square$

Observe that Lemma 1.3.1 holds if $k \in M(\mathbf{R}^p)$ and $\hat{k}(0) = 0$. We now give a variant of the Wiener-Levy Theorem that will be useful to us.

1.3.2 Theorem (Wiener-Levy Theorem)

Assume $f \in L^1(\mathbf{R^p})$ and $\hat{f}(\xi_0) \neq 0$. Then there exists $g \in L^1(\mathbf{R^p})$ and $\delta > 0$ such that

$$\hat{g}(\xi) = \frac{1}{\hat{f}(\xi)} \quad \text{for} \quad |\xi - \xi_0| < \delta.$$

Proof. Let $\psi(t)$ be a smooth function supported in some compact set containing $|\xi - \xi_0| < \delta$ with $\psi(\xi) = 1$ on $|\xi - \xi_0| < \delta$. Then we will prove that there exist a function $\omega \in L^1$ such that

$$\frac{\psi((\xi - \xi_0)/\alpha)}{\hat{f}(\xi)} = \hat{\omega}(\xi)$$

if α is small enough. We assume for simplicity that $\xi_0 = 0$ and that $\hat{f}(0) = \hat{f}(\xi_0) = 1$. Also we may assume the support of ψ is contained in $|\xi| \leq \delta'$ and $\psi = 1$ on $|\xi| \leq \delta = \frac{\delta'}{2}$. Then $\psi(\xi/2\alpha) = 1$ on the support of $\psi(\xi/\alpha)$ and we have

$$\frac{\psi(\xi/\alpha)}{\hat{f}(\xi)} = \frac{\psi(\xi/\alpha)}{1 + (\hat{f}(\xi) - 1)\psi(\xi/2\alpha)}.$$

Set $h(\xi) = \hat{f}(\xi) - 1$. Then $h(0) = 0$. Also we claim that

$$\|h\psi(\cdot/2\alpha)\|_{\hat{L}^1} \to 0 \quad \text{as} \quad \alpha \to 0.$$

Indeed, set $h = \hat{k}$ and $\psi = \hat{u}$. Then $k \in M(\mathbf{R^p})$, $\hat{k}(0) = 0$, the function u belong to the Schwartz class and hence in L^1. Thus

$$\begin{aligned} \|h\psi(\cdot/2\alpha)\|_{\hat{L}^1} &= \|h(\alpha\cdot)\psi(\cdot/2\|_{\hat{L}^1} = \|\hat{k}_\alpha \hat{u}_{2^{-1}}\|_{\hat{L}^1} \\ &= \|\widehat{k_\alpha * u_{2^{-1}}}\|_{\hat{L}^1} = \|k_\alpha * u_{2^{-1}}\|_{L^1}. \end{aligned}$$

Our claim follows from Lemma 1.3.1.

To complete the prove we have to show that if $\|G\|_1 < 1$ then

$$\frac{\psi(\xi)}{1 - \hat{G}(\xi)} \in \hat{L}^1(\mathbf{R^p}).$$

To do this we first observe that

$$\frac{\psi(\xi)}{1-\hat{G}(\xi)} \approx \psi(\xi) \sum_{n=0}^{N} \hat{G}(\xi)^n \tag{1.15}$$

if N is large. We choose μ so that $\hat{\mu} = \psi$. Then the right hand side of (1.15) is the Fourier transform of

$$F_N := \sum_{n=0}^{N} \mu * G^n, \quad G^n = G * G * \cdots * G, \ n - \text{times}.$$

The fact that $\|G^n\|_1 \leq \|G\|_1^n$ and $\|G\|_1 < 1$ implies the infinite series

$$\sum_{n=1}^{\infty} \hat{G}(\xi)^n$$

converges in L^1. The result follows by letting $\hat{G}(\xi) = -h(\xi)\psi(\xi/2\alpha)$. $\square$

We give a comparison result of generalized modulus of continuity.

The measure μ is said to satisfy the Tauberian Condition (TC), if

for each $\xi \in \mathbf{R}^p$, $\xi \neq 0$, there exist a number $b > 0$ such that $\hat{\mu}(b\xi) \neq 0$.

1.3.3 Lemma

If $\mu, \tau \in M(\mathbf{R}^p)$, μ satisfies (TC) and $\hat{\tau}$ vanishes in neighborhoods of the origin and infinity, then for a function f defined on $\mathbf{R}^p$ and $t > 0$, there exist constants C and B depending on μ and τ only such that

$$\omega_\tau(f;t) \leq C\omega_\mu(f;Bt)$$

Proof. Let K be the support of $\hat{\tau}$ and let $\xi_0 \in K$ and $b_0 > 0$ be such that $\hat{\mu}(b_0\xi_0) \neq 0$. Then there exist a neighborhood of ξ_0, N_{ξ_0} and a number $b_0 > 0$ such that $\hat{\mu}(b_0\xi_0) \neq 0$ on N_{ξ_0}. Thus for each $\xi_k \in K$ there is a an open set N_k and a number b_k such that $\hat{\mu}(b_k\xi_k) \neq 0$ on N_k and the N_k is an infinite family of open sets whose union covers the compact set K. By the Heine-Borel Theorem there exists a finite subfamily of open sets, N_k, $k = 1, 2, \cdots, m$, of the infinite family N_k, which covers the compact set K. Thus for each $\xi \in K$ there is a open set N_k, $k = 1, 2, \cdots, m$ and $b_k > 0$ such that $\hat{\mu}(b_k\xi) \neq 0$ on N_k. Now let $\{\varphi_k\}$ be a partition of unity such that $1 = \sum_{k=1}^{m} \varphi_k(\xi)$ on K and supp$\varphi_k \subset N_k$ for each k. Now since $\varphi_k \in C_0^\infty$ and supported in N_k the Wiener-Levy theorem implies there exists a $\nu_k \in M(\mathbf{R}^p)$ such that $\psi_k(\xi) = \hat{\nu}(\xi)\hat{\mu}(b_k\xi)$ for all $\xi \in N_k$.

Thus

$$1 = \sum_{k=1}^{m} \psi_k(\xi) = \sum_{k=1}^{m} \hat{\nu}_k(\xi)\hat{\mu}(b_k\xi).$$

This shows that, for every $\xi \in \mathbf{R}^p$

$$\begin{aligned} \hat{\tau}(\xi) &= \sum_{k=1}^{m} \hat{\nu}_k(\xi)\hat{\mu}(b_k\xi)\hat{\tau}(\xi) \\ &= \sum_{k=1}^{m} \hat{\tau}_k(\xi)\hat{\mu}(b_k\xi) \end{aligned}$$

where $\hat{\nu}_k(\xi)\hat{\tau}(\xi) = \hat{\tau}_k(\xi)$. Thus $\tau = \sum_{k=1}^{m} \tau_k * \mu_{b_k}$. This shows that

$$\begin{aligned} |\tau_u * f| &\leq \sum_{k=0}^{m} |(\tau_k)_u * (\mu_{b_k})_u * f| \\ &\leq \sum_{k=0}^{m} \|(\tau_k)_u\|_M \|(\mu_{b_k})_u * f\| \end{aligned}$$

Hence

$$\omega_\tau(f,t) \leq C \sum_{k=0}^{m} \omega_\mu(f, b_k t) \leq C' \omega_\mu(f, Bt)$$

where $B = \max\{b_k; k = 1, 2, \cdots, m\}$. $\square$

1.3.4 Lemma

If $\mu, \tau \in M(\mathbf{R}^p)$, μ satisfies (TC) and $\hat{\tau}$ vanishes in neighborhoods of the origin, then for a function f on $\mathbf{R}^p$ and $t > 0$, there exist constants C and B depending on μ and

τ only such that

$$\omega_\tau(f,t) \le C \int_0^{Bt} \omega_\mu(f,s)\frac{ds}{s}.$$

Proof. Assume that $\hat{\tau}(\xi) = 0$ for $|\xi| < 1$ and choose ψ as in Lemma 1.2.6, that is smooth and compactly supported in $\frac{1}{2} < |\xi| < 2$ such that

$$1 = \sum_{k=0}^{\infty} \psi(2^{-k}\xi), \quad |\xi| > 1.$$

This implies

$$\hat{\tau}(\xi) = \sum_{k=0}^{\infty} \psi(2^{-k}\xi)\hat{\tau}(\xi) \text{ for all } \xi \in \mathbf{R}^p.$$

Set $\hat{\nu} = \psi$. Then we have

$$\tau = \sum_{k=0}^{\infty} \tau * \nu_{2^{-k}}.$$

We have

$$|\tau_u * f| \le |\sum_{k=0}^{\infty} \tau_u * (\nu_{2^{-k}})_u * f| \le \|\tau_u\|_M \sum_{k=0}^{\infty} \|(\nu_{2^{-k}})_u * f\|.$$

Thus,

$$\omega_\tau(f,t) \le C \sum_{k=0}^{\infty} \omega_{\nu_{2^{-k}}}(f,t).$$

By Lemma 1.3.3 there exist constants C' and B' such that

$$\omega_{\nu_{2^{-k}}}(f,t) \le C'\omega_\mu(f, B'2^{-k}t)$$

and hence,

$$\omega_\tau(f,t) \le C'' \sum_{k=0}^{\infty} \omega_\mu(f, B'2^{-k}t).$$

This completes the proof, once we take account of the elementary inequality

$$\sum_{k=1}^{\infty} \phi(k) \le \int_0^{\infty} \phi(v)dv$$

for functions ϕ continuous and decreasing for $0 \leq \lambda < \infty$, and apply it to $\phi(\lambda) = \omega_\mu(f, B'2^{-\lambda}t)$. Indeed, if $a > 0$, then

$$\begin{aligned}
\int_0^\infty \phi(v)dv &= \int_0^\infty \omega_\mu(f, B'2^{-v}t)dv \\
&= -\int_{B't}^0 \omega_\mu(f,s)\frac{ds}{s\ln 2} \\
&\leq C\int_0^{B't} \omega_\mu(f,s)\frac{dt}{s}
\end{aligned}$$

where we have used the change of variable $s = B'2^{-v}a$.

1.3.5 Theorem

Assume that μ satisfies (TC) and that $\hat{\mu}$ divides $\hat{\tau}$, $\tau \in M(\mathbf{R}^p)$, in some neighborhood of the origin. Then there exist constants C and B which only depend on μ and τ, such that

$$\omega_\tau(f,t) \leq C\int_0^t \omega_\mu(f, Bs)\frac{ds}{s}, \quad t > 0.$$

Proof. Take a smooth function ψ, equal to 1 for $|\xi| < \delta/2$ and supported in $|\xi| < \delta$. Then choose ρ by $\hat{\rho}(\xi) = \psi(\xi)\hat{\tau}(\xi)/\hat{\mu}(\xi)$ and let $\nu \in M(\mathbf{R}^p)$ be given by $\hat{\nu}(\xi) = (1-\psi(\xi))\hat{\tau}(\xi)$. Then $\hat{\nu}(\xi) = 0$ for $|\xi| < \delta$ and

$$\hat{\tau}(\xi) = \hat{\mu}(\xi)\hat{\rho}(\xi) + \hat{\nu}(\xi), \quad \text{for all } \xi \in \mathbf{R}^p.$$

Thus,

$$\begin{aligned}
\omega_\tau(f,t) &\leq \omega_{\mu*\rho}(f,t) + \omega_\nu(f,t) \\
&\leq \|\rho\|_M\omega_\mu(f,t) + \omega_\nu(f,t).
\end{aligned}$$

We apply Lemma 1.3.4 with $\tau = \nu$ to have

$$\omega_\tau(f,t) \leq \|\rho\|_M\omega_\mu(f,t) + C\int_0^{Bt} \omega_\mu(f,s)\frac{ds}{s}$$

for some constants B and C depending on ν and μ. The proof is complete by observing that

$$\omega_\mu(f,t) \leq \int_t^{Ct} \omega_\mu(f,s)\frac{ds}{s}, \text{ for all } C \text{ such that } \ln C \geq 1. \qquad \square$$

1.3.6 Lemma

Let $\mu \in M(\mathbf{R})$ and assume that $\widehat{\mu}(\xi)/\xi$ is in $\widehat{M}(\mathbf{R})$. Assume moreover that $f \in C_0(\mathbf{R})$, $f \in K(\sigma)$ and

$$(1+|x|)\mu \in M(\mathbf{R}). \tag{1.16}$$

Then there exist a constant $C > 0$ such that

$$\sup|\mu_t * f| \leq C\sigma(t). \tag{1.17}$$

Proof. Since $\widehat{\mu}(\xi)/\xi$ is in $\widehat{M}(\mathbf{R})$ we have that $\widehat{\mu}(0) = 0$, that is $\langle \mu, 1\rangle = 0$. This shows that ,

$$\mu_t * f(x) = \langle \mu, f(x - t\cdot)\rangle = \langle \mu, f(x - t\cdot) - f(x)\rangle.$$

Thus using the monotonicity and subadditivity of $\sigma(t)$ we have

$$|\mu_t * f(x)| \leq C\int_{\mathbf{R}} \sigma(|yt|)|d\mu(y)| \leq C\int_{\mathbf{R}}(1+|y|)\sigma(t)|d\mu(y)| \leq C\|(1+|y|)\mu\|\sigma(t)$$

which completes the proof.

1.3.7 Lemma

Let $\mu \in M(\mathbf{R})$ and assume that $\widehat{\mu}(\xi)/\xi$ is in $\widehat{M}(\mathbf{R})$. Assume moreover that $f \in C_0(\mathbf{R})$ and $f \in K(\sigma)$. Then (1.17) holds for some constant C.

Proof. Let ψ be smooth function on $\mathbf{R}$ that is equal to 1 for $|\xi| < 1$ and equals 0 for $|\xi| > 2$. Define h and τ by $\widehat{h} = \psi$ and $\widehat{\tau} = 1 - \psi$. It is clear that τ satisfies the conditions of Lemma 1.3.6. Indeed, since h is in the Schwartz class and $\tau = \delta_0 - h$

we have that τ satisfies (1.16), the other conditions are fulfilled trivially. Decompose μ by writing $\mu = \nu + \rho$ where $\widehat{\nu} = \psi\widehat{\mu}$. We observe that

$$\widehat{\rho}(\xi) = \widehat{\rho}(\xi)\widehat{\tau}(2\xi), \tag{1.18}$$

because $\widehat{\tau}(2\xi) = 1$ on the support of $\widehat{\rho}$. It follows from Lemma 1.3.6 that

$$\sup |\tau_t * f| \leq C\sigma(t)$$

for some constant C. Since equation (1.18) implies $\rho = \rho * \tau_2$ we see that the assertion is proved for ρ.

Consider ν. We note that $\widehat{\nu}(\xi)/\xi$ is in $\widehat{M}(\mathbf{R})$ and is compactly supported. Thus we can choose the function ψ as above then $(e^{-i\xi} - 1)/\xi$ is non zero on the support of ψ. We apply Weiner's theorem to get some $\omega \in M(\mathbf{R})$ such that

$$\frac{\widehat{\nu}(\xi)/\xi}{(e^{-i\xi} - 1)/\xi} = \widehat{\omega}(\xi)$$

which gives

$$\nu = \Delta * \omega,$$

which implies

$$|\nu * f| \leq C\sigma(t),$$

as required.□

Chapter 2

Mappings between Euclidean spaces

We say a function f from $\mathbf{R^p}$ to $\mathbf{R}$ is continuous in the direction $\theta \in \mathbf{R^p} \setminus \{0\}$ if the function $h : \mathbf{R} \to \mathbf{R}$ defined by $h(t) = f(x + t\theta)$ is continuous uniformly with respect to x on compact sets. It is obvious that f must be continuous if it satisfies p conditions of this kind for a set of θ-vectors that spans $\mathbf{R^p}$. However to be able to conclude that *two* real functions on $\mathbf{R^2}$ are continuous we do not need four conditions, but *three* conditions suffice. Infact, if f is continuous in the direction $(1, 0)$, g is continuous in the direction $(0, 1)$, and $f + g$ is continuous in the direction $(1, 1)$, then f and g must be continuous. This makes it natural to ask question on what condition on subsets of $\mathbf{R^2} \times \mathbf{R^2}$ do we have a transfer of regularity properties in given directions to the functions itself. More generally we will be studying the following:

Let Λ be a finite set of pairs $(\theta, \eta) \in \mathbf{R^p} \times \mathbf{R^q}$ and let f be a function from $\mathbf{R^p}$ to $\mathbf{R^q}$. Assume the real-valued function $x \mapsto \langle \eta, f \rangle(x)$ has certain regularity properties (e.g continuity, infinite differentiability, modulus of continuity etc) in the direction θ for every $(\theta, \eta) \in \Lambda$. Then under what conditions on Λ do we have the following: Each regularity property on $x \mapsto \langle \eta, f \rangle(x)$ in the direction θ for every $(\theta, \eta) \in \Lambda$ will imply a corresponding (unrestricted) regularity property for the function f.

In this chapter we will study two (algebraic) conditions on the set Λ which we shall denote by $(\widehat{A})$ and (A). Also if $\eta \in \mathbf{R}^q$ and $\mu \in M(\mathbf{R^p})$ then $\eta\mu$ shall be considered as the q-tuple $(\eta_1\mu, \cdots, \eta_q\mu)$. In [10] the notation $\eta \otimes \mu$ was used where $\otimes$ denotes the tensor product over $\mathbf{R}$.

2.1 The algebraic condition $(\widehat{A})$.

In order to be able to study regularity properties between Euclidean spaces we will need to use some algebraic conditions.

The subset Λ of $\mathbf{R^p} \times \mathbf{R^q}$ is said to satisfy $(\widehat{A})$ if

$$u \in \mathbf{R^p}, v \in \mathbf{R^q} \text{ and } \langle u, \theta\rangle\langle v, \eta\rangle = 0 \text{ for all } (\theta, \eta) \in \Lambda$$

$$\text{implies } \; |u||v| = 0$$

or equivalently as stated in [10]

$$\text{if } \Phi \text{ is a bilinear form } \mathbf{R^p} \times \mathbf{R^q} \to \mathbf{R} \text{ of rank one and } \Phi(\Lambda) = 0\ , \text{ then } \Phi = 0.$$

Another formulation of this condition can be as follows. If we associate the $p\times q$ matrix with the linear form $\alpha_{ij} \mapsto \sum \alpha_{ij}a_{ij}$ on the vector space $M(p,q)$ of all $p \times q$ matrices, then the dual space of $M(p,q)$ will be identified with another copy of $M(p,q)$. Λ will satisfy $(\widehat{A})$ if there is no *rank one* element of $M(p,q)$ that is orthogonal to all the matrices $(\theta_i\eta_j)$ for $(\theta,\eta) \in \Lambda$.

2.1.1 Lemma

If Λ satisfies $(\widehat{A})$, then Λ must contain at least $p + q - 1$ elements.

Proof. Suppose Λ contains only $p+q-2$ elements and let $(\theta^k, \eta^k) \in \Lambda$, $k = 1, 2, \cdots, p+q-2$. Let $\{w_1, \cdots, w_{p-1}\}$ be a set of $p-1$ vectors in $\mathbf{R^p}$. Then there exist a non-zero vector $u \in \mathbf{R^p}$ such that $\langle u, w_i \rangle = 0$, $i = 1, 2, \cdots, p-1$, since the $p-1$ elements cannot span $\mathbf{R^p}$. Similarly if $\{w_1, \cdots, w_{q-1}\}$ be a set of $q-1$ vectors in $\mathbf{R^q}$ then there exist a non-zero vector $v \in \mathbf{R^p}$ such that $\langle v, w_i \rangle = 0$, $i = 1, 2, \cdots, q-1$. This shows that we can find non-zero vector $u \in \mathbf{R^p}$ and $v \in \mathbf{R^q}$ such that $\langle u, \theta^k \rangle \langle v, \eta^k \rangle = 0$ for all $(\theta^k, \eta^k) \in \Lambda$, $k = 1, 2, \cdots, p+q-2$ which contradicts the condition $(\widehat{A})$. Hence Λ cannot contain less than $p+q-1$ elements. $\square$

2.1.2 Remark

We give some simple examples of Λ satisfying $(\widehat{A})$. Lets consider the case $p = q = 2$, then Lemma 2.1.1 shows that Λ must contain at least three elements. Let

$$\Lambda = \{(e_1, e_1), (e_2, e_2), (e_1 + e_2, e_1 + e_2) : e_1 = (1, 0), e_2 = (0, 1)\}.$$

Then Λ contains only three elements and satisfies $(\widehat{A})$. Indeed, Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathbf{R^2}$ be such that $\langle u, \theta \rangle \langle v, \eta \rangle = 0$ for all $(\theta, \eta) \in \Lambda$. That is,

$$\langle u, e_1 \rangle \langle v, e_1 \rangle = 0, \quad \langle u, e_2 \rangle \langle v, e_2 \rangle = 0, \; \langle u, e_1 + e_2 \rangle \langle v, e_1 + e_2 \rangle = 0.$$

Then $u_1 v_1 = 0,\; u_2 v_2 = 0$ and $u_1 v_2 + u_2 v_1 = 0$. We see easily that either u or v is the zero vector. Also the set of 2×2 matrices $(\theta_i \eta_j)$ for $(\theta, \eta) \in \Lambda$ is:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Now if

$$\Lambda = \{(e_1, e_1), (e_2, e_2), (e_1, e_2), (e_2, e_1) : e_1 = (1, 0), e_2 = (0, 1)\}$$

then Λ satisfies $(\widehat{A})$ with four elements. We shall latter see that our last example satisfies a stronger algebraic condition which we are going to study. There are infact infinitely many subsets of $\mathbf{R^p} \times \mathbf{R^q}$ satisfying $(\widehat{A})$ with only $p+q-1$ elements. In a previous studies (Licentiate thesis [1]), we called such sets "minimal sets", and our study was basically on the minimal sets. However as the second example shows, not every set satisfying $(\widehat{A})$ contains a minimal set. The set in question satisfies $(\widehat{A})$ but if you take away one element the condition $(\widehat{A})$ will no longer be satisfied. Thus our results here are more general. However the minimal sets contain some algebraic properties which does not hold in the general case, which we used.

Now suppose Λ satisfies $(\widehat{A})$ and contains only $p+q-1$ elements. If we take a subset D of Λ of p-elements then the complement of D, D', will contain $q-1$ elements. Now if the set of θ-vectors for which (θ, η) in D is linearly dependent then there exist a non-zero vector $u \in \mathbf{R^p}$ such that $\langle u, \theta \rangle = 0$ for all $(\theta, \eta) \in D$. Also there exist a non-zero vector $v \in \mathbf{R^q}$ such that $\langle v, \eta \rangle = 0$ for all $(\theta, \eta) \in D'$. Thus $\langle u, \theta \rangle \langle v, \eta \rangle = 0$ for all $(\theta, \eta) \in \Lambda$. This contradicts the fact that Λ satisfies $(\widehat{A})$. This shows that the θ vectors in D are linearly independent and hence form a basis for $\mathbf{R^p}$. A similar statement holds if D contains q elements. We have proved the following:

2.1.3 Lemma

Suppose $(\theta, \eta) \in \Lambda$ satisfies $(\widehat{A})$ and contains only $p+q-1$ elements. Then every set of p(resp. q) elements of the θ(resp. η) vectors form a basis of $\mathbf{R^p}$ (resp.$\mathbf{R^q}$).

The following is one of the most useful property of the condition $(\widehat{A})$.

2.1.4 Lemma

Assume that Λ satisfies $(\widehat{A})$. Then for every $\xi \neq 0$ the set of η^k such that $(\theta^k, \eta^k) \in \Lambda$ and $\theta^k \cdot \xi \neq 0$ spans $\mathbf{R^q}$;

Proof. If the above statement were false there would exist ξ and y such that $\eta^k \cdot y = 0$ for all k for which $\theta^k \cdot \xi \neq 0$, that is $(\eta^k \cdot y)(\theta^k \cdot \xi)$ is equal to 0 for all k, which means that $(\widehat{A})$ is false. $\square$

2.1.5 Lemma

Assume that Λ is finite subset of $\mathbf{R}^p \times \mathbf{R}^q$ satisfying $(\widehat{A})$ and that s is a natural number. Then there exist a constant C which is independent of u and v such that

$$|v||u|^s \leq C \sum_{(\theta,\eta)\in\Lambda} |\langle \eta, v\rangle||\langle \theta, u\rangle|^s, \quad u \in \mathbf{R}^p,\ v \in \mathbf{R}^q. \tag{2.1}$$

Proof. The function on the right-hand side is positively homogeneous with respect to v of degree 1 and with respect to u of degree s, and so is the left-hand side. Thus it is sufficient to prove (2.1) when $|v| = |u| = 1$. But if $|v| = |u| = 1$ the bilinear form $\Phi(\theta, \eta) = \langle \theta, u\rangle\langle \eta, v\rangle$ cannot vanish on all of Λ in view of $(\widehat{A})$. Hence the continuous function on the right-hand side of (2.1) must have a positive lower bound $\delta > 0$ on the compact set $|u| = |v| = 1$. Choose $C > 1/\delta$ to complete the proof. $\square$

2.1.6 Definition

Let $0 \neq \theta \in \mathbf{R}^p$ and $k \geq 1$. We denote by $C^k(\theta)$ the class of continuous functions defined in $\mathbf{R}^p$ such that the derivatives

$$D^j_\theta f(x) = [(d/dt)^j f(x + t\theta)]_{t=0}$$

exist and are continuous when $j \leq k$.

C^k is the class of k times continuously differentiable functions. $C^\infty(\theta)$ denotes $\bigcap_{k=1}^\infty C^k(\theta)$ and C^∞ denotes $\bigcap_{k=0}^\infty C^k$

2.1.7 Theorem

Assume Λ satisfies $(\widehat{A})$ and that $f : \mathbf{R}^p \to \mathbf{R}^q$ satisfies

$$\langle \eta, f\rangle \in C^\infty(\theta) \quad for \quad each \quad (\theta, \eta) \in \Lambda. \tag{2.2}$$

Then $f \in C^\infty$. Conversely if (2.2) implies that $f \in C^\infty$, then $(\widehat{A})$ holds.

Proof. Take $\psi : \mathbf{R}^p \to \mathbf{R}$, $\psi \in C^\infty$ such that $\psi = 1$ on some open set and $\psi = 0$ outside some compact set. It will be enough to prove that $g = \psi f \in C^\infty$. It is clear that $\langle \eta, g\rangle \in C^\infty(\theta)$ for each $(\theta, \eta) \in \Lambda$. Since g has compact support we can form the Fourier transform $\hat{g}$ of g. By partial integration we obtain in a well known way for any natural number s a constant C_s such that

$$|\langle \eta, \widehat{g}(\xi)\rangle||\langle \theta, \xi\rangle|^s \le C_s,$$

for every $(\theta, \eta) \in \Lambda$ and $\xi \in \mathbf{R}^p$. Indeed, we know that

$$\begin{aligned} \hat{g}(\xi) &= \int_{\mathbf{R}^p} e^{-i\xi\cdot x} g(x)dx \\ &= \int_{\mathbf{R}} e^{-i\xi_1\cdot x_1} \int_{\mathbf{R}} e^{-i\xi_2\cdot x_2} \cdots \int_{\mathbf{R}} e^{-i\xi_p\cdot x_p} g(x)dx_p \cdots dx_2 dx_1. \end{aligned}$$

Let $\theta = (1, 0, \cdots, 0)$ then by the hypothesis

$$(\partial^s/\partial x_1^s)\langle \eta, g\rangle(x) = \langle \eta, (\partial^s/\partial x_1^s) g(x)\rangle,$$

exists, is continuous and compactly supported. Using integration by parts we obtain

$$\begin{aligned} \int_{\mathbf{R}} e^{-i\xi_1\cdot x_1}\langle \eta, g\rangle(x)dx_1 &= \frac{1}{(i\xi_1)^s}\int_{\mathbf{R}} e^{-i\xi_1\cdot x_1}(\partial^s/\partial x_1^s)\langle \eta, g\rangle(x)dx_1 \\ &= \frac{1}{(i\xi_1)^s}\int_{\mathbf{R}} e^{-i\xi_1\cdot x_1}\langle \eta, (\partial^s/\partial x_1^s)g(x)\rangle dx_1 \end{aligned}$$

for any natural number s. We now multiply by $e^{-i(x_2\xi_2+\cdots+x_p\xi_p)}$ and integrate over all the other x-variables we obtain

$$\langle \eta, \widehat{g}(\xi)\rangle = \frac{1}{(i\xi_1)^s}\int_{\mathbf{R^p}} e^{-i\xi\cdot x}\langle \eta, (\partial^s/\partial x_1^s)g(x)\rangle dx.$$

Thus,

$$|\xi_1|^s|\langle \eta, \widehat{g}(\xi)\rangle| = |\langle \theta, \xi\rangle|^s|\langle \eta, \widehat{g}(\xi)\rangle| \le C.$$

But we can always make a rotation of the coordinate system so that θ becomes $(1, 0, \cdots, 0)$ in the new coordinate system. Thus,

$$|\langle \theta, \xi\rangle|^s|\langle \eta, \widehat{g}(\xi)\rangle| \le C \tag{2.3}$$

for all $(\theta, \eta) \in \Lambda$.

Now, since any infinite set satisfying $(\widehat{A})$ contains a finite subset satisfying $(\widehat{A})$, we may assume Λ is a finite subset of $\mathbf{R}^p \times \mathbf{R}^q$. Thus summing over all $(\theta, \eta) \in \Lambda$ in (2.3) we obtain

$$\sum_{(\eta,\theta)\in\Lambda} |\langle \theta, \xi\rangle|^s|\langle \eta, \widehat{g}(\xi)\rangle| \le C_1.$$

We now apply Lemma 2.1.5 with $\xi = u$ and $\widehat{g}(\xi) = v$ to get

$$|\widehat{g}(z)||z|^s \le C'_s,\ z \in \mathbf{R}^p,\ s = 1, 2, \cdots.$$

It is well known that this implies that $g \in C^\infty$.

Conversely, suppose that there exist $u \neq 0$ and $v \neq 0$ such that

$$\langle u, \theta\rangle\langle v, \eta\rangle = 0 \text{ for every } \ (\theta, \eta) \in \Lambda.$$

Let h be an arbitrary continuous function $\mathbf{R} \to \mathbf{R}$, and the function $f : \mathbf{R}^p \to \mathbf{R}^q$ be defined by $f(x) = vh(\langle u, x\rangle)$. Then for every $(\theta, \eta) \in \Lambda$

$$\begin{aligned} D_\theta\langle \eta, f(x)\rangle &= \langle \eta, v\rangle D_\theta h(\langle u, x\rangle) \\ &= \langle \eta, v\rangle\langle u, \theta\rangle h'(\langle u, x\rangle) = 0 \end{aligned}$$

which implies $\langle \eta, f(x) \rangle \in C^\infty(\theta)$ for each $(\theta, \eta) \in \Lambda$. But f does not in general belong to C^∞.

Let $\chi(t)$ be a smooth function on the set of real numbers that is equal to 1 for $|t| > 1$ and equals 0 for $|t| < \frac{1}{2}$ and define the function $\psi(t) = 1 - \chi(t)$ so that $\psi(t) = 1$ on $|t| < \frac{1}{2}$ and supp$\psi \subset [-1, 1]$. Let $\psi_k(\xi)$ denote $\psi(\theta^k \cdot \xi)$ and $\chi_k(\xi)$ denote $\chi(\theta^k \cdot \xi)$. We note from Lemma 1.1.1 that ψ_k, χ_k and the products of these functions are all in $\widehat{M}(\mathbf{R}^p)$. We are now ready to state and prove a theorem on the decomposition of vector valued measures.

2.1.8 Theorem

Assume that $\mu_0 \in M(\mathbf{R}^p)$, and that $\widehat{\mu}_0(\xi) = 0$ in some neighborhood of the origin, and that Λ satisfies $(\widehat{A})$. Then for each $\eta^0 \in \mathbf{R}^q$, there exist $\mu_k \in M_{\theta^k},\ k = 1, 2, \cdots, m \geq p + q - 1$, with

$$\frac{\widehat{\mu}_k(\xi)}{\theta^k \cdot \xi} \in \widehat{M}(\mathbf{R}^p), \tag{2.4}$$

such that

$$\eta^0 \mu_0 = \sum_{k=1}^{m} \eta^k \mu_k. \tag{2.5}$$

Infact, one can take μ_k such that $\widehat{\mu}_k(\xi) = 0$ when ever $|\theta^k \cdot \xi| \leq \delta$ for some $\delta > 0$.

Proof Let η^0 be an arbitrary vector in $\mathbf{R^q}$ and the functions $\chi(t)$ and $\psi(t)$ be define as above. We will show that there exist measures μ_k in M_{θ^k} such that the equation

$$\eta^0 = \sum_{k}^{m} \widehat{\mu}_k(\xi) \eta^k \tag{2.6}$$

holds for $|\xi| > B$ for some sufficiently large B.

For $s = 1, 2, \cdots, 2^m$, let $E_s \subset \{1, 2, \cdots, m\}$ and E_s' the compliment of E_s in

$\{1, 2, \cdots, m\}$. Outside any compact set we write

$$1 = \prod_{i=1}^{m} (\psi_i(\xi) + \chi_i(\xi)) = \sum_{s=1}^{2^m} \prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi) \tag{2.7}$$

We may take $E_1 = \{1, 2, \cdots, m\}$ and $E_{2^m} = \emptyset$ so that

$$\prod_{i=1}^{m} \psi_i(\xi) \quad \text{and} \quad \prod_{i=1}^{m} \chi_i(\xi)$$

are the first and the last terms respectively on the right hand of equation (2.7). If Λ satisfy $(\widehat{A})$ then for each $s = 1, 2, \cdots, 2^m$, we have the following: Either

1. the set of $\{\theta^i\}$, $i \in E_s$ spans $\mathbf{R^p}$ or

2. the set of $\{\eta^j\}$, $j \in E'_s$ spans $\mathbf{R^q}$.

Indeed, if neither of this is not true then we can find $0 \neq v \in \mathbf{R}^q$ and $0 \neq u \in \mathbf{R}^p$ such that $\langle v, \eta^k \rangle \langle u, \theta^k \rangle = 0$ for all $(\theta^k, \eta^k) \in \Lambda$ which implies $(\widehat{A})$ is false.

We also observe that if $\{\theta^i\}_{i \in E_s}$ spans $\mathbf{R^p}$ then the support of $\prod_{i \in E_s} \psi_i(\xi)$ is contained in some compact set ,$\{|\xi| \leq R_s\}$, around the origin. Now for each s for which $\{\theta^i\}_{i \in E_s}$ spans $\mathbf{R^p}$, there is an R_s such that $\prod_{i \in E_s} \psi_i(\xi) = 0$ for all ξ with $|\xi| > R_s$. Let $B = \max_s R_s$. Then for $|\xi| > B$ equation (2.7) will contain the sum of terms of the form

$$\prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi)$$

where the set $\{\eta^j\}_{j \in E'_s}$ span $\mathbf{R}^q$. Now if $\eta^0 \in \mathbf{R^q}$, there exist b_k such that

$$\eta^0 = \sum_{k \in E'_s} b_k \eta^k$$

which implies

$$\begin{aligned} \prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi) \eta^0 &= \prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi) \sum_{k \in E'_s} b_k \eta^k \\ &= \sum_{k \in E'_s} c(\xi) b_k \eta^k, \end{aligned}$$

where

$$c(\xi) = \prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi).$$

Now, let

$$c_k^s(\xi) = b_k \prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s,\ j \neq k} \chi_j(\xi),$$

for $k \in E'_s$. Then $c_k^s(\xi)$ is in $\widehat{M}(\mathbf{R^p})$ and

$$\prod_{i \in E_s} \psi_i(\xi) \prod_{j \in E'_s} \chi_j(\xi)\eta^0 = \sum_{k \in E'_s} c_k^s(\xi)\chi_k(\xi)\eta^k. \tag{2.8}$$

Thus for each s for which the set $\eta^j \in E'_s$ spans $\mathbf{R^q}$, there exist $c_k^s \in \widehat{M}(\mathbf{R^p})$ such that equation (2.8) holds.

Multiplying equation (2.7) by η^0 and summing over s we get

$$\begin{aligned} \eta^0 &= \sum_{s=1}^{2^m} \sum_{k \in E'_s} c_k^s(\xi)\chi(\theta^k \cdot \xi)\eta^k \\ &= \sum_{k=1}^{m} c_k(\xi)\chi(\theta^k \cdot \xi)\eta^k, \quad |\xi| > B, \end{aligned}$$

where

$$c_k(\xi) = \sum_s c_k^s(\xi),$$

which is in $\widehat{M}(\mathbf{R^p})$. Let

$$\widehat{\mu}_k(\xi) = \widehat{\mu}_0(\xi)c_k(\xi)\chi(\theta^k \cdot \xi)$$

then $\mu_k \in \widehat{M}_{\theta^k}$ and

$$\widehat{\mu}_0(\xi)\eta^0 = \sum_{k=1}^{m} \widehat{\mu}_k(\xi)\eta^k, \quad |\xi| > B.$$

To complete the proof we must show that $\mu_k,\ k = 1, 2 \cdots, m$ satisfies (2.4). To this end, let $\Phi(t)$ be a smooth function on the set of real numbers that is equal to 1 for

$|t| > 1/2$ and equals 0 for $|t| < \frac{1}{4}$, then

$$\frac{\Phi(\theta^k \cdot \xi)}{\theta^k \cdot \xi}\widehat{\mu}_k(\xi) = \frac{\widehat{\mu}_k(\xi)}{\theta^k \cdot \xi}, \quad \text{for all } \xi.$$

Thus it suffices to show that $\frac{\Phi(\theta^k \cdot \xi)}{\theta^k \cdot \xi} \in \widehat{M}(\mathbf{R}^p)$. But this follows by Lemma 1.1.5 and thus the theorem is proved. $\square$

Our next Lemma shows that the result above holds for $|\xi| > \delta$ for any $\delta > 0$, precisely we have

2.1.9 Lemma

Suppose there exist B and $\mu_k \in M_{\theta^k}$ such that

$$\eta = \sum_k \widehat{\mu}_k(\xi)\eta^k \quad \text{for} \quad |\xi| > B.$$

Then for any $\delta > 0$ there exist $\nu_k \in M_{\theta^k}$ such that

$$\eta = \sum_k \widehat{\nu}_k(\xi)\eta^k \quad \text{for} \quad |\xi| > \delta.$$

Proof. If $\mu \in M_\theta$ then $\widehat{\mu} \in \widehat{M_\theta}$. For $a \neq 0$ define ν by $\widehat{\nu}(\xi) = \widehat{\mu}(a\xi)$. Thus $\nu \in M_\theta$ and if we let $a = \frac{B}{\delta}$ we get the result. $\square$

2.1.10 Theorem

If Λ satisfies $(\widehat{A})$ then for all $\eta^0 \in \mathbf{R}^q$, $\theta^0 \in \mathbf{R}^p$ and positive integer r, there exist $c_k(\xi)$, $\xi \in \mathbf{R}^\mathbf{p}/\{0\}$, which are smooth and homogeneous of degree $r-1$ such that

$$\eta^0(\theta^0 \cdot \xi)^r = \sum_{k=1}^{m} c_k(\xi)\eta^k(\theta^0 \cdot \xi). \tag{2.9}$$

Proof. Let $\xi^0 \in S^{p-1}$. Then by Lemma 2.1.4 we can choose a set E of numbers k of $\{\eta^k\}$ such that $\theta^k \cdot \xi^0 \neq 0$, for all $k \in E$ and constants b_k such that

$$\eta^0 = \sum_{k \in E} b_k\eta^k. \tag{2.10}$$

By continuity, there exist a open neighborhood, V_0, of ξ^0 such that $\theta^k \cdot \xi \neq 0$ for all $\xi \in V_0$. Since the unit sphere, S^{p-1}, is compact we can find a finite set of open neighborhoods, V_j $j = 1, 2, \cdots, N$, which covers S^{p-1}. Let $\{\varphi_j\}$ be a partition of unity such that $1 = \sum_{j=1}^{N} \varphi_j(\xi)$ on S^{p-1} and supp$\varphi_j \subset V_j$ for each j. Then for each $j = 1, 2, \cdots, N$, equation (2.10) implies there exist E_j such that $\theta^k \cdot \xi \neq 0$ for all $k \in E_j$ and all $\xi \in V_j$ and

$$\varphi_j(\xi/|\xi|)\eta^0 = \sum_{k \in E_j} \varphi_j(\xi/|\xi|) b_k \eta^k \tag{2.11}$$

this implies

$$\varphi_j(\xi/|\xi|)\eta^0(\theta^0 \cdot \xi)^r = \sum_{k \in E_j} d_{k,j}(\xi)\eta^k(\theta^k \cdot \xi) \tag{2.12}$$

where $d_{k,j}(\xi) = \frac{(\theta^0 \cdot \xi)^r}{(\theta^k \cdot \xi)} \varphi_j(\xi/|\xi|)$ is smooth and homogeneous of degree $r - 1$. Putting all of this together we have

$$\begin{aligned} \eta^0(\theta^0 \cdot \xi)^r &= \sum_{j}^{N} \sum_{k=1}^{m} d_{k,j}(\xi)\eta^k(\theta^k \cdot \xi) \\ &= \sum_{k=1}^{m} \sum_{j}^{N} d_{k,j}(\xi)\eta^k(\theta^k \cdot \xi) \end{aligned} \tag{2.13}$$

with $d_{k,j}(\xi) = 0$ if $k \notin E_j$. $\square$

Using Theorems 2.1.7 and 2.1.8 we have the following:

2.1.11 Corollary

Assume Λ satisfies $(\widehat{A})$, and $f \in K(\sigma, \theta)$ for all $(\theta, \eta) \in \Lambda$ and r a positive integer greater than 1. Then there exists a constant C

$$\omega_r(f, t) \leq C\sigma(\epsilon), \;\; |t| \leq \epsilon.$$

Proof. Let $\eta^0 \in \mathbf{R}^q$ and $\theta^0 \in \mathbf{R}^p$ be arbitrary and let μ_0 be the r^{th} order difference measure in the direction θ^0 for $r \geq 2$. Then

$$\widehat{\mu}_0(\xi) = (e^{-i\theta^0\cdot\xi} - 1)^r. \tag{2.14}$$

We claim that

$$\sup |\mu_{0,t} * \langle \eta^0, f\rangle| \leq C\sigma(t). \tag{2.15}$$

To prove (2.15), take a smooth function $\psi(\xi)$ supported in $|\xi| < 2$ and equal to 1 in $|\xi| \leq 1$ and decompose μ_0 as $\mu_0 = \nu_0 + \rho_0$ by taking $\widehat{\nu}_0(\xi) = \psi(\xi)\widehat{\mu}_0(\xi)$ and $\rho_0 = \mu_0 - \nu_0$. The measure ρ_0 satisfies the hypothesis of Theorem 2.1.8 and thus

$$|\rho_{0,t} * \langle \eta^0, f\rangle| = |\sum_{k=1}^{m} \mu_{k,t} * \langle \eta^k, f\rangle| \leq \sum_{k=1}^{m} |\mu_{k,t} * \langle \eta^k, f\rangle|.$$

This shows that

$$|\rho_{0,t} * \langle \eta^0, f\rangle| \leq C\sigma(t),$$

since $\langle \eta^k, f\rangle \in K(\theta^k, \sigma)$ by Lemma 1.3.7. Consider ν_0. Theorem 2.1.10 implies that there exist smooth and homogeneous functions, $c_k(\xi)$, in $\mathbf{R}^p/\{0\}$ of degree $r-1$ such that

$$\eta^0(\theta^0 \cdot \xi)^r = \sum_{k=1}^{m} c_k(\xi)\eta^k(\theta^k \cdot \xi). \tag{2.16}$$

Now, write

$$\widehat{\nu}_0(\xi) = \widehat{\mu}_0(\xi)\psi(\xi) = (\theta^0 \cdot \xi)^r \frac{\widehat{\mu}_0(\xi)}{(\theta^0 \cdot \xi)^r}\psi(\xi) = (\theta^0 \cdot \xi)^r G(\xi)\psi(\xi),$$

where

$$G(\xi) = \frac{\widehat{\mu}_0(\xi)}{(\theta^0 \cdot \xi)^r} = \left(\frac{(e^{-i\theta^o\cdot\xi} - 1)}{(\theta^0 \cdot \xi)}\right)^r$$

which is the Fourier transform of a measure in $M(\mathbf{R}^p)$ by Lemma 1.1.5. Multiplying (2.16) by $G(\xi)\psi(\xi)$ we obtain

$$\eta^0\widehat{\nu}_0(\xi) = \sum_k c_k(\xi)\psi(\xi)\eta^k G(\xi)(\theta^k \cdot \xi).$$

Since ψ is smooth and has compact support it follows from Lemma 1.1.7 that $c_k(\xi)\psi(\xi)G(\xi)$ is the Fourier transform of a measure in $M(\mathbf{R}^p)$, and thus $c_k(\xi)\psi(\xi)G(\xi)(\theta^k \cdot \xi)$ belongs to $\widehat{M_{\theta^k}}$. The claim follows as in the case of ρ above.

Now, observe that (2.15) implies

$$|\Delta^r_{t,\theta^0}\langle \eta^0, f(x)\rangle| \leq C\sigma(t)$$

with θ^0 and η^0 arbitrary. Now let $\theta^0 = (1, 1, \cdots, 1)$ and $\eta^0 = e_k$ then

$$|\Delta^r_t f_k(x)| \leq C\sigma(t)$$

for all $k = 1, 2, \cdots, q$ and this implies there exist a constant C' which may depend on q such that

$$|\Delta^r_t f(x)| \leq C'\sigma(\epsilon), \; |t| \leq \epsilon. \square$$

2.1.12 Corollary

Let f be a function from $\mathbf{R^p}$ to $\mathbf{R^q}$ and $\sigma \in \Sigma$. Let Λ be a subset of $\mathbf{R^p} \times \mathbf{R^q}$ satisfying $(\widehat{A})$ and assume that

$$\langle \eta, f\rangle \in K(\theta, \sigma) \text{ for every } \; (\theta, \eta) \in \Lambda. \tag{2.17}$$

Then $f \in K(\hat{\sigma})$.

Proof. Choose ν by $\widehat{\nu}(\xi) = \chi(\theta^0 \cdot \xi)$. By Lemma 1.2.8 it is enough to prove that

$$\omega_\nu(\langle \eta^0, f\rangle, t) \leq C\sigma(t) \tag{2.18}$$

Using Theorem 2.1.8 with $\nu = \mu_0$ together with Lemma 1.3.7 we get

$$|\nu_{0,t} * \langle \eta^0, f\rangle| = |\sum_{k=1}^{m} \nu_{k,t} * \langle \eta^k, f\rangle| \leq \sum_{k=1}^{m} |\nu_{k,t} * \langle \eta^k, f\rangle| \leq C\sigma(t). \square$$

We end this section with an example which shows that Corollary 2.1.11 is not always true for $r = 1$ and that no stronger conclusion is valid for Corollary 2.1.12. Let $f : \mathbf{R^2} \to \mathbf{R^2}$ given by

$$f(x) = (x_2 \log |x|, -x_1 \log |x|),\ f(0,0) = (0,0), x = (x_1, x_2),$$

$\sigma(\epsilon) = \epsilon$ and $(\theta^k, \eta^k) \in \Lambda$ given by $\theta^1 = \eta^1 = (1,0),\ \theta^2 = \eta^2 = (0,1),\ \theta^3 = \eta^3 = (1,1)$. Since

$$|(d/dx_1) x_2 \log |x|| = |x_1 x_2|/|x|^2 \leq 1 \quad \text{when} \quad |x| \neq 0.$$

Thus

$$|\langle \eta^1, f(x + t\theta^1) - f(x)| = |f_1(x_1 + t, x_2) - f_1(x_1, x_2)| \leq |t|,$$

that is $\langle \eta^1, f \rangle \in K(\theta^1, \sigma)$. Similarly we show that $\langle \eta^2, f \rangle \in K(\theta^2, \sigma)$. Also, if $|x| \neq 0$, then

$$\begin{aligned}
|\eta^3, f(x + t\theta^3) - f(x)\rangle| &= |f_1(x_1 + t, x_2 + t) - f_1(x_1, x_2) + f_2(x_1 + t, x_2 + t) - f_2(x_1, x_2)| \\
&\leq |f_1(x_1 + t, x_2 + t) - f_1(x_1, x_2)| + |f_2(x_1 + t, x_2 + t) - f_2(x_1, x_2)| \\
&= |f_1(x_1 + t, x_2 + t) - f_1(x_1 + t, x_2) + f_1(x_1 + t, x_2) - f_1(x_1, x_2)| \\
&+ |f_2(x_1 + t, x_2 + t) - f_2(x_1, x_2 + t) + f_2(x_1, x_2 + t) - f_2(x_1, x_2)| \\
&\leq |f_1(x_1 + t, x_2 + t) - f_1(x_1 + t, x_2)| + |f_1(x_1 + t, x_2) - f_1(x_1, x_2)| \\
&+ |f_2(x_1 + t, x_2 + t) - f_2(x_1, x_2 + t)| + |f_2(x_1, x_2 + t) - f_2(x_1, x_2)| \\
&\leq c|t|,
\end{aligned}$$

and when $|x| = 0$ the estimate is trivial. Thus $\langle \eta^k, f \rangle \in K(\theta^k, \sigma)$ for each $(\theta^k, \eta^k) \in \Lambda$. By Corollary 2.1.11, $\omega_k(f, t) \leq c\sigma(|t|)$ for $k > 1$ while Corollary 2.1.12 implies that $\omega_1(f, t) \leq C\hat{\sigma}(|t|)$. Now if we choose the compact set $K = \{x \in \mathbf{R^2} : |x| \leq 0.09\}$ then

$$|f(x) - f(0)| = |x| |\log |x|| = \hat{\sigma}(|x|) > |x|.$$

We see that $\omega_1(f,t) \leq c\sigma(|t|)$ is not possible in this case.

2.2 The algebraic condition (A)

The set Λ of pairs $(\theta,\eta) \in \mathbf{R}^p \times \mathbf{R}^q$ is said to satisfy the condition (A) if the set of tensor products $\theta \otimes \eta$ for $(\theta,\eta) \in \Lambda$ spans $\mathbf{R}^p \otimes \mathbf{R}^q$. This is the same as saying that the set of $p \times q$ matrices $(\theta_i \eta_j)$ spans the pq- dimensional vector space $M(p,q)$ of all $p \times q$ matrices.

If we associate the $p \times q$ matrix with the linear form $\alpha_{ij} \mapsto \sum \alpha_{ij} a_{ij}$ on the vector space $M(p,q)$ of all $p \times q$ matrices, then the dual space of $M(p,q)$ will be identified with another copy of $M(p,q)$. Λ will satisfy (A) if there is no non-vanishing element of $M(p,q)$ that is orthogonal to all the matrices $(\theta_i \eta_j)$ for $(\theta,\eta) \in \Lambda$.

Note that the formulation of the condition (A), here is equivalent to that in [10], which is as follows:

$$\text{if } \Phi \text{ is a bilinear form } \mathbf{R^p} \times \mathbf{R^q} \to \mathbf{R} \text{ and } \Phi(\Lambda) = 0 \text{ , then } \Phi = 0.$$

One can establish this equivalence by representing the bilinear forms by $\Phi(\theta,\eta) = \sum a_{ij} \theta_i \eta_j$. We note immediately that if Λ satisfies the condition (A) it must contain at least pq elements. We also observe that the condition (A) is a stronger condition compare with the condition $(\widehat{A})$.

2.2.1 Remark

We give an example of a set satisfying (A). Let $p = q = 2$ and take

$$\Lambda = \{(e_1,e_1),(e_2,e_2),(e_1,e_2),(e_2,e_1) : e_1 = (1,0), e_2 = (0,1)\}$$

then Λ satisfies (A) and the set of 2×2 matrices $(\theta_i \eta_j)$ for $(\theta, \eta) \in \Lambda$ is:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

2.2.2 Theorem

Assume Λ satisfies (A) and that $f : \mathbf{R^p} \to \mathbf{R^q}$, continuous and satisfies

$$\langle \eta, f \rangle \in C^1(\theta) \quad for \quad each \quad (\theta, \eta) \in \Lambda. \tag{2.19}$$

Then $f \in C^1$. Moreover, there exist a constant C depending on Λ only, such that

$$\|Df(x)\| \leq C \sup_{(\theta,\eta)\in\Lambda} |D_\theta \langle \eta, f(x) \rangle| \quad for \quad every \quad x \in \mathbf{R^p}.$$

Here $\|Df(x)\|$ denotes the norm of the differential of f at x considered as an operator from $\mathbf{R^p}$ to $\mathbf{R^q}$. Conversely if (2.19) implies that $f \in C^1$, then (A) holds.

Proof. We have to prove that an arbitrary first partial derivative of f exist and is continuous. Choose bases in $\mathbf{R^p}$ and $\mathbf{R^q}$ such that this derivative is $D_1 f_1$ where $D_1 = \partial/\partial x_1$. Since Λ satisfies (A), there exist b_k and $(\theta^k, \eta^k) \in \Lambda$ such that

$$\sum_{k=1}^{n} b_k \theta_i^k \eta_j^k = \begin{cases} 1 & \text{when } (i,j) = (1,1), \\ 0 & \text{when } (i,j) \neq (1,1). \end{cases} \tag{2.20}$$

Take ψ of class C^1 with compact support such that $\int \psi dx = 1$ and for any $\epsilon > 0$ set

$$f_\epsilon(x) = \int f(x + \epsilon y) \psi(y) dy.$$

For each $(\theta, \eta) \in \Lambda$

$$\begin{aligned} D_\theta \langle \eta, f_\epsilon(x) \rangle &= \frac{d}{dt} \langle \eta, f_\epsilon(x + t\theta) \rangle_{t=0} \\ &= \frac{d}{dt} \int \langle \eta, f(x + t\theta + \epsilon y) \rangle_{t=0} \psi(y)\, dy \end{aligned}$$

which tends to $D_\theta\langle\eta, f(x)\rangle$ as $\epsilon \to 0$. This implies $D_\theta\langle\eta, f_\epsilon(x)\rangle$ converges uniformly on compact sets to $D_\theta\langle\eta, f(x)\rangle$ when $\epsilon \to 0$. Denoting the first component of f_ϵ by $(f_\epsilon)_1$ we have by (2.20)

$$D_1(f_\epsilon)_1 = \sum_{k=1}^{n} b_k D_{\theta^k}\langle\eta^k, f_\epsilon\rangle. \tag{2.21}$$

Indeed

$$\begin{aligned}\sum_{k=1}^{n} b_k D_{\theta^k}\langle\eta^k, f_\epsilon\rangle &= \sum_{k=1}^{n} b_k \frac{d}{dt}\langle\eta^k, f_\epsilon(x+t\theta^k)\rangle|_{t=0}\\ &= \sum_{k=1}^{n} b_k \langle\eta^k, \frac{d}{dt} f_\epsilon(x+t\theta^k)|_{t=0}\rangle\\ &= \sum_{k=1}^{n} b_k \langle\eta^k, D_{\theta^k}(f_\epsilon)\rangle.\end{aligned}$$

Now, since

$$D_{\theta^k}(f_\epsilon) = \sum_{i=1}^{p} \theta_i^k D_i(f_\epsilon),$$

where $D_i = \frac{\partial}{\partial x_i}$, we have that

$$\begin{aligned}\sum_{k=1}^{n} b_k D_{\theta^k}\langle\eta^k, f_\epsilon\rangle &= \sum_{k=1}^{n} b_k \sum_{i=1}^{p} \theta_i^k \langle\eta^k, D_i(f_\epsilon)\rangle\\ &= \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{n} \theta_i^k \eta_j^k D_i(f_\epsilon)_j\\ &= D_1(f_\epsilon)_1,\end{aligned}$$

by (2.20).

Thus, $D_1(f_\epsilon)_1$ is continuous and converges uniformly on compact sets to some function g when $\epsilon \to 0$. Since f_1 is continuous and $D_1(f_\epsilon)_1$ converges uniformly on compact sets to f_1 we must conclude that f_1 is differentiable with respect to x_1 and that $D_1 f_1 = g$. We also obtain equation (2.21) with $f_\epsilon = f$. Since the constants b_k

only depend on Λ we have

$$|D_1 f_1| \le C \sum_{(\theta,\eta)\in\Lambda} |D_\theta \langle \eta, f\rangle|.$$

Using a similar arguments we obtain

$$|D_k f_j| \le C \sum_{(\theta,\eta)\in\Lambda} |D_\theta \langle \eta, f\rangle| \text{ for each } k = 1, 2, \cdots, n \text{ and } , j = 1, 2 \cdots, n.$$

Thus

$$\begin{aligned} \|Df(x)\| &= \max_{k,j} |D_k f_j| \\ &\le C \sup_{(\theta,\eta)\in\Lambda} |D_\theta \langle \eta, f(x)\rangle| \end{aligned}$$

which the estimate for $\|Df(x)\|$.

Conversely, suppose Λ does not satisfy (A). We shall show that there exists a function, $f : \mathbf{R^p} \to \mathbf{R^q}$ with $\langle \eta, f\rangle \in C^1(\theta)$ for all $(\eta, \theta) \in \Lambda$ but f is not differentiable at the origin. Since Λ does not satisfy (A), there exists a non-zero linear operator B from $\mathbf{R^p}$ to $\mathbf{R^q}$ such that $\langle B\theta, \eta\rangle = 0$ for each $(\theta, \eta) \in \Lambda$. Consider the function, f, defined by

$$f(x) = (Bx) \log |\log |x||, 0 < |x| < \frac{1}{2}, \quad f(0) = 0.$$

Then

$$\lim_{t\to 0} \frac{f(tx) - f(0)}{t} = \lim_{t\to 0} (Bx) \log |\log |tx||$$

which does not exist, thus f is not differentiable at the origin.

On the other hand, for all $(\theta, \eta) \in \Lambda$

$$\begin{aligned} D_\theta \langle \eta, f(x)\rangle &= \frac{d}{dt} \langle \eta, f(x + t\theta)\rangle|_{t=0} \\ &= \frac{d}{dt} \langle \eta, (B(x + t\theta)) \log |\log |x + t\theta||\rangle|_{t=0} \\ &= \langle \eta, Bx\rangle \frac{d}{dt} \log |\log |x + t\theta||\ |_{t=0}, \end{aligned}$$

since $\langle B\theta, \eta\rangle = 0$. Also,

$$\frac{d}{dt}\log|\log|x+t\theta||_{\{t=0\}} = \frac{\log|x|}{|\log|x||^2}\frac{\langle\theta, x\rangle}{|x|^2}.$$

Thus,

$$D_\theta\langle\eta, f(x)\rangle = \frac{\log|x|}{|\log|x||^2}\frac{\langle\eta, Bx\rangle\langle\theta, x\rangle}{|x|^2} = \frac{\log|x|}{|\log|x||^2}\langle\eta, B\frac{x}{|x|}\rangle\langle\theta, \frac{x}{|x|}\rangle.$$

Also,

$$|D_\theta\langle\eta, f(x)\rangle| \leq \|B\||\theta||\eta|\frac{1}{|\log|x||}$$

which tends to zero as $x \to 0$. Thus

$$\lim_{x\to 0} D_\theta\langle\eta, f(x)\rangle = 0.$$

Hence, $\langle\eta, f\rangle$, is $C^1(\theta)$ for all $(\eta, \theta) \in \Lambda$. $\square$

2.2.3 Remark

If we replace $C^1(\theta)$ by $C^k(\theta)$, $k > 1$, the assertion of Theorem 2.2.2 becomes false. Indeed, Let $f : \mathbf{R}^2 \to \mathbf{R}$ be define by

$$f(x) = x_1x_2\log|\log|x||, 0 < |x| < \frac{1}{2}, \quad f(0) = 0, \quad x = (x_1, x_2)$$

then $D_1^2 f$ and $D_2^2 f$ exist and are continuous. Also $D_1 f(0, x_2) = x_2\log|\log|x_2||$ which shows that $D_2D_1f(0,0)$ does not exist.

The most important consequences of (A) are deduced from the theorem on the decomposition of vector valued measures (Theorem 2.1.8).

2.2.4 Theorem

Assume that Λ satisfies (A) and let $\theta^0 \in \mathbf{R}^p \setminus \{0\}$, $\eta^0 \in \mathbf{R}^q$, $\mu_0 \in N_{\theta^0}$. Then there exist $\mu_k \in N_{\theta^k}$, $k = 1, 2, \cdots, m \geq pq$ such that

$$\eta^0 \mu_0 = \sum_{k=1}^{m} \eta^k \mu_k. \tag{2.22}$$

Proof. Take a smooth function $\psi(\xi)$ supported in $|\xi| < 2$ and equal to 1 in $|\xi| \leq 1$ and decompose μ_0 as $\mu_0 = \nu_0 + \rho_0$ by taking $\widehat{\nu}_0(\xi) = \psi(\xi)\widehat{\mu}_0(\xi)$ and $\rho_0 = \mu_0 - \nu_0$. The fact that ρ_0 can be expressed in the form (2.22) follows from Theorem 2.1.8. Indeed, since

$$\widehat{\rho}_0(\xi) = \widehat{\mu}_0(\xi) - \widehat{\nu}_0(\xi) = \widehat{\mu}_0(\xi)(1 - \psi(\xi))$$

equals zero for $|\xi| \leq 1$ and the fact that Λ satisfies (A) implies Λ satisfies $(\widehat{A})$. It remains to consider ν_0. Now, because Λ satisfies (A), there exist constants b_k and $(\theta^k, \eta^k) \in \Lambda$ such that

$$\eta^0 \otimes \theta^0 = \sum_k b_k \eta^k \otimes \theta^k$$

or equivalently

$$\eta^0(\theta^0 \cdot \xi) = \sum_k b_k \eta^k (\theta^k \cdot \xi), \ \xi \in \mathbf{R}^p. \tag{2.23}$$

Since $\mu_0 \in N_{\theta^0}(\mathbf{R}^p)$,

$$\frac{\widehat{\mu}_0(\xi)}{\theta^0 \cdot \xi}$$

is the Fourier transform of a measure in $M(\mathbf{R}^p)$ and thus the function

$$G(\xi) = \frac{\widehat{\nu}_0(\xi)}{\theta^0 \cdot \xi} = \frac{\widehat{\mu}_0(\xi)}{\theta^0 \cdot \xi}\psi(\xi) \tag{2.24}$$

is the Fourier transform of a measure in $M(\mathbf{R}^p)$ since $\psi(\xi)$ is smooth and has compact support. Multiplying (2.23) by $G(\xi)$ gives

$$\eta^0 \widehat{\nu}_0(\xi) = \sum_k b_k \eta^k G(\xi)(\theta^k \cdot \xi)$$

where $G(\xi)(\theta^k \cdot \xi)$ belongs to N_{θ^k} which completes the proof.

2.2.5 Corollary

Let f be a function from $\mathbf{R^p}$ to $\mathbf{R^q}$ and $\sigma \in \Sigma$. Let Λ be a subset of $\mathbf{R^p} \times \mathbf{R^q}$ satisfying (A) and assume that

$$\langle \eta, f \rangle \in K(\theta, \sigma) \text{ for every } \ (\theta, \eta) \in \Lambda. \tag{2.25}$$

Then $f \in K(\sigma)$.

Proof Let $\theta^0 \in \mathbf{R}^p$ and $\eta^0 \in \mathbf{R}^q$ be arbitrary vectors. Then by applying Theorem 2.2.4 to $\mu_{0,t}$ with $\mu_0(\phi) = \phi(\theta^0) - \phi(0)$ we get

$$|\mu_{0,t} * \langle \eta^0, f \rangle| = |\sum_{k=1}^{m} \mu_{k,t} * \langle \eta^k, f \rangle| \leq \sum_{k=1}^{m} |\mu_{k,t} * \langle \eta^k, f \rangle|.$$

We now apply Lemma 1.3.7 to get

$$|\mu_{0,t} * \langle \eta^0, f \rangle| \leq C\sigma(t),$$

since $\langle \eta^k, f \rangle \in K(\theta^k, \sigma)$. Now, if $\mu_0(\phi) = \phi(\theta^0) - \phi(0)$ then $\mu_{0,t}(\phi) = \phi(t\theta^0) - \phi(0)$ and $\mu_{0,t} * f(x) = f(x + t\theta^0) - f(x)$ and since θ^0 and η^0 are arbitrary, we see that if we apply this result for $\theta^0 = (1, 1, \cdots, 1)$ and η^0 a set of bases for $\mathbf{R^q}$ we have

$$|f(x + t) \ - f(x)| \leq C\sigma(t)$$

as required.

2.3 Theorem on Modulus of Continuity

2.3.1 Theorem

Let f be a function from $\mathbf{R^p}$ to $\mathbf{R^q}$ and $\sigma \in \Sigma$. Let Λ be a subset of $\mathbf{R^p} \times \mathbf{R^q}$ satisfying (A) and assume that

$$\langle \eta, f \rangle \in K(\theta, \sigma) \text{ for every } \ (\theta, \eta) \in \Lambda. \tag{2.26}$$

Then $f \in K(\sigma)$. Moreover, if Λ satisfies $(\widehat{A})$ and (2.26) holds, then $f \in K(\hat{\sigma})$. Conversely, if (2.26) implies $f \in K(\tau)$, then either Λ satisfies $(\widehat{A})$ and $\tau(t) \geq C\hat{\sigma}(t)$ or Λ satisfies (A) and $\tau(t) \geq C\sigma(t)$

Proof. The if part is just Corollaries 2.1.12 and 2.2.5.

Conversely, if Λ does not satisfies $(\widehat{A})$ then (2.26) does not imply $f \in K(\sigma)$ for any $\sigma \in \sum$. Indeed if Λ does not satisfies $(\widehat{A})$ then there exist non-zero vectors $u \in \mathbf{R}^p$ and $v \in \mathbf{R}^q$ such that $\langle u, \theta\rangle\langle v, \eta\rangle = 0$ for all $(\theta, \eta) \in \Lambda$. Choose such vectors u and v and consider the function $f(x) = v\langle u, x\rangle$. It is easy to see that $\langle \eta, f(x)\rangle \in K(\theta, \sigma)$ for all $(\theta, \eta) \in \Lambda$ and all $\sigma \in \sum$. Now if K is a compact set, for which $x, y, x+y \in K$ with $y = \frac{u}{a|u|^2}$, for some $a > 0$ then $|f(x+y) - f(y)| = |v|/a$ which is a non zero constant.

Furthermore if Λ does not satisfies (A) and (2.26) implies $f \in K(\tau)$ for some $\tau \in \sum$ then by the previous argument Λ must satisfies $(\widehat{A})$. We must show that $f \in K(\tau)$ always holds if $\hat{\sigma}(t) \leq C\tau(t)$. Since (A) does not hold there exist a non-trivial bilinear form Φ which vanishes on Λ. We can represent Φ in the form $\Phi(\theta, \eta) = \langle B\theta, \eta\rangle$, where B is a linear operator from $\mathbf{R}^p$ to $\mathbf{R}^q$. Set

$$f(x) = (Bx/|Bx|)\hat{\sigma}(|Bx|), \quad x \in \mathbf{R}^p.$$

Then $f \in K(\tau)$ implies $\hat{\sigma}(t) \leq C\tau(t)$. We claim that $\langle \eta, f\rangle \in K(\theta, \sigma)$ whenever $\langle B\theta, \eta\rangle = 0$; this will complete the proof. We may assume $|Bx| < |B(x + t\theta)|$. Then

$$\begin{aligned} |\langle \eta, f(x+t\theta) - f(x)\rangle| &= \left| \langle \eta, Bx\rangle \left(\frac{\hat{\sigma}(|B(x+t\theta)|)}{|B(x+t\theta)|} - \frac{\hat{\sigma}(|Bx|)}{|Bx|} \right) \right| \\ &\leq |\langle \eta, Bx\rangle| \int_{|Bx|}^{|B(x+t\theta)|} s^{-2}\sigma(s)ds. \end{aligned}$$

Using the fact that σ is increasing and that $\sigma(st) \leq (1+s)\sigma(t)$ we obtain if $a < b$,

$$\begin{aligned}\int_a^b s^{-2}\sigma(s)ds &\leq \sigma(b)\int_a^b s^{-2}ds = \sigma(b)(b-a)/(ab) = \sigma(\frac{b}{b-a}(b-a))(b-a)/(ab) \\ &\leq \sigma(b-a)\left(1+\frac{b}{b-a})\right)\frac{b-a}{ab}.\end{aligned}$$

With $a = |Bx|$, $b = |B(x+t\theta)|$, we have that $b-a \leq |Bt\theta|$ and hence

$$|\langle \eta, f(x+t\theta) - f(x)\rangle| \leq 2|\eta|\sigma(|Bt\theta|) \leq 2|\eta|(1+|B\theta)\sigma(|t|).\square$$

2.3.2 Remark

As in Boman [10] the proof of Theorem 2.3.1 is deduced from Theorems 2.1.8 and 2.2.4 which gives the representation of vector valued measures as a finite sums of measures of the form $\eta\nu_\theta$ where $\nu_\theta \in N_\theta(\mathbf{R}^p)$ and $(\theta, \eta) \in \Lambda$. The novelties about our approach here relative to Boman [10] are as follows: In the proof of Theorem 2.2.4 the measure μ_0 is written as a sum, $\mu_0 = \nu_0 + \nu_1$, of a measure ν_0 whose Fourier transform has compact support and a measure ν_1 whose Fourier transform vanishes in a neighborhood of the origin, and it is observed that the representation for ν_0 is easy to prove and that the representation for ν_1 is just an immediate consequence of Theorem 2.1.8. The measures μ_k constructed in this way will not be compactly supported like those constructed in the proof of Theorem 3 in Boman [10], but this is not needed for the proof of Theorem 2.3.1. Moreover, Theorem 2.1.8 is somewhat weaker than Theorem 5 of Boman [10] in that $\hat{\mu}_0$ is assumed to vanish in some neighborhood of the origin, but the proof is much simpler, the main new idea being the use of the special partition of unity (2.7). Finally, the measure μ in Lemma 1.2.7 with Fourier transform vanishing in a neighborhood of the origin has replaced the iterated convolutions occurring in Theorem 5 of Boman [10] (typically q-th order difference measures) and therefore Marchaud's inequality ([30]) has been replaced by the easier and perhaps more fundamental Lemma 1.2.7

Part II

Toeplitz Operators on the Bergman Spaces, $L_a^p(\mathbf{B}_n)$, of the Unit Ball in $\mathbf{C}^n$

Chapter 3

Toeplitz operators on $L^p_a(B_n)$ for $p > 1$.

3.1 Introduction

Throughout this chapter, n is a fixed positive integer. We denote by $\mathbf{B}_n$ the unit ball of $\mathbf{C}^n$ and by ν the Lebesgue volume measure on $\mathbf{B}_n$, normalized so that $\nu(\mathbf{B}_n) = 1$. For $p \in [1, \infty]$, the Bergman space $L^p_a = L^p_a(\mathbf{B}_n)$ is the closed subspace of the Lebesgue space $L^p(\mathbf{B}_n, d\nu)$ consisting of analytic functions on $\mathbf{B}_n$. When $p = 2$, the Bergman space L^2_a is a closed subspace of the Hilbert space $L^2(\mathbf{B}_n, d\nu)$. We denote by $\langle ., . \rangle$ the usual inner product in $L^2(\mathbf{B}_n, d\nu)$ and by $||.||_p$ the norm in $L^p(\mathbf{B}_n, d\nu)$, $\quad p \in [1, \infty]$. The orthogonal projection P from $L^2(\mathbf{B}_n, d\nu)$ unto L^2_a is called the Bergman projector. It is well-known that the Bergman projection $P\phi$ of a function $\phi \in L^2(\mathbf{B}_n, d\nu)$ is given by

$$P\phi(z) = \langle \phi, K_z \rangle \quad (z \in \mathbf{B}_n)$$

where

$$K_z(w) = \frac{1}{(1 - (w \cdot z))^{n+1}}$$

is called the Bergman kernel of $\mathbf{B}_n$. Here,

$$(w \cdot z) := \sum_{j=1}^{n} w_j \overline{z_j}, \ \ z = (z_1, \cdots, z_n), \ \ w = (w_1, \cdots, w_n).$$

The normalized Bergman kernel k_z is given by

$$k_z(w) := \frac{K_z(w)}{||K_z||_2} = \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-(w \cdot z))^{n+1}}. \tag{3.1}$$

Given $f \in L^{\mathbf{1}}(\mathbf{B}_n, d\nu)$, the Toeplitz operator T_f is densely defined on $L^p_a(\mathbf{B}_n)$ by

$$(T_f g)(w) = \int_{\mathbf{B}_n} \frac{f(z)g(z)}{(1-(w \cdot z))^{n+1}} d\nu(z) = P(fg)(w),$$

for $g \in L^\infty_a$ and $w \in \mathbf{B}_n$, that is $T_f g = P(fg)$. Note that the above formula makes sense and defines an analytic function on $\mathbf{B}_n$.

3.1.1 Definition

Let A be a linear operator on L^p_a, $\quad p \in (1, \infty)$, the Berezin transform $\widetilde{A}$ of A is the function defined by

$$\widetilde{A}(z) := \langle Ak_z, k_z \rangle$$

where k_z is the normalized kernel defined by (3.1). For a function $f \in L^1(\mathbf{B}_n, d\nu)$, the Berezin transform $\widetilde{T_f}$ of T_f will simply be denoted $\widetilde{f}$ and will be called the Berezin transform of f. Explicitly,

$$\widetilde{f}(z) := \widetilde{T_f}(z) = \int_{\mathbf{B}_n} f(w) \frac{(1-|z|^2)^{n+1}}{|1-(z.w)|^{2n+2}} d\nu(w).$$

Let $p \in (1, \infty)$, our interest is to determine conditions on the symbols $f \in L^1(\mathbf{B}_n, d\nu)$ which ensure the boundedness (resp. the compactness) on L^p_a of the associated Toeplitz operator T_f. It is easily checked that if A is a bounded operator on L^p_a, then its Berezin transform $\widetilde{A}(z)$ is a bounded function on $\mathbf{B}_n$; moreover, if A

is also compact on L^p_a, then $\widetilde{A}(z) \to 0$ as $z \to \partial\mathbf{B}_n$. But the converses of these two implications are false. Furthermore, for a symbol $f \in L^1(\mathbf{B}_n, d\nu)$, if T_f is bounded on $L^p_a, \quad p \in (1,\infty)$, then $\sup_{z\in\mathbf{B}_n} ||T_f(k^p_z)||_p < \infty$, where $k^p_z(w) := \frac{(1-|z|^2)^{\frac{n+1}{p'}}}{(1-(w\cdot z))^{n+1}}$ satisfies $||k^p_z||_p = 1$ and $k^2_z = k_z$. However, for $p = 2$ and $n = 1$, F. Nazarov produced an unpublished counterexample for the converse implication.

In our study we will extend to the unit ball $\mathbf{B}_n, \quad n \geq 1$ the space BT of symbols introduced by Miao and Zheng [31]. Explicitly, a symbol $f \in L^1(\mathbf{B}_n, d\nu)$ belongs to BT if the measure $|f|d\nu$ is a Carleson measure for Bergman spaces on the unit ball. For BT symbols, the associated Toeplitz operators are bounded on L^p_a for all $p \in (1,\infty)$. Using this, we derive some new classes of L^1 symbols for which known necessary conditions for boundedness of the Toeplitz operators on L^p_a are also sufficient. For example, we obtain a class of L^1 functions, X_1, containing the space BMO^1 and non-negative L^1 functions such that T_f is bounded if and only if $\tilde{f}(z)$ is a bounded function on $\mathbf{B}_n$, provided $f \in X_1$. An example shows that there exist functions f in X_1 such that T_f is not bounded.

We also exhibit a class X_2 of L^1 symbols f for which T_f is bounded on L^2_a if

$$\sup_{z\in\mathbf{B}_n} ||T_f(k_z)||_2 < \infty. \tag{3.2}$$

In other words, X_2 is a class of symbols for which this well known necessary condition (3.2) is also sufficient. Moreover, this class of symbols can be described without any reference to the space BT.

We also mention that our results on compactness extend the result of Lu Yu-feng [27], whose result was an extension of the result of Axler and Zheng[5] to the unit ball and the polydisk in $\mathbf{C^n}$. Finally, we notice that the condition $f \in BT$ is equivalent to some Schur's estimates related to some linear operator S_f defined on $L^p(\mathbf{B}_n)$. Let us mention that beyond Toeplitz operators T_f, $\quad f \in L^1(\mathbf{B}_n, d\nu)$, we study general linear operators A on L^p_a.

3.2 Preliminaries

For $w \in \mathbf{B}_n$, let $P_w(z) = \frac{(z \cdot w)}{|w|^2} w$ which is the orthogonal projection from $\mathbf{C^n}$ onto the subspace spanned by w and $Q_w = I - P_w$. Then the mapping $\varphi_w : \mathbf{B}_n \to \mathbf{B}_n$ given by

$$\varphi_w(z) = \frac{w - P_w(z) - \sqrt{(1 - |w|^2)} Q_w(z)}{1 - (z \cdot w)}$$

is an automorphism of $\mathbf{B}_n$ such that $\varphi_w(0) = w$ and $\varphi_w^{-1} = \varphi_w$. More about the mappings φ_w are described in section 2.2 of [37] and section 1.2 of [47]. The following identities hold:

$$1 - (\varphi_w(z) \cdot w) = \frac{1 - |w|^2}{1 - (z \cdot w)}, \quad 1 - |\varphi_w(z)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - (z \cdot w)|^2}. \tag{3.3}$$

For $w \in \mathbf{B}_n$, the real Jacobian of the function φ_w is given by

$$|J_{\varphi_w}(z)|^2 = \frac{(1 - |w|^2)^{n+1}}{|1 - (w \cdot z)|^{2n+2}}.$$

This gives the change of variable formula

$$\int_{B_n} f(\varphi_w(z)) |k_w(z)|^2 \, d\nu(z) = \int_{B_n} f(z) \, d\nu(z) \tag{3.4}$$

for every $f \in L^1(\mathbf{B}_n, d\nu)$. It then follows easily that the transformation U_w defined by $U_w f := (f \circ \varphi_w) k_w$ is an isometry on L^2_a, that is,

$$\|U_w f\|_2^2 = \int_{\mathbf{B}_n} |f(\varphi_w(z))|^2 |k_w(z)|^2 \, d\nu(z) = \int_{\mathbf{B}_n} |f(z)|^2 \, d\nu(z) = \|f\|_2^2,$$

for all $f \in L^2_a(\mathbf{B}_n)$. Using the first identity in (3.3) we have

$$k_w(\varphi_w(z)) = \frac{(1 - |w|^2)^{(n+1)/2}}{(1 - (\varphi_w(z) \cdot w)^{n+1}} = \frac{(1 - (z \cdot w)^{n+1}}{(1 - |w|^2)^{(n+1)/2}} = \frac{1}{k_w(z)}.$$

This implies

$$(U_w(U_wf))(z) = (U_wf)(\varphi_w(z))k_w(z) = f(z)k_w(\varphi_w(z))k_w(z) = f(z),$$

for all $z \in \mathbf{B}_n$ and $f \in L^2_a(\mathbf{B}_n)$. Hence $U_w \circ U_w$ is the identity on L^2_a and $U_w^* = U_w$, i.e. U_w is unitary.

If A is a bounded operator on L^2_a, then for $z \in \mathbf{B}_n$, we define a bounded operator A_z on L^2_a by

$$A_z := U_z A U_z.$$

We shall need the following lemma:

3.2.1 Lemma

(1) For all $f \in L^1(\mathbf{B}_n, d\nu)$ and $z \in \mathbf{B}_n$, we have

$$\widetilde{f}(z) = \int_{\mathbf{B}_n} f \circ \varphi_z(w) d\nu(w).$$

(2) For all $w \in \mathbf{B}_n$ and all $f \in L^1(\mathbf{B}_n, d\nu)$ such that T_f is bounded on L^2_a, we have that

$$T_{f\circ\phi_w} U_w = U_w T_f.$$

(3) Let $z, w \in \mathbf{B}_n$. Then

$$U_z K_w = \overline{k_z(w)} K_{\varphi_z(w)} \tag{3.5}$$

and

$$U_z k_w = \alpha k_{\varphi_z(w)}, \tag{3.6}$$

where $|\alpha| = 1$.

(4) If A is a bounded operator on L^p_a, then for every $z \in \mathbf{B}_n$, we have

$$\widetilde{A} \circ \varphi_z = \widetilde{A_z}.$$

Proof (1) Using (3.4) we have

$$\begin{aligned}\tilde{f}(z) &= \langle T_f k_z, k_z\rangle = \langle f k_z, k_z\rangle \\ &= \int_{\mathbf{B}_n} f(w)|k_z(w)|^2\, d\nu(w) \\ &= \int_{\mathbf{B}_n} f\circ\varphi_z(w)\, d\nu(w).\end{aligned}$$

(2) For every $g, h \in L^2_a$, we have:

$$\begin{aligned}\langle U_w T_f g, U_w h\rangle &= \langle T_f g, h\rangle = \langle fg, h\rangle \\ &= \int_{\mathbf{B}_n} f(z)g(z)\overline{h(z)}d\nu(z) \\ &= \int_{\mathbf{B}_n} f(\varphi_w(z))g(\varphi_w(z))\overline{h(\varphi_w(z))}|k_w(z)|^2 d\nu(z) \\ &= \langle f\circ\varphi_w U_w g, U_w h\rangle = \langle T_{f\circ\varphi_w} U_w g, U_w h\rangle.\end{aligned}$$

(3) For every $g \in L^2_a$, we have:

$$\langle g, U_z K_w\rangle = \langle U_z g, K_w\rangle = U_z g(w) = g(\varphi_z(w))k_z(w) = \langle g, \overline{k_z(w)}K_{\varphi_z(w)}\rangle.$$

This proves (3.5). Also,

$$U_z k_w = (1-|w|^2)^{\frac{n+1}{2}} U_z K_w = (1-|w|^2)^{\frac{n+1}{2}}\overline{k_z(w)}K_{\varphi_z(w)},$$

where the latter equality follows from (3.5). Thus

$$U_z k_w = k_{\varphi_z(w)} K_w(z) \frac{(1-|w|^2)^{\frac{n+1}{2}}(1-|z|^2)^{\frac{n+1}{2}}}{(1-|\varphi_z(w)|^2)^{\frac{n+1}{2}}} = k_{\varphi_z(w)} K_w(z)|K_z(w)|^{-1}.$$

The latter equality follows from the equality,

$$\frac{1}{(1-|\varphi_z(w)|^2)^{\frac{n+1}{2}}} = |K_z(w)|^{-1}(1-|w|^2)^{-\frac{n+1}{2}}(1-|z|^2)^{-\frac{n+1}{2}} \tag{3.7}$$

obtained easily using (3.3).

(4) For $z \in \mathbf{B}_n$

$$\widetilde{A_z}(w) = \langle A_z k_w, k_w\rangle = \langle A U_z k_w, U_z k_w\rangle.$$

Use the relation in (3.6) to get

$$\begin{aligned}\widetilde{A}(\varphi_z(w)) &= \langle Ak_{\varphi_z(w)}, k_{\varphi_z(w)}\rangle \\ &= \langle AU_z k_w, U_z k_w\rangle \\ &= \langle A_z k_w, k_w\rangle = \widetilde{A_z}(w)\end{aligned}$$

as required. □

Let $d\sigma$ denote the surface measure on the unit sphere $\mathbf{S}_n$, normalized so that $\sigma(\mathbf{S}_n) = 1$. Our next lemma which is [47, Theorem 1.12] or [37, Proposition 1.4.10] is always very useful to estimate integral operators whose kernels is a power of the Bergman kernel. We just state this very useful result.

3.2.2 Lemma

Suppose c is real and $t > -1$. Then the integrals

$$I_c(z) = \int_{\mathbf{S}_n} \frac{d\sigma(\xi)}{|1-(z\cdot\xi)|^{n+c}}, \qquad z \in B_n,$$

and

$$J_{c,t}(z) = \int_{\mathbf{B}_n} \frac{(1-|w|^2)^t}{|1-(z\cdot w)|^{n+1+t+c}}\, d\nu(w), \qquad z \in B_n,$$

have the following asymptotic properties.

1. If $c < 0$ then I_c and $J_{c,t}$ are both bounded in B_n.

2. If $c = 0$, then

$$I_c \approx J_{c,t} \approx \log\frac{1}{1-|z|^2}, \quad \text{as } |z| \to 1^-$$

3. If $c > 0$, then

$$I_c \approx J_{c,t} \approx \frac{1}{(1-|z|^2)^c}, \quad \text{as} \quad |z| \to 1^-$$

Here and elsewhere, $A \approx B$ implies there exist constants $C, K > 0$ such that $CA \leq B \leq KA$.

Let $1 < p < \infty$. Then p' is the conjugate exponent of p whenever $\frac{1}{p} + \frac{1}{p'} = 1$.

3.2.3 Lemma

Let $1 < p < \infty$, $p_1 = \min\{p, p'\}$, $n \geq 1$ and $s > (p_1 + n)/(p_1 - 1)$. Suppose A is a bounded operator on L^p_a and

$$\max\left\{\frac{n+1}{sp'}, \frac{n+1}{sp}\right\} < \epsilon < \min\left\{\frac{1}{ps'}, \frac{1}{p's'}\right\}. \tag{3.8}$$

Then there exists a constant C such that

$$\int_{\mathbf{B}_n} |(AK_z)(w)|(1-|w|^2)^{-p\epsilon}\, d\nu(w) \leq C\|A_z 1\|_s (1-|z|^2)^{-p\epsilon} \tag{3.9}$$

and

$$\int_{\mathbf{B}_n} |(AK_z)(w)|(1-|z|^2)^{-p'\epsilon}\, d\nu(z) \leq C\|A^*_w 1\|_s (1-|w|^2)^{-p'\epsilon} \tag{3.10}$$

Proof We first observe that

$$\left\{\frac{n+1}{sp'}, \frac{n+1}{sp}\right\} < \min\left\{\frac{1}{ps'}, \frac{1}{p's'}\right\}$$

where p' and s' are the conjugate exponents of p and s respectively. Indeed, if we assume, without loss of generality, that $p \geq 2$, that is $p_1 = p'$. We then have to show that $\frac{n+1}{sp'} < \frac{1}{ps'}$, but this inequality holds for $s' < (p' + n)/(n+1)$ or equivalently for $s > (p' + n)/(p' - 1)$. We now prove our Lemma. Let ϵ be as in (3.8).

Fix $z \in \mathbf{B}_n$. We have

$$AK_z = \frac{AU_z 1}{(1-|z|^2)^{(n+1)/2}} = \frac{U_z A_z U_z 1}{(1-|z|^2)^{(n+1)/2}} = \frac{(A_z 1) \circ \varphi_z)k_z}{(1-|z|^2)^{(n+1)/2}}$$

where the second equality comes from the definition of A_z, and the third equality from the definition of U_z. Thus

$$\int_{\mathbf{B}_n} \frac{|(AK_z)(w)|}{(1-|w|^2)^{p\epsilon}}\, d\nu(w) = \frac{1}{(1-|z|^2)^{(n+1)/2}} \int_{\mathbf{B}_n} \frac{|(A_z 1)\circ\varphi_z(w)|k_z(w)|}{(1-|w|^2)^{p\epsilon}}\, d\nu(w).$$

In the latter integral, we make the change of variable $w = \varphi_z$ to obtain,

$$\begin{aligned}
& \int_{\mathbf{B}_n} \frac{|(AK_z)(w)|}{(1-|w|^2)^{p\epsilon}}\, d\nu(w) \\
= \; & \frac{1}{(1-|z|^2)^{(n+1)/2}} \int_{\mathbf{B}_n} \frac{|(A_z 1)(w)|k_z(\varphi_z(w))|}{(1-|\varphi_z(w)|^2)^{p\epsilon}} |k_z(w)|^2\, d\nu(w) \\
= \; & \frac{1}{(1-|z|^2)^{(n+1)/2}} \int_{\mathbf{B}_n} \frac{|(A_z 1)(w)|k_z(w)|}{(1-|\varphi_z(w)|^2)^{p\epsilon}}\, d\nu(w) \\
= \; & \frac{1}{(1-|z|^2)^{\epsilon p}} \int_{\mathbf{B}_n} \frac{|A_z 1(w)|}{|1-(z\cdot w)|^{n+1-2p\epsilon}(1-|w|^2)^{p\epsilon}}\, d\nu(w),
\end{aligned}$$

where we have used (3.7) to get the second and third equality respectively. Apply Hölder's inequality on the right hand side above, we get

$$\int_{\mathbf{B}_n} \frac{|(AK_z)(w)|}{(1-|w|^2)^{p\epsilon}}\, d\nu(w) \leq \frac{\|A_z 1\|_s}{(1-|z|^2)^{\epsilon p}} \left(\int_{\mathbf{B}_n} \frac{(1-|w|^2)^{-s'p\epsilon}}{|1-(z\cdot w)|^{(n+1-2p\epsilon)s'}}\, d\nu(w) \right)^{\frac{1}{s'}}.$$

Since $\frac{n+1}{sp} < \epsilon < \frac{1}{ps'}$, Lemma 3.2.2, shows that the last integral on the right is uniformly bounded with respect to $z \in \mathbf{B}_n$. This gives the estimate (3.9). We finally show the estimate (3.10). Since A^* is bounded on $L_a^{p'}$, the identity

$$(A^* K_w)(z) = \langle A^* K_w, K_z \rangle = \langle K_w, AK_z \rangle = \overline{(AK_z)(w)},$$

reduces the estimate (3.10) to estimate (3.9) with A replaced by A^*. $\square$

Let us recall Schur's lemma which is a widely used method to show that an integral operator with kernel Q is bounded on the Lebesgue space $L^p(X, \mathcal{M}, d\mu), 1 < p < \infty$. For $p \in (1, \infty)$, we denote p' the conjugate exponent of p.

3.2.4 Lemma (Schur's Lemma, cf. [22])

For $p \in (1,\infty)$, suppose that there are a positive measurable function ϕ on the measurable space $(X, \mathcal{M})$ and two positive constants C_1 and C_2 such that

$$\int_X |Q(x,y)|\phi(y)^{p'} d\mu(y) \leq C_1 \phi(x)^{p'}$$

and

$$\int_X |Q(x,y)|\phi(x)^{p} d\mu(x) \leq C_2 \phi(y)^{p}.$$

Then the integral operator $\mathcal{Q}$ defined by

$$\mathcal{Q}\psi(x) := \int_X Q(x,y)\psi(y) d\mu(y)$$

is bounded on $L^p(X, \mathcal{M}, d\mu)$ with $||\mathcal{Q}|| \leq C_1^{\frac{1}{p}} C_2^{\frac{1}{p'}}$.

3.2.5 Lemma

Let A be a bounded operator on L^p_a for some $p \in (1,\infty)$. We suppose further that $\sup_{z \in \mathbf{B}_n} ||A_z 1||_s < \infty$ for some $s > 1$. Then the following are equivalent:

(1) $\widetilde{A}(z) \to 0$ as $z \to \partial\mathbf{B}_n$;

(2) for every $q \in [1, s)$, $||A_z 1||_q \to 0$ as $z \to \partial\mathbf{B}_n$;

(3) $||A_z 1||_1 \to 0$ as $z \to \partial\mathbf{B}_n$.

Proof (1) $\Rightarrow$ (2). Suppose that $\widetilde{A}(z) \to 0$ as $z \in \partial\mathbf{B}_n$. Fix $q \in [1, s)$. We will show that $\|A_z 1\|_q \to 0$ as $z \to \partial\mathbf{B}_n$. Let α, β be multi-indices with non-negative integers and $z \in \mathbf{B}_n$. Then

$$\begin{aligned} |\langle A_z w^\alpha, w^\beta\rangle| &= |\langle AU_z w^\alpha, U_z w^\beta\rangle| \\ &= (1-|z|^2)^{n+1}|\langle A[w^\alpha \circ \varphi_z K_z], w^\beta \circ \varphi_z K_z\rangle| \\ &\leq (1-|z|^2)^{n+1}\|A\|_p \|w^\alpha \circ \varphi_z K_z\|_p \|w^\beta \circ \varphi_z K_z\|_{p'} \\ &\leq (1-|z|^2)^{n+1}\|A\|_p \|K_z\|_p \|K_z\|_{p'} \\ &\leq C(1-|z|^2)^{n+1}\|A\|_p (1-|z|^2)^{-(n+1)/p'}(1-|z|^2)^{-(n+1)/p} \\ &= C\|A\|_p \end{aligned} \tag{3.11}$$

where the first inequality comes from Hölder's inequality and the second comes from the fact that $|w^\alpha \circ \varphi_z| < 1$ and $|w^\beta \circ \varphi_z| < 1$ for all multi-indices α and β with non-negative integers.

First we show that $\langle A_z 1, w^m\rangle \to 0$ as $z \to \partial\mathbf{B}_n$ for every multi-index m with nonnegative integers. If this is not true, then there is a sequence $z_k \in \mathbf{B}_n$ such that

$$\langle A_{z_k} 1, w^m\rangle \to a_{0m}$$

and $z_k \to \partial\mathbf{B}_n$ for some non-zero constant a_{0m} and some m. Since (3.11) implies $|\langle A_z w^\alpha, w^\beta\rangle|$ is uniformly bounded for $z \in \mathbf{B}_n$ we may assume that for each α and β

$$\langle A_{z_k} w^\alpha, w^\beta\rangle \to a_{\alpha\beta}$$

as $z_k \to \partial\mathbf{B}_n$ for some constant $a_{\alpha\beta}$.

We shall also make use of the power series representation of the normalized Bergman kernel. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{N^n}$, write $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$,

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$$

and

$$\partial^\alpha f = \partial^{|\alpha|} f = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n} f}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}}.$$

Then

$$\begin{aligned} k_z(w) &= (1-|z|^2)^{(n+1)/2)} \sum_{j=0}^{\infty} \frac{(j+n)!}{n!j!} (w \cdot z)^j \\ &= (1-|z|^2)^{(n+1)/2)} \sum_{j=0}^{\infty} \frac{(j+n)!}{n!j!} \sum_{|\alpha|=j} \frac{j!}{\alpha!} w^\alpha \overline{z}^\alpha \\ &= (1-|z|^2)^{(n+1)/2)} \sum_{|\alpha|=0}^{\infty} \frac{(|\alpha|+n)!}{n!\alpha!} w^\alpha \overline{z}^\alpha \end{aligned} \tag{3.12}$$

Set $C_{\alpha,n} = \frac{(|\alpha|+n)!}{n!\alpha!}$. Then for $z \in \mathbf{B}_n$ we have

$$\widetilde{A}(z) = (1-|z|^2)^{n+1} \sum_{|\alpha|,|\beta|=0}^{\infty} C_{\alpha,n} C_{\beta,n} \langle Aw^\alpha, w^\beta \rangle \overline{z}^\alpha z^\beta \tag{3.13}$$

by first multiplying both sides (3.12) by A and then take the inner product with k_z. Thus using the relation equality $\widetilde{A}(\varphi_z(v)) = \widetilde{A_z}(v)$ (Lemma 3.2.1(4)) we have immediately that

$$\widetilde{A}(\varphi_z(v)) = (1-|v|^2)^{n+1} \sum_{|\alpha|,|\beta|=0}^{\infty} C_{\alpha,n} C_{\beta,n} \langle A_z w^\alpha, w^\beta \rangle \overline{v}^\alpha v^\beta. \tag{3.14}$$

For each $v \in \mathbf{B}_n$, $\varphi_{z_k}(v) \to \partial \mathbf{B}_n$ as $z_k \to \partial \mathbf{B}_n$. Thus $\widetilde{A}(\varphi_{z_k}(v)) \to 0$ as $z_k \to \partial \mathbf{B}_n$. Replacing z by z_k in (3.14) and taking the limit as $z_k \to \partial \mathbf{B}_n$ for (3.14), we get

$$(1-|v|^2)^{n+1} \sum_{|\alpha|,|\beta|=0}^{\infty} C_{\alpha,n} C_{\beta,n} a_{\alpha\beta} \overline{v}^\alpha v^\beta = 0$$

for each $v \in \mathbf{B}_n$ (note that the interchange of limit and infinite sum is justified by the fact that for each fixed $v \in \mathbf{B}_n$, the series of (3.14) converges uniformly for $z \in \mathbf{B}_n$). Let

$$f(v) = \sum_{|\alpha|,|\beta|=0}^{\infty} C_{\alpha,n} C_{\beta,n} a_{\alpha\beta} \overline{v}^\alpha v^\beta.$$

Then $f(v) = 0$ for all $v \in \mathbf{B}_n$. This implies

$$\left[\frac{\partial^\beta}{\partial v^\beta}\frac{\partial^\alpha}{\partial \overline{v}^\alpha} f\right](0) = 0$$

for each α and β. On the other hand, we have

$$\left[\frac{\partial^\beta}{\partial v^\beta}\frac{\partial^\alpha}{\partial \overline{v}^\alpha} f\right](0) = C_{\alpha,n}C_{\beta,n}\alpha!\beta!a_{\alpha\beta}$$

for each α and β. Thus $a_{\alpha\beta} = 0$, in particular $a_{0m} = 0$ which is a contradiction. Hence we obtain

$$\lim_{z\to\partial B_n} \langle A_z 1, w^m\rangle = 0.$$

For $v \in \mathbf{B}_n$, we have

$$(A_z 1)(v) = \sum_{|m|=0} C_{m,n}\langle A_z 1, w^m\rangle v^m.$$

It is clear that for each fixed $v \in \mathbf{B}_n$, the power series above converges uniformly for $z \in B_n$. This gives

$$\lim_{z\to\partial \mathbf{B}_n} (A_z 1)(v) = 0$$

for each $v \in \mathbf{B}_n$. Thus

$$\lim_{z\to\partial \mathbf{B}_n} |(A_z 1)(v)|^q = 0$$

for each $v \in \mathbf{B}_n$. Let $r = s/q$. Then $r > 1$. Thus

$$\int_{\mathbf{B}_n} (|(A_z 1)(v)|^q)^r \, d\nu(v) = \|A_z 1\|_s^s \le \sup_{z\in\mathbf{B}_n} \|A_z 1\|_s^s < \infty.$$

This implies that $\{|A_z 1|^q\}_{z\in\mathbf{B}_n}$ is uniformly integrable. By Vitali's Theorem (cf Corollary 13 page 183 of [35]) or [38] page 134-135 exercises 9-11, we have that

$$\lim_{z\to\partial \mathbf{B}_n} \|(A_z 1)(v)\|_q = 0$$

which shows that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Suppose for every $q \in [1, s)$, $\|A_z\|_q \to 0$ as $z \to \partial \mathbf{B}_n$. This implies $\|A_z\|_1 \to 0$ as $z \to \partial \mathbf{B}_n$.

(3) $\Rightarrow$ (1). By assertion (4) of Lemma 3.2.1, we get:

$$|\widetilde{A}(z)| = |\widetilde{A}(\varphi_z(0))| = |\widetilde{A}_z(0)| = |\langle A_z 1, 1\rangle| \leq ||A_z 1||_1. \quad \square$$

We end this section with some notions on the Carleson measures on the Bergman spaces. Recall that for $r > 0$ and $z \in \mathbf{B}_n$ the set

$$D(z, r) = \{w \in \mathbf{B}_n : \beta(z, w) < r\}$$

is the Bergman ball centered at z with radius r. For fixed $r > 0$, we have the following well known identities:

$\nu(D(z, r)) \approx \nu(D(w, r))$, $1 - |z|^2 \approx |1 - (z \cdot w)|$ when $\beta(z, w) < r$, and $\nu(D(z, r)) \approx (1 - |z|^2)^{n+1}$.

Our next lemma is Theorem 2.23 of [47].

3.2.6 Lemma

There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in B_n with the following properties:

1. $\mathbf{B}_n = \bigcup_k^\infty D(a_k, r)$.
2. The sets $D(a_k, r/4)$ are mutually disjoint.
3. Each point $z \in \mathbf{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$.

The following is Lemma 2.2.4 of [47].

3.2.7 Lemma

Suppose $r > 0$, $p > 0$. Then there exists a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{\nu(D(z,r))} \int_{D(z,r)} |f(w)|^p \, d\nu(w)$$

for all f analytic in $\mathbf{B}_n$ and all $z \in \mathbf{B}_n$.

3.2.8 Definition

A positive Borel measure μ on $\mathbf{B}_n$ is called a Carleson measure for the Bergman space L_a^p, or simply a Carleson measure, if there exists a constant $C > 0$ such that

$$\int_{\mathbf{B}_n} |f(z)|^p d\mu(z) \leq C \int_{\mathbf{B}_n} |f(z)|^p d\nu(z) \tag{3.15}$$

for all $f \in L_a^p$.

The infimum of all constants C which satisfy (3.15) is called the Carleson measure constant of μ and will be denoted by $C(\mu)$. The next theorem recalls a characterization of Carleson measures for Bergman spaces which is [47, Theorem 2.25].. We fix $r > 0$ and write $D(z) = D(z,r)$ for every $z \in \mathbf{B}_n$.

3.2.9 Theorem

Let μ be a positive Borel measure on $\mathbf{B}_n$. The following four assertions are equivalent:

(1) for some $p \in [1, \infty)$, μ is a Carleson measure for the Bergman space L_a^p;

(2) there exists a positive constant C such that

$$\int_{\mathbf{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-(w.z)|^{2(n+1)}} d\mu(w) \leq C$$

for all $z \in \mathbf{B}_n$;

(3) there exists a positive constant C such that

$$\int_{D(z)} d\mu(w) \leq C(1-|z|^2)^{n+1}$$

for all $z \in \mathbf{B}_n$;

(4) for all $p \in [1, \infty)$, μ is a Carleson measure for the Bergman space L^p_a.

Moreover, the Carleson measure constant $C(\mu)$ of μ is smaller than the constant C of assertion (2).

Proof (1) $\Rightarrow$ (2). Suppose μ is a Carleson measure for the Bergman space L^p_a for some $p \in [1, +\infty)$. Then for $z \in \mathbf{B}_n$ we have

$$\begin{aligned}
\int_{\mathbf{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2(n+1)}} d\mu(w) &= (1-|z|^2)^{n+1} \int_{\mathbf{B}_n} \left(\frac{1}{|1-(w\cdot z)|^{\frac{2(n+1)}{p}}} \right)^p d\mu(w) \\
&\leq C(\mu)(1-|z|^2)^{n+1} \int_{\mathbf{B}_n} \frac{1}{|1-(w\cdot z)|^{2(n+1)}} d\nu(w) \\
&\leq C'(\mu)
\end{aligned}$$

since $(1-(w\cdot z))^{-\frac{2(n+1)}{p}}$ is in L^p_a and the last inequality follows from Lemma 3.2.2. If (2) is true, then

$$\int_{D(z)} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2(n+1)}} d\mu(w) \leq C$$

since $D(z) \subset \mathbf{B}_n$. Also, because $w \in D(z)$ we have

$$\int_{D(z)} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2(n+1)}} d\mu(w) \approx (\nu(D(z)))^{-1} \int_{D(z)} d\mu(w).$$

Thus

$$\mu(D(z)) \leq C\nu(D(z)),$$

that is (2) $\Rightarrow$ (3).

Suppose (3) is true and let f be analytic in $\mathbf{B}_n$ and $r > 0$. Then Lemma 3.2.6 implies there exists sequences $\{a_k\}$ in $\mathbf{B}_n$ such that

$$\begin{aligned}\int_{\mathbf{B}_n} |f(z)|^p d\mu(z) &\leq \sum_{k=1}^{\infty} \int_{D(a_k,r)} |f(z)|^p d\mu(z) \\ &\leq \sum_{k=1}^{\infty} \mu(D(a_k,r)) \sup\{|f(z)|^p : z \in D(a_k,r)\}.\end{aligned}$$

By Lemma 3.2.7, there exists a constant $C' > 0$ such that

$$\begin{aligned}|f(z)|^p &\leq \frac{C'}{\nu(D(z,r))} \int_{D(z,r)} |f(w)|^p d\nu(w) \\ &\leq \frac{C'}{\nu(D(a_k,r))} \int_{D(a_k,2r)} |f(w)|^p d\nu(w)\end{aligned}$$

where we have used the fact that $\nu(D(z,r)) \approx \nu(D(a_k,r))$ since $\beta(z,a_k) < r$. Thus

$$\sup\{|f(z)|^p : z \in D(a_k,r)\} \leq \frac{C'}{\nu(D(a_k,r))} \int_{D(a_k,2r)} |f(w)|^p d\nu(w)$$

for all $k \geq 1$. It follows from assertion (3) that

$$\sum_{k=1}^{\infty} \mu(D(a_k,r)) \sup\{|f(z)|^p : z \in D(a_k,r)\} \leq C'' \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p d\nu(z).$$

That is,

$$\int_{\mathbf{B}_n} |f(z)|^p d\mu(z) \leq C'' \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(z)|^p d\nu(z)$$

for all analytic $f \in B_n$. Since every point in $\mathbf{B}_n$ belongs to at most N of the sets $D(a_k, 4r)$, we must have

$$\int_{\mathbf{B}_n} |f(z)|^p d\mu(z) \leq C'' N \int_{\mathbf{B}_n} |f(z)|^p d\nu(z)$$

for all analytic f in $\mathbf{B}_n$. That is (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1) is trivial. $\square$

3.3 An Associated operator on the Lebesgue space $L^p(\mathbf{B}_n)$

In this section , we clarify some results obtained by N. Zorboska [52] and Miao-Zheng [31]. Let $p \in (1, \infty)$ and $f \in L^1(\mathbf{B}_n)$. To a Toeplitz operator T_f, we associate an integral operator, S_f, on the Lebesgue space $L^p := L^p(\mathbf{B}_n, d\nu)$ by

$$S_f = T_f \circ P.$$

The boundedness of P in L^p implies that

$$T_f \text{ is bounded on } L^p_a \iff S_f \text{ is bounded on } L^p.$$

We then apply Schur's Lemma with test functions $(1 - |z|^2)^{-\epsilon}$ to get a sufficient condition for the boundedness of S_f on $L^p(\mathbf{B}_n, d\nu)$ and hence for T_f on L^p_a. On the unit disk $\mathbf{B}_1$ of the complex plane, N. Zorboska [52] and J. Miao and D. Zheng [31] proved respectively for $p = 2$ and for general $p \in (1, \infty)$ that the boundedness of T_f is implied by extra integrability conditions

$$\sup_{z \in \mathbf{B}_1} ||T_{f \circ \phi_z} 1||_s < \infty \quad and \quad \sup_{z \in \mathbf{B}_1} ||T_{\bar{f} \circ \phi_z} 1||_s < \infty.$$

Here $s > \frac{3}{p_1 - 1}$ with $p_1 = \min(p, p')$, and $\phi_z(w) := \frac{z - w}{1 - \bar{z}w}$.

We show here that these extra integrability conditions can be weakened to

$$\sup_{z \in \mathbf{B}_1} ||T_{f \circ \phi_z} 1||_s < \infty \quad and \quad \sup_{z \in \mathbf{B}_1} ||T_{\bar{f} \circ \phi_z} 1||_s < \infty$$

with $s > \frac{p_1 + 1}{p_1 - 1}$.

Let A be a linear operator on L^p_a, densely defined on L^∞_a with values in the space of analytic functions on $\mathbf{B}_n$. If A extends to a bounded operator on L^p_a, then the operator

$$S := AP$$

extends to a bounded operator from $L^p(\mathbf{B}_n, d\nu)$ to L^p_a since P is a bounded operator from $L^p(\mathbf{B}_n, d\nu)$ to L^p_a. In this case, the adjoint operator $A^\star$ (with respect to the duality pairing $\langle , \rangle$) of A is bounded on $L^{p'}_a$ and satisfies

$$A^\star K_z(w) = \langle A^\star K_z, K_w\rangle = \langle K_z, AK_w\rangle = \overline{AK_w(z)} \tag{3.16}$$

for all $z, w \in \mathbf{B}_n$. The latter equality holds because $AK_w \in L^p_a$ since A is bounded on L^p_a and $K_w \in L^p_a$. Therefore, for every $l \in \mathcal{C}^\infty_c(\mathbf{B}_n)$, we obtain the formula

$$Sl(z) = \int_{\mathbf{B}_n} l(w) AK_w(z) d\nu(w). \tag{3.17}$$

In fact,

$$Sl(z) = \langle Sl, K_z\rangle = \langle APl, K_z\rangle = \langle Pl, A^\star K_z\rangle = \langle l, A^\star K_z\rangle$$

and using (3.16), we get (3.17). In other words, S is an integral operator with kernel $(z, w) \longmapsto AK_w(z)$.

Conversely, let A be a linear operator on L^p_a densely defined on L^∞_a with values in the space of analytic functions on $\mathbf{B}_n$. We suppose that for every $z \in \mathbf{B}_n$, the function η_z defined on $\mathbf{B}_n$ by

$$\eta_z(w) := AK_w(z)$$

is antianalytic on $\mathbf{B}_n$. Then we have the following.

3.3.1 Proposition

Let A be a linear operator on L^p_a densely defined on L^∞_a with values in the space of analytic functions on $\mathbf{B}_n$. If S extends to a bounded operator from the Lebesgue space $L^p(\mathbf{B}_n, d\nu)$ to L^p_a, then A extends to a bounded operator on L^p_a. Furthermore, the restriction of S to L^p_a coincides with A i.e. $S = AP$.

Proof For every $z \in \mathbf{B}_n$, we get

$$\left(\int_{\mathbf{B}_n} |AK_w(z)|^{p'} d\nu(w)\right)^{\frac{1}{p'}} = \sup_{l\in\mathcal{C}_c^\infty(\mathbf{B}_n),||l||_p\le 1} \left|\int_{\mathbf{B}_n} l(w)AK_w(z)d\nu(w)\right| = \sup_{l\in\mathcal{C}_c^\infty(\mathbf{B}_n),||l||_p\le 1} |Sl(z)|.$$

Since $Sl \in L_a^p$, by the mean-value property, there exists a constant $C(z)$ such that

$$|Sl(z)| \le C(z)||Sl||_p \le C(z)||S||$$

and hence the function $\eta_z : w \mapsto \overline{AK_w(z)}$ belongs to $L_a^{p'}$.

We need to prove that for every $g \in L_a^p$, the following identity

$$Sg = Ag$$

is valid. Since the linear span of $\{K_\zeta : \zeta \in \mathbf{B}_n\}$ is dense in L_a^p, it suffices to show that $SK_\zeta = AK_\zeta$, for every $\zeta \in \mathbf{B}_n$.

Now for every $z \in \mathbf{B}_n$, since $\overline{\eta_z} \in L_a^{p'}$, we obtain

$$SK_\zeta(z) = \int_{\mathbf{B}_n} K_\zeta(w)AK_w(z)d\nu(w) = AK_\zeta(z). \qquad \square$$

Let us consider the particular case of Toeplitz operators, that is $A = T_f$ with $f \in L^1$. The assumptions of the previous proposition are in particular fulfilled. Indeed, for fixed z, we obtain

$$\eta_z(w) = T_f K_w(z) = \overline{P(K_z\bar{f})(w)}$$

and so the function $\eta_z(w) = T_f K_w(z)$ is antianalytic on $\mathbf{B}_n$ since the function $K_z\bar{f}$ is integrable on $\mathbf{B}_n$. On the other hand, the function $z \longmapsto T_f K_w(z)$ is analytic on $\mathbf{B}_n$ for every $w \in \mathbf{B}_n$. The associated operator $S = S_f$ is given on the dense subspace $\mathcal{C}_c^\infty(\mathbf{B}_n)$ of $L^p(\mathbf{B}_n, d\nu)$ consisting of $\mathcal{C}^\infty$ functions on $\mathbf{B}_n$ with compact support, by

$$S_f l(z) = \int_{\mathbf{B}_n} l(\zeta)T_f K_\zeta(z)d\nu(\zeta) \tag{3.18}$$

and $S_f l$ is an analytic function on $\mathbf{B}_n$. The right hand side integral in (3.18) is absolutely convergent for all $z \in \mathbf{B}_n$, since

$$\int_{\mathbf{B}_n} |l(\zeta)| \left(\int_{\mathbf{B}_n} \frac{|f(w)||K_\zeta(w)|}{|1-(z\cdot w)|^{n+1}} d\nu(w) \right) d\nu(\zeta) \leq \frac{C(l)}{(1-|z|)^{n+1}} ||l||_\infty ||f||_1.$$

This inequality follows from the fact that $|K_\zeta(w)| \leq C(l)$ for all $\zeta \in Supp(l)$ and $w \in \mathbf{B}_n$ on the one hand, and $\frac{1}{|1-(z\cdot w)|^{n+1}} \leq \frac{1}{(1-|z|)^{n+1}}$ for all $z, w \in \mathbf{B}_n$ on the other hand. So by Fubini's Theorem, for all $l \in \mathcal{C}_c^\infty(\mathbf{B}_n)$, we obtain

$$S_f l(z) = \int_{\mathbf{B}_n} \left(\int_{\mathbf{B}_n} \frac{l(\zeta)}{(1-(w.\zeta))^{n+1}} d\nu(\zeta) \right) f(w)\overline{K_z(w)} d\nu(w) = (T_f \circ P) l(z).$$

It is then easy to get the following proposition.

3.3.2 Proposition

Let $f \in L^1(\mathbf{B}_n, d\nu)$ and $p \in (1, \infty)$. Then the following two assertions are equivalent:

1. T_f extends to a bounded operator on L_a^p;

2. The operator S_f defined in (3.18) extends to a bounded operator from the Lebesgue space $L^p(\mathbf{B}_n, d\nu)$ to L_a^p.

Furthermore, $S_f = T_f \circ P$ and $||T_f|| \leq ||S_f|| \leq ||T_f|| ||P||$, where $||T_f||$, $||S_f||$, and $||P||$ denote the operator norms of T_f, S_f and P on L_a^p, $L^p(\mathbf{B}_n, d\nu)$ and $L^p(\mathbf{B}_n, d\nu)$ respectively.

For $p \in (1, \infty)$, let A be a linear operator on L_a^p, densely defined on L_a^∞ with values in the space of analytic functions on $\mathbf{B}_n$, such that the function $\eta_z(w) = AK_w(z)$ is antianalytic on $\mathbf{B}_n$ for every $z \in \mathbf{B}_n$ and such that the function $z \mapsto AK_w(z)$ is analytic on $\mathbf{B}_n$. According to Schur's Lemma, if there exist a positive ϵ and positive constants C_1 and C_2 such that

$$\int_{\mathbf{B}_n} |(AK_w)(z)|(1-|w|^2)^{-\epsilon p'} d\nu(w) \leq C_1 (1-|z|^2)^{-\epsilon p'} \qquad (3.19)$$

and

$$\int_{\mathbf{B}_n} |(AK_w)(z)|(1-|z|^2)^{-\epsilon p} d\nu(z) \le C_2(1-|w|^2)^{-\epsilon p}, \tag{3.20}$$

then the associated operator S defined in (3.17) is bounded from $L^p(\mathbf{B}_n, d\nu)$ to L^p_a with $||S|| \le (C_1)^{\frac{1}{p}}(C_2)^{\frac{1}{p'}}$. Hence, by Proposition 3.3.2, the operator A is also bounded on L^p_a with $||A|| \le ||S|| \le (C_1)^{\frac{1}{p}}(C_2)^{\frac{1}{p'}}$. Furthermore, if $f \in L^1(\mathbf{B}_n, d\nu)$ and $A = T_f$, estimates (3.19) and (3.20) imply that the associated operator S_f defined in (3.18) is bounded on $L^p(\mathbf{B}_n, d\nu)$, and hence by Proposition 3.3.2, T_f is also bounded on L^p_a with $||T_f|| \le ||S_f|| \le (C_1)^{\frac{1}{p}}(C_2)^{\frac{1}{p'}}$.

We end this section by giving some improvements to the indices of the work of [52] and [31]. We give a simple application on operator norm.

3.3.3 Proposition

Suppose $1 < p < \infty$, $p_1 = \min(p, p')$ and $n \ge 1$. Suppose further that A is a bounded operator on $L^p_a(B_n)$ and such that

$$C_1 = \sup_{z\in\mathbf{B}_n} \|A_z 1\|_s < \infty, \; C_2 = \sup_{z\in\mathbf{B}_n} \|A^\star_z 1\|_s < \infty,$$

for some s with $s > (p_1 + n)/(p_1 - 1)$. Then there exists a constant C such that

$$\|A\| \le C(C_1)^{1/p}(C_2)^{1/p'}.$$

Proof If $h \in L^\infty_a$ then

$$\begin{aligned} (Ah)(z) &= \langle Ah, K_z\rangle = \langle h, A^*K_z)\rangle \\ &= \int_{\mathbf{B}_n} h(w)\overline{(A^*K_z)}(w)\, d\nu(w) \\ &= \int_{\mathbf{B}_n} h(w)(AK_w)(z)\, d\nu(w). \end{aligned} \tag{3.21}$$

By Schur's Lemma, if there exist a positive measurable function g on $\mathbf{B}_n$ and constants c_1, c_2 such that

$$\int_{\mathbf{B}_n} |(AK_z)(w)| g(w)^p \, d\nu(w) \leq c_1 g(z)^p$$

for all $z \in \mathbf{B}_n$ and

$$\int_{\mathbf{B}_n} |(AK_z)(w)| g(z)^{p'} \, d\nu(z) \leq c_2 g(w)^{p'}$$

for all $w \in \mathbf{B}_n$, where p' is the conjugate exponent of p then A is bounded on $L_a^p(\mathbf{B}_n)$ and $\|A\| \leq (c_1)^{1/p}(c_2)^{1/p'}$. We choose $g(w) = (1-|w|^2)^{-\epsilon}$ with $\max\{\frac{n+1}{sp'}, \frac{n+1}{sp}\} < \epsilon < \min\{\frac{1}{ps'}, \frac{1}{p's'}\}$ and apply Lemma 3.2.3 to get the result. □

The next question that arises is whether the index $\frac{p_1+n}{p_1-1}$ in Proposition 3.3.3 is sharp. We do not know whether the index $\frac{p_1+n}{p_1-1}$ is sharp for $p \in (1, \infty)$ even for $p = 2$ and $n = 1$. However, we can prove that Proposition 3.3.3 is not valid for $p = 2$ and $n = 1$ if we take $s < 3$. More precisely, we prove the following Lemma.

3.3.4 Lemma

There exists no positive constant $C = C(s)$ such that every bounded operator A on L_a^2 satisfies

$$||A|| \leq C(\sup_{z\in\mathbf{B}_1} ||A_z 1||_s)^{\frac{1}{2}}(\sup_{z\in\mathbf{B}_1} ||A_z^\star 1||_s)^{\frac{1}{2}}. \tag{3.22}$$

Proof

Let $b = \{b_n\}$ be a sequence of complex numbers, we consider a linear operator A_b defined by

$$A_b\left(\sum_{n=0}^{\infty} a_n z_n\right) = \sum_{n=0}^{\infty} a_n b_n z_n.$$

It is well known that $||A|| = \sup_{z\in mathbf B_1} |b_n|$. Also the adjoint operator $A_b^\star$ of A_b is

$$A_b^\star\left(\sum_{n=0}^{\infty} a_n z_n\right) = \sum_{n=0}^{\infty} a_n \overline{b_n} z_n.$$

It is not difficult to see that for a linear operator A,

$$||A_z 1||_s^s = (1-|z|^2)^2 \int_{\mathbf{B}_1} |AK_z(w)|^s |1-\overline{z}w|^{2(s-2)} d\lambda(w).$$

Hence, if we take $b_m = \delta_{m,n}$, then $||A_b|| = 1$ and

$$||(A_b)_z 1||_s^s = ||(A_b^\star)_z 1||_s^s = (1-|z|^2)^2 (n+1)^s |z|^{ns} \int_{\mathbf{B}_1} |w|^{ns} |1-\overline{z}w|^{2(s-2)} d\lambda(w)$$

We may assume $s \in [2,3)$. It is then enough to show that there exists no constant C such that for every positive integer n, the following estimate holds:

$$1 \leq C \sup_{z \in \mathbf{B}_1} \left\{ (1-|z|)^2 (n+1)^s |z|^{ns} \int_{\mathbf{B}_1} |w|^{ns} d\lambda(w) \right\}. \tag{3.23}$$

The right hand side of the (3.23) is

$$C \frac{(n+1)^s}{2+ns} \sup_{z \in \mathbf{B}_1} \{(1-|z|)^2 |z|^{ns}\}.$$

Since the latter supremum is attained at $|z| = \frac{ns}{ns+2}$, the inequality (3.23) would implies that for any n,

$$1 \leq C \frac{(n+1)^s}{2+ns} \frac{4}{(2+ns)^2} \left(\frac{ns}{2+ns} \right)^{ns} \leq 4C \frac{(n+1)^s}{(2+ns)^3}. \tag{3.24}$$

Letting $n \longrightarrow \infty$, we are led to a contradiction. □

If we take, in particular, $A = T_f$, then by assertion (2) of Lemma 3.2.1, we obtain $A_z 1 = T_{f \circ \phi_z} 1$ and Propositions 3.3.2 and 3.3.3, gives the following result.

3.3.5 Theorem

Let $p \in (1,\infty)$, $p_1 = \min(p,p')$, $s > \frac{p_1+n}{p_1-1}$. Let p' and s' are conjugate exponents of p and s respectively and let $f \in L^1(\mathbf{B}_n, d\nu)$. Then for every positive number ϵ

satisfying
$$\max\left\{\frac{n+1}{sp'},\frac{n+1}{sp}\right\}<\epsilon<\min\left\{\frac{1}{s'p},\frac{1}{s'p'}\right\},$$
there exists a positive constant $C(\epsilon,p,s)$ such that, if $C_1=\sup_{z\in\mathbf{B}_n}||T_{f\circ\phi_z}1||_s<\infty$ and $C_2=\sup_{z\in\mathbf{B}_n}||T_{\bar{f}\circ\phi_z}1||_s<\infty$, then
$$\int_{\mathbf{B}_n}|(T_fK_w)(z)|(1-|w|^2)^{-p'\epsilon}d\nu(w)\le CC_1(1-|z|^2)^{-p'\epsilon}\tag{3.25}$$
and
$$\int_{\mathbf{B}_n}|(T_fK_w)(z)|(1-|z|^2)^{-p\epsilon}d\nu(z)\le CC_2(1-|w|^2)^{-p\epsilon}.\tag{3.26}$$
Moreover, T_f is bounded on L^p_a with $||T_f||\le C(C_1)^{\frac{1}{p}}(C_2)^{\frac{1}{p'}}$.

3.3.6 Remark

We recall that $||T_{\psi\circ\phi_z}1||_2=||T_\psi k_z||_2$ for every $\psi\in L^1(\mathbf{B}_n,d\nu)$ (assertion (1) of Lemma 3.2.1). For $p=2$, even in dimension one, we do not know whether T_f is bounded on L^2_a if the following two conditions hold.
$$\sup_{z\in\mathbf{B}_n}||T_fk_z||_2<\infty\quad and\quad \sup_{z\in\mathbf{B}_n}||T_{\bar{f}}k_z||_2<\infty.$$
F. Nazarov showed that one single condition is not sufficient for T_f be bounded on L^2_a.

3.4 The space BT and related spaces of symbols.

The space $BT=BT(\mathbf{B}_n)$ is the space of functions $f\in L^1(\mathbf{B}_n,d\nu)$ such that
$$||f||_{BT}:=\sup_{z\in\mathbf{B}_n}\widetilde{|f|}(z)<\infty.$$
This is an extension to several complex variables of the space BT introduced by J. Miao and D. Zheng [31] for $n=1$. We first give the relation between the space BT and

the space of Carleson measures for Bergman spaces. An easy application of Theorem 3.2.9 gives the following:

3.4.1 Lemma

If $f \in BT$ if and only if the measure $\mu = |f|\,d\nu$ is a Carleson type measure on the Bergman spaces L^p_a.

Proof If $f \in BT$ then clearly

$$\int_{B_n} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2(n+1)}}\,d\mu(w) \le \sup_{z\in\mathbf{B}_n} \widetilde{|f|}(z) < \infty.$$

Thus by Theorem 3.2.9 $\mu = |f|\,d\nu$ is a Carleson measure.

Conversely, if $\mu = |f|\,d\nu$ is a Carleson type measure. Then

$$\begin{aligned}
\widetilde{|f|}(z) &= \int_{\mathbf{B}_n} |f(w)| \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2n+2}}\,d\nu(w)\\
&= \int_{\mathbf{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2n+2}}\,d\mu(w)\\
&\le C(\mu)\int_{\mathbf{B}_n} \frac{(1-|z|^2)^{n+1}}{|1-(w\cdot z)|^{2n+2}}\,d\nu(w) = CC(\mu),
\end{aligned}$$

where $C(\mu)$ is the Carleson measure constant. This shows that $f \in BT$. $\square$

We next state the following generalization to several complex variables of a result of [31]:

3.4.2 Theorem

Let $f \in BT$ and $p \in (1,\infty)$. Then the Toeplitz operator T_f is bounded on L^p_a. Moreover, there exists a positive constant C such that

$$||T_f|| \le C||f||_{BT}$$

for every $f \in BT$.

Proof The proof is the same as for $n = 1$. Let $h \in L_a^{p'}$ and $g \in L_a^p$. Then

$$\langle T_f g, h\rangle = \langle fg, h\rangle$$

for all $g \in L_a^p$ and $h \in L_a^{p'}$. Thus

$$|\langle T_f g, h\rangle| \leq \int_{\mathbf{B}_n} |gh|(|f|d\nu) \leq C(f)||gh||_1 \leq C(f)||g||_p||h||_{p'},$$

where $C(f)$ is the Carleson measure constant of the Carleson measure $|f|d\nu$. This implies that T_f is bounded on L_a^p with $||T_f|| \leq C(f)$. The announced estimate for $||T_f||$ follows from the fact that $C(f) \leq ||f||_{BT}$ according to the last assertion of Theorem 3.2.9. □

An alternative proof of Theorem 3.4.2 can be provided using Schur's lemma. According to Proposition 3.3.2, it is sufficient to show that the integral operator S_f defined in (3.18) is bounded on the Lebesgue space $L^p(\mathbf{B}_n, d\nu)$. By Schur's lemma, it is enough to prove that f satisfies the following property:

(SL) there exist a real number ϵ and a positive constant C such that for every $f \in BT$, there is a constant C_f for which the following two estimates hold:

$$\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} \frac{|f(w)|(1-|\zeta|^2)^{-\epsilon p'}}{|(1-(w\cdot\zeta))^{n+1}(1-(z\cdot w))^{n+1}|}d\nu(w)\right)d\nu(\zeta) \leq CC_f(1-|z|^2)^{-\epsilon p'}$$

and

$$\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} \frac{|f(w)|(1-|z|^2)^{-\epsilon p}}{|(1-(w\cdot\zeta))^{n+1}(1-(z\cdot w))^{n+1}|}d\nu(w)\right)d\nu(z) \leq CC_f(1-|\zeta|^2)^{-\epsilon p}.$$

In fact, we have another characterisation of the space BT.

3.4.3 Proposition

For $f \in L^1(\mathbf{B}_n, d\nu)$, the following two assertions are equivalent:

(1) $f \in BT$;

(2) f satisfies property (SL).

Proof (1) $\Rightarrow$ (2). We prove assertion (2) with $C_f = ||f||_{BT}$. Since $f \in BT$, we get:

$$\int_{\mathbf{B}_n} \frac{|f(w)|}{|(1-(w\cdot\zeta))^{n+1}(1-(w\cdot z))^{n+1}|} d\nu(w) \leq \\ ||f||_{BT} \int_{\mathbf{B}_n} \frac{1}{|(1-(w\cdot\zeta))^{n+1}(1-(w\cdot z))^{n+1}|} d\nu(w).$$

The conclusion follows from an iterative application of the well-known fact that, for positive ϵ such that $0 < \epsilon < \min(\frac{1}{p}, \frac{1}{p'})$, there exists a positive constant C such that the following estimates

$$\frac{1}{C}(1-|w|^2)^{-\epsilon p'} \leq \int_{\mathbf{B}_n} \frac{(1-|z|^2)^{-\epsilon p'}}{|1-(z\cdot w)|^{n+1}} d\nu(z) \leq C(1-|w|^2)^{-\epsilon p'} \tag{3.27}$$

and

$$\frac{1}{C}(1-|z|^2)^{-\epsilon p} \leq \int_{\mathbf{B}_n} \frac{(1-|w|^2)^{-\epsilon p}}{|1-(z\cdot w)|^{n+1}} d\nu(w) \leq C(1-|z|^2)^{-\epsilon p}. \tag{3.28}$$

hold simultaneously.

(2) $\Rightarrow$ (1) According to the left inequality of (3.27), property (SL) implies that

$$\int_{\mathbf{B}_n} \frac{|f(w)|(1-|w|^2)^{-\epsilon p'}}{|1-(z\cdot w)|^{n+1}} d\nu(w) \leq CC_f(1-|z|^2)^{-\epsilon p'}$$

for all $z \in \mathbf{B}_n$. Since the functions $w \mapsto (1-|w|^2)^{-\epsilon p'}$ and $w \mapsto |1-(z\cdot w)|^{n+1}$ are almost constant on the Bergman ball $D(z)$, we obtain,

$$\begin{aligned} C_1 \frac{(1-|z|^2)^{-\epsilon p'}}{(1-|z|^2)^{n+1}} \int_{D(z)} |f(w)| d\nu(w) &\leq \int_{D(z)} \frac{|f(w)|(1-|w|^2)^{-\epsilon p'}}{|1-(z\cdot w)|^{n+1}} d\nu(w) \\ &\leq \int_{\mathbf{B}_n} \frac{|f(w)|(1-|w|^2)^{-\epsilon p'}}{|1-(z\cdot w)|^{n+1}} d\nu(w) \leq CC_f(1-|z|^2)^{-\epsilon p'}. \end{aligned}$$

Hence,

$$\int_{D(z)} |f(w)|d\nu(w) \leq CC_1^{-1}C_f(1-|z|^2)^{n+1}.$$

Theorem 3.2.9 then gives that the measure $|f|d\nu$ is a Carleson measure, i.e. $f \in BT$. $\square$

Let $1 \leq p < \infty$, the space $BMO^p = BMO^p(\mathbf{B}_n)$ consists of functions $f \in L^1$ such that

$$\sup_{z \in \mathbf{B}_n} ||f \circ \phi_z - \widetilde{f}(z)||_p < \infty.$$

It is clear that for $1 \leq p \leq q$, $BMO^q \subset BMO^p \subset BMO^1$.

In some sense, Proposition 3.4.3 says that to go further in the research of the complete criterion for boundedness of Toeplitz operator for general symbols, one should use more than only Schur's estimates. Nevertheless, when we can relate the set of symbols to this BT class in a clever and simple way, we are able to obtain new sets of symbols for which a boundedness criterion for T_f can be given. Such an idea appeared implicitly in [52], where it is showed that if $f \in BMO^1$ and if $\widetilde{f}$ is in L^∞, then $f \in BT$. The best result so far known for boundedness of Toeplitz operators is contained in the following lemma.

3.4.4 Lemma

Let $p \in (1, \infty)$. For $f \in BMO^1$ (resp. for f nonnegative and integrable on $\mathbf{B}_n$), the following three properties are equivalent.

(1) T_f is bounded on L^p_a.

(2) $\tilde{f}$ is bounded on L^p_a.

(3) $f \in BT$.

A natural question is to know if in the above Lemma, we can replace BMO^1, and/or the nonnegative L^1 functions by a bigger set of symbols. To do this, we use a simple procedure to construct a subset X of $L^1(\mathbf{B}_n, d\nu)$ such that for an exponent $p \in (1, \infty)$, the associated Toeplitz operator T_f is bounded on L^p_a for $f \in X$ only if $f \in BT$. We proceed as follows. For $f \in L^1(\mathbf{B}_n, d\nu)$, we suppose that there exists a positive function $K(f)$ on $\mathbf{B}_n$ with $K(f)(z)$ finite at each $z \in \mathbf{B}_n$, such that the following implication holds:

$$T_f \quad \textit{bounded} \quad \textit{on} \quad L^p_a \Rightarrow \sup_{z \in \mathbf{B}_n} K(f)(z) < \infty.$$

Usually, the necessary condition $\sup_{z \in \mathbf{B}_n} K(f)(z) < \infty$ is not sufficient in general. We define the set $X^{(q)}_K, \quad 1 \leq q < \infty$, to be the set of all functions $f \in L^q(\mathbf{B}_n, d\nu)$ such that

$$\sup_{z \in \mathbf{B}_n} |\widetilde{|f|^q}(z) - K(f)(z)| < \infty. \tag{3.29}$$

Now, if T_f is bounded on L^p_a and if $f \in X_K$, then $f \in BT$ by (3.29) This gives the following result.

3.4.5 Theorem

Let $f \in L^1$. Then the following are equivalent:

(1) T_f is bounded on L^p_a.

(2) $K(f)$ is bounded on L^p_a.

(3) $f \in BT$.

We study two special cases which show that in the above construction, we obtain bigger sets which strictly contains previously known sets of symbols.

Case 1: Take $K(f) = K_1(f) = \widetilde{|f|}$. We denote by X_1 the corresponding space $X_K^{(1)}$. Since for $f \in L^1(\mathbf{B}_n, d\nu)$, we necessarily have $K_1(f)(z) = |\widetilde{f}(z)| < \infty$ for every $z \in \mathbf{B}_n$, the space X_1 therefore consists of L^1 functions f such that

$$\sup_{z\in\mathbf{B}_n} \widetilde{|f|}(z) - |\widetilde{f}(z)| < \infty.$$

3.4.6 Lemma

The following inclusions hold.

(1) If $f \in L^1(\mathbf{B}_n, d\nu)$ and if f is a nonnegative function, then $f \in X_1$.

(2) $BT \subsetneq BMO^1 \subsetneq X_1$.

Proof (1) Clear from the definition of X_1 since $\widetilde{|f|} = \widetilde{f}$ if f is a nonnegative function.

(2) The first inclusion follows from the inequality $||f \circ \varphi_z - \widetilde{f}(z)||_1 \leq 2\widetilde{|f|}(z)$ which is valid for all $f \in L^1(\mathbf{B}_n, d\nu)$ and $z \in \mathbf{B}_n$. The inclusion is strict because for $a \in \partial\mathbf{B}_n$, the function $f(z) = \log(1-(z.a))$ belongs to BMO^1, but not to BT. Indeed if $f \in BT$ then $\widetilde{f} \in L^\infty$, but $\widetilde{f}(w) = \log(1-(w.a))$ does not belong to L^∞.

For $z \in \mathbf{B}_n$, we get that

$$\begin{aligned} \widetilde{|f|}(z) - |\widetilde{f}(z)| &= \int_{\mathbf{B}_n} [|f(w)| - |\widetilde{f}(z)|]|k_z(w)|^2 d\nu(z) \\ &\leq \int_{\mathbf{B}_n} [|f(w) - \widetilde{f}(z)|]|k_z(w)|^2 d\nu(z) \\ &= \int_{\mathbf{B}_n} |f \circ \varphi_z(w) - \widetilde{f}(z)| d\nu(w) = ||f \circ \varphi_z - \widetilde{f}(z)||_1. \end{aligned}$$

This gives the second inclusion. This inclusion is strict because there are nonnegative L^1 functions on $\mathbf{B}_n$ which do not belong to BMO^1.$\square$

We give an example of a function in X_1, with T_f not bounded on L^p_a.

3.4.7 Example

There exists a function $f \in X_1$ such that T_f is not bounded on L^p_a.

Define $f(z) = f(|z|)$ by $f(x) = 0$ if $x \in [0, \frac{1}{2})$, and for $x \in [1 - \frac{1}{2^k}; 1 - \frac{1}{2^{k+1}})$, $k = 1, 2, \cdots$

$$f(x) = \begin{cases} 2^{\frac{3}{2}k} & \text{if } 1 - \frac{1}{2^k} \leq x \leq 1 - \frac{1}{2^k} + \frac{1}{2^{2k}}; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int_{\mathbf{B}_1} |f(z)| d\lambda(z) &= \int_0^1 f(s) s\, ds \\ &= \int_{\frac{1}{2}}^1 f(s) s ds \\ &\approx \sum_{k=1}^{\infty} 2^{\frac{3}{2}k} \int_{1-\frac{1}{2^k}}^{1-\frac{1}{2^{k+1}}} f(s) ds \\ &= \sum_{k=1}^{\infty} 2^{\frac{3}{2}k} \int_{1-\frac{1}{2^k}}^{1-\frac{1}{2^k}+\frac{1}{2^{2k}}} ds \\ &= \sum_{k=1}^{\infty} \left(\frac{\sqrt{2}}{2} \right)^k < \infty. \end{aligned}$$

Thus $f \in L^1$. For $\xi \in \partial\mathbf{B}_1$ and $r \in [0, 1)$ we denote that Carleson square by

$$S(\xi, r) = \{z \in \mathbf{B}_1 : r < |z| < 1,\ \arg \xi - \frac{1-r}{2} < \arg z < \arg \xi + \frac{1-r}{2}\}.$$

Then by the comment after Theorem 2.2 of [29] or Lemma 2.1 of [31], a positive Borel measure μ is a Carleson measure if and only if

$$\sup\{\mu(S(\xi, r))/\lambda(S(\xi, r)) : \xi \in \partial\mathbf{B}_1,\quad r \in (0, 1)\} < \infty.$$

Now, let $d\mu = f d\lambda$, then

$$\lambda((S(\xi, r)) = \frac{1}{\pi} \int_r^1 s ds \int_{-(1-r)/2}^{(1-r)/2} d\theta = \frac{(1-r)^2(1+r)}{2\pi}.$$

Thus

$$\begin{aligned}
\frac{\mu(S(\xi,r))}{\lambda(S(\xi,r))} &= \frac{1}{\lambda(S(\xi,r))}\int_{S(\xi,r)} f(z)d\lambda(z) \\
&= \frac{2}{(1-r^2)}\int_r^1 f(s)sds
\end{aligned}$$

For $n \in \mathbf{N}$, take $r = 1 - \frac{1}{2^n}$ then

$$\begin{aligned}
\frac{\mu(S(\xi,r))}{\lambda(S(\xi,r))} &= \frac{2^{2n}}{2^{n+1}-1}\int_{1-\frac{1}{2^n}}^1 f(s)sds \\
&\approx \frac{2^{2n}}{2^{n+1}-1}\sum_{k=n}^{\infty}\int_{1-\frac{1}{2^k}}^{1-\frac{1}{2^{k+1}}} f(s)ds \\
&= \frac{2^{2n}}{2^{n+1}-1}\sum_{k=n}^{\infty} 2^{\frac{3}{2}k}\int_{1-\frac{1}{2^k}}^{1-\frac{1}{2^k}+\frac{1}{2^{2k}}} ds \\
&= \frac{2^{2n}}{2^{n+1}-1}\sum_{k=n}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^k \\
&= C\frac{2^{\frac{3}{2}n}}{2^{n+1}-1}.
\end{aligned}$$

As n tends to ∞, $\frac{\mu(S(\xi,r))}{\lambda(S(\xi,r))}$ is unbounded and thus f does not belong to BT. This implies T_f is not bounded on L_a^p. Lemma 3.4.6 clearly shows that $f \in X_1$.

3.4.8 Corollary

Let $p \in (1,\infty)$. Let $f \in X_1$ (in particular, $f \in BMO^1$, or f is a nonnegative L^1 function). Then following two properties are equivalent:

(1) T_f is bounded on L_a^p;

(2) the function $\widetilde{f}$ is bounded on $\mathbf{B}_n$.

Case 2. Take $K(f)(z) = K_2(f)(z) = \|T_f k_z\|_2^2$. We denote by X_2 the corresponding space X_K^2. we point out that $K_1(f) \leq K_2(f)$ for every $f \in L^1(\mathbf{B}_n, d\nu)$. Since for every $f \in L^2(\mathbf{B}_n, d\nu)$ we have $\|T_f k_z\|_2 < \infty$ for all $z \in \mathbf{B}_n$. The space X_2 therefore consists of L^2 functions f such that

$$\sup_{z\in\mathbf{B}_n} \widetilde{|f|^2}(z) - ||T_f k_z||_2^2 < \infty.$$

We just recall that this case refers to a necessary condition which was proved insufficient for $p = 2$ and $n = 1$ by F. Nazarov.

3.4.9 Lemma

$BMO^2(\mathbf{B}_n) \subsetneq X_2$ and X_2 is not contained in BMO^1.

Proof We first show that $f \in BMO^2$ is equivalent to the following:

$$\sup_{z\in\mathbf{B}_n} \{\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2\} < \infty.$$

Direct calculation shows that

$$\int_{\mathbf{B}_n}\int_{\mathbf{B}_n} |f(u) - f(v)|^2 |k_z(u)|^2 |k_z(v)|^2 \, d\nu(v)\, d\nu(u) = 2(\widetilde{|f|^2}(z) - |\widetilde{f(z)}|^2). \tag{3.30}$$

Indeed,

$$\begin{aligned} & \int_{\mathbf{B}_n} |k_z(u)|^2 \langle (f(u) - f)k_z, (f(u) - f)k_z\rangle \, d\nu(u) \\ = & \int_{\mathbf{B}_n} |k_z(u)|^2 [|f(u)|^2 - f(u)\langle k_z, fk_z\rangle - \overline{f(u)}\langle fk_z, k_z\rangle \ + \widetilde{|f|^2}(z) \, d\nu(u) \\ = & \ 2(\widetilde{|f|^2}(z) - |\widetilde{f(z)}|^2). \end{aligned}$$

Also, the double integral in equation (3.30) is equal to

$$\begin{aligned}
& \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} |f\circ\varphi_z(u) - f\circ\varphi_z(v)|^2\, d\nu(v)\, d\nu(u) \\
= & \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} |f\circ\varphi_z(u)|^2 - \overline{f\circ\varphi_z(u)} f\circ\varphi_z(v) - f\circ\varphi_z(u)\overline{f\circ\varphi_z(v)} \\
+ & \ |f\circ\varphi_z(v)|^2\, d\nu(v)\, d\nu(u) \\
= & \int_{\mathbf{B}_n} |f\circ\varphi_z(u)|^2 - \overline{f\circ\varphi_z(u)}\int_{B_n} f\circ\varphi_z(v)\, d\nu(v) \\
- & \ f\circ\varphi_z(u)\overline{\int_{\mathbf{B}_n} f\circ\varphi_z(v)\, d\nu(v)} + \widetilde{|f|^2}(z)\, d\nu(u) \\
= & \int_{\mathbf{B}_n} |f\circ\varphi_z(u) - \widetilde{f(z)}|^2\, d\nu(u) - |\widetilde{f(z)}|^2 + \widetilde{|f|^2}(z)
\end{aligned}$$

which gives the equivalence. We recall that

$$|\widetilde{f}(z)| = |P(f\circ\varphi_z)(0)| \le ||P(f\circ\varphi_z)||_1 \le ||P(f\circ\varphi_z)||_2 = ||T_f k_z||_2,$$

here, the former identity follows from assertion (1) of Lemma 3.2.1, while the latter identity follows from assertion (3) of Lemma 3.2.1. Thus if $f \in BMO^2$ then

$$\sup_{z\in\mathbf{B}_n}\{\widetilde{|f|^2}(z) - \|T_{f\circ\varphi_z}1\|_2^2\} \le \sup_{z\in\mathbf{B}_n}\{\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2\} < \infty$$

that is $f \in X_2$. This inclusion is strict. Otherwise, $BMO^2 = X_2$ and hence we have the Bloch space equals $X_2 \cap H(\mathbf{B}_n)$, where $H(\mathbf{B}_n)$ denote the space of analytic function on $\mathbf{B}_n$. But $L_a^2 \subset X_2 \cap H(\mathbf{B}_n)$, this shows that the Bloch space contains L_a^2 which is false. The fact that X_2 does not belong to BMO^1 is clear from the previous statement above. □

We next show that the space X_2 can be described without reference to the space BT.

3.4.10 Proposition

For $f \in L^2$, the following two properties are equivalent:

(1) $f \in X_2$;

(2) The following estimate holds:

$$\sup_{z\in\mathbf{B}_n} \int_{B_n} \left| \int_{\mathbf{B}_n} \frac{f\circ\varphi_z(\xi) - f\circ\varphi_z(w)}{(1-(w\cdot z))^{n+1}}\, d\nu(w) \right|^2 d\nu(\xi) < \infty.$$

Proof We have

$$\begin{aligned}
& \widetilde{|f|^2}(z) - \|P(f\circ\varphi_z)\|_2^2 \\
= \quad & \|f\circ\varphi_z\|_2^2 - \|P(f\circ\varphi_z)\|_2^2 \\
= \quad & \langle f\circ\varphi_z, f\circ\varphi_z\rangle - \langle P(f\circ\varphi_z), P(f\circ\varphi_z)\rangle \\
= \quad & \langle f\circ\varphi_z, f\circ\varphi_z\rangle - 2\langle P(f\circ\varphi_z), P(f\circ\varphi_z)\rangle + \langle P(f\circ\varphi_z), P(f\circ\varphi_z)\rangle \\
= \quad & \langle f\circ\varphi_z, f\circ\varphi_z\rangle - \langle P(f\circ\varphi_z), (f\circ\varphi_z)\rangle \\
- \quad & \langle (f\circ\varphi_z), P(f\circ\varphi_z)\rangle + \langle P(f\circ\varphi_z), P(f\circ\varphi_z)\rangle \\
= \quad & \langle f\circ\varphi_z - P(f\circ\varphi_z), f\circ\varphi_z - P(f\circ\varphi_z)\rangle \\
= \quad & \|(I-P)(f\circ\varphi_z)\|_2^2.
\end{aligned}$$

Thus we have the equivalence. □

Similar argument as the one used for the case of functions in X_K yields the following.

3.4.11 Corollary

Let $f \in X_2$. The following three assertions are equivalent:

(1) T_f is bounded on L_a^2;

(2) $\sup\limits_{z\in\mathbf{B}_n} ||T_f k_z||_2 < \infty$.

(3) $\sup\limits_{z\in\mathbf{B}_n} ||P(f \circ \phi_z)||_2 < \infty$

Other cases can be obtained from the following theorem:

3.4.12 Theorem

Let $f \in L^1$ and $1 < p < \infty$, suppose that T_f is a bounded operator on $L_a^p(B_n)$ and p' the conjugate exponent of p. Then the following hold

1. $\widetilde{f}$ is bounded;
2. $\sup_{z\in\mathbf{B}_n} \|T_f \tilde{k}_z^p\|_p < \infty$,
3. $\sup_{z\in\mathbf{B}_n} \|P(f \circ \varphi_z)(\cdot)(1-(\ \cdot z))^{(n+1)(p-2)/p}\|_p < \infty$,
4. $\sup_{z\in\mathbf{B}_n} \|P(f \circ \varphi_z)\|_p < \infty$ if $1 < p \le 2$
5. $\sup_{z\in\mathbf{B}_n} \|P(f \circ \varphi_z)\|_1 < \infty$
6. $\sup_{z\in\mathbf{B}_n} \|P(f \circ \varphi_z)\|_q,\ qp < p+q,\ p > q$

where $\tilde{k}_z^p(w) = \frac{(1-|z|^2)^{(n+1)/p'}}{(1-(w\cdot z))^{n+1}}$.

Proof

1) It is known that there exist positive constants C, C' such that

$$|g(z)| \le C\|g\|_p(1-|z|^2)^{-(n+1)/p}, g \in L_a^p(B_n) \tag{3.31}$$

and

$$\|K_z\|_p \leq C'(1-|z|^2)^{-(n+1)/p'},\ \frac{1}{p}+\frac{1}{p'}=1.$$

If $z \in \mathbf{B}_n$ then using these two estimates we have

$$\begin{aligned}
|\widetilde{f}(z)| &= \left|\left(T_f \frac{(1-|z|^2)^{n+1}}{(1-(_ \cdot z))^{n+1}}\right)(z)\right| \\
&= (1-|z|^2)^{n+1}|(T_f K_z)(z)| \\
&\leq C(1-|z|^2)^{n+1}\|T_f K_z\|_p (1-|z|^2)^{-(n+1)/p} \\
&\leq C(1-|z|^2)^{n+1}\|T_f\|\|K_z\|_p (1-|z|^2)^{-(n+1)/p} \\
&\leq C\|T_f\|
\end{aligned}$$

2) Follows from the fact that the L^p-norm of $\tilde{k}_z^p$ is 1.

3) Observe that

$$\begin{aligned}
(T_f \tilde{k}_z^p)(w) &= (1-|z|^2)^{(n+1)/p'}\langle T_f K_z, K_w\rangle \\
&= (1-|z|^2)^{(n+1)/p'}\langle U_z U_z T_f U_z U_z K_z, K_w\rangle \\
&= (1-|z|^2)^{(n+1)/p'}\langle T_{f\circ\varphi_z} U_z K_z, U_z K_w\rangle \\
&= (1-|z|^2)^{(n+1)(1/p'-1/2)}\langle T_{f\circ\varphi_z} 1, \overline{k_z(w)} K_{\varphi_z(w)}\rangle \\
&= \tilde{k}_z^p(w)(T_{f\circ\varphi_z}1)(\varphi_z(w))
\end{aligned}$$

where the fourth equality comes from the relation (3.5).

Thus

$$\begin{aligned}
\|T_f \tilde{k}_z^p\|_p^p &= \int_{\mathbf{B}_n} |P(f\circ\varphi_z)(\varphi_z(w))|^p |\tilde{k}_z^p(w)|^p \, d\nu(w) \\
&= \int_{\mathbf{B}_n} |P(f\circ\varphi_z)(w)|^p |\tilde{k}_z^p(\varphi_z(w))|^p |k_z(w)|^2 \, d\nu(w) \\
&= \int_{\mathbf{B}_n} |P(f\circ\varphi_z)(w)|^p |1-(w\cdot z)|^{(n+1)(p-2)} \, d\nu(w)
\end{aligned}$$

where the second equality is by making a change of variable $w = \varphi_z$. Use assertion 2) to get the result.

4) A consequence of 3).

5) Using Hölder's inequality we have

$$\begin{aligned}
& \|P(f\circ\varphi_z)\|_1 \\
= & \int_{\mathbf{B}_n} |P(f\circ\varphi_z)(w)||1-(w\cdot z)|^{(n+1)(p-2)/p}|1-(w\cdot z)|^{-(n+1)(p-2)/p}\,d\nu(w) \\
\leq & \|P(f\circ\varphi_z)|1-(w\cdot z)|^{(n+1)(p-2)}\|_p \left\{\int_{\mathbf{B}_n} |1-(w\cdot z)|^{-(n+1)(p-2)p'/p}\,d\nu(w)\right\}^{1/p'}.
\end{aligned}$$

If $p \leq 2$ the last integral is clearly bounded. Now for $p > 2$ the last integral will be bounded provided $(n+1)(p-2)p'/p - (n+1) < 0$, if and only if $(p-2)p'/p - 1) < 0$ if and only if $-p' + 1 < 0$. This last inequality is true since $p' > 1$.

6) Let $s = p/q$ and s' the conjugate exponent of s, that is $s' = p/(p-q)$. Then by the Hölder's inequality we have

$$\begin{aligned}
\|P(f\circ\varphi_z)\|_q^q &= \int_{\mathbf{B}_n} |P(f\circ\varphi_z)(w)|^q \frac{|1-(w\cdot z)|^{(n+1)(p-2)/s}}{|1-(w\cdot z)|^{(n+1)(p-2)/s}}\,d\nu(w) \\
&\leq \|P(f\circ\varphi_z)|1-(\ \cdot z)|^{(n+1)(p-2)}\|_p \left\{\int_{\mathbf{B}_n} \frac{d\nu(w)}{|1-(w\cdot z)|^{\frac{(n+1)(p-2)s'}{s}}}\right\}^{1/s'}.
\end{aligned}$$

Now $s'/s = s' - 1 = q/(p-q)$ and thus the last integral is bounded if and only if $(p-2)q/(p-q) - 1 < 0$ or equivalently $pq < p + q$, which completes the proof. □

3.5 Compactness of Toeplitz operators

In this section, we focus our attention to the problem of determining classes of symbols for which we have a characterization of compact Toeplitz operators on L^p_a. We begin with a general result from which we shall characterize compactness on L^p_a for symbols in the space X_K defined in the previous section. The main result of this section is the following theorem which improves earlier one-dimensional results of [31] and [52]. We state it here and give the proof at the end of this section.

3.5.1 Theorem

Suppose that $1 < p < \infty$ and write $p_1 = \min(p, p')$. Suppose further that A is a bounded operator on L_a^p such that

$$\sup_{z \in \mathbf{B}_n} ||A_z 1||_s < \infty, \quad and \quad \sup_{z \in \mathbf{B}_n} ||A_z^\star 1||_s < \infty \tag{3.32}$$

for some s satisfying $s > \frac{p_1+n}{p_1-1}$. Then the following four assertions are equivalent:

(1) A is compact on L_a^p;

(2) $\widetilde{A}(z) \to 0$ as z tends to $\partial \mathbf{B}_n$;

(3) for every $q \in [1, s)$, $||A_z 1||_q \to 0$ as $z \to \partial \mathbf{B}_n$;

(4) $||A_z 1||_1 \to 0$ as $z \to \partial \mathbf{B}_n$.

In [31], for $n = 1, \quad p = 2$, J. Miao and D. Zheng showed that the bound $\frac{p_1+n}{p_1-1} = 3$ is sharp for this theorem. Namely, they produced a bounded operator A on L_a^2 satisfying

$$\sup_{z \in \mathbf{B}_n} ||A_z 1||_3 < \infty, \quad and \quad \sup_{z \in \mathbf{B}_n} ||A_z^* 1||_3 < \infty$$

and $\widetilde{A}(z) \to 0$ as z tends to $\partial \mathbf{B}_n$, but it is not compact on L_a^2.

To proof Theorem 3.5.1, we will require the following lemma which is given in Appendix C of [34] and is Exercise 7 on page 181 of [19] and is also stated in [31].

3.5.2 Lemma

Suppose $1 < p < \infty$ and $K(z,w)$ is a measurable function on $\mathbf{B}_n \times \mathbf{B}_n$ such that

$$\int_{\mathbf{B}_n} \left(\int_{\mathbf{B}_n} |K(z,w)|^p \, d\nu(w) \right)^{p'-1} d\nu(z) < \infty.$$

Then the integral operator T defined by

$$Tf(w) = \int_{\mathbf{B}_n} f(z) K(z,w) \, d\nu(z)$$

is compact on $L^p(\mathbf{B}_n, d\nu)$.

3.5.3 Proof of Theorem 3.5.1

The method used is the same as the one used to prove Theorem 1.1 in [31]. Suppose A is compact. If

$$k_z^p(w) = \frac{(1-|z|^2)^{(n+1)/p'}}{(1-(w \cdot z))^{n+1}}$$

then k_z^p tends to 0 weakly in L_a^p as $z \to \partial \mathbf{B}_n$. Hence

$$\langle A k_z^p, k_z^{p'} \rangle \to 0$$

as $z \to \partial \mathbf{B}_n$. Since

$$\widetilde{A}(z) = \langle A k_z^p, k_z^{p'} \rangle$$

for $z \in \mathbf{B}_n$, we have $\widetilde{A}(z) \to 0$ as $z \to \partial \mathbf{B}_n$. This proves the implication (1) $\Rightarrow$ (2). Recall that by Lemma 3.2.5, the properties (2), (3) and (4) are equivalent. We next show the implication (3) $\Rightarrow$ (1). Under assumption (3), we want to show that A is compact on L_a^p. Fix q such that $(p_1+n)/(p_1-1) < q < s$ in the rest of the proof. Suppose $h \in L_a^p$ and $z \in \mathbf{B}_n$. Then

$$(Ah)(z) = \int_{\mathbf{B}_n} h(w)(AK_w)(z) \, d\nu(w).$$

For $r \in (0,1)$, define an operator $A^{[r]}$ on $L^p_a(\mathbf{B}_n)$ by

$$(A^{[r]}h)(z) = \int_{r\overline{\mathbf{B}_n}} h(w)(AK_w)(z)\,d\nu(w).$$

Then $A^{[r]}$ is an integral operator with kernel $(AK_w)(z)\chi_{r\overline{\mathbf{B}_n}}(w)$. In view of Lemma 3.5.2, this operator is compact on $L^p(\mathbf{B}_n, d\nu)$ since

$$\begin{aligned}\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n}|(AK_w)(z)\chi_{r\mathbf{B}_n}(w)|^p\,d\nu(z)\right)^{p'-1}d\nu(w) &\leq \int_{r\overline{\mathbf{B}_n}}\|A\|^{p'}\|K_w\|_p^{p'}\,d\nu(w)\\ &\leq C\|A\|^{p'}\int_{r\overline{\mathbf{B}_n}}\frac{d\nu(w)}{(1-|w|^2)^{n+1}}.\end{aligned}$$

The latter quantity is finite since A is bounded on L^p_a and the last integral is over a compact set. This proves that $A^{[r]}$ is a compact operator on L^p_a. To conclude, we only need to show that

$$\|A - A^{[r]}\|_p \to 0 \text{ as } r \to 1^-.$$

Now, for $h \in L^p_a$, we have

$$(A - A^{[r]})h(z) = \int_{\mathbf{B}_n}\chi_{\mathbf{B}_n/r\overline{\mathbf{B}_n}}(w)h(w)(AK_w)(z)\,d\nu(w).$$

This implies that $A - A^{[r]}$ is an integral operator on $L^p(\mathbf{B}_n)$ with kernel

$$(AK_w)(z)\chi_{\mathbf{B}_n/r\overline{\mathbf{B}_n}}(w).$$

We apply Shur's Lemma. Since the hypotheses of Proposition 3.3.3 are fulfilled with s replaced by q, the proof of this proposition implies

$$\begin{aligned}\int_{\mathbf{B}_n}|(AK_w)(z)\chi_{\mathbf{B}_n/r\overline{\mathbf{B}_n}}(w)|(1-|w|^2)^{-p'\epsilon}d\nu(w) &\leq \int_{\mathbf{B}_n}|(AK_w)(z)|(1-|w|^2)^{-p'\epsilon}d\nu(w)\\ &\leq C||A_z^\star 1||_q(1-|z|^2)^{-p'\epsilon}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{B}_n}|(AK_w)(z)|\chi_{B_n/r\overline{B_n}}(w)(1-|z|^2)^{-p\epsilon}d\nu(z) &\leq C\chi_{\mathbf{B}_n/r\overline{\mathbf{B}_n}}(w)||A_w 1||_q(1-|w|^2)^{-p\epsilon}\\ &\leq \sup_{r\leq|w|<1}||A_w 1||_q(1-|w|^2)^{-p\epsilon}.\end{aligned}$$

By Schur's Lemma, the following conclusion holds:

$$||A - A^{[r]}||_p \leq C(c_1)^{\frac{1}{p}}(c_2)^{\frac{1}{p'}},$$

where $c_1 = \sup\limits_{z \in \mathbf{B}_n} \{||A_z^\star 1||_q\}$ and $c_2 = \sup\{||A_z^\star 1||_q : r \leq |z| < 1\}$. We have, by assertion (3), that $c_2 \to 0$ as $r \to 1^-$. The hypotheses of the theorem give $c_1 < \infty$. Thus $\|A - A^{[r]}\|_p \to 0$ as $r \to 1^-$, completing the proof. □

The following lemma will enable us establish some corollaries of Theorem 3.5.1.

3.5.4 Lemma

If $f \in BT$ then

$$\sup_{z \in \mathbf{B}_n} \|T_{f \circ \varphi_z} 1\|_s < \infty, \quad \sup_{z \in \mathbf{B}_n} \|T_{\overline{f} \circ \varphi_z} 1\|_s < \infty$$

for $s > 0$.

Proof It is enough to prove these estimates for $s > 1$. So let $s > 1$ and $z \in \mathbf{B}_n$, we have

$$\begin{aligned}
\|T_{f \circ \varphi_z} 1\|_s &= \|P(f \circ \varphi_z)\|_s \\
&= \sup_{\|h\|_{s'} \leq 1} |\langle P(f \circ \varphi_z), h\rangle| \\
&= \sup_{\|h\|_{s'} \leq 1} |\langle f \circ \varphi_z, h\rangle| \\
&= \sup_{\|h\|_{s'} \leq 1} |\langle f, h \circ \varphi_z |k_z|^2\rangle| \\
&\leq \sup_{\|h\|_{s'} \leq 1} \int_{\mathbf{B}_n} |h \circ \varphi_z(\zeta)(k_z(\zeta))^2||f| d\nu(\zeta) \\
&\leq \sup_{\|h\|_{s'} \leq 1} C(f) \int_{\mathbf{B}_n} |h \circ \varphi_z(\zeta)||k_z(\zeta)|^2 |d\nu(\zeta) \\
&= C(f) \sup_{\|h\|_{s'} \leq 1} \int_{\mathbf{B}_n} |h(\zeta)| d\nu(\zeta) \\
&\leq C(f).
\end{aligned}$$

The other inequality follows the same way. □

3.5.5 Remark

We recall that by assertion (2) of Lemma 3.2.1, for $A = T_f, \quad f \in L^1(\mathbf{B}_n, d\nu)$, we have $A_z = T_{f\circ\phi_z}$ if T_f is bounded on L_a^2. Hence, by Lemma 3.5.4, for every $f \in BT$, the associated Toeplitz operator T_f satisfies the hypotheses of Theorem 3.5.1. We therefore obtain the following corollary:

3.5.6 Corollary

Suppose that $1 < p < \infty$. Then for $f \in BT, \quad T_f$ is compact on L_a^p if and only if $\widetilde{f}(z) \to 0$ as z tends to $\partial\mathbf{B}_n$.

3.5.7 Corollary

Let $f \in X_1$ (in particular, let $f \in BMO^1$ or let f be a nonnegative function on $\mathbf{B}_n$) be such that T_f is bounded on $L_a^p, \quad 1 < p < \infty$. Then T_f is compact on L_a^p if and only if $\widetilde{f}(z) \to 0$ as z tends to $\partial\mathbf{B}_n$.

3.5.8 Corollary

Let f belong to the set X_2 defined at the end of Section 3. Suppose that T_f is bounded on L_a^2. Then the following four assertions are equivalent:

(1) T_f is compact on L_a^2;

(2) $\lim\limits_{z\to\partial\mathbf{B}_n} ||T_f k_z||_2 = 0$;

(3) $\lim\limits_{z\to\partial\mathbf{B}_n} \widetilde{f}(z) = 0$.

Proof The implication (4) $\Rightarrow$ (1) follows from Corollary 3.5.6. The implication (1) $\Rightarrow$ (2) follows from the fact that $\{k_z\}$ converges weakly to 0 on L_a^2 when z tends to $\partial\mathbf{B}_n$. The implication (2) $\Rightarrow$ (3) is elementary to get, while the implication (3) $\Rightarrow$ (4)

is trivial. □

Chapter 4

Toeplitz Operators on $L^1_a(\mathbf{B}_n)$

4.1 Introduction

In this chapter we will be treating the question of boundedness and compactness of the Toeplitz operator on the Bergman space $L^1_a(\mathbf{B}_n)$. Unlike the case in the previous chapter i.e when $p > 1$, the case $p = 1$ produces a new phenomenon. For example, in [51] Zhu showed that a Toeplitz operator $T_{\bar{f}}$ with antianalytic symbols is bounded on L^1_a if and only if $f \in L^\infty \cap LB$, where LB is the Logarithmic Bloch space defined below. At the same time, for $p > 1$, it is well known that $T_{\overline{f}}$ is bounded on L^p_a if and only if f is bounded. So the study of T_μ on L^1_a deserves a particular attention. The study of Toeplitz operators on L^1_a has been considered in [45, 46].

For $\alpha > 0$, we let

$$\tilde{K}^\alpha_\xi(z) = \frac{d(\alpha)}{(1-(z\cdot\xi))^{n+1+\alpha}}$$

where

$$d(\alpha) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)}$$

and

$$\tilde{k}^\alpha_\xi(z) = \frac{\tilde{K}_\xi(z)}{\|\tilde{K}_\xi\|_1} = \frac{(1-|\xi|^2)^\alpha}{((1-(z\cdot\xi))^{n+1+\alpha}}.$$

Given a complex Borel measure μ on $\mathbf{B}_n$, the Bergman projection $P\mu$ of μ is defined by

$$(P\mu)(w) = \int_{\mathbf{B}_n} \frac{d\mu(z)}{(1-(w\cdot z))^{n+1}},\ w \in \mathbf{B}_n.$$

The Toeplitz operator T_μ is densely defined on L^1_a by

$$(T_\mu g)(w) = \int_{\mathbf{B}_n} \frac{g(z)}{(1-(w\cdot z))^{n+1}} d\mu(z) = P(\mu g)(w),$$

for $g \in L^\infty_a$ and $w \in \mathbf{B}_n$. Note that the formula

$$T_\mu g = P(g\mu), \quad g \in L^\infty_a$$

makes sense, and defines an analytic function on $\mathbf{B}_n$ and the operator T_μ is in general unbounded on L^1_a. For $d\mu = f d\nu$, with $f \in L^1$ we write $T_\mu = T_f$.

4.2 Boundedness

We will need the following Lemma which is Proposition 1.14 of [47] and Theorems 1 and 2 of [48].

4.2.1 Lemma

Let α and t be any two parameters with the property that neither $n+1+c$ nor $n+1+c+t$ is a negative integer. Then there exists a unique linear operator $\mathcal{D}^{c,t}$ on $H(\mathbf{B}_n)$ with the following properties:

1. $\mathcal{D}^{c,t}$ is continuous on $H(\mathbf{B}_n)$ with respect to the topology of uniform convergence on compact sets of $\mathbb{C}^n$ contained in $\mathbf{B}_n$;

2. $\mathcal{D}^{c,t}_z[(1-(z\cdot w))^{-(n+1+c)}] = (1-(z\cdot w))^{-(n+1+c+t)}$ for every $w \in B_n$.

3. $\mathcal{D}^{c,t}$ is invertible on $H(\mathbf{B}_n)$.

Proof Let $f \in H(\mathbf{B}_n)$ and $f(z) = \sum_{k=0}^{\infty} f_k(z)$, be the homogeneous expansion of f. We define

$$\mathcal{D}^{c,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+c)\Gamma(n+1+k+c+t)}{\Gamma(n+1+c+t)\Gamma(n+1+k+c)} f_k(z),$$

and

$$\mathcal{D}_{c,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+c+t)\Gamma(n+1+k+c)}{\Gamma(n+1+c)\Gamma(n+1+k+c+t)} f_k(z).$$

Then $\mathcal{D}^{c,t}$ and $\mathcal{D}_{c,t}$ are continuous on $H(\mathbf{B}_n)$ if we equip $H(\mathbf{B}_n)$ with the topology of uniform convergence on compact sets. Also $\mathcal{D}^{c,t}$ is invertible with inverse $\mathcal{D}_{c,t}$. Finally we recall that

$$\frac{1}{(1-(z\cdot w))^{n+1+c}} = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+k+c)}{k!\Gamma(n+1+c)} (z\cdot w)^k \quad z, w \in \mathbf{B}_n,$$

is actually a homogeneous expansion. Thus

$$\begin{aligned}
& \mathcal{D}^{c,t}{}_z[(1-(z\cdot w))^{-(n+1+c)}] \\
= & \sum_{k=0}^{\infty} \frac{\Gamma(n+1+k+c)}{k!\Gamma(n+1+c)} \frac{\Gamma(n+1+c)\Gamma(n+1+k+c+t)}{\Gamma(n+1+c+t)\Gamma(n+1+k+c)} (z\cdot w)^k \\
= & \sum_{k=0}^{\infty} \frac{\Gamma(n+1+k+c+t)}{k!\Gamma(n+1+c+t)} (z\cdot w)^k \\
= & (1-(z\cdot w))^{-(n+1+c+t)}. \qquad \square
\end{aligned}$$

We will need the following Lemma which is Theorem 2.19 of [47].

4.2.2 Lemma

Suppose $p > 0$ and $c > 0$. There exist constants A and B such that

$$A \int_{\mathbf{B}_n} |f(z)|^p d\nu(z) \leq \int_{\mathbf{B}_n} |(1-|z|^2)^c \mathcal{D}^c f(z)|^p d\nu(z) \leq B \int_{\mathbf{B}_n} |f(z)|^p d\nu(z) \qquad (4.1)$$

We shall denote $\mathcal{D}^{c,0}$ simply by $\mathcal{D}^c$.

4.2.3 Lemma

For every $h \in L^1_a$, the function $\mathcal{D}^{c,0}h = \mathcal{D}^c h$, $c > 0$, has the following expression,

$$\mathcal{D}^c h(z) = \int_{\mathbf{B}_n} \frac{h(w)}{(1-(z\cdot w))^{(n+1+c)}} d\nu(w), \quad (z \in \mathbf{B}_n).$$

Moreover, there exists a constant C such that all $h \in L^1_a$ and $g \in L^\infty_a$,

$$\int_{\mathbf{B}_n} (1-|z|^2)^c \overline{\mathcal{D}^c h(z)} g(z) d\nu(z) = C \int_{\mathbf{B}_n} \overline{h(z)} g(z) d\nu(z).$$

Proof We first prove the lemma for all $h \in L^2_a$ and $g \in L^\infty_a$. Let $\{a_k\}$ be a $r-$lattice as described in Lemma 3.2.6. By the atomic decomposition theorem (cf. e.g. Theorem 2.30 of [47]), for every $h \in L^2_a$, there exists a sequence $\{c_k\}$ of complex numbers belonging to the sequence space l^2 such that

$$h(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(z\cdot a_k))^{(n+1)}}, \quad (z \in \mathbf{B}_n),$$

where the series converges in the norm topology of L^2_a. Then this series converges uniformly on compact sets of $\mathbb{C}^n$ contained in $\mathbf{B}_n$ to its sum $h(z)$. Next, the series

$$\sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(z\cdot w))^{(n+1+c)}}$$

converges in the norm topology of the weighted Bergman space $L^2_a((1-|z|^2)^{2c} d\nu(z))$, and thus converges uniformly on compact sets of $\mathbb{C}^n$ contained in $\mathbf{B}_n$ to its sum.

We recall that $\mathcal{D}^c{}_z[(1-(z\cdot w))^{-(n+1)}] = (1-(z\cdot w))^{-(n+1+c)}$ for every $w \in \mathbf{B}_n$. This implies the partial sums

$$\sum_{k=1}^{N} c_k (1-|a_k|^2)^{\frac{n+1}{2}} \mathcal{D}^c{}_z \left[\frac{1}{(1-(z\cdot a_k))^{n+1}}\right] = \mathcal{D}_z \left[\sum_{k=1}^{N} c_k \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(z\cdot a_k))^{n+1}}\right]$$

converges uniformly on compact sets of $\mathbb{C}^n$ contained in $\mathbf{B}_n$ to the analytic function

$$\sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(z\cdot a_k))^{n+1+c}}$$

as $N \to \infty$. Since $\mathcal{D}^c$ is continuous in $H(\mathbf{B}_n)$, we conclude that

$$\mathcal{D}^c h(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(z\cdot a_k))^{n+1+c}}. \tag{4.2}$$

Hence

$$\begin{aligned}
\mathcal{D}^c h(z) &= \sum_{k=1}^{\infty} c_k (1-|a_k|^2) \int_{\mathbf{B}_n} \frac{1}{(1-(w\cdot a_k))^{n+1}(1-(z\cdot w))^{n+1+c}} d\nu(w) \\
&= \int_{\mathbf{B}_n} \left\{ \sum_{k=1}^{\infty} c_k \frac{1-|a_k|^2}{(1-(w\cdot a_k))^{n+1}} \right\} \frac{1}{(1-(z\cdot w))^{n+1+c}} d\nu(w) \\
&= \int_{\mathbf{B}_n} \frac{h(w)}{(1-(z\cdot w))^{n+1+c}} d\nu(w).
\end{aligned}$$

Next the convergence in $L^2_a(1-|z|^2)^{2c}d\nu(z))$ of the series in the right hand side of (4.2) implies that

$$\int_{B_n} (1-|z|^2)^c \overline{\mathcal{D}^c h(z)} g(z) d\nu(z) = \sum_{k=1}^{\infty} \overline{c_k} (1-|a_k|^2)^{\frac{n+1}{2}} \int_{B_n} \frac{(1-|z|^2)^c}{(1-(a_k\cdot z))^{n+1+c}} g(z) d\nu(z).$$

Since for $w \in \mathbf{B}_n$ there exists a constant C such that

$$\int_{\mathbf{B}_n} \frac{(1-|z|^2)^c}{(1-(a_k\cdot w))^{n+1+c}} g(z) d\nu(z) = Cg(a_k) = C \int_{\mathbf{B}_n} \frac{g(w)}{(1-(a_k\cdot w))^{n+1}} d\nu(z),$$

for every $g \in L^\infty_a$. This implies that

$$\begin{aligned}
\int_{\mathbf{B}_n} (1-|z|^2)^c \overline{\mathcal{D}^c h(z)} g(z) d\nu(z) &= C \sum_{k=1}^{\infty} \overline{c_k} (1-|a_k|^2) \int_{\mathbf{B}_n} \frac{g(z)}{(1-(a_k\cdot z))^{n+1}} d\nu(z) \\
&= C \int_{\mathbf{B}_n} \left\{ \sum_{k=1}^{\infty} \overline{c_k} \frac{(1-|a_k|^2)^{\frac{n+1}{2}}}{(1-(a_k\cdot z))^{n+1}} \right\} g(z) d\nu(z) \\
&= C \int_{\mathbf{B}_n} \bar{h}(z) g(z) d\nu(z).
\end{aligned}$$

We next consider the general case when $h \in L^1_a$. The announced conclusions follow from the density of L^2_a in L^1_a and from the existence of a constant C such that

$$\int_{\mathbf{B}_n} (1-|z|^2)^c |\mathcal{D}^c h(z)| d\nu(z) \leq C \int_{\mathbf{B}_n} |h(z)| d\nu(z)$$

for all analytic functions h on $\mathbf{B}_n$. Indeed, if $h \in L_a^1$ then

$$h(z) = d(s)\int_{\mathbf{B}_n} \frac{h(w)(1-|w|^2)^s}{(1-(z\cdot w))^{n+1+s}} d\nu(w), \quad s>0, z\in \mathbf{B}_n.$$

This shows that

$$\mathcal{D}^c h(z) = d(s)\int_{\mathbf{B}_n} \frac{h(w)(1-|w|^2)^s}{(1-(z\cdot w))^{n+1+s+c}} d\nu(w), \quad s>0,\ z\in \mathbf{B}_n.$$

Applying Fubini's theorem we have,

$$\begin{aligned}
\int_{\mathbf{B}_n}(1-|z|^2)^c|\mathcal{D}^c h(z)|d\nu(z) &\leq C'\int_{\mathbf{B}_n}\int_{B_n}\frac{|h(w)|(1-|z|^2)^c(1-|w|^2)^s}{|1-(z\cdot w)|^{n+1+s+c}}d\nu(w)d\nu(z)\\
&\leq C'\int_{\mathbf{B}_n}|h(w)|(1-|w|^2)^s\int_{B_n}\frac{(1-|z|^2)^c}{|1-(z\cdot w)|^{n+1+s+c}}d\nu(z)d\nu(w)\\
&\leq C\int_{\mathbf{B}_n}|h(w)|d\nu(w)
\end{aligned}$$

The latter result is just Lemma 4.2.2. This completes finishes the proof of the lemma.

□

For $c>0$, we denote by P_c the orthogonal projector from $L^2((1-|z|^2)^c d\nu(z))$ onto the weighted Bergman space $L_a^2((1-|z|^2)^c d\nu(z))$ (P_c is a weighted Bergman projector in $\mathbf{B}_n$). For every $\phi \in L^2((1-|z|^2)^c d\nu(z))$, and $z\in\mathbf{B}_n$ we have

$$P_c\phi(z) = (1+c)\int_{\mathbf{B}_n}\frac{(1-|\zeta|^c}{(1-(z\cdot\zeta))^{n+1+c}}\phi(\zeta)d\nu(\zeta).$$

We also denote by $\mathcal{D}(\mathbf{B}_n)$ the space of $\mathcal{C}^\infty$ functions with compact support in $\mathbf{B}_n$. We shall need the following lemma.

4.2.4 Lemma

The space $P_c(\mathcal{D}(\mathbf{B}_n))$ is a dense subspace of L_a^1.

Proof It is easy to check that $P_c(\mathcal{D}(\mathbf{B}_n)) \subset L^1_a$. Since the dual space of L^1_a with respect to the usual duality pairing $<,>$ in $L^2(\mathbf{B}_n, d\nu)$ is the Bloch space B^∞, it suffices to show that every $h \in B^\infty$ such that

$$\int_{\mathbf{B}_n} P_c\phi(z)\bar{h}(z)d\nu(z) = 0 \qquad \forall \phi \in \mathcal{D}(\mathbf{B}_n)$$

vanishes identically. An application of Fubini's theorem and Lemma 4.2.3 gives

$$\begin{aligned} 0 &= \int_{\mathbf{B}_n} P_c\phi(z)\bar{h}(z)d\nu(z) = \int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} \frac{(1-|\zeta|^2)^2}{(1-(z\cdot\zeta))^{n+1+c}}\phi(\zeta)d\nu(\zeta)\right)\bar{h}(z)d\nu(z) \\ &= \int_{\mathbf{B}_n} \phi(\zeta)\overline{\mathcal{D}^c h(\zeta)}(1-|\zeta|^2)^c d\nu(\zeta), \end{aligned}$$

for all $\phi \in \mathcal{D}(\mathbf{B}_n)$. We conclude that $\mathcal{D}^c h \equiv 0$ on $\mathbf{B}_n$. Using the invertibility of $\mathcal{D}^c$ on $H(\mathbf{B}_n)$, we obtain that $h \equiv 0$ on $\mathbf{B}_n$. $\square$

We have the following boundedness result.

4.2.5 Theorem

Let A be a linear operator defined on L^∞_a with values in the space of analytic functions on $\mathbf{B}_n$ and let $c > 0$. Then the implication (1) $\Rightarrow$ (2) holds for the following two assertions.

1. A extends to a bounded operator on L^1_a;

2. the following estimate holds:

$$\sup_{\xi \in \mathbf{B}_n} \|A\tilde{k}^c_\xi\|_1 < \infty. \tag{4.3}$$

The converse (2) $\Rightarrow$ (1) also holds in the following two cases.

(a) The operator A satisfies the property that

$$\int_{B_n} (Ak_\xi^c)(z)g(\xi)d\nu(\xi) = CAg(z)$$

for some absolute constant C and for all $z \in \mathbf{B}_n$ and $g \in P_c(\mathcal{D}(\mathbf{B}_n))$.

(b) $A = T_\mu$, μ a complex Borel measure on $\mathbf{B}_n$.

Moreover, in such cases if $C_1 = \sup_{\xi\in\mathbf{B}_n} \|A\tilde{k}_\xi^c\|_1$, there exists a constant C such that

$$\|A\| \leq CC_1.$$

Proof If A is bounded on L_a^1 then

$$\|A\tilde{k}_z^c\|_1 \leq \|A\|\|\tilde{k}_z^c\|_1$$

and since

$$\|\tilde{k}_z^c\|_1 = \int_{\mathbf{B}_n} \frac{(1-|z|^2)^\alpha}{|1-(w\cdot z)|^{n+1+\alpha}}\, d\nu(w)$$

which is bounded in z by Lemma 3.2.2.

Conversely, suppose (4.3) holds.

Case (a): By our assumption on A, we have

$$\begin{aligned}
\int_{\mathbf{B}_n} |Ag(z)|d\nu(z) &\leq C^{-1}\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} |A\tilde{k}_\zeta^c(z)||g(\zeta)|d\nu(\zeta)\right)d\nu(z)\\
&= C^{-1}\int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n} |A\tilde{k}_\zeta^c(z)|d\nu(z)\right)|g(\zeta)|d\nu(\zeta)\\
&= C^{-1}\sup_{\zeta\in\mathbf{B}_n} ||A\tilde{k}_\zeta^c||_1||g||_1.
\end{aligned}$$

This shows the implication (2) $\Longrightarrow$ (1) for the case (a).

Case (b): Let μ be a complex Borel measure on $\mathbf{B}_n$. From case (a), it is enough to prove that if $z \in \mathbf{B}_n$ and g in the dense subspace $P_c(\mathcal{D}(\mathbf{B}_n))$ of L^1_a, then

$$\int_{\mathbf{B}_n} (T_\mu \tilde{k}^{(c)}_\zeta)(z) g(\zeta) d\nu(\zeta) = d(c)^{-1} T_\mu g(z). \tag{4.4}$$

Let $h \in L^1_a(\mathbf{B}_n, (1-|z|^2)^c d\nu(z)) = L^1_a((1-|z|^2)^c d\nu(z))$ and $g = P_c\phi$ with $\phi \in \mathcal{D}(\mathbf{B}_n)$. Then

$$\int_{\mathbf{B}_n} \bar{h}(\zeta) g(\zeta)(1-|\zeta|^2)^c d\nu(\zeta) = \int_{\mathbf{B}_n} \bar{h}(\zeta)\phi(\zeta)(1-|\zeta|^2)^c d\nu(\zeta). \tag{4.5}$$

Fix $z \in \mathbf{B}_n$ and take

$$h_z(\zeta) := \overline{(T_\mu \tilde{K}^c_\zeta)(z)} = d(c) \int_{\mathbf{B}_n} \frac{1}{(1-(w\cdot z))^{n+1}(1-(\zeta\cdot w))^{n+1+c}} d\overline{\mu}(w).$$

It is clear that the function h_z is analytic and for every $\zeta \in \mathbf{B}_n$, and the function $z \mapsto h_z(\zeta)$ is antianalytic. By the mean value property, there exists a constant C_z such that

$$|h_z(\zeta)| \le C_z ||T_\mu \tilde{K}^{(c)}_\zeta||_1$$

and hence

$$\int_{\mathbf{B}_n} |h_z(\zeta)|(1-|\zeta|^2)^c d\nu(\zeta) \le C_z \sup_{\zeta\in\mathbf{B}_n} ||T_\mu \tilde{k}^{(c)}_\zeta||_1 < \infty.$$

In the latter inequality, we applied assertion (2).

For every ϕ in the space $\mathcal{D}(\mathbf{B}_n)$, we have

$$\int_{\mathbf{B}_n} \left(\int_{\mathbf{B}_n} \frac{(1-|\zeta|^2)^c}{|1-(w\cdot\zeta)|^{n+1+c}} |\phi(\zeta)| d\nu(\zeta) \right) \frac{d|\mu|(w)}{|1-(z\cdot w)|^{n+1}} \quad \le \quad \frac{C(\phi)}{(1-|z|^2)^{n+1}} \int_{\mathbf{B}_n} d|\mu|(w) < \infty$$

for every $z \in \mathbf{B}_n$. By identity (4.5) and Fubini's Theorem, we obtain that for every $g = P_c\phi$ in the dense subspace $P_c(\mathcal{D}(\mathbf{B}_n))$ of L^1_a and for every $z \in \mathbf{B}_n$,

$$\begin{aligned}
\int_{\mathbf{B}_n} (T_\mu \tilde{k}^c_\zeta)(z) g(\zeta) d\nu(\zeta) &= \int_{\mathbf{B}_n} (T_\mu \tilde{k}^c_\zeta)(z)\phi(\zeta) d\nu(\zeta) \\
&= \int_{\mathbf{B}_n} \left(\int_{\mathbf{B}_n} \frac{1}{(1-(z\cdot w)^{n+1}} \frac{(1-|\zeta|^2)^c}{(1-(w\cdot\zeta)^{n+1+c}} d\mu(w) \right) \phi(\zeta) d\nu(\zeta) \\
&= \int_{\mathbf{B}_n} \left(\int_{\mathbf{B}_n} \frac{(1-|\zeta|^2)^c}{(1-(w\cdot\zeta))^{n+1+c}} \phi(\zeta) d\nu(\zeta) \right) \frac{1}{(1-z\bar{w})^2} d\mu(w) \\
&= d(c)^{-1} \int_{\mathbf{B}_n} \frac{g(w)}{(1-(z\cdot w))^{n+1}} d\mu(w) = d(c)^{-1} T_\mu g(z).
\end{aligned}$$

This proves identity (4.4) and so the implication (2) $\Rightarrow$ (1) is proved for case (b). $\square$

Our next lemma shows that our necessary condition, when $A = T_\mu$, in Theorem 4.2.5 is remarkably strong.

4.2.6 Lemma

Let $c > 0$ and μ is a complex Borel measure in $\mathbf{B}_n$. Then there exists a constant C such that for all $z, \xi \in \mathbf{B}_n$,

$$\left| \int_{\mathbf{B}_n} \frac{(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}} \frac{d\mu(w)}{(1-(z\cdot w))^{n+1+c}} \right| \leq \frac{C}{(1-|z|^2)^{n+1+c}} \|T_\mu \tilde{k}^c_\xi\|_1. \tag{4.6}$$

Proof Let $\xi, z \in \mathbf{B}_n$. Then equation (3.31) implies

$$\begin{aligned}
&\left| \int_{\mathbf{B}_n} \frac{(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}} \frac{d\mu(w)}{(1-(z\cdot w))^{n+1+c}} \right| \\
\leq &\frac{C}{(1-|z|^2)^{n+1+c}} \int_{\mathbf{B}_n} (1-|\xi|^2)^c \left| \int_{\mathbf{B}_n} \frac{(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}} \frac{d\mu(w)}{(1-(\zeta\cdot w))^{n+1+c}} \right| d\nu(\zeta).
\end{aligned}$$

We apply (4.1) to get that the last equation is atmost

$$\begin{aligned}
&\frac{B}{(1-|z|^2)^{n+1+c}} \int_{\mathbf{B}_n} \left| \int_{\mathbf{B}_n} \frac{(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}} \frac{d\mu(w)}{(1-(\zeta\cdot w))^{n+1}} \right| d\nu(\zeta) \\
= &\frac{B}{(1-|z|^2)^{n+1+c}} \|T_\mu \tilde{k}^c_\xi\|_1,
\end{aligned}$$

which completes the proof. $\square$

4.2.7 Remark

If we take $\xi = z$ in (4.6), we see that for each $\xi \in B_n$,

$$\left| \int_{\mathbf{B}_n} \frac{(1-|\xi|^2)^{n+1+2c}}{|1-(w\cdot\xi)|^{2(n+1+c)}} d\mu(w) \right| \leq C \sup_{\xi\in\mathbf{B}_n} \|T_\mu \tilde{k}_\xi^c\|_1.$$

Thus for positive measures, this clearly shows that our necessary condition implies that μ must be a Carleson measure.

Let $\bigtriangledown f(z)$ be the holomorphic gradient of f at z, that is

$$\bigtriangledown f(z) = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_n} \right).$$

We recall the definition of the Bloch space, $B^\infty(\mathbf{B}_n)$, to be the space of analytic functions f in $\mathbf{B}_n$ for which

$$\|f\|_{B^\infty} = \sup_{z\in\mathbf{B}_n} (1-|z|^2)| \bigtriangledown f(z)| < \infty.$$

We define the Logarithmic Bloch space, LB, to be the space of analytic functions, f, on the unit ball such that

$$\sup_{z\in\mathbf{B}_n} (1-|z|^2)| \bigtriangledown f(z)| \log\left(\frac{2}{1-|z|^2} \right) < \infty.$$

We have the following characterization with anti-analytic symbols.

4.2.8 Theorem

Let $f \in L_a^1$. The following three assertions are equivalent;

(1) $T_{\bar{f}}$ is bounded on L^1_a;

(2) the following estimate holds:

$$\sup_{z\in\mathbf{B}_n} ||T_{\bar{f}}\tilde{k}^c_z||_1 < \infty$$

for some (all) $c > 0$;

(3) f belongs to $L^\infty_a \cap LB$.

Proof The implication (1) $\Rightarrow$ (2). was shown in the proof of Theorem 1.1. The equivalence (1). $\Leftrightarrow$ (3). was proved by K. Zhu [51]. It suffices to prove the implication (2) $\Rightarrow$ (1). Apply Theorem 4.2.5, since the measure $\overline{f}d\nu = d\mu$ is a complex measure on $\mathbf{B}_n$. $\square$

We extend some results on boundedness of [45] to the unit ball.

We will need the following Lemma.

4.2.9 Lemma

For all $z, w \in \mathbf{B}_n$ we have

$$(1) \qquad |w - \varphi_w(z)|^2 = \frac{(1-|w|^2)(|z|^2 - |(z\cdot w)|^2}{|1-(z\cdot w)|^2}.$$

Consequently,

$$(2) \qquad |w - \varphi_w(z)| \geq \frac{(1-|w|^2)|z|}{|1-(z\cdot w)|},$$

and

$$(3) \qquad |w - \varphi_w(z)|^2 \leq \frac{2(1-|w|^2)}{|1-(z\cdot w)|}.$$

Proof. Observe that

$$(w-\varphi_w(z))(1-(z\cdot w)) = (1-|w|^2)P_w(z) + \sqrt{(1-|w|^2)}Q_w(z).$$

Since $(P_w(z) \cdot Q_z(w)) = 0$ we have

$$|(1-|w|^2)P_w(z) + \sqrt{(1-|w|^2)}Q_w(z)|^2 = (1-|w|^2)^2|P_w(z)|^2 + (1-|w|^2)|Q_w(z)|^2.$$

Now, using the identities $|Q_w(z)|^2 = |z|^2 - \frac{|(z\cdot w)|^2}{|w|^2}$ and $|P_w(z)|^2 = \frac{|(z\cdot w)|^2}{|w|^2}$ we have

$$(1-|w|^2)^2|P_w(z)|^2 + (1-|w|^2)|Q_w(z)|^2 = |z|^2 - |(z\cdot w)|^2(1-|w|^2)$$

which gives (1). The inequality (2) follows from (1) and the fact that $|(z\cdot w)| \le |z||w|$. The inequality (3) follows from (1) and the estimates

$$|z|^2 - |(z\cdot w)|^2 \le 1 - |(z\cdot w)|^2 = (1+|(z\cdot w)|)(1-|(z\cdot w)|) \le 2(1-|(z\cdot w)|). \qquad \square$$

Let k be a non-negative integer and let

$$\tau_k(w,z) := \frac{1}{B(n,k)} \sum_{p=0}^{n-1} \begin{pmatrix} n-1 \\ p \end{pmatrix} \frac{(-1)^p}{k+p}(1-|\varphi_w(z)|^2)^p,$$

where $B(n,k)$ is the Bessel function

$$B(n,k) = \frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)}.$$

We associate to every complex Borel measure μ on $\mathbf{B}_n$ the locally integrable function $R_i(\mu)$, $i = 1, 2, \cdots, n$, defined on $\mathbf{B}_n$ by and

$$R_i(\mu)(w) := (1-|w|^2)\overline{\int_{\mathbf{B}_n} \frac{(\overline{w_i}-\overline{z_i})(1-(z\cdot w))^{n-1}}{|w-z|^{2n}(1-(w\cdot z))^2}\tau_k(w,z)d\mu(z)},\ w \in \mathbf{B}_n.$$

We say that μ satisfies condition (R) if the measure $|R_i(\mu)(w)|d\nu(w)$, $i = 1, 2, \cdots, n$, is a Carleson measure for Bergman spaces. We simply say that $f \in L^1$ satisfies condition (R) when the measure $d\mu = f d\nu$ satisfies condition (R).

4.2.10 Remark

When $n = 1$ the function $R(\mu)$ is given by

$$R(\mu)(w) := (1 - |w|^2) \int_{\mathbf{B}_1} \frac{d\mu(z)}{\overline{(z-w)}(1 - z\bar{w})^2},$$

which was introduced in [45].

4.2.11 Lemma

If μ is a complex measure on $\mathbf{B}_n$ such that $|\mu|$ is a Carleson measure for Bergman spaces then μ satisfies the condition (R) and hence

$$R(\mu)(w) = \sum_{i=1}^{n} |R_i(\mu)(w)| d\nu(w)$$

is a Carleson measure for Bergman spaces.

Proof. We fix $r > 0$. The question is to prove that if $|\mu|$ is a Carleson measure for Bergman spaces then

$$\sup_{z \in \mathbf{B}_n} \frac{1}{\nu(D(\xi, r))} \int_{D(\xi,r)} |R_i(\mu)(w)| d\nu(w) < \infty, \quad i = 1, 2, \cdots, n. \tag{4.7}$$

First observe if we let $w = \varphi_z(a)$ in assertion (2) of Lemma 4.2.9 we have

$$|z - w| \geq |1 - (z \cdot w)||\varphi_z(w)|.$$

Applying Fubini's Theorem and using (2) we obtain

$$
\begin{aligned}
& \frac{1}{\nu(D(\xi,r))}\int_{D(\xi,r)} |R_i(\mu)(w)|d\nu(w) \\
= & \frac{1}{\nu(D(\xi,r))}\int_{D(\xi,r)} \left|(1-|w|^2)\int_{\mathbf{B}_n} \frac{(\overline{w_i}-\overline{z_i})(1-(z\cdot w))^{n-1}}{|w-z|^{2n}(1-(w\cdot z))^2}\tau_k(w,z)d\mu(z)\right| d\nu(w) \\
\leq & \frac{C}{\nu(D(\xi,r))}\int_{\mathbf{B}_n}\int_{D(\xi,r)} \frac{|1-(z\cdot w)|^{n-3}(1-|w|^2)}{|w-z|^{2n-1}}d\nu(w)d|\mu|(z) \\
\leq & \frac{C}{\nu(D(\xi,r))}\int_{\mathbf{B}_n}\int_{D(\xi,r)} \frac{|1-(z\cdot w)|^{n-3}(1-|w|^2)}{|1-(z\cdot w)|^{2n-1}|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z) \\
= & \frac{C}{\nu(D(\xi,r))}\left(\int_{D(\xi,3r)}+\int_{\mathbf{B}_n/D(\xi,3r)}\right)\int_{D(\xi,r)} \frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z) \\
= & I+J,
\end{aligned}
$$

where

$$
I=\frac{C}{\nu(D(\xi,r))}\int_{D(\xi,3r)}\int_{D(\xi,r)} \frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z)
$$

and

$$
J=\frac{C}{\nu(D(\xi,r))}\int_{\mathbf{B}_n/D(\xi,3r)}\int_{D(\xi,r)} \frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z).
$$

Now if $z\in D(\xi,3r)$ and $w\in D(\xi,r)$, we know that $\nu(D(\xi,r))\approx\nu(D(z,r))\approx(1-|\xi|^2)^{n+1}$ and that $|1-(z\cdot w)|\approx(1-|\xi|^2)\approx(1-|w|^2)$ and we have $D(\xi,r)\subset D(z,4r)$. Thus

$$
\begin{aligned}
I &\leq \frac{C}{(1-|\xi|^2)^{2n+2}}\int_{D(\xi,3r)}\int_{D(z,4r)} \frac{1}{|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z) \\
&= \frac{C}{(1-|\xi|^2)^{2n+2}}\int_{D(\xi,3r)}\left\{\int_{D(0,4r)} \frac{1}{|w|^{2n-1}}|J_{\varphi_z}(w)|^2 d\nu(w)\right\}d|\mu|(z) \\
&\leq \frac{C_r}{(1-|\xi|^2)^{n+1}}\int_{D(\xi,3r)}\int_{D(0,4r)} \frac{1}{|w|^{2n-1}}d\nu(w)d|\mu|(z) \\
&\leq \frac{C|\mu|(D(\xi,3r))}{(1-|\xi|^2)^{n+1}}<\infty,
\end{aligned}
$$

here we have made the change of variable $w = \varphi_z$ to get the first equality.

We now consider J.

If $z \notin D(\xi, 3r)$ and $w \in D(\xi, r)$, we have $\beta(z, w) \geq 2r$. This implies $|\varphi_z(w)| \geq \frac{e^{4r}-1}{e^{4r}+1} > 0$. This and the fact that $|\mu|$ is a Carleson measure gives

$$\begin{aligned}
J &= \frac{C}{\nu(D(\xi,r))}\int_{\mathbf{B}_n/D(\xi,3r)}\int_{D(\xi,r)}\frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}|\varphi_z(w)|^{2n-1}}d\nu(w)d|\mu|(z)\\
&\leq C_r\frac{1}{\nu(D(\xi,r))}\int_{\mathbf{B}_n/D(\xi,3r)}\int_{D(\xi,r)}\frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}}d\nu(w)d|\mu|(z)\\
&\leq \frac{C_rC(|\mu|)}{\nu(D(\xi,r))}\int_{D(\xi,r)}\int_{\mathbf{B}_n}\frac{(1-|w|^2)}{|1-(z\cdot w)|^{n+2}}d\nu(z)d\nu(w)\\
&\leq \frac{C_r'C(|\mu|)}{\nu(D(\xi,r))}\int_{D(\xi,r)}d\nu(w) < \infty.
\end{aligned}$$

This shows that (4.7) holds for every $i = 1, 2, \cdots, n$. $\square$

We introduce the standard volume form on $\mathbb{C}^n$,

$$d\nu(\xi) = \left(\frac{1}{2i\pi}\right)^n \bigwedge_{i=1}^{n}(d\overline{\xi_i}\wedge d\xi_i), \tag{4.8}$$

where

$$\bigwedge_{i=1}^{n}(d\overline{\xi_i}\wedge d\xi) = (d\overline{\xi_1}\wedge d\xi_1)\wedge(d\overline{\xi_2}\wedge d\xi_2)\wedge\cdots\wedge(d\overline{\xi_n}\wedge d\xi_n).$$

Also if u is continuously differentiable on $\mathbb{C}^n$ then

$$\overline{\partial}u(\xi) := \sum_{i=1}^{n}\frac{\partial u}{\partial\overline{\xi_i}}d\overline{\xi_i},$$

where

$$\frac{\partial}{\partial\overline{\xi_i}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right),\ \xi_i = x_i + iy_i,\ i = 1, 2, \cdots, n,$$

and

$$\frac{\partial}{\partial\xi_i} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right),\ \xi_i = x_i + iy_i,\ i = 1, 2, \cdots, n.$$

We recall that if $u \in C^1(\overline{\mathbf{B}_1})$, and $z \in \mathbf{B}_1$,the unit disk of $\mathbb{C}$, then

$$\begin{aligned} u(z) &= \frac{k}{2i\pi}\int_{\mathbf{B}_1} u(\xi)\frac{(1-|\xi|^2)^{k-1}}{(1-\overline{\xi}z)^{k+1}}d\overline{\xi}\wedge d\xi \\ &+ \frac{1}{2i\pi}\int_{\mathbf{B}_1}\frac{\partial u}{\partial\overline{\xi}}(\xi)\frac{(1-|\xi|^2)^k}{(z-\xi)(1-\overline{\xi}z)^{k+1}}d\overline{\xi}\wedge d\xi \\ &= P_{k-1}u(z) + \frac{1}{2i\pi}\int_{\mathbf{B}_1}\frac{\partial u}{\partial\overline{\xi}}(\xi)\frac{(1-|\xi|^2)^k}{(z-\xi)(1-\overline{\xi}z)^{k+1}}d\overline{\xi}\wedge d\xi, \end{aligned}$$

where $k0$ a non-zero positive integer. We note that the second integral is zero if u is analytic on $\mathbf{B}_1$. The extension of the above formula to the unit ball in $\mathbb{C}^n$ is given by P.Charpentier [17] which we shall present below. If $u \in C^1(\overline{\mathbf{B}_n})$, $k > 0$ and $z \in \mathbf{B}_n$ then

$$u(z) = P_{k-1}u(z) + \int_{\mathbf{B}_n}\overline{\partial}u \wedge C_k(\xi, z), \tag{4.9}$$

where

$$C_k(\xi, z) = \Psi_k(\xi, z)C_0(\xi, z),$$

and

$$\Psi_k(\xi, z) = \left(\frac{(1-|\xi|^2)}{(1-(z\cdot\xi))}\right)^k \tau_k(\xi, z),$$

$$C_0(\xi, z) = d(n)\frac{(1-(\xi\cdot z))^{n-1}}{|\xi - z|^{2n}}\left\{\sum_{i=1}^{n}(-1)^{i-1}(\overline{\xi_i}-\overline{z_i})\bigwedge_{j\neq i}d\overline{\xi_j}\right\}\bigwedge_{i=1}^{n}d\xi_i,$$

with $d(n) = -(-1)^{n(n-1)/2}\frac{(n-1)!}{(2i\pi)^n}$.

4.2.12 Lemma

Let $u \in C^1(\overline{\mathbf{B}_n})$, and $k > 0$. Then there exists a constant, $C = C(n)$, such that

$$\int_{\mathbf{B}_n}\overline{\partial}u\wedge C_k(\xi, z) = C\sum_{i=1}^{n}\int_{\mathbf{B}_n}\frac{\partial u}{\partial\overline{\xi_i}}(\xi)(\overline{\xi_i}-\overline{z_i})\Psi_k(\xi, z)\frac{(1-(\xi\cdot z))^{n-1}}{|\xi - z|^{2n}}\,d\nu(\xi). \tag{4.10}$$

Proof.

$$\begin{aligned}
& \left\{\sum_{i=1}^{n}(-1)^{i-1}(\overline{\xi_i}-\overline{z_i})\bigwedge_{j\neq i}d\overline{\xi_j}\right\}\bigwedge_{i=1}^{n}d\xi_i \\
= & \left((\overline{\xi_1}-\overline{z_1})\bigwedge_{j=2}^{n}d\overline{\xi_j}-(\overline{\xi_2}-\overline{z_2})\bigwedge_{j=1,j\neq 2}^{n}d\overline{\xi_j}+\cdots+(-1)^{n-1}(\overline{\xi_n}-\overline{z_n})\bigwedge_{j=1}^{n-1}d\overline{\xi_j}\right)\bigwedge_{i=1}^{n}d\xi_i \\
= & (-1)^{n(n-1)/2}(\overline{\xi_1}-\overline{z_1})d\xi_1\bigwedge_{j=2}^{n}(d\overline{\xi_j}\wedge d\xi_j)+(-1)^{n(n-1)/2}(\overline{\xi_2}-\overline{z_2})d\overline{\xi_1}\wedge d\xi_1\wedge d\xi_2 \\
& \bigwedge_{j=3}^{n}(d\overline{\xi_j}\wedge d\xi_j)+\cdots+(-1)^{n(n-1)/2}(\overline{\xi_n}-\overline{z_n})\bigwedge_{j=1}^{n-1}(d\overline{\xi_j}\wedge d\xi_j)\wedge d\xi_n \\
= & (-1)^{n(n-1)/2}(\overline{\xi_1}-\overline{z_1})d\xi_1\bigwedge_{j=2}^{n}(d\overline{\xi_j}\wedge d\xi_j) \\
+ & (-1)^{n(n-1)/2}\sum_{i=2}^{n}(\overline{\xi_i}-\overline{z_i})\bigwedge_{j=1}^{i-1}(d\overline{\xi_j}\wedge d\xi_j)\wedge d\xi_i\bigwedge_{j=i+1}^{n}(d\overline{\xi_j}\wedge d\xi_j).
\end{aligned}$$

Let $G(\xi,z)=(-1)^{n(n-1)/2}d(n)\frac{(1-(\xi\cdot z))^{n-1}}{|\xi-z|^{2n}}$ and let's use the notation $(\overline{\xi_i}-\overline{z_i})\bigwedge_{j=1}^{0}(d\overline{\xi_j}\wedge d\xi_j)=(\overline{\xi_i}-\overline{z_i})$. Then

$$C_0(\xi,z)=G(\xi,z)\sum_{i=1}^{n}(\overline{\xi_i}-\overline{z_i})\bigwedge_{j=1}^{i-1}(d\overline{\xi_j}\wedge d\xi_j)\wedge d\xi_i\bigwedge_{j=i+1}^{n}(d\overline{\xi_j}\wedge d\xi_j),$$

and thus

$$C_k(\xi,z)=\Psi_k(\xi,z)G(\xi,z)\sum_{i=1}^{n}(\overline{\xi_i}-\overline{z_i})\bigwedge_{j=1}^{i-1}(d\overline{\xi_j}\wedge d\xi_j)\wedge d\xi_i\bigwedge_{j=i+1}^{n}(d\overline{\xi_j}\wedge d\xi_j),$$

which is an $(n,n-1)$ form. Also since

$$\overline{\partial}u(\xi)=\sum_{i=1}^{n}\frac{\partial u}{\partial\overline{\xi_i}}d\overline{\xi_i}$$

and the fact that $d\xi_i\wedge d\xi_i=0$, $d\overline{\xi_i}\wedge d\overline{\xi_i}=0$, $d\xi_i\wedge d\xi_j=-d\xi_j\wedge d\xi_i$ and $d\overline{\xi_i}\wedge d\overline{\xi_j}=$

$-d\overline{\xi_j} \wedge d\overline{\xi_i}$ we obtain

$$\begin{aligned}
&\overline{\partial} u(\xi) \wedge C_k(\xi, z) \\
= \;& \sum_{i=1}^{n} \frac{\partial u}{\partial \overline{\xi_i}} d\overline{\xi_i} \wedge \Psi_k(\xi, z) G(\xi, z) \sum_{i=1}^{n} (\overline{\xi_i} - \overline{z_i}) \bigwedge_{j=1}^{i-1} (d\overline{\xi_j} \wedge d\xi_j) \wedge d\xi_i \bigwedge_{j=i+1}^{n} (d\overline{\xi_j} \wedge d\xi_j) \\
= \;& \Psi_k(\xi, z) G(\xi, z) \sum_{i=1}^{n} \frac{\partial u}{\partial \overline{\xi_i}} (\overline{\xi_i} - \overline{z_i}) \bigwedge_{j=1}^{n} (d\overline{\xi_j} \wedge d\xi_j).
\end{aligned}$$

Now using (4.8) we see that

$$\overline{\partial} u(\xi) \wedge C_k(\xi, z) = C_n \Psi_k(\xi, z) G(\xi, z) \sum_{i=1}^{n} \frac{\partial u}{\partial \overline{\xi_i}} (\overline{\xi_i} - \overline{z_i}) \, d\nu(\xi).$$

Substituting this gives (4.10). $\square$

4.2.13 Lemma

Let $h \in L_a^\infty$ and $g \in B^\infty$. Then there exists a constant, $C = C(n)$, such that

$$(I - P_1)(\overline{g} h)(z) = \sum_{i=1}^{n} \int_{\mathbf{B}_n} h(\xi) \frac{\partial \overline{g}}{\partial \overline{\xi_i}} (\xi) (\overline{\xi_i} - \overline{z_i}) \Psi_2(\xi, z) \frac{(1 - (\xi \cdot z))^{n-1}}{|\xi - z|^{2n}} \, d\nu(\xi), \;\; z \in \mathbf{B}_n.$$

Proof Let $r \in (0,1)$ and $z \in \mathbf{B}_n$. Set $g_r(z) = g(rz)$ and $h_r(z) = h(rz)$. Then the function $u = \overline{g_r} h_r$ is in $C^1(\overline{\mathbf{B}_n})$ and thus by (4.9)

$$(I - P_1) u(z) = \int_{\mathbf{B}_n} \overline{\partial} u \wedge C_2(\xi, z).$$

Now since h is analytic we see that

$$\overline{\partial} u = \overline{\partial}(\overline{g_r} h_r) = h_r \overline{\partial}(\overline{g_r}),$$

and thus Lemma 4.2.12 shows that

$$(I - P_1)(\overline{g_r} h_r)(z) = C \sum_{i=1}^{n} \int_{\mathbf{B}_n} h(r\xi) r \frac{\partial \overline{g}}{\partial \overline{\xi_i}} (r\xi) (\overline{\xi_i} - \overline{z_i}) \Psi_2(\xi, z) \frac{(1 - (\xi \cdot z))^{n-1}}{|\xi - z|^{2n}} \, d\nu(\xi). \tag{4.11}$$

Now since,

$$|\Psi_2(\xi, z)| \leq |C_n| \frac{(1-|\xi|^2)^2}{|1-(\xi \cdot z)|^2},$$

we have that

$$\begin{aligned} & \left| h(r\xi) r \frac{\partial \overline{g}}{\partial \overline{\xi_i}}(r\xi)(\overline{\xi_i} - \overline{z_i}) \Psi_2(\xi, z) \frac{(1-(\xi \cdot z))^{n-1}}{|\xi - z|^{2n}} \right| \\ \leq \; & |C_n||h(r\xi)| r \left| \frac{\partial \overline{g}}{\partial \overline{\xi_i}}(r\xi) \right| \frac{(1-|\xi|^2)^2}{|1-(\xi \cdot z)|^2} \frac{|1-(\xi \cdot z)|^{n-1}}{|\xi - z|^{2n-1}} \\ \leq \; & C' \|g\|_{B^\infty} \|h\|_\infty \frac{(1-|\xi|^2)|1-(\xi \cdot z)|^{n-3}}{|\xi - z|^{2n-1}}. \end{aligned}$$

Now the latter function is integrable with respect to $d\nu(\xi)$ for each $z \in \mathbf{B}_n$. Thus dominated convergence gives the convergence of the right hand side of (4.11). For the left hand side, we note that $\overline{g_r} h_r \to \overline{g} h$ pointwise and also in L^2 and hence in $L^2(\mathbf{B}_n, (1-|z|^2) d\nu(z))$, so $P_1(\overline{g_r} h_r) \to P_1(\overline{g} h)$ in $L^2(\mathbf{B}_n, (1-|z|^2) d\nu(z))$ as $r \to 1^-$. That is,

$$(I - P_1)\overline{g_r} h_r(z) \to (I - P_1)\overline{g} h(z), \quad \text{as } r \to 1, \quad \text{for each} \quad z \in \mathbf{B}_n. \quad \square$$

We now give an extension of Theorem 2.1 of [45] to the unit ball of $\mathbb{C}^n$.

4.2.14 Theorem

Suppose μ is a complex measure which satisfies the condition (R). Then T_μ is bounded on B^∞ if and only if $P(\mu) \in LB$.

Moreover, there exists a positive constant C such that for every complex Borel measure μ satisfying the condition (R), the following estimate holds:

$$\|P(\mu)\|_{LB} \leq C(\|T_\mu\| + Carl(R(\mu))), \tag{4.12}$$

where $Carl(R(\mu))$ denotes the Carleson measure constant for the measure $R(\mu)(\xi) d\nu(\xi)$.

Proof. Let $g \in B^\infty$ and $h \in L_a^\infty$. Since $(L_a^1)^* = B^\infty$ with respect to the usual L^2- duality pairing, we have

$$\begin{aligned}
\langle h, T_\mu g\rangle &= \int_{\mathbf{B}_n} h(w)\overline{T_\mu g}(w)d\nu(w) \\
&= \int_{\mathbf{B}_n} h(w)\overline{\int_{B_n} g(z)K_z(w)d\mu(z)}d\nu(w) \\
&= \int_{\mathbf{B}_n} \overline{g(z)}\int_{B_n} h(w)K_w(z)d\nu(w)d\overline{\mu}(z) \\
&= \int_{\mathbf{B}_n} \overline{g(z)}h(z)d\overline{\mu}(z) \\
&= \int_{\mathbf{B}_n} P_1(\overline{g}h)(z)d\overline{\mu}(z) + \int_{\mathbf{B}_n} (I-P_1)(\overline{g}h)(z)d\overline{\mu}(z) \qquad (4.13) \\
&= I_1 + I_2.
\end{aligned}$$

By Lemma 4.2.13 we have

$$\begin{aligned}
I_2 &= \int_{\mathbf{B}_n} C(n)\sum_{i=1}^n \int_{\mathbf{B}_n} h(\xi)\frac{\partial\overline{g}}{\partial\overline{\xi_i}}(\xi)(\overline{\xi_i}-\overline{z_i})\Psi_2(\xi,z)\frac{(1-(\xi\cdot z))^{n-1}}{|\xi-z|^{2n}}\,d\nu(\xi)d\overline{\mu}(z) \\
&= C(n)\sum_{i=1}^n \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} h(\xi)\frac{\partial\overline{g}}{\partial\overline{\xi_i}}(\xi)(\overline{\xi_i}-\overline{z_i})\tau(\xi,z)\frac{(1-|\xi|^2)^2(1-(\xi\cdot z))^{n-1}}{|\xi-z|^{2n}(1-(z\cdot\xi))^2}\,d\nu(\xi)d\overline{\mu}(z) \\
&= C(n)\sum_{i=1}^n \int_{\mathbf{B}_n} h(\xi)\frac{\partial\overline{g}}{\partial\overline{\xi_i}}(\xi)(1-|\xi|^2)\overline{R_i(\mu)}(\xi)\,d\nu(\xi).
\end{aligned}$$

Thus, Lemma 4.2.12 implies

$$\begin{aligned}
|I_2| &\le C'(n)\|g\|_{B^\infty}\int_{\mathbf{B}_n}|h(\xi)|\sum_{i=1}^n |R_i(\mu)(\xi)|d\nu(\xi) \\
&\le C'(n)C(\mu)\|g\|_{B^\infty}\|h\|_1
\end{aligned}$$

where $C(\mu)$ is the Carleson measure constant for the measure $\sum_{i=1}^n |R_i(\mu)(\xi)|d\nu(\xi)$.

On the other hand, by Fubini's Theorem

$$\begin{aligned} I_1 &= \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} h(\xi)\overline{g}(\xi)\frac{(1-|\xi|^2)}{(1-(z\cdot\xi))^{n+2}}d\nu(\xi)d\overline{\mu}(\xi) \\ &= \int_{\mathbf{B}_n} h(\xi)\overline{g}(\xi)(1-|\xi|^2)\int_{\mathbf{B}_n}\frac{d\mu(\xi)}{(1-(\xi\cdot z))^{n+2}}d\nu(\xi) \\ &= \int_{\mathbf{B}_n} h(\xi)\overline{g}(\xi)(1-|\xi|^2)Q(\mu)(\xi)d\nu(\xi), \end{aligned}$$

where Q is given by

$$Q(\mu)(\xi) = \int_{B_n}\frac{d\mu(z)}{(1-(\xi\cdot z))^{n+2}}. \tag{4.14}$$

Thus $T_\mu(g) \in B^\infty$ if and only if

$$\overline{g}(\xi)(1-|\xi|^2)Q(\mu)(\xi) \in L^\infty. \tag{4.15}$$

Now, the relation between $Q(\mu)$ and $P(\mu)$ is

$$Q(\mu)(\xi) = P(\mu)(\xi)(n+1) + \sum_{i=1}^{n}\xi_i\frac{\partial}{\partial\xi_i}P(\mu)(\xi). \tag{4.16}$$

Thus if $P(\mu) \in LB$, then

$$Q(\mu)(\xi)(1-|\xi|^2)\log\frac{2}{1-|\xi|^2} \in L^\infty$$

which shows that (4.15) holds.

Conversely, if T_μ is bounded on B^∞, then by (4.15) there exists a constant $C > 0$, independent of g, such that

$$(1-|\xi|^2)|g(\xi)Q(\mu)(\xi)| \leq C\|g\|_{B^\infty}. \tag{4.17}$$

Using $g_a(w) = \log\frac{2}{(1-(w\cdot a))}$, $a \in \mathbf{B}_n$ we get

$$Q(\mu)(\xi)(1-|\xi|^2)\log\frac{2}{1-|\xi|^2} \in L^\infty, \tag{4.18}$$

since $\|g_a\|_{B^\infty}$ is uniformly bounded. Now the boundedness of T_μ on B^∞, implies $P(\mu) \in B^\infty$ and thus $(1-|\xi|^2)\log\frac{2}{1-|\xi|^2}|P(\mu)(\xi)| \leq C\|P(\mu)\|_{B^\infty} < \infty$, that is $(1-|\xi|^2)\log\frac{2}{1-|\xi|^2}P(\mu)(\xi) \in L^\infty$. Equations (4.16) and (4.18) shows that $(1-|\xi|^2)\log\frac{2}{1-|\xi|^2}|\bigtriangledown P(\mu)(\xi)| \in L^\infty$ that is $P(\mu) \in LB$.

To get the estimate (4.12), we observe using (4.13), Lemma 4.2.13 and the definition of the operator P_1 that

$$T_\mu g = P\left(C(n)\sum_{i=1}^{n}\frac{\partial\overline{g}(\xi)}{\partial\overline{\xi_i}}(1-|\xi|^2)R_i(\mu)\right) + P(g(1-|\xi|^2)Q(\mu)). \tag{4.19}$$

Let L denote the operator on the space of bounded analytic functions, L_a^∞, defined by

$$L(g) := P(g(1-|\xi|^2)Q(\mu)),\ g \in L_a^\infty.$$

Then T_μ is bounded on B^∞ if and only if the operator L extends to a bounded operator on B^∞ with equivalent norms. Thus for every $g \in B^\infty$, with $\|g\|_{B^\infty} = 1$, we have

$$\begin{aligned}\|P(\mu)\|_{LB} &\leq C''\sup_{a\in\mathbf{B}_n}\|L(g_a)\|_{B^\infty}\\ &\leq C'\sup_{a\in\mathbf{B}_n}\{\|T_\mu g_a\|_{B^\infty} + C(n)Carl(R(\mu))\|g_a\|_{B^\infty}\}\\ &\leq C(\|T_\mu\| + C(n)Carl(R(\mu))).\end{aligned}$$

□

By the duality between B^∞ and L_a^1 with respect to the usual pairing in $L^2(\mathbf{B}_n, d\nu)$, if T_μ is bounded on B^∞, the adjoint operator of T_μ is $T_{\bar{\mu}}$ and is bounded on L_a^1. This gives the following corollary.

4.2.15 Corollary

Let μ be a complex measure satisfying the condition (R). Then $T_{\overline{\mu}}$ is bounded on L_a^1 if and only if $P(\mu) \in LB$.

4.2.16 Corollary

Let μ be a positive measure on $\mathbf{B}_n$ and let $c > 0$. Then the following four assertions are equivalent:

(i) The Toeplitz operator T_μ is bounded on L_a^1;

(ii) For every positive constant c, there is a constant A such that

$$\sup_{\xi \in \mathbf{B}_n} \|T_\mu \tilde{k}_\xi^c\|_1 \leq A.$$

(iii) There is a constant A such that

$$\sup_{\xi \in \mathbf{B}_n} \|T_\mu \tilde{k}_\xi^1\|_1 < \infty.$$

(iv) μ is a Carleson measure for Bergman spaces and $P(\mu) \in LB$.

Proof. It is clear that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$. It suffices to show that $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$. For the implication $(iii) \Rightarrow (iv)$ Remark 4.2.7 shows that μ is a Carleson measure for Bergman spaces. Thus we have to show that $P(\mu) \in LB$. We shall use the following result:

4.2.17 Lemma

Let μ be a positive Carleson measure for Bergman spaces. Then the operator

$$S_\mu(h)(z) = (1 - |z|^2) \int_{\mathbf{B}_n} \frac{h(z) - h(w)}{(1 - (z \cdot w))^{n+2}} d\mu(w)$$

is bounded from B^∞ to L^∞.

Proof Using the Carleson condition, Lemma 4.2.6 and a change of variable $w = \varphi_z$ we have

$$
\begin{aligned}
|S_\mu(h)(z)| &\leq (1-|z|^2)\int_{\mathbf{B}_n} \frac{|h(z)-h(w)|}{|1-(z\cdot w)|^{n+2}}d\mu(w) \\
&\leq C(\mu)(1-|z|^2)\int_{\mathbf{B}_n} \frac{|h(z)-h(w)|}{|1-(z\cdot w)|^{n+2}}d\nu(w) \\
&\leq C(\mu)(1-|z|^2)\|h\|_{B^\infty}\int_{\mathbf{B}_n} \frac{\beta(z,w)}{|1-(z\cdot w)|^{n+2}}d\nu(w) \\
&\leq C(\mu)\|h\|_{B^\infty}\int_{\mathbf{B}_n} \frac{\beta(0,w)}{|1-(z\cdot w)|^{n}}d\nu(w) \\
&\leq C\|h\|_{B^\infty}
\end{aligned}
$$

by lemma 3.2.2, and this proves the Lemma.

We continue our proof of $(iii) \Rightarrow (iv)$. Let $h \in B^\infty$ and $z \in \mathbf{B}_n$. Then

$$
\begin{aligned}
\left\langle T_\mu \tilde{k}_z^1, h\right\rangle &= \int_{\mathbf{B}_n} T_\mu \tilde{k}_z^1(w)\overline{h(w)}d\nu(w) \\
&= \int_{\mathbf{B}_n}\left(\int_{\mathbf{B}_n}\frac{(1-|z|^2)}{(1-(\zeta\cdot z))^{n+2}}\frac{d\mu(\zeta)}{(1-(w\dot{\zeta}))^{n+1}}\right)\overline{h(w)}d\nu(w) \\
&= (1-|z|^2)\int_{\mathbf{B}_n}\frac{1}{(1-(\zeta\cdot z))^{n+2}}\overline{\int_{\mathbf{B}_n}\frac{h(w)d\nu(w)}{(1-(\zeta\cdot w))^{n+1}}}d\mu(\zeta) \\
&= (1-|z|^2)\overline{L_\mu(h)(z)}
\end{aligned}
$$

where $L_\mu(h)(z) = \int_{\mathbf{B}_n}\frac{h(\zeta)}{(1-(z\cdot\zeta))^{n+2}}d\mu(\zeta)$, and $L_\mu(1)(z) = Q(\mu)(z)$. It is then easy to obtain this identity

$$
(1-|z|^2)\overline{h(z)\,L_\mu(1)(z)} = \left\langle T_\mu \tilde{k}_z^1, h\right\rangle + \overline{S_\mu(h)(z)} \tag{4.20}
$$

for $z \in \mathbf{B}_n$ and $h \in B^\infty$. So that using Lemma 4.2.17 we get

$$
(1-|z|^2)|h(z)L_\mu(1)(z)| \leq C||h||_{B^\infty} \tag{4.21}
$$

for $z \in \mathbf{B}_n$ and $h \in B^\infty$. A similar argument as in the proof of the converse part of Theorem 4.2.14 we see that $P(\mu) \in LB$. The implication $(iv) \Rightarrow (i)$ follows from Corollary 4.2.15 since μ is a Carleson measure and thus satisfies the condition (R). $\square$

4.2.18 Remark

We just want to point out here that the condition $P(\mu) \in LB$ is not superfluous in assertion (iv). If this is the case, every Carleson measure μ for the Bergman spaces would satisfy $P(\mu) \in LB$. In particular, for every bounded non-negative function on B_n, we have that $P(f) \in LB$. Thus if f is a real and bounded function on $\mathbf{B}_n$ then $f = f^+ + f^-$ with $P(f^+) \in LB$ and $P(f^-) \in LB$. This shows that $P(f) \in LB$ for all bounded complex functions on $\mathbf{B}_n$. This will imply B^∞ is contained in LB which is false.

4.2.19 Corollary

Suppose that μ is a complex Borel measure on $\mathbf{B}_n$ such that T_μ is bounded on L^1_a. Then for every $z \in \mathbf{B}_n$, the Toeplitz operator $T_{K_z\overline{\mu}}$ is bounded on the Bloch space B^∞.

We suppose further that the measure $K_z\bar{\mu}$ satisfies condition (R) for every $z \in \mathbf{B}_n$ with the following uniform condition:

$$\forall r \in (0,1), \qquad \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu)) < \infty$$

(this is the case when $|\mu|$ is a Carleson measure for Bergman spaces). Then for every $r \in (0,1)$, there exists a constant $C = C(r)$ such that

$$\sup_{z \in r\mathbf{B}_n} ||P(K_z\bar{\mu})||_{LB} \leq C(||T_\mu|| + \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu))),$$

where $||T_\mu||$ denotes the norm operator of T_μ on L_a^1 and $Carl(R(K_z\mu))$ denotes the Carleson constant of the Carleson measure $|R(K_z\mu)|d\nu$.

Proof It is easy to check that for every $g \in B^\infty$ and for every $z \in \mathbf{B}_n$, the function K_zg belongs to B^∞ and there exists a constant $C(z)$ such that $||K_zg||_{B^\infty} \leq C(z)||g||_{B^\infty}$. Hence, for all $g \in B^\infty, \quad h \in L_a^2$ and $z \in \mathbf{B}_n$, we get:

$$\begin{aligned} |\langle T_{K_z\bar{\mu}}g, h\rangle| &= |\langle K_zg, T_\mu h\rangle| \leq \|K_zg\|_{B^\infty}\|T_\mu h\|_1 \\ &\leq C(z)||g||_{B^\infty}||T_\mu||||h||_1. \end{aligned}$$

For every $r \in (0,1)$, there exists a constant $C(r)$ such that

$$\sup_{z\in r\mathbf{B}_n} ||K_zg||_{B^\infty} \leq C(r)||g||_{B^\infty}.$$

Hence for all $g \in B^\infty, \quad h \in L_a^2$ and $z \in r\mathbf{B}_n$, we have that

$$|\langle T_{K_z\bar{\mu}}g, h\rangle| \leq C(r)\|g\|_{B^\infty}||T_\mu||\|h\|_1.$$

If we denote by $||T_{K_z\bar{\mu}}||'$ the operator norm of $T_{K_z\bar{\mu}}$ on B^∞, we obtain

$$||T_{K_z\bar{\mu}}||' \leq C(r)||T_\mu||.$$

Since the measure $K_z\bar{\mu}$ satisfies condition (R) for every $z \in \mathbf{B}_n$, the conclusion follows from inequality

$$||P(K_z\bar{\mu})||_{LB} \leq C(||T_{K_z\bar{\mu}}||' + Carl(R(K_z\bar{\mu})).$$

□

4.3 Duality

In this section we will be extending duality results of Shields and Williams [40], to the case $n > 1$. We note that Ren and Xiao [36] had extended this results with weights that are slightly different from that of Shields and Williams [40]. Since our method to prove the compactness result could not make use of the weights used by Ren and Xiao [36] we have to extend it with weights that will be useful in our study. Most of the lemmas are found in [40] and [36] with some slight modifications in some cases.

4.3.1 Lemma

For $\alpha > -1$, and $n = 0, 1, \cdots$,

$$\int_0^1 r^n (1-r)^\alpha dr = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} = n!/(\alpha+1)(\alpha+2)\cdots(\alpha+n+1).$$

Proof The proof is by induction on n. The result is trivial for $n = 0$. Assume that it holds for $n = k$ and all $\alpha > -1$. Integrating by parts

$$\int_0^1 r^{k+1}(1-r)^\alpha dr = (k+1)(\alpha+1)^{-1} \int_0^1 r^k (1-r)^{\alpha+1} dr.$$

Now we apply the induction hypothesis on the right side to see that the result holds for $n = k+1$ and hence for all $n = 0, 1, \cdots$. $\square$

4.3.2 Lemma

For $\delta > -1$ and $m > 1 + \delta$ we have

$$\int_0^1 (1-\rho r)^{-m}(1-r)^\delta dr \le C(1-\rho)^{1+\delta-m}, \ 0 < \rho < 1.$$

Proof Integrating by parts,

$$\begin{aligned}
\int_0^1 (1-\rho r)^{-m}(1-r)^\delta dr &= \frac{1}{1+\delta} + \frac{m\rho}{1+\delta}\int_0^1 (1-\rho r)^{-m-1}(1-r)^{\delta+1} dr \\
&\le \frac{1}{1+\delta} + \frac{m\rho}{1+\delta}\int_0^1 (1-\rho r)^{-m+\delta} dr \\
&= \frac{1}{\delta+1-m} - \frac{m}{(\delta+1)(\delta+1-m)}(1-\rho)^{\delta+1-m} \\
&= \frac{1}{1+\delta}(1-\rho)^{\delta+1-m}
\end{aligned}$$

where the first inequality is from the fact that $(1-r)^{\delta+1} \leq (1-\rho r)^{\delta+1}$. $\square$

4.3.3 Definition of some spaces of holomorphic functions

Let ϕ, ψ be positive and continuous functions on $[0,1)$ with

$$\lim_{r\to 1} \phi(r) = 0 \quad \text{and} \quad \int_0^1 \psi(r)dr < \infty.$$

Let $H(\mathbf{B}_n)$ denote the space of holomorphic function on the unit ball in $\mathbf{C^n}$. For $f \in H(\mathbf{B}_n)$, let

$$\|f\|_\phi = \sup_{z\in\mathbf{B}_n} |f(z)|\phi(|z|) = \sup_{0\leq r<1} M_\infty(f,r)\phi(r),$$

$$\|f\|_\psi = \int_{\mathbf{B}_n} |f(z)|\psi(|z|)\, d\nu(z) = 2n\int_0^1 r^{2n-1}M_1(f,r)\psi(r)dr$$

where

$$M_\infty(f,r) = \max_{|z|=r}|f(z)| \text{ and } M_1(f,r) = \int_{\mathbf{S}_n} |f(r\xi)|d\sigma(\xi).$$

We define the following spaces of holomorphic functions.

$$A_\infty(\phi) = \{f \in H(\mathbf{B}_n) : \|f\|_\phi < \infty\},$$

$$A_0(\phi) = \{f \in H(\mathbf{B}_n) : \sup_{0\leq r<1} M_\infty(f,r)\phi(r) = 0,\}$$

$$A^1(\psi) = \{f \in H(\mathbf{B}_n) : \|f\|_\psi < \infty\}.$$

Clearly $A_0(\phi) \subset A_\infty(\phi)$ so we may use the norm $\|f\|_\phi$ on $A_0(\phi)$. These three spaces are all norm linear spaces with the indicated norms.

If $L^1_\psi(\mathbf{B}_n) = L^1(\psi)$ denotes the Banach space of measurable functions f such that $\|f\|_\psi = \int_{\mathbf{B}_n} |f|\, d\nu_\psi < \infty$, where $d\nu_\psi(z) = \psi(|z|)\, d\nu(z)$ then $A^1(\psi)$ is the closed subspace of $L^1(\psi)$ consisting of all analytic functions.

4.3.4 Lemma

Let A denote any of the above three normed spaces.

(i) If D is a bounded subset of A, then the functions in D are uniformly bounded on each compact subset of $\mathbf{B}_n$.

(ii) If f_n is a Cauchy in A, then it converges uniformly on each compact subset of $\mathbf{B}_n$.

(iii) Point evaluation at any point of $\mathbf{B}_n$ is a bounded linear functional on A.

(iv) A is a Banach space.

(v) $A_0(\phi)$ is a closed subspace of $A_\infty(\phi)$.

Proof. (i) This is obvious for $A_0(\phi)$ and $A_\infty(\phi)$. Suppose D is a bounded subset of $A^1(\psi)$, and $f \in D$. For $|z| \leq R < 1$, the Cauchy integral formulae gives

$$f(z) = \int_{\mathbf{S}_n} f(\rho\xi)(1 - \rho^{-1}(z \cdot \xi))^{-n}\, d\sigma(\xi), \tag{4.22}$$

where $\rho = (1+R)/2$. Indeed, if $f \in H(\mathbf{B}_n)$ then the function $g : \xi \mapsto f(\frac{1+R}{2}\xi)$ is in the Ball algebra (that is g is analytic on $\mathbf{B}_n$ and continuous up to the boundary $\mathbf{S}_n$). This implies by the Cauchy integral formulae that

$$g(\xi) = \int_{\mathbf{S}_n} \frac{g(w)d\sigma(w)}{(1 - (\xi \cdot w))^n}.$$

So if $\rho = (1+R)/2$ we have that

$$f(\rho\xi) = \int_{\mathbf{S}_n} \frac{f(\rho w)d\sigma(w)}{(1 - (\xi \cdot w))^n}.$$

If $z = \rho\xi \in B(0;\rho)$ then

$$f(z) = \int_{\mathbf{S}_n} \frac{f(\rho w)d\sigma(w)}{(1 - \rho^{-1}(z \cdot w))^n},$$

which gives (4.22).

Hence

$$|f(z)| \le ((1+R)/(1-R))^n M_1(f,\rho).$$

Also,

$$M_1(f,\rho)\int_\rho^1 \psi(r)dr \le \int_\rho^1 M_1(f,r)\psi(r)dr \le \|f\|_\psi,$$

since $M_1(f,r)$ is an increasing function of r by Corollary 4.21 of [47]. The gives the result.

(ii) and (iii) follow from the above estimate for $f(z)$.

(iv) It is only necessary to establish completeness, and this is easy for $A_\infty(\phi)$ and $A_0(\phi)$. If f_n is a Cauchy sequence in $A^1(\psi)$, then it converges uniformly on compact sets to a holomorphic function f, by (ii). Also $f \in A^1(\psi)$ by Fatou's lemma. Thus $A^1(\psi)$ is complete.

(v) This follows from (iv). □

Define $f_r(z) = f(rz)$, $0 \le r < 1$, for $f \in H(\mathbf{B}_n)$.

4.3.5 Lemma

(i) For $f \in A^1(\psi)$ or $A_0(\phi)$, $f_\rho \to f$ in norm as $\rho \to 1$.

(ii) The polynomials are dense in $A^1(\psi)$ and $A_0(\phi)$.

Proof This is obvious for $A_0(\phi)$. For $f \in A^1(\psi)$ and $\epsilon > 0$ choose $R < 1$ so that

$$\int_{|z|>R} |f(z)|\psi(|z|)\, d\nu(z) = 2n\int_R^1 r^{2n-1}M_1(f,r)dr < \epsilon.$$

Since $M_1(f_\rho, r) = M_1(f, r\rho) \le M_1(f, r)$, we have

$$\int_{|z|>R} |f_\rho(z)|\psi(|z|)\, d\nu(z) < \epsilon.$$

Choose ρ so that $|f(z) - f_\rho(z)| < \epsilon$ on $|z| \le R$. Then

$$\int_{\mathbf{B}_n} |f(z) - f_\rho(z)|\psi(|z|)\, d\nu(z) \le \epsilon \int_{|z|\le R} \psi(|z|)\, d\nu(z) + 2\epsilon \le \epsilon(\|\psi\|_1 + 2).$$

(ii) In either $A_0(\phi)$ or $A^1(\psi)$, if $\epsilon > 0$ is given, choose ρ so that $\|f - f_\rho\| < \epsilon$, which is possible by (i). Since the power series of f converges uniformly to f on every compact subset of $\mathbf{B}_n$, we may choose a polynomial P so that $|f_\rho(z) - P(z)| < \epsilon$ on all of $\mathbf{B}_n$. The result follows from $\|f - P\| \le \|f - f_\rho\| + \|f_\rho - P\|$.

4.3.6 Definition

The positive continuous function ϕ will be called normal if there exist $0 < a < b$ and $r_0 < 1$ such that

$$\frac{\phi(r)}{(1-r^2)^a} \searrow 0 \text{ and } \frac{\phi(r)}{(1-r^2)^b} \nearrow \infty \quad (r_0 \le r \to 1^-). \tag{4.23}$$

4.3.7 Definition

The functions $\{\phi, \psi\}$ is called a normal pair if ϕ is normal, ψ is integrable on $(0,1)$, and if for some b satisfying (4.23), there exists $\alpha > b - 1$ such that

$$\phi(r)\psi(r) = (1-r^2)^\alpha, \quad 0 \le r < 1. \tag{4.24}$$

The following is Lemma 7 of [40].

4.3.8 Lemma

If ϕ is normal then there exist ψ such that $\{\phi, \psi\}$ is a normal pair.

Proof Choose k such that $\epsilon(r) = (1-r)^k/\phi(r) \to 0$ as $r \to 1^-$. Then

$$\psi(r) = (1-r^2)^\alpha/\phi(r) = \epsilon(r)(1-r^2)^\alpha/(1-r)^k$$

which is integrable if we choose $\alpha > k-1$. □

The following lemma, (Lemma 8 of [40]) is basic for everything that follows in this section.

4.3.9 Lemma

If $\{\phi, \psi\}$ is a normal pair and if $m \geq \alpha + 1$ then

$$I = \int_0^1 (1-\rho r)^{-m}\psi(r)dr \leq C\frac{1}{\phi(\rho)}(1-\rho)^{1+\alpha-m}, \quad 0 \leq \rho < 1.$$

Proof With the notations in (4.23) and (4.24) we have

$$\int_0^1 (1-\rho r)^{-m}\psi(r)dr = \int_0^{r_0} (1-\rho r)^{-m}\psi(r)dr + \int_{r_0}^1 (1-\rho r)^{-m}\psi(r)dr = I_1 + I_2.$$

Assume that $\rho > r_0$. Then

$$\begin{aligned}
I_2 &= \int_{r_0}^1 (1-\rho r)^{-m}\frac{1}{\phi(r)}(1-r)^\alpha dr \\
&= \int_{r_0}^\rho (1-\rho r)^{-m}\frac{1}{\phi(r)}(1-r)^\alpha dr + \int_\rho^1 (1-\rho r)^{-m}\frac{1}{\phi(r)}(1-r)^\alpha dr \\
&\leq C\int_{r_0}^\rho (1-\rho r)^{-m}\frac{(1-r)^a}{\phi(r)}(1-r)^{\alpha-a}dr + C\int_\rho^1 (1-\rho r)^{-m}\frac{(1-r)^b}{\phi(r)}(1-r)^{\alpha-b}dr \\
&\leq C\frac{(1-\rho)^a}{\phi(\rho)}\int_{r_0}^\rho (1-\rho r)^{-m}(1-r)^{\alpha-a}dr + C\frac{(1-\rho)^b}{\phi(\rho)}\int_\rho^1 (1-\rho r)^{-m}(1-r)^{\alpha-b}dr
\end{aligned}$$

and the result follows from Lemma 4.3.2, since $\alpha - a > \alpha - b > -1$, and $m > 1 + \alpha - a$. Similarly,

$$I_1 = \int_0^{r_0} (1-\rho r)^{-m}\frac{(1-r)^\alpha}{\phi(r)}dr \leq C\frac{(1-\rho)^a}{\phi(\rho)}\int_0^{r_0} (1-\rho r)^{-m}(1-r)^{\alpha-a}dr$$

and the required estimate follows as above.

Finally, if $\rho \leq r_0$ a similar argument gives the required result. $\square$

From now to the end of this section $\{\phi, \psi\}$ will be a normal pair as defined above. We shall use the following pairing between $A_\infty(\phi)$ and $A^1(\psi)$.

$$[f, g] = \int_{\mathbf{B}_n} f(z)g(\overline{z})\phi(|z|)\psi(|z|)\, d\nu(z), \quad f \in A_\infty(\phi),\ g \in A^1(\psi). \tag{4.25}$$

Note that $[f, g]$ is unchanged if $f(z)g(\overline{z})$ is replaced by $f(\overline{z})g(z)$.

4.3.10 Lemma

Let $f = \sum_\beta a_\beta z^\beta$ and $g = \sum_\beta b_\beta z^\beta$ be polynomials, where the summation is over all multi-indices $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ of non-negative integers. Then

$$[f, g] = \sum_\beta a_\beta b_\beta \frac{n!\beta!\Gamma(n + |\beta|)\Gamma(\alpha + 1)}{\Gamma(n + |\beta| + \alpha + 1)}.$$

Proof

$$\begin{aligned}
[f, g] &= \sum_{\beta,\theta} a_\theta b_\beta \int_{\mathbf{B}_n} z^\theta \overline{z}^\beta (1 - |z|^2)^\alpha \, d\nu(z) \\
&= \sum_\beta a_\beta b_\beta \int_{\mathbf{B}_n} |z^\beta|^2 (1 - |z|^2)^\alpha \, d\nu(z) \\
&= \sum_\beta a_\beta b_\beta 2n \int_0^1 r^{2n+2|\beta|-1} \int_{\mathbf{S}_n} |\xi^\beta|^2 (1 - r^2)^\alpha \, d\sigma(\xi)\, dr
\end{aligned}$$

By Lemma 1.11 of [47],

$$\int_{\mathbf{S}_n} |\xi^\beta|^2 \, d\sigma(\xi) = \frac{(n-1)!\beta!}{(n - 1 + |\beta|)!}. \tag{4.26}$$

Using (4.26) and Lemma 4.3.1 we have

$$\begin{aligned}
[f,g] &= \sum_\beta a_\beta b_\beta \frac{(n-1)!\beta!}{(n-1+|\beta|)!} 2n \int_0^1 r^{2n+2|\beta|-1}(1-r^2)^\alpha \, dr \\
&= \sum_\beta a_\beta b_\beta \frac{(n-1)!\beta!}{(n-1+|\beta|)!} n \int_0^1 r^{n+|\beta|-1}(1-r)^\alpha \, dr \\
&= \sum_\beta a_\beta b_\beta \frac{n!\beta!}{(n-1+|\beta|)!} \frac{\Gamma(n+|\beta|)\Gamma(\alpha+1)}{\Gamma(n+|\beta|+\alpha+1)} \\
&= \sum_\beta a_\beta b_\beta \frac{n!\beta!\Gamma(n+|\beta|)\Gamma(\alpha+1)}{\Gamma(n+|\beta|+\alpha+1)}. \qquad \square
\end{aligned}$$

Using Lemmas 4.3.5 and 4.3.10 we obtain the following.

4.3.11 Lemma

For $f = \sum_\beta a_\beta z^\beta \in A_\infty(\phi)$ and $g = \sum_\beta b_\beta z^\beta \in A^1(\psi)$, we have

(i) $[f_\rho, g] = [f, g_\rho] = \sum a_\beta b_\beta \rho^{|\beta|} \dfrac{n!\beta!\Gamma(n+|\beta|)\Gamma(\alpha+1)}{\Gamma(n+|\beta|+\alpha+1)}$,

(ii) $[f,g] = \lim_{\rho\to 1} [f_\rho, g]$.

4.3.12 Lemma

For $\alpha > -1$ we let

$$J_w(z) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)} \frac{1}{(1-(w\cdot\overline{z}))^{n+1+\alpha}}, \quad w, z \in \mathbf{B}_n.$$

Then

(i) J_w is both in $A_0(\phi)$ and $A^1(\psi)$;

(ii) $g(w) = [J_w, g]$, for all $g \in A^1(\psi)$;

(iii) $f(w) = [f, J_w]$, for all $f \in A_\infty(\phi)$.

Proof (i) J_w is holomorphic for $|z| < |w|^{-1}$ and so is both in A_0 and A^1.

(ii) Let $b_{n,\alpha} = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)}$ and $a_\beta = \frac{\Gamma(n+1+\alpha+|\beta|)}{\Gamma(n+1+\alpha)\beta!}$. Then

$$\frac{1}{(1-(w\cdot\overline{z}))^{n+1+\alpha}} = \sum_{k=0}^{\infty}\sum_{|\beta|=k} a_\beta w^\beta z^\beta. \tag{4.27}$$

Also, for any multi-index θ of non-negative integers we have

$$\begin{aligned}
[J_w, z^\theta] &= b_{n,\alpha}\sum_{k=0}^{\infty}\sum_{|\beta|=k} a_\beta w^\beta [z^\beta, z^\theta] \\
&= b_{n,\alpha}\sum_{k=0}^{\infty}\sum_{|\beta|=k} a_\beta w^\beta \int_{\mathbf{B}_n} z^\beta \overline{z}^\theta (1-|z|^2)^\alpha \, d\nu(z) \\
&= b_{n,\alpha} a_\theta w^\theta \int_{\mathbf{B}_n} |z^\theta|^2 (1-|z|^2)^\alpha \, d\nu(z) \\
&= b_{n,\alpha} a_\theta w^\theta 2n \int_0^1 r^{2n+2|\theta|-1}(1-r^2)^\alpha \, dr \int_{\mathbf{S}_n} |\xi^\theta|^2 \, d\sigma(\xi) \\
&= b_{n,\alpha} a_\theta w^\theta n \int_0^1 r^{n+|\theta|-1}(1-r)^\alpha dr \int_{\mathbf{S}_n} |\xi^\theta|^2 \, d\sigma(\xi).
\end{aligned}$$

Using (4.26) and Lemma 4.3.1 we see that

$$[J_w, z^\theta] = w^\theta,$$

for all multi-indices θ. The result follows for all polynomials. The general case follows from Lemmas 4.3.4(iii) and 4.3.5(iii) (if two bounded linear functional agree on a dense set then they agree identically) which proves (ii).

(iii) Let $f \in A_\infty(\phi)$. It is easily known that $\int_{\mathbf{B}_n} |f(z)|(1-|z|^2)^\alpha \, d\nu(z) < \infty$. This shows that f is in the weighted Bergman space $L^1_a(\mathbf{B}_n, d(\alpha)(1-|z|^2)^\alpha \, d\nu)$. The result follows by Theorem 2.2 of [47]. □

Let $C_0(\mathbf{B}_n)$ denote the Banach space of continuous functions on the closed ball that vanish on the boundary, with supremum norm. Also let $L^1(\mathbf{B}_n)$ and $L^\infty(\mathbf{B}_n)$

denote, respectively, the usual Banach spaces of integrable and essentially bounded measurable functions associated with Lebesgue measure on $\mathbf{B}_n$. The maps

$$T_0 : A_0(\phi) \to C_0(\mathbf{B}_n), \quad T_\infty : A_\infty(\phi) \to L^\infty(\mathbf{B}_n), \quad T_1 : A^1(\psi) \to L^1(\mathbf{B}_n) \tag{4.28}$$

defined by $T_0 f = \phi f$, $T_\infty f = \phi f$, $T_1 g = \psi g$ are isometries. We use the following notation for the ranges of these operators.

Notation. $TA_0 = T_0 A_0(\phi)$, $TA_\infty = T_\infty A_\infty(\phi)$, $TA^1 = T_1 A^1(\psi)$.

Thus TA_0 is a subspace of $C_0(\mathbf{B}_n)$, TA^1 is a subspace of $L^1(\mathbf{B}_n)$, and TA_∞ is a subspace of $L^\infty(\mathbf{B}_n)$. These subspaces are closed by Lemma 4.3.4.

Let $M(\mathbf{B}_n)$ denote the Banach space of complex valued, bounded Borel measures on $\mathbf{B}_n$ with the variation norm. The map

$$M : A^1(\psi) \to M(\mathbf{B}_n) \tag{4.29}$$

defined by $Mg = g\psi\, d\nu$ is an isometry of $A^1(\psi)$ unto a closed subspace of $M(\mathbf{B}_n)$, which we denote by MA^1.

We shall need the following:

4.3.13 Lemma

$$\int_{\mathbf{B}_n} |J_w(\overline{z})| \psi(|z|)\, d\nu(z) \leq \frac{C}{\phi(|w|).} \tag{4.30}$$

Proof

$$\int_{\mathbf{B}_n} |J_w(\overline{z})| \psi(|z|)\, d\nu(z) = \int_0^1 \psi(r) r^{2n-1} M_1(J_w, r) dr.$$

By Lemma 3.2.2

$$\int_{\mathbf{S}_n} \frac{1}{(1-(rw\cdot\xi))^{n+1+\alpha}}\, d\sigma(\xi) \sim \frac{1}{(1-r^2|w|^2)^{\alpha+1}} \quad \alpha > -1.$$

Thus,

$$M_1(J_w, r) \sim C\frac{1}{(1 - r^2|w|^2)^{\alpha+1}}.$$

From this we have

$$\int_{\mathbf{B}_n} |J_w(\overline{z})|\psi(|z|)\, d\nu(z) \leq C2n \int_0^1 \psi(r)\frac{1}{(1 - r^2|w|^2)^{\alpha+1}}dr.$$

Applying Lemma 4.3.9 we obtain

$$\int_{\mathbf{B}_n} |J_w(\overline{z})|\psi(|z|)\, d\nu(z) \leq C'\frac{1}{\phi(|w|)}. \qquad \square$$

4.3.14 Theorem

(i) The transformation P defined by

$$(Ph)(w) = \int_{\mathbf{B}_n} J_w(\overline{z})h(z)\psi(|z|)\, d\nu(z), \quad h \in L^\infty(\mathbf{B}_n),\ w \in \mathbf{B}_n, \tag{4.31}$$

is a bounded operator mapping $L^\infty(\mathbf{B}_n)$ onto $A_\infty(\phi)$. The operator $T_\infty P$ is a bounded projection of $L^\infty(\mathbf{B}_n)$ onto the subspace TA_∞.

(ii) The operator $P_0 = P|_{C_0(\mathbf{B}_n)}$ is a bounded operator mapping $C_0(\mathbf{B}_n)$ onto $A_0(\phi)$; the operator T_0P_0 is a bounded projection of $C_0(\mathbf{B}_n)$ onto the subspace TA_0.

(iii) The transformation defined by

$$(Q\mu)(w) = \int_{\mathbf{B}_n} J_w(\overline{z})\phi(|z|)\, d\mu(z), \quad \mu \in M(\mathbf{B}_n),\ w \in \mathbf{B}_n, \tag{4.32}$$

is a bounded operator mapping $M(\mathbf{B}_n)$ unto $A^1(\psi)$; the operator MQ is a bounded projection of $M(\mathbf{B}_n)$ onto the subspace MA^1.

(iv) The transformation $Q_1 = Q|_{L^1(\mathbf{B}_n)}$ is a bounded operator mapping $L^1(\mathbf{B}_n)$ unto $A^1(\psi)$; the operator T_1Q_1 is a bounded projection of $L^1(\mathbf{B}_n)$ unto the subspace TA^1.

Proof (i) For $w \in \mathbf{B}_n$ and $h \in L^\infty(\mathbf{B}_n)$ we have

$$\begin{aligned} |(Ph)(w)| &\leq \int_{B_n} |J_w(\overline{z})||h(z)|\psi(|z|)\,d\nu(z) \\ &\leq \|h\|_\infty \int_{\mathbf{B}_n} |J_w(\overline{z})|\psi(|z|)\,d\nu(z). \end{aligned}$$

Lemma 4.3.13 shows that

$$\sup_{w\in\mathbf{B}_n} |(Ph)(w)|\phi(|w|) < \infty,$$

that is, P is a bounded operator mapping $L^\infty(\mathbf{B}_n)$ into $A_\infty(\phi)$. Now let $f \in A_\infty(\phi)$ be given. Then from Lemma 4.3.12(iii),

$$\begin{aligned} (P(T_\infty f))(w) &= \int_{\mathbf{B}_n} (T_\infty f)(z) J_w(\overline{z})\psi(|z|)\,d\nu(z) \\ &= \int_{\mathbf{B}_n} f(z)J_w(\overline{z})\phi(|z|)\psi(|z|)\,d\nu(z) \\ &= [f, J_w] = f(w). \end{aligned}$$

Thus $PT_\infty = I$, the identity on $A_\infty(\phi)$, and so P is onto and $T_\infty P$ is a bounded projection of L^∞ onto the subspace TA_∞. This proves (i).

(ii) This follows from (i) if it can be shown that $h \in C_0(\mathbf{B}_n)$ implies $P_0 h \in A_0(\phi)$. Given $\epsilon > 0$, choose $R \in (0,1)$, such that $|h(z)| < \epsilon$ for $|z| > R$. Then

$$|(Ph)(w)| \leq \left(\int_{|z|\leq R} + \int_{|z|>R}\right) |h(z)||J_w(\overline{z})\psi(|z|)\,d\nu(z) = I_1 + I_2.$$

From Lemma 4.3.13, $I_2 \leq C\epsilon/\phi(|w|)$. Also, $I_1 \leq c_R\|h\|_\infty$ where c_R is some constant depending on R. Hence

$$|(Ph)(w)|\phi(|w|) \leq c_R\|h\|_\infty\phi(|w|) + c\epsilon, w \in \mathbf{B}_n.$$

Thus, by the definition of ϕ,

$$\lim_{|w|\to 1^-} \sup |(Ph)(w)|\phi(|w|) \leq c\epsilon,$$

that is, $P_0 h \in A_0(\phi)$.

(iii) For $\mu \in M(\mathbf{B}_n)$,

$$\int_{\mathbf{B}_n} |(Q\mu)(w)|\psi(|w|)\, d\nu(w) \le \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} |J_w(\overline{z})\psi(|w|)\phi(|z|)\, d\nu(w)\, d|\mu|(z).$$

In a similar manner to that of Lemma 4.3.13, we have

$$\int_{\mathbf{B}_n} |J_w(\overline{z})|\psi(|w|)\, d\nu(w) \le \frac{C}{\phi(|z|)}. \tag{4.33}$$

Thus Q is a bounded operator from $M(\mathbf{B}_n)$ into $A^1(\psi)$. Now let $g \in A^1(\psi)$. From Lemma 4.3.12(ii) we have $(Q(T_1 g))(w) = (Q(\psi g)) = [J_w, g] = g(w)$. Thus $QT_1 = I$, the identity on $A^1(\psi)$, and so Q is onto and $T_1 Q$ is a projection.

(iv) This follows from (iii). $\square$

Let $\langle J_w \rangle$ denote the vector space spanned by the functions J_w, $w \in \mathbf{B}_n$.

4.3.15 Lemma

$\langle J_w \rangle$ is dense in $A^1(\psi)$ and in $A_0(\phi)$

Proof Consider first $A^1(\psi)$. It is equivalent to showing that $\langle T_1 J_w \rangle$ is dense in TA^1. By the Hahn Banach theorem and the Riesz representation theorem, it suffices to show that if $h \in L^\infty$ and if

$$\int_{\mathbf{B}_n} J_w(z)\psi(|z|)h(\overline{z})\, d\nu(z) = 0 \tag{4.34}$$

for all $w \in \mathbf{B}_n$, then h annihilates all of TA^1. Using (4.27), equation (4.34) shows that

$$0 = b_{n,\alpha} \sum_{k=0}^{\infty} \sum_{|\beta|=k} a_\beta w^\beta \int_{\mathbf{B}_n} z^\beta \psi(|z|)h(\overline{z})\, d\nu(z)$$

for all $w \in \mathbf{B}_n$. Thus h annihilates polynomials and the result follow from Lemma 4.3.5 (ii).

The proof of $A_0(\phi)$ is similar, by using the duality between $C_0(\mathbf{B}_n)$ and $M(\mathbf{B}_n)$ given by the pairing

$$(f, \mu) = \int_{\mathbf{B}_n} f(\overline{z}) d\mu(z), \ f \in C_0(\mathbf{B}_n), \ \mu \in M(\mathbf{B}_n). \tag{4.35}$$

We now come to the main result of this section.

4.3.16 Theorem

Using the pairing in (4.25), we have

(i) $A_0(\phi)^* \cong A^1(\psi)$,

(ii) $A^1(\psi)^* \cong A_\infty(\phi)$.

More precisely, if $g \in A^1(\psi)$ and if we define $\lambda_g = [f, g]$, $f \in A_0(\phi)$ then $\lambda_g \in A_0(\phi)^*$ and $\|\lambda_g\| \leq \|g\|_\psi$. Conversely, given $\lambda \in A_0(\phi)^*$ then there is a unique $g \in A^1(\psi)$ such that $\lambda = \lambda_g$. Also, $\|g\|_\psi \leq \|Q\| \|\lambda\|$.

Furthermore, if $f \in A_\infty(\phi)$ and if we define $\lambda_f(g) = [f, g]$, $g \in A^1(\psi)$, then $\lambda_f \in A^1(\psi)^*$ and $\|\lambda_f\| \leq \|f\|_\phi$. Conversely, given $\lambda \in A^1(\psi)^*$ then there is a unique $f \in A_\infty(\phi)$ such that $\lambda = \lambda_f$. Also, $\|f\|_\phi \leq \|P\| \|\lambda\|$.

Proof It is trivial that if $g \in A^1(\psi)$ then $\lambda_g \in A_0(\phi)^*$. We also have uniqueness: if $\lambda_g(f) = 0$ for all $f \in A_0(\phi)$, then $g = 0$. Indeed, from Lemma 4.3.12(ii) $g(w) = \lambda_g(J_w)$.

Now let $\lambda \in A_0(\phi)^*$ be given. Since T_0 is an isometric embedding of $A_0(\phi)$ into $C_0(\mathbf{B}_n)$, there exists $\mu \in M(\mathbf{B}_n)$ with $\|\mu\| = \|\lambda\|$ and, by (4.35), $\lambda(f) = (T_0 f, \mu) =$

$(\phi f, \mu)$ for all $f \in A_0(\phi)$ by the Riesz representation theorem. Let $g(w) = \lambda(J_w)$. Then

$$g(w) = \lambda(J_w) = \int_{\mathbf{B}_n} J_w(\overline{z})\phi(|z|)\, d\mu(z) = (Q\mu)(w).$$

By Theorem 4.3.14, $g \in A^1(\psi)$ and $\|g\|_\psi \leq \|Q\|\|\mu\| = \|Q\|\|\lambda\|$. From Lemma 4.3.12(ii) we see that $\lambda_g(J_w) = g(w)$ for $w \in \mathbf{B}_n$. Hence $\lambda = \lambda_g$ on $\langle J_w \rangle$ and hence also on $A_0(\phi)$ by Lemma 4.3.15.

(ii) The proof of the first part and the proof of the uniqueness of f are the same as in the proof of (i).

Now let $\lambda \in A^1(\psi)^*$. T_1 is an isometric embedding of $A^1(\psi)$ into $L^1(\mathbf{B}_n)$. There exists $h \in L^\infty(\mathbf{B}_n)$ with $\|h\| = \|\lambda\|$ and

$$\lambda(g) = \int_{\mathbf{B}_n} g(\overline{z})h(z)\psi(|z|)\, d\nu(z)$$

for all $g \in A^1(\psi)$, by the Riesz representation theorem. Let $f(w) = \lambda(J_w)$. Then

$$f(w) = \int_{\mathbf{B}_n} J_w(\overline{z})h(z)\psi(|z|)\, d\nu(z) = (Ph)(w).$$

By Theorem 4.3.14(i), $f \in A_\infty$ and $\|f\|_\phi \leq \|P\|\|h\| = \|P\|\|\lambda\|$. From Lemma 4.3.12(iii) we see that $\lambda_f(J_w) = f(w)$ for $w \in \mathbf{B}_n$. Hence, $\lambda = \lambda_f$ on $\langle J_w \rangle$, and hence also on $A^1(\psi)$ by Lemma 4.3.15. $\square$

4.3.17 Remark

We observe that for $z, w \in \mathbf{B}_n$, we have

$$\overline{K_w(z)} = \overline{K_w^\alpha(z)} = c(\alpha)\frac{1}{(1-(w\cdot z))^{n+1+\alpha}} = J_w(\overline{z})$$

where $c(\alpha) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)}$. Yu T [46] noted that the duality Theorem 4.3.16 holds with the pairing between $A^1(\psi)$ and $A_\infty(\phi)$ given by

$$[f, g] = \int_{\mathbf{B}_n} f(z)\overline{g(z)}\phi(|z|)\psi(|z|)\, d\nu(z). \tag{4.36}$$

And also that if $f \in A_\infty(\phi)$ then using the pairing in (4.36) we have

$$\begin{aligned}[f, K_w] &= \int_{\mathbf{B}_n} f(z)\overline{K_w(z)}\phi(|z|)\psi(|z|)\, d\nu(z) \\ &= \int_{\mathbf{B}_n} f(z)J_w(\overline{z})\phi(|z|)\psi(|z|)\, d\nu(z) \\ &= [f, J_w] = f(w)\end{aligned}$$

where the last equality comes from Lemma 4.3.12(iii). Thus K_w also reproduces functions in $A_\infty(\phi)$. We normalize the kernel K_w by

$$k_w^\psi(z) = \frac{K_w(z)}{\|K_w\|_\psi}.$$

From now henceforth we will be using duality paring given in (4.36).

4.4 Compactness on the Weighted Bergman space $L_a^1(\psi)$

We now extend the compactness given in [46] for $n = 1$ to the case $n \geq 2$. Define an operator Q' on $L^1(\psi)$ by

$$(Q'f)(w) = [f, K_w] = \int_{\mathbf{B}_n} f(z)\overline{K_w(z)}\phi(|z|)\psi(|z|)\, d\nu(z).$$

Then Q' is a bounded operator from $L^1(\psi)$ onto $A^1(\psi)$ and by Lemma 4.3.12, K_w is a reproducing kernel of $A^1(\psi)$. We have the following estimates for K_w:

4.4.1 Lemma

There exist constants c and C such that

$$c/\phi(|w|) \leq \|K_w\|_\psi \leq C/\phi(|w|), \quad w \in \mathbf{B}_n.$$

Proof If $|w| \leq r_0 < 1$ the first inequality obviously holds. If $|w| > r_0$, using the equations (4.23) we have

$$\frac{\phi(|z|)}{(1-|z|^2)^a} \leq \frac{\phi(|w|)}{(1-|w|^2)^a},\ 1 > |z| > |w|$$

and

$$\frac{\phi(|z|)}{(1-|z|^2)^b} \leq \frac{\phi(|w|)}{(1-|w|^2)^b},\ r_0 \leq |z| < |w|.$$

Thus (4.24) gives

$$\frac{(1-|z|^2)^{\alpha-a}}{\psi(|z|)} \leq \frac{(1-|w|^2)^{\alpha-a}}{\psi(|w|)},\ 1 > |z| > |w|,$$

$$\frac{(1-|z|^2)^{\alpha-b}}{\psi(|z|)} \leq \frac{(1-|w|^2)^{\alpha-b}}{\psi(|w|)},\ r_0 \leq |z| < |w|.$$

Hence,

$$\begin{aligned}
\|K_w\|_\psi &= c(\alpha)\int_{B_n} \frac{1}{|1-(z\cdot w)|^{n+1+\alpha}}\psi(|z|)\,d\nu(z) \\
&\geq \frac{c(\alpha)\psi(|w|)}{(1-|w|^2)^{\alpha-b}}\int_{r_0\leq|z|\leq|w|} \frac{(1-|z|^2)^{\alpha-b}}{|1-(z\cdot w)|^{n+1+\alpha}}\,d\nu(z) \\
&+ \frac{c(\alpha)\psi(|w|)}{(1-|w|^2)^{\alpha-b}}\int_{|w|<|z|<1} \frac{(1-|z|^2)^{\alpha-a}}{|1-(z\cdot w)|^{n+1+\alpha}(1-|w|^2)^{b-a}}\,d\nu(z).
\end{aligned}$$

Also, since $(1-|z|^2)(1-|w|^2) \leq |1-(z\cdot w)|^2$ we have

$$\frac{1}{(1-|w|^2)^{b-a}} \geq \frac{(1-|z|^2)^{b-a}}{|1-(z\cdot w)|^{2(b-a)}}.$$

So there exist a positive constant c' such that

$$\begin{aligned}
\|K_w\|_\psi &\geq \frac{c'\psi(|w|)}{(1-|w|^2)^{\alpha-b}}\int_{r_0\leq|z|} \frac{(1-|z|^2)^{\alpha+b-2a}}{|1-(z\cdot w)|^{n+1+\alpha+2b-2a}}\,d\nu(z) \\
&= \frac{c'\psi(|w|)}{(1-|w|^2)^{\alpha-b}}\left(\int_{\mathbf{B}_n} - \int_{|z|\leq r_0}\right)\frac{(1-|z|^2)^{\alpha+b-2a}}{|1-(z\cdot w)|^{n+1+\alpha+2b-2a}}\,d\nu(z) \\
&=: \frac{c'\psi(|w|)}{(1-|w|^2)^{\alpha-b}}(I_1(w)-I_2(w)).
\end{aligned}$$

Now by Lemma 3.2.2, $I_1(w) \sim (1-|w|^2)^{-b}$ as $|w| \to 1^-$; and I_2 is bounded. Thus it is easy to see that there exist a constant c such that

$$\|K_w\|_\psi \geq c/\phi(|w|),$$

which is the first inequality. Finally,

$$\begin{aligned} \|K_w\|_\psi &= \int_{\mathbf{B}_n} |K_w(z)| d\nu(z) \\ &= \int_{\mathbf{B}_n} |J_w(\overline{z})| d\nu(z) \\ &\leq C/\phi(|w|) \end{aligned}$$

by Lemma 4.3.13, which gives the second inequality. □

4.4.2 Lemma

k_z^ψ converges weakly* to 0 in $A^1(\psi)$ as $z \to \partial \mathbf{B}_n$.

Proof For $g \in A_0(\phi)$, by the reproducing property of K_w, we have

$$[k_w^\psi, g] = \frac{[K_w, g]}{\|K_w\|_\psi} = \frac{\overline{g(w)}}{\|K_w\|_\psi}.$$

Lemma 4.4.1 implies $|[k_w^\psi, g]| \leq C|g(w)|\phi(|w|)$ and hence from the definition of $A_0(\phi)$ we have that $[k_w^\psi, g] \to 0$ as $w \to \partial B_n$. □

We give the main result of this section.

4.4.3 Theorem

Suppose that A is a bounded operator on $A^1(\psi)$ and let $A^{\star\star}$ denote the adjoint of A. Then A is compact and $A_0(\phi)$ is an invariant subspace of $A^{\star\star}$ if and only if $\|Ak_w^\psi\|_\psi \to 0$ as $w \to \partial \mathbf{B}_n$.

Proof Suppose A is compact and $A_0(\phi)$ is an invariant subspace of $A^{\star\star}$. If $\|Ak_w^\psi\|_\psi \to$

$a \neq 0$ as $w \to \partial\mathbf{B}_n$, then there exist a constant $\delta > 0$ and a sequence $\{w_p\}$ in $\mathbf{B}_n$ such that

$$w_p \to \partial\mathbf{B}_n \text{ and } \|Ak^{\psi}_{w_p}\|_{\psi} > \delta. \tag{4.37}$$

Since $\{k^{\psi}_{w_p}\}$ is a bounded sequence in $A^1(\psi)$ and A is compact, there exists a subsequence of $\{k^{\psi}_{w_p}\}$, also denoted by $\{k^{\psi}_{w_p}\}$, such that $\{Ak^{\psi}_{w_p}\}$ converges in $A^1(\psi)$. By Lemma 4.4.2, $w_p \to \partial B_n$ implies $k^{\psi}_{w_p}$ tend weakly* to 0. Since $A_0(\phi)$ is an invariant subspace of $A^{\star\star}$, we have for any $g \in A_0(\phi)$,

$$[Ak^{\psi}_{w_p}, g] = [k^{\psi}_{w_p}, A^{\star\star}g] \to 0.$$

Thus $Ak^{\psi}_{w_p}$ tends weakly* to 0. Since $\{Ak^{\psi}_{w_p}\}$ converges in $A^1(\psi)$, it must converge to its weak*-limit, that is 0. This contradicts (4.37).

Conversely, suppose $\|Ak^{\psi}_{w}\|_{\psi} \to 0$ as $w \to \partial\mathbf{B}_n$. Since K_z is in $A_0(\phi)$ the reproducing property of K_w gives

$$A^{\star\star}K_z(w) = [A^{\star\star}K_z, K_w] = \overline{[AK_w, K_z]} = \overline{AK_w(z)}.$$

So for $f \in A^1(\psi)$,

$$(Af)(w) = [Af, K_w] = [f, A^{\star\star}K_w],$$

thus

$$(Af)(w) = \int_{\mathbf{B}_n} f(z)(AK_z)(w)\psi(|z|)\phi(|z|)\,d\nu(z).$$

For $0 < t < 1$, define a compact supporting continuous function η_t on $\mathbf{B}_n$ by

$$\eta_t(z) = \begin{cases} 1, & |z| \leq t \\ \frac{1+t}{1-t} - \frac{2|z|}{1-t}, & t < |z| \leq (1+t)/2 \\ 0, & (1+t)/2 < |z| < 1. \end{cases}$$

For any $0 < r, t < 1$, define the integral operators $A_{[r]}$ on $A^1(\psi)$ and $A_{[r,t]}$ from $A^1(\psi)$ to $L^1(\psi)$ as follows;

$$A_{[r]}f(w) = \int_{\mathbf{B}_n} f(z)(AK_z)(w)\eta_r(z)\psi(|z|)\phi(|z|)\,d\nu(z),$$
$$A_{[r,t]}f(w) = \int_{\mathbf{B}_n} f(z)(AK_z)(w)\eta_r(z)\eta_t(w)\psi(|z|)\phi(|z|)\,d\nu(z).$$

Using Lemma 4.4.1, we have

$$\begin{aligned}\|(A - A_{[r]})f\|_\psi &\leq \int_{\mathbf{B}_n} |f(z)|\psi(|z|)\int_{\mathbf{B}_n} |(AK_z)(w)|\phi(|z|)(1-\eta_r(z))\psi(|w|)\,d\nu(w)\,d\nu(z)\\ &\leq C\|f\|_\psi \sup_{z\in\mathbf{B}_n}(1-\eta_r(z))\|Ak_z^\psi\|_\psi.\end{aligned}$$

Since $\|Ak_z^\psi\|_\psi \to 0$ as $z \to \partial\mathbf{B}_n$, we have that $\sup_{z\in\mathbf{B}_n}(1-\eta_r(z))\|Ak_z^\psi\|_\psi \to 0$ as $r \to 1^-$. Thus

$$\|A - A_{[r]}\|_\psi \to 0 \quad \text{as} \quad r \to 1^-. \tag{4.38}$$

Seeing $A_{[r]}$ as an operator from $A^1(\psi)$ to $L^1(\psi)$, if we show that it is compact, then it is also compact as an operator on $A^1(\psi)$. Similar to the above, we have

$$\|A_{[r]} - A_{[r,t]}\|_\psi \leq C \sup_{z\in\frac{1+r}{2}\mathbf{B}_n}\int_{\mathbf{B}_n} |(Ak_z^\psi)(w)|(1-\eta_t(w))\psi(|w|)\,d\nu(w) \tag{4.39}$$

We will prove that

$$\sup_{z\in\frac{1+r}{2}\mathbf{B}_n}\int_{\mathbf{B}_n} |(Ak_z^\psi)(w)|(1-\eta_t(w))\psi(|w|)\,d\nu(w) \to 0$$

as $t \to 1^-$ for fixed $r < 1$.

Let

$$g_t(z) = \int_{\mathbf{B}_n} |(Ak_z^\psi)(w)|(1-\eta_t(w))\psi(|w|)\,d\nu(w).$$

Firstly we will show that the set $\{g_t : 0 < t < 1\}$ is equicontinuous and uniformly bounded on $\frac{1+r}{2}\overline{\mathbf{B}_n}$. Let $a_k = \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!}$. Then

$$K_z(w) = c(\alpha)\sum_{k=0}^{\infty} a_k(w\cdot z)^k.$$

Thus for $u, v \in \frac{1+r}{2}\overline{\mathbf{B}_n}$, we have

$$\begin{aligned}
|K_u(w) - K_v(w)| &= \left| c(\alpha) \sum_{k=0}^{\infty} a_k ((w \cdot u)^k - (w \cdot v)^k) \right| \\
&\leq c(\alpha) \sum_{k=0}^{\infty} a_k |(w \cdot u)^k - (w \cdot v)^k| \\
&\leq c(\alpha) \sum_{k=0}^{\infty} a_k |(w \cdot u) - (w \cdot v)| \sum_{s=0}^{k-1} |(w \cdot u)|^k |(w \cdot v)|^{s-k} \\
&\leq c(\alpha) \sum_{k=1}^{\infty} a_k k r^{k-1} |u - v|.
\end{aligned}$$

The last series above is convergent in $|r| < 1$. So for any $\epsilon > 0$, there exists a constant $\delta_1 > 0$ such that $|K_u(w) - K_v(w)| < \epsilon$ for any $u, v \in \frac{1+r}{2}\overline{\mathbf{B}_n}$ with $|u - v| < \delta_1$ and for every $w \in \mathbf{B}_n$, and so the function $z \mapsto \|K_z\|_\psi$ is $\|K_z\|_\psi$ is uniformly continuous on $\frac{1+r}{2}\overline{\mathbf{B}_n}$. Thus for any $\epsilon > 0$ there exist δ_2 such that $|k_u^\psi(w) - k_v^\psi(w)| < \epsilon$ for any $u, v \in \frac{1+r}{2}\overline{\mathbf{B}_n}$ with $|u - v| < \delta_2$, whence

$$\begin{aligned}
|g_t(u) - g_t(v)| &\leq \int_{\mathbf{B}_n} |Ak_u^\psi(w) - Ak_v^\psi(w)|(1 - \eta_t(w))\psi(|w|)\, d\nu(w) \\
&\leq \|A\| \int_{\mathbf{B}_n} |k_u^\psi(w) - k_v^\psi(w)|\psi(|w|)\, d\nu(w) \\
&\leq \epsilon \|A\| \|1\|_\psi.
\end{aligned}$$

Since ϵ is arbitrary, $\{g_t : 0 < t < 1\}$ is equicontinuous. It is obvious that $\{g_t : 0 < t < 1\}$ is uniformly bounded, since A is bounded on $A^1(\psi)$.

For $z \in \frac{1+r}{2}\overline{\mathbf{B}_n}$, Lebesgue's dominated convergence theorem implies that $g_t(z) \to 0$ as $t \to 1^-$. It follows from Ascoli's theorem that $\{g_t : 0 < t < 1\}$ is relatively compact in $C(\overline{\frac{1+r}{2}\mathbf{B}_n})$, the Banach space of continuous functions on $\overline{\frac{1+r}{2}\mathbf{B}_n}$, so has a unique accumulation point 0. Therefore $g_t(z) \to 0$ in $C(\overline{\frac{1+r}{2}\mathbf{B}_n})$ as $t \to 1^-$. So (4.39) implies that

$$\|A_{[r]} - A_{[r,t]}\|_\psi \to 0 \quad \text{as} \quad t \to 1^-. \tag{4.40}$$

We now show that the operator $A_{[r,t]}$ is compact. By Appendix C of [34], it suffices to show that there exist a constant $C > 0$, such that

$$\psi(|z|)\phi(|z|)\int_{\mathbf{B}_n}|Ak_\xi(z)|\eta_t(z)\eta_r(\xi)\,d\nu(\xi) < C$$

and

$$\int_{\mathbf{B}_n}|Ak_\xi(z)|\eta_t(z)\eta_r(\xi)\psi(|z|)\phi(|z|)\,d\nu(z) < C,$$

but this follows easily from the fact that A is bounded and the definition of the function η. Thus (4.40) implies $A_{[r]}$ is compact and (4.38) implies A is compact.$\square$

The following example due to Yu[46] shows that the condition $A_0(\phi)$ is an invariant subspace of A^{**} cannot be ignored in general.

Suppose that $f \in A_\infty(\phi) \setminus A_0(\phi)$. Then by definition of $A_\infty(\phi)$, there exists a sequence $\{z_n\}$ in $\mathbf{B}_n$ and an $\epsilon > 0$ such that

$$|f(z_n)|\phi(|z_n|) \geq \epsilon \text{ whenever } \; z_n \to \partial\mathbf{B}_n.$$

Now if $0 \neq g \in A^1(\psi)$. Then there exists $h \in A_\infty(\phi)$ such that $[g,h] \neq 0$. Let $A = g \otimes f$ be defined by $Au = (g \otimes f)u = [f,u]g$ for $u \in A^1(\psi)$. Then A is a finite rank operator and hence compact. However

$$\begin{aligned}
|[Ak_{z_n},h]| &= |[(g\otimes f)k_{z_n},h]| \\
&= |[[f,k_{z_n}]g,h]| \\
&= |[g,h]||[f,k_{z_n}]| \\
&= |[g,h]|\frac{|f(z_n)|}{\|K_{z_n}\|_\phi} \\
&\geq C|[g,h]||f(z_n)|\phi(|z_n|) \\
&\geq C\epsilon|[g,h]|
\end{aligned}$$

where the first inequality comes from Lemma 4.4.1. This shows that $\|Ak_{z_n}\|_\phi$ does not tend to zero.

We now consider the Toeplitz operator. If we take $\psi = 1$, and $\alpha = n+1$ then by letting

$$\tilde{k}^1_\xi(z) := c(n+1)\frac{(1-|\xi|^2)^{n+1}}{(1-(z\cdot\xi))^{2(n+1)}}$$

and

$$\tilde{K}_\xi(w) := \frac{1}{(1-(z\cdot\xi))^{2(n+1)}}$$

then we have the following (cf Yu[46]).

4.4.4 Theorem

Let $f \in L^1(\mathbf{B}_n)$ and T_f bounded on $L^1_a(\mathbf{B}_n)$. Then T_f is compact and $A_0(\phi)$ is an invariant subspace of $T_f^{\star\star}$ if and only if $\|T_f\tilde{k}^1_\xi\|_1 \to 0$ as $\xi \to \partial\mathbf{B}_n$.

4.5 Compactness on the unweighted space L^1_a

We begin by presenting results on a more general operator A before considering the Toeplitz operator. We state a version of the Ascoli theorem adapted to $L^p(\mathbb{R}^n)$ spaces. We write $(\tau_h f)(x) = f(x+h)$. Let $U \in \mathbb{R}^n$ we write $O \subset\subset U$ to imply the closure of O is contained in U.

4.5.1 Theorem [M. Riesz-Frechet-Kolmogrov]

Let U be an open subset of $\mathbb{R}^n$ and let $O \subset\subset U$. Let $\mathcal{F}$ be a bounded subset of $L^p(U)$, $1 \le p < \infty$. Suppose that

$$\text{for all } \epsilon > 0 \text{ there exist } \delta > 0,\ \delta < \operatorname{dist}(O, U') \text{ such that },$$

$$\|\tau_h f - f\|_{L^p(O)} < \epsilon \text{ for all } h \in \mathbb{R}^n,\ \text{with } |h| < \delta \text{ and all } f \in \mathcal{F}.$$

Then $\mathcal{F}|_{L^p(O)}$ is relatively compact in $L^p(O)$, where U' is the complement of U.

For a proof of this theorem see [15] pages 72-73.

4.5.2 Theorem

Let U be an open subset of $\mathbb{R}^n$ and $\mathcal{F}$ be a bounded subset of $L^p(U)$, $1 \leq p < \infty$. Then $\mathcal{F}$ is relatively compact in $L^p(U)$ if and only if

(1) for all $\epsilon > 0$ and all $O \subset\subset U$ there exist $\delta > 0,\ \delta < \text{dist}(O, U^{'})$ such that

$$\|\tau_h f - f\|_{L^p(O)} < \epsilon \text{ for all } h \in \mathbb{R}^n, \text{ with } |h| < \delta \text{ and all } f \in \mathcal{F}.$$

(2) for all $\epsilon > 0$ there exist $O \subset\subset U$ such that $\|f\|_{L^p(U/O)} < \epsilon$ and all $f \in \mathcal{F}$.

Where U' is the complement of U.

Proof. Suppose (1) and (2). By the Riesz-Frechet-Kolmogrov theorem 4.5.1, $\mathcal{F}|_{L^p(O)}$ is relatively compact in $L^p(O)$. That is there exist a finite number of open balls in $L^p(O)$ with radius r such that

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{k} B(g_i, r), \quad \text{with} \quad g_i \in L^p(O).$$

Let

$$h_i(x) = \begin{cases} g_i(x) & \text{when } x \in O, \\ 0 & \text{when } x \in U/O. \end{cases}$$

Then for each $f \in \overline{\mathcal{F}}$ there exists a $g_i \in L^p(O)$, $i = 1, 2, \cdots, k$, such that

$$\|f - g_i\|_{L^p(O)} < r.$$

Thus for $f \in \overline{\mathcal{F}}$ assertion (2) implies

$$\|f - h_i\|_{L^p(U)} \leq \|f - g_i\|_{L^p(O)} + \|f\|_{L^p(U/O)} < 2r.$$

That is

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{k} B(h_i, 2r), \quad \text{with} \quad h_i \in L^p(U).$$

Conversely suppose $\overline{\mathcal{F}}$ is compact in $L^p(U)$. Let g_i, $i = 1, 2 \cdots, k$ be compactly supported, $C^\infty(U)$, functions be the center of the finite number of balls with radius ϵ in $L^p(U)$ which covers $\overline{\mathcal{F}}$. Let $O \subset\subset U$ and $f \in \overline{\mathcal{F}}$. Then there exist g_i, $i = 1, 2, \cdots, k$ such that $f \in B(g_i, \epsilon/3)$ and

$$\|\tau_h f - f\|_{L^p(O)} \leq \|\tau_h f - \tau_h g_i\|_{L^p(O)} + \|\tau_h g_i - g_i\|_{L^p(O)} + \|g_i - f\|_{L^p(O)}.$$

Choose $\delta > 0$, such that $|h| < \delta$ and $h + x, x \in O$. Then

$$\|\tau_h f - f\|_{L^p(O)} < \epsilon$$

which is (1) with the $g_i \in C_0^\infty(U)$. The result follows by density.

Assertion (2) is well known property of L^p functions.□

We present our first result on compactness of a general bounded operator in L_a^1.

4.5.3 Theorem

Let A be a bounded operator on L_a^1. The following two assertions are equivalent:

1. The operator A is compact on L_a^1;

2. For every $\epsilon > 0$, there exists $R \in (0, 1)$ such that

   $$\int_{R \leq |z| < 1} |(A\tilde{k}_\zeta^c)(z)| d\nu(z) < \epsilon$$

   for every $\zeta \in \mathbf{B}_n$.

Proof Let $\mathcal{F} := \{Ag : g \in L^1_a, ||g||_1 \leq 1\}$. Since A is bounded on L^1_a, the set $\mathcal{F}$ is a bounded subset of L^1_a and hence a bounded subset of $L^1(\mathbf{B}_n, d\nu)$. Moreover, the compactness of $\mathcal{F}$ in L^1_a is equivalent to the compactness of $\mathcal{F}$ in $L^1(\mathbf{B}_n, d\nu)$. According to Theorem 4.5.2, it suffices to show that the following two properties are equivalent:

1. For every $\epsilon > 0$, there exists $R \in (0,1)$ such that
$$\int_{R\leq|z|<1} |(A\tilde{k}^c_\zeta)(z)|d\nu(z) < \epsilon;$$
2. a) For all $\epsilon > 0$ and $R \in (0,1)$, there exists $\delta \in (0, 1-R)$ such that
$$\int_{|z|<R} |\phi(z+h) - \phi(z)|d\nu(z) < \epsilon$$
for all $\phi \in \mathcal{F}$ and all $h \in \mathbb{C}^n$ such that $|h| < \delta$ and

   b) For every $\epsilon > 0$, there exists $R \in (0,1)$ such that $\int_{R\leq|z|<1} |\phi(z)|d\nu(z) < \epsilon$ for every $\phi \in \mathcal{F}$.

The implication 2. $\Rightarrow$ 1. is obtained by taking $g = A\tilde{k}^c_\zeta$ in part b) of assertion 2.

We next prove the implication 1. $\Rightarrow$ 2. We first point out that part a) of assertion 2. is valid for every bounded subset $\mathcal{F}$ of L^1_a. In fact, the closed subset $\omega = \{z \in \mathbf{B}_n : |z| \leq \frac{1+R}{2}\}$ is a compact subset of $\mathbf{B}_n$ and hence on this set, the Bergman distance β on $\mathbf{B}_n$ is equivalent to the Euclidean distance. On the other hand, it is well known (cf [8] that for ϕ analytic on $\mathbf{B}_n$, $\quad \delta \in (0,1)$ and $z, \zeta \in \mathbf{B}_n$ such that $\beta(z,\zeta) < \delta$, the following estimate holds:
$$|\phi(z) - \phi(\zeta)| \leq C\delta \int_{\beta(z,w)<1} |\phi(w)| \frac{d\nu(w)}{(1-|w|^2)^{n+1}}.$$
We recall that the measure $\frac{d\nu(w)}{(1-|w|^2)^{n+1}}$ is invariant under automorphisms of $\mathbf{B}_n$. On ω, there exist two constants A and B such that $A|z-\zeta| \leq \beta(z,\zeta) \leq B|z-\zeta|$ for all

$z, \zeta \in \omega$. We suppose that $\delta < \frac{A(1-R)}{2}$. Now, for all $h \in \mathbb{C}^n$ such that $|h| < \frac{\delta}{A}$ and all $z \in \mathbb{C}^n$ such that $|z| < R$, it is easy to check that z and $z + h$ both lie in ω. Moreover for every $h \in \mathbb{C}^n$ such that $|h| < \frac{\delta}{B}$, for every ϕ analytic on $\mathbf{B}_n$ and every z such that $|z| < R$, we obtain:

$$|\phi(z+h) - \phi(z)| \leq C(R)\delta ||\phi||_1.$$

We set $C = \sup_{\phi \in \mathcal{F}} ||\phi||_1$ and get

$$\int_{|z|<R} |\phi(z+h) - \phi(z)| d\nu(z) \leq CC(R)\delta.$$

Part a) of assertion 2. follows when we take $\delta < \frac{\epsilon}{CC(R)R^2}$.

We next prove that assertion 1. implies part b) of assertion 2. By the atomic decomposition theorem (cf. e.g. Theorem 2.30 of [47]), for every $g \in L^1_a$, there exists a sequence $\{c_k\}$ of complex numbers belonging to the sequence space l^1 such that

$$g(z) = \sum_{k=1}^{\infty} c_k \tilde{k}^c_{a_k}(z) \qquad (z \in \mathbf{B}_n).$$

This series converges to g in the norm topology of L^1_a. Moreover, there exists a constant C such that for every $g \in L^1_a$, the following estimate holds:

$$\sum_{k=1}^{\infty} |c_k| \leq C||g||_1.$$

Here, the sequence $\{a_k\}$ is again an $r-$lattice (Lemma 3.2.6). Since A is bounded on L^1_a, we see that

$$\begin{aligned}
\int_{R\le|z|<1} |Ag(\zeta)|d\nu(\zeta) &= \int_{R\le|z|<1} |A(\sum_{k=1}^{\infty} c_k \tilde{k}^c_{a_k})(\zeta)|d\nu(\zeta) \\
&= \int_{R\le|z|<1} \sum_{k=1}^{\infty} c_k A(\tilde{k}^c_{a_k})(\zeta)|d\nu(\zeta) \\
&\le \int_{R\le|z|<1} \sum_{k=1}^{\infty} |c_k||A(\tilde{k}^c_{a_k})|(\zeta)|d\nu(\zeta) \\
&= \sum_{k=1}^{\infty} |c_k| \int_{R\le|z|<1} |A(\tilde{k}^c_{a_k})|(\zeta)|d\nu(\zeta).
\end{aligned}$$

Assertion 1. implies that

$$\int_{R\le|z|<1} |Ag(\zeta)|d\nu(\zeta) \le \epsilon \sum_{k=1}^{\infty} |c_k| \le C\epsilon||g||_1 \le C\epsilon,$$

because $||g||_1 \le 1$. $\square$

We shall also a need the inner product formula in $L^2_a(\mathbf{B}_n)$ given in [41, section 4.2].

4.5.4 Lemma

Let $F, G \in L^2_a(\mathbf{B}_n)$. Then there exist constants a_1, a_2 and a_3 such that

$$\begin{aligned}
\langle F, G\rangle &= a_1 \int_{\mathbf{B}_n} \bigtriangledown F(z)\overline{\bigtriangledown G(z)}(1-|z|^2)^2\, d\nu(z) \\
&+ a_2 \int_{\mathbf{B}_n} \bigtriangledown F(z)\overline{\bigtriangledown G(z)}(1-|z|^2)^3\, d\nu(z) \\
&+ a_3 \int_{\mathbf{B}_n} F(z)\overline{G(z)}(1-|z|^2)^2\, d\nu(z).
\end{aligned}$$

This leads us to the following.

4.5.5 Theorem

Let $c > 0$. Suppose that the complex measure μ is such that $K_z\bar{\mu}$ satisfies condition (R) for every $z \in \mathbf{B}_n$ with the following uniform condition:

$$\forall r \in (0,1), \qquad \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu)) < \infty$$

(in particular if $|\mu|$ is a Carleson measure for Bergman spaces) Suppose further that T_μ is bounded on L^1_a. Then T_μ is compact on L^1_a if and only if $\|T_\mu \tilde{k}^c_\zeta\|_1 \to \ 0$ as $\zeta \to \partial\mathbf{B}_n$.

Proof Take $\psi(r) = 1$, $\phi(r) = (1-r^2)^c$, $c > 0$. Then by Theorem 4.4.4 it suffices to prove that $A_0(\phi)$ is an invariant subspace of the adjoint operator $T^{\star\star}_\mu$ of T_μ with respect to the duality pairing $[,]$ defined in (4.25). We just suppose T_μ is bounded on L^1_a. Then $T^{\star\star}_\mu$ is bounded on $A_\infty(\phi)$. Since the weighted Bergman kernel $\tilde{K}^c_\xi(z) = \frac{d(c)}{(1-(z\cdot\xi))^{n+1+c}}$ reproduces $A_\infty(\varphi)-$functions in the sense that for every $h \in A_\infty(\phi)$,

$$h(\xi) = [h, \tilde{K}^c_\xi], \qquad (\xi \in \mathbf{B}_n).$$

Thus, for every $h \in A_\infty(\phi)$ and for every $\xi \in \mathbf{B}_n$, we obtain,

$$\begin{aligned} T^{\star\star}_\mu h(\xi) &= [T^{\star\star}_\mu h, \tilde{K}^c_\xi] = [h, T_\mu \tilde{K}^c_\xi] \\ &= d(c)\int_{\mathbf{B}_n} \overline{\left(\int_{\mathbf{B}_n} \frac{K_w(z)}{(1-(w\cdot\xi))^{n+1+c}} d\mu(w)\right)} h(z)(1-|z|^2)^c d\nu(z). \end{aligned}$$

We need to show that $T^{\star\star}_\mu h \in A_0(\phi)$ if $h \in A_0(\varphi)$. We fix $\epsilon > 0$ arbitrary. Then there exists $r = r(\epsilon) \in (0,1)$ such that

$$(1-|z|^2)^c |h(z)| < \epsilon \quad \text{whenever} \quad r < |z| < 1. \tag{4.41}$$

We write,

$$\frac{1}{d(c)} T^{\star\star}_\mu h(\xi)(1-|\xi|^2)^c = I + II$$

where

$$\begin{aligned} I &= \int_{r\leq|z|<1} \overline{\left(\int_{\mathbf{B}_n} \frac{K_w(z)(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}}\, d\mu(w)\right)} h(z)(1-|z|^2)^c\, d\nu(z) \\ &= \int_{r\leq|z|<1} \overline{T_\mu \tilde{k}^c_\xi(z)} h(z)(1-|z|^2)^c\, d\nu(z) \end{aligned}$$

and

$$II = \int_{|z|<r} \overline{\left(\int_{\mathbf{B}_n} \frac{K_w(z)(1-|\xi|^2)^c}{(1-(w\cdot\xi))^{n+1+c}}\, d\mu(w)\right)} h(z)(1-|z|^2)^c\, d\nu(z). \tag{4.42}$$

Concerning I, we deduce from (4.41) that

$$|I| \leq \int_{r\leq|z|<1} |T_\mu \tilde{k}^c_\xi(z)||h(z)|(1-|z|^2)^c\, d\lambda(z) \leq C\epsilon, \tag{4.43}$$

with $C = \sup_{\xi\in\mathbf{B}_n} \|T_\mu \tilde{k}^c_\xi\|_1 < \infty$, since T_μ is bounded on L^1_a.

Now for II, we first study the inner integral. We observe that

$$\overline{\int_{\mathbf{B}_n} \frac{K_w(z)}{(1-(w\cdot\xi))^{n+1+c}}\, d\mu(w)} = \frac{1}{d(c)} \langle T_{\overline{\mu}} K_z, \tilde{K_\xi} \rangle.$$

Lemma 4.5.4 implies

$$\langle T_{\overline{\mu}} K_z, \tilde{K_\xi} \rangle = J_1 + J_2 + J_3$$

where

$$\begin{aligned} J_1 &= \frac{3}{d(c)} \int_{\mathbf{B}_n} (1-|w|^2)^2 T_{\overline{\mu}} K_z(w) \overline{\tilde{K_\xi}(w)}\, d\nu(w) \\ J_2 &= \frac{1}{2(d(c))} \int_{\mathbf{B}_n} (1-|w|^2)^2 (\bigtriangledown T_{\overline{\mu}} K_z)(w) \overline{\bigtriangledown \tilde{K_\xi}(w)}\, d\nu(w) \\ J_3 &= \frac{1}{3(d(c))} \int_{\mathbf{B}_n} (1-|w|^2)^3 (\bigtriangledown T_{\overline{\mu}} K_z)(w)) \overline{\bigtriangledown \tilde{K_\xi}(w)}\, d\nu(w). \end{aligned}$$

Now, since T_μ is bounded on L^1_a, Corollary 4.2.19 implies that there exists a constant $C(r)$ such that

$$\sup_{|z|<r} ||P(K_z\bar{\mu})||_{LB} \leq C(r)(||T_\mu|| + \sup_{z\in r\mathbf{B}_n} Carl(R(K_z\mu))).$$

So

$$\begin{aligned}
|J_1| &\leq 3\int_{\mathbf{B}_n}(1-|w|^2)^2|P(K_z\bar{\mu})(w)|\frac{1}{|1-(w\cdot\xi)|^{n+1+c}}\,d\nu(w)\\
&\leq 3\int_{\mathbf{B}_n}\frac{(1-|w|^2)^2|P(K_z\bar{\mu})(w)-P(K_z\bar{\mu})(0)|}{|1-(w\cdot\xi)|^{n+1+c}}d\nu(w)\\
&+ 3|P(K_z\bar{\mu})(0)|\int_{\mathbf{B}_n}\frac{(1-|w|^2)^2}{|1-(w\cdot\xi)|^{n+1+c}}d\nu(w)\\
&\leq C\|P(K_z\bar{\mu})\|_{B^\infty}\left\{\int_{\mathbf{B}_n}\frac{(1-|w|^2)^2\beta(0,w)}{|1-(w\cdot\xi)|^{n+1+c}}\,d\nu(w)+\int_{\mathbf{B}_n}\frac{(1-|w|^2)^2}{|1-(w\cdot\xi)|^{n+1+c}}d\nu(w)\right\}.
\end{aligned}$$

It is well known, see for example [8], that for every $\nu > 0$, there exists a constant $C(\nu)$ such that

$$\beta(0,w)\leq\frac{C(\nu)}{(1-|w|^2)^\nu},$$

for every $w\in\mathbf{B}_n$. Hence,

$$|J_1|\leq C(\nu)||P(K_z\bar{\mu})||_{B^\infty}\int_{\mathbf{B}_n}\frac{(1-|w|^2)^{2-\nu}}{|1-(w\cdot\xi)|^{n+2+c}}d\nu(w).$$

Since $||g||_{B^\infty}\leq\frac{||g||_{LB}}{log2}$ for every $g\in B^\infty$, we obtain by Corollary 4.2.19 that

$$|J_1|\leq C'(\nu)||P(K_z\bar{\mu})||_{LB}\int_{\mathbf{B}_n}\frac{(1-|w|^2)^{2-\nu}}{|1-(w\cdot\xi)|^{n+2+c}}d\nu(w)$$

$$\leq C(r,\nu)(||T_\mu||+\sup_{z\in r\mathbf{B}_n}Carl(R(K_z\mu)))\int_{\mathbf{B}_n}\frac{(1-|w|^2)^{2-\nu}}{|1-(w\cdot\xi)|^{n+2+c}}d\nu(w).$$

Applying Lemma 3.2.2, the conclusion for $|J_1|$ is the following:

1. If $c<2$, we take ν such that $\nu<2-c$ and we get,

$$|J_1|\leq C(r,\nu)(||T_\mu||+\sup_{z\in r\mathbf{B}_n}Carl(R(K_z\mu)));$$

2. If $c=2$, we take $\nu\in(0,1)$ and we get,

$$|J_1|\leq C(r,\nu)(||T_\mu||+\sup_{z\in r\mathbf{B}_n}Carl(R(K_z\mu)))\frac{1}{(1-|\xi|^2)^\nu};$$

3. If $c > 2$, we take $\nu \in (0,1)$ and we get,

$$|J_1| \leq C(r,\nu)(||T_\mu|| + \sup_{z\in r\mathbf{B}_n} Carl(R(K_z\mu)))\frac{1}{(1-|\xi|^2)^{c-2+\nu}}.$$

Also,

$$\begin{aligned}
2|J_2| &\leq \frac{1}{d(c)}\int_{\mathbf{B}_n}(1-|w|^2)^2|(\bigtriangledown P(K_z\bar{\mu})(w))||(\bigtriangledown \tilde{K}_\xi^c)(w)|d\nu(w)\\
&\leq C\int_{\mathbf{B}_n}\log(\frac{2}{1-|w|^2})\frac{(1-|w|^2)^2|(\bigtriangledown P(K_z\bar{\mu})(w))|}{\log(\frac{2}{1-|w|^2})}\frac{1}{|1-(w\cdot\xi)|^{n+2+c}}\,d\nu(w)\\
&\leq C\|P(K_z\bar{\mu})\|_{LB}\int_{\mathbf{B}_n}\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{(1-|w|^2)}{|1-(w\cdot\xi)|^{n+2+c}}\,d\nu(w)\\
&\leq d'(c))C(r)(||T_\mu|| + \sup_{z\in r\mathbf{B}_n} Carl(R(K_z\mu)))\\
&\quad \int_{\mathbf{B}_n}(1-|w|^2)\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1}{|1-(w\cdot\xi)|^{n+2+c}}\,d\nu(w)
\end{aligned}$$

The latter inequality comes from Corollary 4.2.19. There exists $s \in (0,1)$ such that $\frac{1}{log(\frac{2}{1-|w|^2})} < \epsilon$ whenever $s < |w| < 1$. We fix such an s. Then

$$\begin{aligned}
&\int_{\mathbf{B}_n}(1-|w|^2)\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1}{|1-(w\cdot\xi)|^{n+2+c}}\,d\nu(w)\\
&= \left\{\int_{s\overline{\mathbf{B}_n}} + \int_{\mathbf{B}_n/s\overline{B_n}}\right\}(1-|w|^2)\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1}{|1-(w\cdot\xi)|^{n+2+c}}\,d\nu(w)\\
&\leq C_s + \frac{C\epsilon}{(1-|\xi|^2)^c}
\end{aligned}$$

with $C_s = \frac{1}{(log2)(1-s)^{3+c}}$. This implies

$$|J_2| \leq \frac{1}{2}(2+c)C(r)(||T_\mu|| + \sup_{z\in r\mathbf{B}_n} Carl(R(K_z\mu)))\left\{C_s + \frac{C\epsilon}{(1-|\xi|^2)^c}\right\}.$$

In a similar manner, we obtain,

$$|J_3| \leq \frac{1}{3}(2+c)C(r)(||T_\mu|| + \sup_{z\in r\mathbf{B}_n} Carl(R(K_z\mu)))\left\{C_s + \frac{C\epsilon}{(1-|\xi|^2)^c}\right\}.$$

Thus,

$$|II| \leq ||h||_{A_\infty(\phi)}(|J_1| + |J_2| + |J_3|)(1 - |\xi|)^c.$$

Given $\epsilon > 0$, there exists an $s \in (0,1)$ such that

1. if $c < 2$, then for ν positive such that $\nu < 2 - c$, we get,

$$|II| \leq C(c, r, \nu)(||T_\mu|| + \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu)))$$

$$||h||_{A_\infty(\phi)} \left[1 + C_s + \frac{C\epsilon}{(1 - |\xi|^2)^c}\right](1 - |\xi|)^c;$$

2. if $c = 2$, then for $\nu \in (0,1)$, we get,

$$|II| \leq C(c, r, \nu)(||T_\mu|| + \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu)))$$

$$||h||_{A_\infty(\phi)} \left[\frac{1}{(1 - |\xi|^2)^\nu} + C_s + \frac{C\epsilon}{(1 - |\xi|^2)^c}\right](1 - |\xi|)^c$$

3. if $c > 2$, then for $\nu \in (0,1)$, we get:

$$|II| \leq C(c, r, \nu)(||T_\mu|| + \sup_{z \in r\mathbf{B}_n} Carl(R(K_z\mu)))$$

$$||h||_{A_\infty(\phi)} \left[\frac{1}{(1 - |\xi|^2)^{c-2+\nu}} + C_s + \frac{C\epsilon}{(1 - |\xi|^2)^c}\right](1 - |\xi|)^c.$$

Combining these estimates when $|\xi| \to 1^-$with (4.43) easily implies the desired conclusion. $\square$

4.5.6 Corollary

Let μ be a positive measure on $\mathbf{B}_n$ such that the Toeplitz operator T_μ is bounded on L^1_a and let $c > 0$. The following three assertions are equivalent:

1. The Toeplitz operator T_μ is compact on L^1_a;

2. The following estimate holds:

$$\lim_{\zeta \to \partial \mathbf{B}_n} ||T_\mu \tilde{k}^c_\zeta||_1 = 0;$$

3. For every $\epsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R \leq |z| < 1} |(T_\mu \tilde{k}^c_\zeta)(z)| d\nu(z) < \epsilon$$

for every $\zeta \in \mathbf{B}_n$.

Proof The Toeplitz operator T_μ is bounded on L^1_a. It follows from Corollary 2.5 that μ is a Carleson measure for Bergman spaces. The proof of the equivalence 1. $\Leftrightarrow$ 2. follows from a direct application of Theorem 4.5.5. The equivalence 1. $\Leftrightarrow$ 3. is a direct application of Theorem 4.5.3. $\Box$

4.5.7 Corollary

Suppose $c > 0$. Let $f \in L^1_a$ be such that $T_{\overline{f}}$ is a bounded operator on L^1_a . Then the following five assertions are equivalent:

1. The Toeplitz operator $T_{\overline{f}}$ is compact on L^1_a;

2. $\|T_{\overline{f}} \tilde{k}^c_\xi\|_1 \to\ 0$ as $\xi \to \partial \mathbf{B}_n$;

3. For every $\epsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R \leq |z| < 1} |(T_{\bar{f}} \tilde{k}^c_\xi)(z)| d\lambda(z) < \epsilon$$

for every $\xi \in \mathbf{B}_n$.

4. $\|T_{\overline{f}} \tilde{k}^1_\xi\|_1 \to\ 0$ as $\xi \to \partial \mathbf{B}_n$;

5. f vanishes identically.

Let us mention that using duality, property (1) is equivalent to "f is a compact multiplier of B^∞. The latter was shown in [32] to be equivalent to (5) in the case $n = 1$.

Proof. The proof goes along the following implications: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) and (1) $\Leftrightarrow$ (3). From Theorem 4.2.8 $T_{\overline{f}}$ bounded on L^1_a implies f is bounded. We apply Theorem 4.5.5 to get the equivalence (1) $\Leftrightarrow$ (2). Theorem 4.5.3 gives (1) $\Leftrightarrow$ (3). Taking $c = 1$ we have (2) $\Rightarrow$ (4). Suppose (4) holds. Using Lemmas 4.2.6 and 4.2.3 with $z = 0$ we have

$$(1 - |\xi|^2)|\mathcal{D}^1(f)(\xi)| \to 0 \ \text{ as } \ \xi \to \mathbf{B}_n, \tag{4.44}$$

where

$$\mathcal{D}^1 f(\xi) = \int_{\mathbf{B}_n} \frac{f(w)}{(1 - (\xi \cdot w))^{n+2}} d\nu(w).$$

On the other hand, observing that $\mathcal{D}^1 = I + (n + 2) \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$ we have for some absolute constant C and for all $\xi, z \in \mathbf{B}_n$

$$\left| \frac{f(\xi)}{(1 - (\xi \cdot z))^{n+3}} \right| \leq C \left| \mathcal{D}^1 \left\{ \frac{f}{(1 - (\cdot \cdot z))^{n+2}} \right\} (\xi) \right| + C \left| \frac{\mathcal{D}^1(f)(w)}{(1 - (\xi \cdot z))^{n+2}} \right|. \tag{4.45}$$

Multiplying (4.45) by $(1 - |\xi|^2)(1 - |z|^2)^{n+2}$ and then using again Lemmas 4.2.6 and 4.2.3, equation (4.44) we have, taking $z = \xi$, that $|f(\xi)| \to 0$ as $\xi \to \mathbf{B}_n$. Hence (5) holds. The implication (5) $\Rightarrow$ (2) is obvious. This completes the proof of the Corollary.$\Box$

Conculsion

In the first part, we develop a new type of partition of unity, which enables us to decompose vector valued measures in terms of measures whose Fourier transform is a measure satisfying some satisfactory properties in a precise direction. With this we are able to give a simplified proof of the theorem on the modulus of continuity first proved by Boman [10] in 1967.

In the second part, we set out with the following two problems:

(1) Characterize the symbols $f \in L^1(\mathbf{B}_n, d\nu)$ whose associated Toeplitz operator, T_f, extend to bounded operators on the Bergman spaces, $L^p_a(\mathbf{B}_n, d\nu)$.

(2) Characterize the symbols $f \in L^1(\mathbf{B}_n, d\nu)$ whose associated Toeplitz operator, T_f, extend to compact operators on the Bergman spaces, $L^p_a(\mathbf{B}_n, d\nu)$.

For $p > 1$ we are able to improve the existing results on the following two problems for the case $p > 1$. While for the $p = 1$ we where able to give new results characterizing boundedness and compactness not only of the Toeplitz operator but also for more general operators on the Bergman space L^1_a.

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