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Computations in the Grothendieck Group of Stacks

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Avhandlingen kommer att presenteras måndagen den 27/2 2012, kl. 13.00 i rum 306, hus 6, Matematiska institutionen, Stockholms universitet, Kräftriket.

To Torsten Ekedahl

It has truly been a privilege to learn geometry from a man of such great knowledge. I am proud of having been your student.

Abstract

Given an algebraic group, one may consider the class of its classifying stack in the Grothendieck group of stacks. This is an invariant studied by Ekedahl. For certain connected groups, called the special groups by Serre and Grothendieck, the invariant simply gives the inverse of the class of the group itself. It is natural to ask whether the same is true for other connected groups. We investigate this for the groups PGL_2 and PGL_3 under mild restrictions on the choice of base field.

In the case of PGL_2, the question turns out to have a positive answer. In the case of PGL_3, we reduce the question to the computation of the invariant for the normaliser of a maximal torus in PGL_3. The reduction involves determining the class of a certain gerbe over the moduli stack of elliptic curves.

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Daniel Bergh

Contents

1 Introduction

The main results of this thesis are computations of an algebro-geometric invariant for groups in the case of the projective general linear groups PGL_2 and PGL_3. In this introduction, I will put these results into context. My aim is to provide a non-technical overview intended for a broader mathematical audience, not necessarily specialised in algebraic geometry. The definition of the invariant is given in Section 1.5 following a preprint of Ekedahl [Eke09a]. The results themselves are stated in Section 1.6.

1.1 Euler characteristics

The main objects of study in algebraic geometry are varieties — geometric objects defined by algebraic equations. When studying varieties, a simple, yet effective, invariant is the classical Euler characteristic. It owes much of its effectiveness to its additivity properties with respect to closed subsets. We capture these properties in the following definition.

Definition. A (generalised) *Euler characteristic* is a function χ from the set of isomorphism classes of varieties over some fixed base field k to some ring R. The function is supposed to satisfy the relations

- $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ for each pair of varieties $Z \subset X$, where Z is closed in X,
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for arbitrary varieties X and Y.

These relations are referred to as the *scissors relation* and the *multiplicative relation* respectively.

The most general Euler characteristic one can think of is the one taking values in the *Grothendieck group of varieties* $\mathrm{K}_0(\mathrm{Var}_k)$. This is defined as the ring generated by isomorphism classes $\{X\}$ of varieties subject to the relations needed to make the assignment $X \mapsto \{X\}$ an Euler characteristic in the sense of the definition above. By construction, every Euler characteristic factors via $\mathrm{K}_0(\mathrm{Var}_k)$. Hence the Euler characteristic taking values in $\mathrm{K}_0(\mathrm{Var}_k)$ deserves to be referred to as the *universal Euler characteristic.*

The class $\{\emptyset\}$ of the empty variety is the zero element in $\mathrm{K}_0(\mathrm{Var}_k)$ and the class $\{*\}$ of a point is the multiplicative identity. The class $\{\mathbb{A}^1_k\}$ of the affine line also plays a particular role. It is usually called the *Lefschetz class*, and we will denote it by $\mathbb{L}$.

In modern times, the study of $\mathrm{K}_0(\mathrm{Var}_k)$ gained interest due to its appearance in the field of *motivic integration* introduced by Kontsevich. In this context, it is common to formally invert the Lefschetz class $\mathbb{L}$ and take the completion with respect to a dimension filtration. The resulting ring is usually denoted by $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$.

We shall neither be directly concerned with motivic integration nor the ring $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$ in this work. But in several of the articles referenced in this introduction, the results are stated in terms of the ring $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$. It is worth noting that the classes of the general linear groups GL_n are invertible in $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$. This is a property which will have significance later on.

As always when introducing algebraic structures by giving relations and generators, one should make sure that the structure not just collapses to something trivial. In the case of $\mathrm{K}_0(\mathrm{Var}_k)$ this is asserted by various concrete examples of Euler characteristics. Here are some:

EC1 The classical topological Euler characteristic gives an Euler characteristic from varieties over $\mathbb{C}$ to the integers.

EC2 When working with varieties over a finite field $\mathbb{F}_q$, we get an Euler characteristic to the integers by simply counting the number of rational points on the variety. Saying that this function is an Euler characteristic is nothing but a fancy way to express the obvious fact that if we cut a variety in halves, the total number of points on the variety equals the sum of the number of points on each half.

EC3 The previous example may be expanded to counting the number of $\mathbb{F}_{q^n}$-points for each positive integer n. This gives an Euler characteristic taking values in the ring of integer valued functions on the positive natural numbers.

EC4 A technically more involved Euler characteristic may be considered if we work with varieties over a field k which is not algebraically closed. Then we may also take advantage of the arithmetic information. Given a variety X, we take the cohomology of $X_{\overline{k}}$. More precisely, we should take the étale cohomology with compact support and with coefficients in $\mathbb{Q}_\ell$. The cohomology groups so obtained have a natural structure of continuous $\mathrm{Gal}(\overline{k}/k)$-representations. Taking the alternating sums of these representations in the representation ring for $\mathrm{Gal}(\overline{k}/k)$ gives our desired Euler characteristic. Since the representation ring for the trivial group is just $\mathbb{Z}$, this example degenerates into the first one if $k = \mathbb{C}$, due to the correspondence principle for étale cohomology.

Actually all of the examples above fit into a common framework. The classical Euler characteristic may be understood as taking the alternating sum of the Betti numbers, i.e. of the dimensions of cohomology groups for X for a suitable cohomology theory. This gives an Euler characteristic in our sense since the inclusion $Z \subset X$ induces a long exact sequence on cohomology.

But taking dimensions is a rather coarse invariant. Usually cohomology groups have richer structure than just being vector spaces, as for instance suggested in the fourth example. If the cohomology groups for the cohomology theory considered belong to some abelian category $\mathcal{A}$, we get an Euler characteristic by taking the alternating sum in the Grothendieck group $\mathrm{K}_0(\mathcal{A})$. The group $\mathrm{K}_0(\mathcal{A})$ is defined as the group generated by isomorphism classes in $\mathcal{A}$ subject to the relations $\{B\} = \{A\} + \{C\}$ for short exact sequences

$$0 \to A \to B \to C \to 0.$$

We shall mention a few more Euler characteristics fitting into this picture.

EC5 In [Eke09a], Ekedahl considers an *Euler characteristic of mixed Galois representations* taking values in a ring we will call $\mathrm{K}_0(\mathrm{Coh}_k)$. This is a variant of the fourth example with the extra feature that it manages to extract useful arithmetic information for arbitrary base fields.

EC6 Ekedahl also considers an *Euler characteristic of mixed Galois representations with torsion* taking values in a different Grothendieck group $\mathrm{L}_0(\mathrm{Coh}_k)$. Here only relations for *split* exact sequences are added. This has the added feature that it preserves torsion in the cohomology groups. The construction depends on an alternative presentation of $\mathrm{K}_0(\mathrm{Var}_k)$ given by Heinloth-Bittner [Bit04] using only smooth and proper varieties as generators. Since this result uses resolutions of singularities and the weak factorisation theorem by Abramovich, Karu, Matsuki and Włodarczyk [AKMW02, Wło03], it is currently only known to work over fields of characteristic zero.

EC7 In [BD07], Behrend and Dhillon consider an *Euler characteristic of effective Voevodsky motives*. This takes values in the Grothendieck group of effective Voevodsky motives $\mathrm{K}_0\left(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})\right)$. Strictly speaking, the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q})$ is not known to be abelian. However, it is triangulated, which makes the construction work anyway.

1.2 Fibre bundles and torsors

In geometry, a *fibre bundle* with fibre F is a map of spaces $E \to S$ which locally on S looks like the projection of a Cartesian product $F \times S$ to the second factor. It is common to restrict transformations gluing the bundle together over different coordinate patches to lie in a group G acting as automorphisms on the fibre F. The group G is called the *structure group* of the bundle. The prototypical example of a fibre bundle is of course a rank n vector bundle. Here the fibre is an n-dimensional vector space and the structure group is GL_n.

In differential or complex analytic geometry it is fairly clear what we mean by a fibre bundle looking like a product *locally*. In algebraic geometry the question is more subtle. In many cases, it seems like the Zariski topology is simply to coarse to capture the geometry of fibre bundles. As an illustration of this, we give the following example:

Example 1.1. Choose coordinates s and t for the affine plane $\mathbb{A}^2_{\mathbb{C}}$ and consider the Zariski open subset S defined by removing the coordinate axes. Let C be the plane projective curve over S defined by the equation

$$s \cdot x^2 + t \cdot y^2 + z^2 = 0.$$

Since s and t do not vanish on S, the fibre over each point in S is a non-singular conic. Such a curve is isomorphic to the projective line provided that it has a rational point, which is true over each closed point of S. This gives a hint that we might want to view C as a $\mathbb{P}^1$-fibred bundle over S. But over the generic point of S we have no rational points since the defining equation of C has no solutions in the function field $\mathbb{C}(s,t)$. As a consequence, there can be no Zariski open subset $U \subset S$ over which C is isomorphic to $\mathbb{P}^1 \times U$. Hence C is not a fibre bundle over S according to the naïve definition.

Instead, we may formally adjoin the square roots of s and t to the coordinate ring of S. This corresponds to a variety S' surjecting onto S. Over S' the defining equation of C *does* have solutions and hence C *is* isomorphic to $\mathbb{P}^1 \times S'$ over S'. A surjection as the one described here is called an *étale covering.* Note that in the classical topology the space S' is a degree 4 covering space over S. In particular, this implies that C is a fibre bundle over S in the complex analytic sense.

Such phenomena eventually led Grothendieck to reformulate topology to deal with coverings instead of open subsets. When referring to fibre bundles in algebraic geometry, we usually mean with respect to the étale topology, i.e. with respect to coverings as in the example above.

For some structure groups, being a fibre bundle in the generalised sense described above actually implies being a fibre bundle in the Zariski sense. This is for instance true for the general linear groups GL_n. Hence trying to generalise vector bundles using étale coverings gives nothing new. Groups with this properties are called *special.* The example above shows that the automorphism group of $\mathbb{P}^1$, namely PGL_2, is *not* special. The same turns out to be true for all projective linear groups PGL_n.

A fibre bundle may be viewed as a twisted product of the fibre and the base space. The amount of twist is described by a geometric object called a *torsor* for the structure group G. Torsors are fibre bundles in their own right. They are fibred by the group G viewed as a G-space by translation. In other branches of geometry, torsors are often called *principal homogeneous spaces.* For a vector bundle, we get the associated torsor by taking the *frame bundle.* A fibre bundle with fibre F and structure group G is completely described by its associated G-torsor. This essentially reduces the study of fibre bundles to the study of torsors.

1.3 Multiplicativity relations for torsors

Let $E \to B$ be a fibre bundle of topological spaces with fibre F. Then the multiplicativity relation $\chi(E) = \chi(F) \cdot \chi(B)$ holds for the classical Euler characteristic in quite general circumstances. A natural question to ask is to what extent this holds for Euler characteristics in the algebraic setting. Reformulated in terms of torsors, this gives the following question:

Given a G-torsor $T \to S$, for some group G, when does the multiplicativity relation $\{T\} = \{G\}\{S\}$ hold in $\mathrm{K}_0(\mathrm{Var}_k)$?

Of course the relation holds, by definition, if $T \to S$ is the trivial torsor, i.e. if T is simply the product $G \times S$. Multiplicativity is also quite easily seen to hold if the torsor trivialises in the Zariski topology. In particular, it holds for G-torsors if G is special. In contrast to the case with the classical Euler characteristic, it does not hold in general if G is not connected. We illustrate this by a simple example.

Example 1.2. In algebraic group theory, it is common to denote the group of units $\mathbb{C}^\times$ in the ring of complex numbers by $\mathbb{G}_m$. The square map $x \mapsto x^2$ from $\mathbb{G}_m$ to itself is a group homomorphism with kernel $\{-1, 1\}$, which is isomorphic to Σ_2. Since the square map is surjective, it makes $\mathbb{G}_m$ a Σ_2-torsor over itself. Now, considered as a variety, the group $\mathbb{G}_m$ is the affine line with a point removed. Hence its class in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ is $\mathbb{L} - 1$. The class of Σ_2 is 2 since the group is just a disjoint union of two points. One can show that $2(\mathbb{L} - 1) \neq \mathbb{L} - 1$ in $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$, which shows that multiplicativity does not hold in this case.

The question is more delicate in the case of connected non-special groups. Here we do have multiplicativity for the Euler characteristic **EC5** of mixed Galois motives [Eke09a, p. 6]. For the Euler characteristic **EC7** of Voevodsky motives, we have multiplicativity for torsors for split connected affine groups [BD07, A.9]. This made Behrend and Dhillon raise the question whether multiplicativity actually holds already in $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$ [BD07, Remark 3.3]. Ekedahl gave this a negative answer in [Eke08]. In fact he showed, by using the Euler characteristic **EC6**, that for each non-special connected affine group G there is a G-torsor for which multiplicativity does not hold.

1.4 Universal torsors and stacks

One way to obtain torsors for a group G, is to take a space X on which G acts freely and then take the quotient X/G. This makes X a G-torsor over X/G via the quotient map.

In homotopy theory, this may be used to construct a universal G-torsor. This is obtained by choosing a *contractible* space $\mathrm{E}G$ on which G acts freely. Such a space always exists. The resulting quotient $\mathrm{E}G/G$ has well-defined homotopy type and is denoted by $\mathrm{B}G$. The space $\mathrm{B}G$ has the property that given any space X, the homotopy classes of maps $X \to \mathrm{B}G$ are in natural correspondence with the homotopy classes of G-torsors over X. Due to this property, the space $\mathrm{B}G$ is called the *classifying space* for G.

We would like to do a similar construction in the algebraic setting. This can be achieved if we enlarge our algebraic objects under consideration to include so called *algebraic stacks*. The class of algebraic stacks includes the usual varieties, but also other, more general objects.

One way to produce stacks is to take the *stack quotient* $[X/G]$ of a variety X by an algebraic group G acting on X. The ordinary quotient X/G may be thought of as the space X together with an equivalence relation on X. Two points are considered equivalent if there is a group element in G transporting one of the points to the other. In contrast, the stack quotient $[X/G]$ may be understood as a groupoid structure on X. Not only do we recall that there exists a group element taking one point to the other, we also recall which. For free group actions, there is a unique group element with this property and the two concepts of quotients essentially coincide. However, for a group action which is not free, the stack quotient is a stack which genuinely fails to be variety. Although this heuristic picture gives a general clue to what a stack quotient is, it should be noted that it is a simplification which fails to capture any topological or algebraic aspects.

The nice thing about the stack quotient is that it always *behaves* as if it were free, even if the group action is not. More precisely, the variety X will always be a G-torsor over $[X/G]$. The price we have to pay by considering stacks instead of varieties is increased complexity. The algebraic stacks form a 2-category. Not only do we need to keep track on ordinary maps between stacks, but also 2-maps between maps.

In order to get a classifying space in the algebraic setting, we may now make an even more straightforward construction than in homotopy theory. Instead of letting G act freely on something that homotopically *looks* like a point, we simply let G act trivially on an *actual* point. The resulting stack quotient has similar classifying properties with respect to torsors as in the homotopy theory case. Therefore, we call it the *classifying stack* for G and denote it by $\mathrm{B}G$. The quotient map from the one-point space $*$ to $\mathrm{B}G$ makes $*$ the *universal torsor* over this stack.

1.5 Universal Euler characteristics for stacks

Once we have agreed that stacks are natural objects to study, we would like to be able to do the same things with them as we do with varieties. In particular, we would like to have an universal Euler characteristic for algebraic stacks. This leads us to consider the *Grothendieck group of stacks*, which has been done independently by several authors. Toën in [Toë05], Joyce in [Joy07], Behrend–Dhillon in [BD07] and Ekedahl in [Eke09a, Eke08, Eke09b]. We will follow Ekedahl's axiomatisation:

Definition. The *Grothendieck group of stacks* is the abelian group $\mathrm{K}_0(\mathrm{Stack}_k)$ generated by equivalence classes $\{\mathcal{X}\}$ of algebraic stacks $\mathcal{X}$ subject to the relations

GS1 $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ if $\mathcal{Z}$ is a closed substack of $\mathcal{X}$,

GS2 $\{\mathcal{E}\} = \{\mathbb{A}^n \times \mathcal{X}\}$ if $\mathcal{E} \to \mathcal{X}$ is a rank n vector bundle of stacks.

The stacks under consideration are assumed to have affine stabiliser groups and be of finite type over a base field k. The group $\mathrm{K}_0(\mathrm{Stack}_k)$ has a natural ring structure with multiplication defined by $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} = \{\mathcal{X} \times \mathcal{Y}\}$.

Note that axiom **GS2** is redundant for varieties. For stacks this is not the case. In fact the axiom is equivalent to requiring that we have multiplicativity relations for all GL_n-torsors. Since for instance $\mathrm{B}\,\mathrm{GL}_n$ has no non-trivial closed substacks, we cannot apply the same cutting and pasting arguments as we did for varieties.

A consequence of the axioms is that each stack which may be described as a stack quotient $[X/\mathrm{GL}_n]$ has the class $\{X\}\{\mathrm{GL}_n\}^{-1}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$. Not every algebraic stack can be written in this form, but due to a result by Kresch [Kre99], every stack we shall consider admits a finite stratification by such quotients. From this, it follows that $\mathrm{K}_0(\mathrm{Stack}_k)$ is the localisation of the ring $\mathrm{K}_0(\mathrm{Var}_k)$ where the classes $\{\mathrm{GL}_n\}$ have been inverted for all n. Since all these classes are invertible in $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$, we get a canonical ring homomorphism $\mathrm{K}_0(\mathrm{Stack}_k) \to \widehat{\mathrm{K}}_0(\mathrm{Var}_k)$ and we may actually talk about the class of a stack in $\widehat{\mathrm{K}}_0(\mathrm{Var}_k)$.

Given a group G, we can consider the class of its classifying stack $\mathrm{B}G$ in $\mathrm{K}_0(\mathrm{Stack}_k)$. Ekedahl studies this invariant for finite groups in [Eke09b]. Using the cohomological Euler characteristic **EC5** taking values in the Grothendieck group of mixed Galois representations, one always get $\chi(\mathrm{B}G) = 1$ when G is a finite group. Ekedahl proves that $\{\mathrm{B}\Sigma_n\} = 1$ already in $\mathrm{K}_0(\mathrm{Stack})$ for all symmetric groups Σ_n. But he also gives examples of finite groups G such that $\{\mathrm{B}G\} \neq 1$.

1.6 The main results

As already noted, the multiplicativity relation for torsors does not hold in general for non-special groups. But it may still hold for particular torsors, an obvious candidate for investigation being the universal one. Since the total space of a universal torsor is a single point, which has the class 1 in $\mathrm{K}_0(\mathrm{Stack}_k)$, this is the same thing as asking if the class of the classifying stack $\mathrm{B}G$ for a group G is the inverse of the class of the group itself.

Since the groups PGL_n are non-special and connected, they provide a natural starting point for our studies. We obtain the following results for the cases $n = 2$ and $n = 3$.

Theorem A. *Let k be a field of characteristic not equal to 2 Then the class of the classifying stack $\mathrm{B}\,\mathrm{PGL}_2$ is the inverse of the class of PGL_2 in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Theorem B. *Let k be a field of characteristic not equal to 2 or 3 containing all third roots of unity. Then the class of the classifying stack $\mathrm{B}\,\mathrm{PGL}_3$ is*

$$\{\mathrm{PGL}_3\}^{-1} + \left(\{\mathrm{B}N_3\} - \frac{\mathbb{L}^3}{(\mathbb{L}-1)^2(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)}\right)\frac{\mathbb{L}-1}{\mathbb{L}^{10}-1}$$

in $\mathrm{K}_0(\mathrm{Stack}_k)$. Here N_3 denotes the normaliser of the maximal torus in PGL_3.

The groups PGL_n themselves have classes which are polynomials in $\mathbb{L}$ in $\mathrm{K}_0(\mathrm{Var}_k)$. More precisely, the classes are given by

$$(\mathbb{L}-1)^{-1}\prod_{i=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^i).$$

Recall that the cohomological Euler characteristic **EC5** of Ekedahl's, taking values in $\widehat{\mathrm{K}}_0(\mathrm{Coh}_k)$ is multiplicative with respect to torsors for connected groups. Furthermore, the induced map $\mathrm{K}_0(\mathrm{Stack}_k) \to \widehat{\mathrm{K}}_0(\mathrm{Coh}_k)$ is injective on rational functions in $\mathbb{L}$. Hence the question of $\{\mathrm{B}\,\mathrm{PGL}_n\}$ being the inverse of $\{\mathrm{PGL}_n\}$ is equivalent to the question of $\{\mathrm{B}\,\mathrm{PGL}_n\}$ being a rational function in $\mathbb{L}$. Therefore Theorem B, although not giving a definitive answer regarding the question of multiplicativity of $\mathrm{B}\,\mathrm{PGL}_3$, reduces the problem to the question of rationality in $\mathbb{L}$ of the class of $\mathrm{B}N_3$.

The group N_3 is 2-dimensional and may be explicitly described as the group of 3×3 monomial matrices modulo the scalar matrices. The lack of low-dimensional faithful representations for this group makes it harder to attack than the other groups arising in the computations.

It should be remarked that the techniques used for computing the results above do not extend well to corresponding computations for projective general linear groups of higher dimension. In particular, the results give no evidence that the multiplicativity relations should be true for higher n. That we repeatedly get rational functions in $\mathbb{L}$ in our computations seems to stem from the fact that we are getting away with working with representations of low dimension and that the orbit spaces of such representations tend to be rational. If I am allowed to speculate, I would rather guess that the multiplicativity relation will fail, if not for $n = 3$ so for higher n.

1.7 Outline

Among the preliminaries in Section 2 are a couple of standard facts about stacks, algebraic groups, torsors and moduli spaces. They will be used later in the work, but are not typically found in introductory text books about scheme theory. This will serve the two-fold purpose of establishing notation and reviewing the facts to non-experts.

In Section 3, we develop some of the basic properties of the Grothendieck group of stacks. The main purpose is to make the text self-contained. Apart from Proposition 3.11 and 3.12 these results already occur, either explicitly as propositions or as simple consequences of such, in for instance the preprint by Ekedahl [Eke09a].

Section 3 also introduces the main techniques for computing the class of a classifying stack. They are similar to the ones used by Ekedahl in [Eke09b] to compute the classes of classifying stacks for some finite groups. The strategy is to find a suitable linear representation V of the group G in question. This gives a relation between the class of $\mathrm{B}G$ with the class of the stack quotient $[\mathbb{P}(V)/G]$. This stack quotient may in turn be stratified into pieces which will hopefully be easier to understand.

In Section 4, these techniques are applied to the projective linear groups PGL_2 and PGL_3. We will use geometrical representations of the groups acting on the space of hypersurfaces in projective space. In the case of PGL_2, this quickly leads to the result in Theorem A. The case of PGL_3 requires more work. Here we will use the natural representation on the space of planar cubic curves. This space may be subdivided into two according to whether the curves are singular or not. The computation of the singular part is reduced to the computation of the group of automorphisms of curves induced by projective transformations. These automorphism groups are determined in Appendix A.

Next, in Section 5, we study the stack $[H_{\mathrm{ns}}/\mathrm{PGL}_3]$ of non-singular planar cubics up to projective equivalence. This is related to the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves. The main result of this section is that $[H_{\mathrm{ns}}/\mathrm{PGL}_3]$ is a gerbe over $\mathcal{M}_{1,1}$. More precisely, the stack $[H_{\mathrm{ns}}/\mathrm{PGL}_3]$ is equivalent to the classifying stack of the 3-torsion subgroup $\mathcal{E}[3]$ of the universal curve $\mathcal{E}$ over $\mathcal{M}_{1,1}$. This equivalence might well have been noted by others, but I am not aware of any references.

Finally, in Section 6, we use the equivalence from the previous section to compute the class of $[H_{\mathrm{ns}}/\mathrm{PGL}_3]$. The key to this computation is to embed the 3-torsion subgroup of an arbitrary family of elliptic curves into a special algebraic torus and to compute the class of the quotient. We show that under mild hypotheses on the base field, the class of $[H_{\mathrm{ns}}/\mathrm{PGL}_3]$ equals the class of $\mathcal{M}_{1,1}$, which in turn is shown to be equal to $\mathbb{L}$.

2 Preliminaries

2.1 Notation and conventions

Our schemes and algebraic spaces are by default assumed to be of finite type over a base field k. We denote the categories of these objects by Sch_k and Space_k respectively. Some of the results are true in a more general setting without the proofs being harder. In these cases, we will write the conditions explicitly. By *variety* we mean a reduced scheme of finite type over a field k. The category of varieties over k is denoted by Var_k.

When working with moduli problems, algebraic groups and actions of algebraic groups, it is often extremely useful to adopt the *functor of points* perspective. When doing so, we shall use the term *sheaf* to mean a sheaf of sets on the site of schemes with the fppf topology over some base scheme. The symbols $\emptyset$ and $*$ denote the initial and terminal objects respectively in this sheaf category. We shall generally make no distinction between a scheme and the functor it represents. The terms *injective*, *surjective* and *image* will be used with their sheaf theoretic meanings. If we mean something else, we shall clarify this by writing for instance *schematic image* or *surjective on geometric points*.

When talking about *groups*, we will usually mean group objects in whatever category we are working with. When we need to be more specific, we write *sheaf of groups*, *group scheme* or *group varieties* for group objects in the category of sheaves, schemes or varieties respectively.

When working with stacks, we shall often describe them as categories fibred in groupoids over the site of schemes with the fppf topology. Algebraic stacks are assumed to be of finite type over the base field and have affine stabilisers. We denote the 2-category of such objects by Stack_k. The assumptions allow us to invoke the following result by Kresch [Kre99].

Proposition 2.1 (Kresch). *A finite type algebraic stack with affine stabilisers has a non-empty open substack which is the global quotient of a scheme by* GL_n.

Our assumptions ensures that every algebraic stack admits a finite stratification by locally closed substacks which are global quotients by GL_n.

2.2 Groups of multiplicative type

We recall some terminology and basic facts about groups of multiplicative type. The standard reference for this is [DG64] exposé VIII-X. A more elementary treatment in the case where the base is affine is given in [Wat79].

Given a group G, we may consider its *Cartier dual*, which we denote by $G^\vee$. This is defined as the sheaf $\underline{\mathrm{Hom}}_{gr}(G, \mathbb{G}_\mathrm{m})$. There is a natural homomorphism from G to its bidual, and we say that G is *reflexive* provided that this is an isomorphism. Since taking the Cartier dual respects base change, it is clear that reflexivity is a local property.

The constant abelian groups are reflexive, and groups isomorphic to their duals are called *diagonalisable*. Diagonalisable groups are always representable by schemes. They are affine and of finite type if and only if they come from finitely generated abelian groups.

Since we shall only be interested in this case, we will henceforth drop the modifiers *finite type* and *finitely generated* respectively in this context. The dual of $\mathbb{Z}/n\mathbb{Z}$ for a positive integer n is the group μ_n of n-th roots of unity and the dual of $\mathbb{Z}$ is $\mathbb{G}_{\mathrm{m}}$. Since the Cartier dual respects products, this gives the full classification of diagonalisable groups.

A group which is étale locally diagonalisable is said to be of *multiplicative type*. The groups of multiplicative type form an abelian category and the Cartier dual gives an involutive exact anti-equivalence to the abelian category of locally constant abelian groups.

A locally constant group is said to be *split* if it is actually constant. Similarly a group of multiplicative type is said to be *split* if it is diagonalisable. Note that the terminology is relative to in which class we consider the group. For instance, the group μ_3 over the base $\mathbb{Q}$ is non-split as a locally constant group but split as a group of multiplicative type.

A group which splits after a finite étale base change is called *isotrivial*. Over a connected base S a choice of geometric point $\overline{s}$ gives an exact equivalence of the category of isotrivial locally constant groups and the category of continuous $\pi_1(S, \overline{s})$-representations on finitely generated abelian groups. Here $\pi_1(S, \overline{s})$ denotes the étale fundamental group and the abelian groups are considered topological groups endowed with the discrete topology.

A group of multiplicative type which locally corresponds to a torsion free abelian group is called a *torus*. A torus is called *quasi-trivial* if it corresponds to a permutation representation of the étale fundamental group of the base.

2.3 Torsors

The theory of torsors is worked out in great detail in for instance Giraud's book on non-abelian cohomology [Gir71]. A shorter and more basic introduction is given in Milne's book on étale cohomology [Mil80, Section III.4]. In this section, we recall some of the facts that will be important for this work.

Let G be a sheaf of groups on a site $\mathcal{C}$. By a *pseudo-torsor* for G we mean a sheaf of G-sets T on which G acts freely and transitively. We will usually assume that G acts on T from the right, although the left action case of course is completely analogous. One way of expressing that the action is free and transitive is to say that the map $T \times G \to T \times T$ defined on generalised points by $(t, g) \mapsto (t, t \cdot g)$ is an isomorphism. If in addition the map $T \to *$ is surjective, we call T a *torsor*.

Torsors will often be considered in the relative setting. If $G \to S$ is a group, we say that $T \to S$ is a (pseudo-)torsor provided that it is a (pseudo-)torsor over the site $\mathcal{C}/S$. If $G \to *$ is a group, we will frequently abuse language and say that $T \to S$ is a (pseudo-)torsor for G when we actually mean that it is a (pseudo-)torsor for $G \times S \to S$.

Let X and Y be right and left G-spaces respectively. If the action of G is free on either X or Y, we may form the *contraction product* $X \times^G Y$. This is defined as the quotient of the product $X \times Y$ by the equivalence relation $(t \cdot g, x) \sim (t, g \cdot x)$ for any $g \in G$. If Y is a torsor, we obtain an object which is locally isomorphic to X. We call this the X-*fibration* associated to the torsor Y.

In general, there is no natural way to define a nontrivial G-action on $X \times^G Y$. However, if X also admits an action from the left by a group H, then so does $X \times^G Y$. The corresponding fact is of course true for right actions on Y. For an abelian group A, we may utilise this to form *contraction powers* T^n for A-torsors T. This is defined as the n-fold contraction product. We may also define negative powers by letting T^{-1} be defined as the sheaf $\underline{\mathrm{Isom}}_A(T, A)$ of A-equivariant isomorphisms. One verifies that we have a natural isomorphism $(T^n)^m \cong T^{n+m}$ for all integers n and m with this definition.

The torsors for a given group G form a category $\mathrm{B}G$. The objects are torsors $T \to S$ and the morphisms are Cartesian squares

$$\begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that the morphism $T' \to T$ is G-equivariant in the obvious sense. The forgetful functor to $\mathcal{C}$ taking $T \to S$ to S makes this a category fibred in groupoids over $\mathcal{C}$. In fact $\mathrm{B}G$ is a stack equivalent to the stack quotient $[*/G]$. It is called the *classifying stack* for G. Geometrically, it may be viewed as the moduli stack of G-torsors. The set of isomorphism classes of torsors over S is denoted by $\mathrm{H}^1(S, G)$. If G is abelian, this coincides with the usual definition of cohomology as a derived functor.

Given an arbitrary stack $\mathcal{X}$ and an object $X \in \mathcal{X}(*)$, we may consider the full substack of objects locally isomorphic to X. This stack is equivalent to the classifying stack $\mathrm{B}\underline{\mathrm{Aut}}(X)$. Given an object $Y \to S$ locally isomorphic to X, the associated torsor is given by $\underline{\mathrm{Isom}}(Y, f^*X)$, where f denotes the structure map $S \to *$. If the stack $\mathcal{X}$ is embedded in the stack of sheaves over $\mathcal{C}$, the 2-inverse of this functor is given by taking the X-fibration associated to a given torsor. This is possible to define even if $\mathcal{X}$ is not embedded in the stack of sheaves by using descent along torsors as described in [Vis05].

For an abelian group A, the stack $\mathrm{B}A$ has an alternative description in terms of extensions by the constant group $\mathbb{Z}$. Consider the category $\mathcal{E}\mathrm{xt}^1(\mathbb{Z}, A)$ of short exact sequences

$$0 \to A_S \to E_S \to \mathbb{Z}_S \to 0$$

of abelian groups over a base S, which we allow to vary. The morphisms are given by morphisms of complexes. We have a functor $\mathcal{E}\mathrm{xt}^1(\mathbb{Z}, A) \to \mathrm{B}A$ taking such a sequence as above to the degree 1 part of E_S, i.e. the subsheaf of elements in E_S mapping to 1. This functor admits a 2-inverse taking an A_S-torsor T_S to the exact sequence

$$0 \to A_S \to \coprod_i T_S^i \to \mathbb{Z} \to 0,$$

where T_S^i denote the i-th contraction power. This makes the two categories $\mathrm{B}A$ and $\mathcal{E}\mathrm{xt}^1(\mathbb{Z}, A)$ equivalent. A proof is given in [SGA72, Expose VII]. It is interesting to note that the decategorification of this functor induces an isomorphism $\mathrm{H}^1(S, A) \to \mathrm{Ext}^1(\mathbb{Z}, A)$.

So far we have only discussed torsors in the general setting of sheaves. When working with schemes the issue of representability enters. For affine group schemes every torsor is representable by an affine scheme. This follows by effective flat descent for affine schemes and the fact that every torsor is locally isomorphic to an affine group. For arbitrary group schemes the situation is more delicate since we do not have effective descent for schemes in general. However, we do have effective descent for algebraic spaces, so each torsor will at least be an algebraic space.

2.4 Special groups

In [CGS58] Serre and Grothendieck study a class of group varieties which they call the *special* groups. They also give a complete classification of these groups over an algebraically closed field. The class is closed under extensions and only contains affine and connected groups. All connected soluble groups are special, but among the semisimple groups only products of SL_n and Sp_{2n} are.

Since we need to work over a general base, we will use a slightly different definition of special group.

Definition. An algebraic group $G \to S$ is called *special* if for each G-torsor $T \to S'$ over an S-scheme S' there is a non-empty open subscheme $U \subset S'$ such that T is trivial over U. A group scheme over a more general base is considered special if the base change to every scheme is.

Ekedahl remarks in [Eke09a] that this definition coincides with the one given by Serre and Grothendieck if we only consider group varieties over an algebraically closed field. But over a general base the classification is more subtle. It is still true that the class of special groups is closed under extensions and contains $\mathbb{G}_{\mathrm{a}}$, GL_n, SL_n and Sp_{2n}. But there are non-split tori which are not special. This is, however, not the case for quasi-split tori.

Proposition 2.2. *Let S be an arbitrary scheme. Then any quasi-trivial torus $T \to S$ is special.*

Proof. First note that T is isomorphic to the group of units of a quasi-coherent sheaf of $\mathscr{O}_S$-algebras $\mathscr{A}$. Indeed, let $S' \to S$ be a Galois extension splitting T with corresponding Galois group Γ. Then, since T is quasi-trivial, it corresponds to the Γ-equivariant sheaf of groups $\mathscr{O}_{S'}^{\times} \times \cdots \times \mathscr{O}_{S'}^{\times}$ where Γ acts by permuting the factors. This is on the other hand the group of units in the Γ-equivariant sheaf of $\mathscr{O}_{S'}$-algebras $\mathscr{O}_{S'} \times \cdots \times \mathscr{O}_{S'}$ with the corresponding permutation action. This sheaf of algebras descends to our desired $\mathscr{A}$ on S.

Now, by flat descent for quasi-coherent sheaves, the fibred category of sheaves of $\mathscr{A}$-modules is a stack for the fppf topology. Therefore the $\underline{\mathrm{Aut}}_{\mathscr{A}-\mathrm{mod}}(\mathscr{A}) = \mathscr{A}^{\times}$-torsors classify the rank 1 locally free sheaves for $\mathscr{A}$. But such sheaves, being quasi-coherent, trivialise Zariski locally, so the same holds for the torsors. $\square$

2.5 Elliptic curves

Fix a scheme S. By a *curve* C/S we shall always mean a smooth, proper morphism $C \to S$ of algebraic spaces of relative dimension 1 with connected geometric fibres. We say that C/S is of genus g provided that all its geometric fibres are.

Given an algebraic space X/S, we may consider its *Picard sheaf* $\underline{\mathrm{Pic}}_{X/S}$ (see [Kle05]). This is the fppf sheafification of the functor taking a point $T \to S$ to the Picard group $\mathrm{Pic}(X_T/T)$. In the case when we have a genus g curve C/S, there is a degree function $\deg\colon \underline{\mathrm{Pic}}_{C/S} \to \mathbb{Z}_S$ to the constant sheaf $\mathbb{Z}_S$. For projective curves C/S, this is a consequence of the Hilbert polynomial being constant over flat families. In general, it follows from the fact that curves are étale locally projective. The degree function gives rise to the exact sequence

$$0 \to \underline{\mathrm{Pic}}^0_{C/S} \to \underline{\mathrm{Pic}}_{C/S} \overset{\deg}{\to} \mathbb{Z}_S \to 0$$

of sheaves of abelian groups. The degree d part of $\underline{\mathrm{Pic}}_{C/S}$, i.e. $\deg^{-1}(d)$, will be denoted by $\underline{\mathrm{Pic}}^d_{C/S}$. The sheaf of groups $\underline{\mathrm{Pic}}^0_{C/S}$ is called the *Jacobian* of C/S.

A global section $s \in C(S)$ of a curve C/S gives rise to a (relative) effective Cartier divisor which we denote by $[s]$ (see [KM85, 1.2]). The corresponding line bundle, i.e. the inverse of the ideal sheaf of $[s]$, is denoted by $\mathscr{O}_C([s])$ and has degree 1. The map $s \mapsto \mathscr{O}_X([s])$ extends to a map of sheaves $C \to \underline{\mathrm{Pic}}^1 C/S$. For genus 1 curves, this is an isomorphism (see [KM85, 2.1]), making C/S a torsor for its Jacobian.

Proposition 2.3. *Let $\pi\colon C \to S$ be a genus 1 curve and $\mathscr{L}$ a line bundle of degree 3. Then $\mathscr{L}$ is very ample in the sense of [DG61, §4] and $\pi_*(\mathscr{L})$ is locally free of rank three.*

Proof. If S is the spectrum of a field k, it follows from Riemann-Roch that the dimension of $\mathrm{H}^0(C, \mathscr{L})$ is 3 and that $\mathrm{H}^1(C, \mathscr{L})$ vanishes. Hence it follows that $\pi_*(\mathscr{L})$ is locally free of rank 3 in the general case and that π_* commutes with any pull-back by the cohomology and base change theorem. Thus surjectivity of the canonical map $\pi^*\pi_*\mathscr{L} \to \mathscr{L}$ follows from the fibrewise surjectivity and we get an induced map $\iota\colon C \to \mathbb{P}(\pi_*\mathscr{L})$. This is proper since C is proper and $\mathbb{P}(\pi_*\mathscr{L})$ is separated over S. Therefore, as a consequence of Zariski's Main Theorem [DG66, Thm. 8.12.6], we may also check that it is a closed immersion fibrewise. Hence the proposition follows. □

An *elliptic curve* is a pair $(E/S, 0_E \in E(S))$, where E/S genus 1 curve. By abuse of notation, we often omit the *zero section* 0_E from the notation, and simply write E/S. Since E/S is a torsor for $\underline{\mathrm{Pic}}^0_{C/S}$, the section 0_E induces an isomorphism $\underline{\mathrm{Pic}}^0_{C/S} \to E$ giving E/S a canonical structure of abelian scheme by transport of structure.

Given an integer n, the sheaf of n-torsion subgroups $E[n]$ of an elliptic curve E/S is finite and flat of finite presentation. If, in addition, the integer n is invertible in $\mathscr{O}_S$, the group $E[n]$ is étale over S and locally isomorphic to the constant sheaf of groups $(\mathbb{Z}/n\mathbb{Z})^2_S$. There is an alternating perfect pairing $e\colon E[n] \times_S E[n] \to \mu_n$ called the *Weil pairing* [Mum08].

The elliptic curves over a given base scheme S are parameterised by the *moduli stack of elliptic curves*, which we denote by $\mathcal{M}_{1,1}$. As a category fibred in groupoids over Sch_S, it has elliptic curves E/T as objects, with T being a scheme over S. A morphism $E'/T' \to E/T$ is defined to be a Cartesian square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ T' & \longrightarrow & T \end{array}$$

such that the section of E/S pulls back to the section of E'/S'. We have a morphism of stacks $\mathcal{E} \to \mathcal{M}_{1,1}$, where $\mathcal{E}$ is the *universal elliptic curve*. It is defined similarly as $\mathcal{M}_{1,1}$, but each of its objects E/T has an additional section $\sigma\colon T \to E$. The map $\mathcal{E} \to \mathcal{M}_{1,1}$ is the functor forgetting this section. This map has a right inverse $0_{\mathcal{E}}\colon \mathcal{M}_{1,1} \to \mathcal{E}$ which just duplicates the zero section. The universal elliptic curve has the property that for each elliptic curve E/T, we have an essentially unique morphism $T \to \mathcal{M}_{1,1}$ inducing a 2-Cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{E} \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & \mathcal{M}_{1,1}. \end{array}$$

Hence the map $\mathcal{E} \to \mathcal{M}_{1,1}$ is schematic, and we may view $\mathcal{E}/\mathcal{M}_{1,1}$ as an elliptic curve over the stack $\mathcal{M}_{1,1}$. Just as elliptic curves over schemes, this has the structure of a group scheme.

3 The Grothendieck group of stacks

3.1 The Grothendieck group of varieties

Let k be an arbitrary field. The *Grothendieck group of varieties*, denoted by $\mathrm{K}_0(\mathrm{Space}_k)$, is the abelian group generated by isomorphism classes of objects in Space_k subject to the relations

$$\{X\} = \{X \setminus Z\} + \{Z\},$$

where Z denotes a closed subspace of X. This relation is sometimes referred to as the *scissor relation*. The group admits the structure of a commutative ring with identity. The product is defined on generators by $\{X\} \cdot \{Y\} = \{X \times Y\}$ and the identity is given by the class of the base $\{*\}$. Also the class of the affine line $\{\mathbb{A}^1_k\}$ plays an important role. It is called the *Lefschetz class* and will be denoted by $\mathbb{L}$.

Note that similar constructions are possible if we instead of the category Space_k start with the category Var_k of varieties or the category Sch_k of finite type schemes over k. However, the natural ring homomorphisms

$$\mathrm{K}_0(\mathrm{Var}_k) \to \mathrm{K}_0(\mathrm{Sch}_k) \to \mathrm{K}_0(\mathrm{Space}_k),$$

induced by the corresponding inclusions of categories, turn out to be isomorphisms.

Note also that the finiteness hypothesis is essential for interesting results. Indeed, otherwise we could play the *Eilenberg Swindle* and consider the the disjoint union $\coprod X$ of an infinite number of copies a schemes X. Removing one copy of X would give an isomorphic scheme, and the scissors relation would force the class of X to be zero.

The following basic fact about special fibrations will be used repeatedly:

Proposition 3.1. *Let $E \to S$ be a fibration of algebraic spaces with fibre F. Assume that the fibration is associated to a torsor for a special group. Then $\{E\} = \{F\}\{S\}$ in $K_0(\mathrm{Space}_k)$.*

Proof. Since the fibration is associated to a special group G, there is a non-empty open subset $U \subset S$ over which $E_U \to U$ is isomorphic to $U \times F \to U$. Then $\{E_U\} = \{F\}\{U\}$ in $K_0(\mathrm{Space}_k)$. If we let Z be any closed subscheme of S with complement U, then also $E_Z \to Z$ is associated to a G-torsor. Under the hypothesis that $\{E_Z\} = \{F\}\{Z\}$, we may therefore conclude that indeed $\{E\} = \{F\}\{S\}$ since $\{E\} = \{E_U\} + \{E_Z\}$ and $\{S\} = \{U\} + \{Z\}$ by the scissors relation. Hence the result follows by noetherian induction on S, the statement for $S = \emptyset$ being trivial. □

3.2 The Grothendieck group of stacks

Next we consider the extension of these notions to algebraic stacks. By the *Grothendieck group of algebraic stacks*, $\mathrm{K}_0(\mathrm{Stack}_k)$, we mean the abelian group presented by generators $\{\mathcal{X}\}$ being equivalence classes of objects $\mathcal{X}$ in Stack_k, subject to the relations

GS1 $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ if $\mathcal{Z}$ is a closed substack of $\mathcal{X}$,

GS2 $\{\mathcal{E}\} = \{\mathbb{A}^n \times \mathcal{X}\}$ if $\mathcal{E} \to \mathcal{X}$ is a vector bundle of constant rank n.

Note that due to Proposition 3.1, axiom **GS2** would be redundant in the definition of $\mathrm{K}_0(\mathrm{Space}_k)$.

Lemma 3.2. *Let n be a natural number and $\mathcal{T} \to \mathcal{S}$ be a GL_n-torsor of algebraic stacks. Then we have the relation $\{\mathcal{T}\} = \{\mathrm{GL}_n\}\{\mathcal{S}\}$ with $\{\mathrm{GL}_n\} = \prod_{i=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^i)$ in $\mathrm{K}_0(\mathrm{Stack}_k)$. In particular, since $* \to \mathrm{B}\,\mathrm{GL}_n$ is a GL_n-torsor, we have that $1 = \{\mathrm{GL}_n\}\{\mathrm{B}\,\mathrm{GL}_n\}$, so $\{\mathrm{GL}_n\}$ is invertible in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. Let $\mathcal{E} \to \mathcal{S}$ be the vector bundle associated to the GL_n-torsor $\mathcal{T} \to \mathcal{S}$. For $1 \le i \le n$, consider the map $\mathcal{E}^i \to \bigwedge^i \mathcal{E}$ from the i-th fibre power to the i-th exterior power over $\mathcal{S}$ taking an i-tuple of sections to their exterior product. Define the stack $\mathcal{F}_i$ to be the complement of the pullback of the zero section along this map in $\mathcal{E}^i$. Informally, we may think of this as the stack of i-tuples of linearly independent vectors in $\mathcal{E}$. In particular $\mathcal{F}_0 \simeq \mathcal{S}$ and $\mathcal{F}_n$, being the frame bundle of $\mathcal{E}$, is isomorphic to $\mathcal{T}$ as GL_n-torsor over $\mathcal{S}$.

The stack $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}_i$ is a rank n vector bundle over $\mathcal{F}_i$. The map $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}_i \to \bigwedge^{i+1} \mathcal{E} \times_{\mathcal{S}} \mathcal{F}_i$ defined by $(v, v_1, \dots, v_i) \mapsto (v \wedge v_1 \wedge \cdots \wedge v_i, (v_1, \dots, v_i))$ may be viewed as a morphism of $\mathscr{O}_{\mathcal{F}_i}$-modules. Its kernel is a rank i vector bundle over $\mathcal{F}_i$ whose complement in $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}_i$

is $\mathcal{F}_{i+1}$. Hence the class of $\mathcal{F}_{i+1}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$ is $(\mathbb{L}^n - \mathbb{L}^i) \cdot \{\mathcal{F}_i\}$, as seen by using axiom **GS1** once and axiom **GS2** twice. By induction on i, we therefore get the relation

$$\{\mathcal{T}\} = \prod_{i=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^i)\{\mathcal{S}\}.$$

The statement about the class of GL_n follows from the special case where $\mathcal{T} = \mathrm{GL}_n$ and $\mathcal{S} = *$. The statement about a general GL_n-torsor follows by substituting this back into the displayed equation. □

Proposition 3.3. *Let $\mathcal{X} \to \mathcal{S}$ be a morphism of stacks, and let C be a fixed element in $\mathrm{K}_0(\mathrm{Stack}_k)$. Assume that for each morphism $S \to \mathcal{S}$ with S a scheme, we have the relation $\{\mathcal{X}_S\} = C \cdot \{S\}$. Then we also have the relation $\{\mathcal{X}\} = C \cdot \{\mathcal{S}\}$.*

Proof. If $\mathcal{Z}$ is a closed substack of $\mathcal{S}$ with complement $\mathcal{U}$, it is enough to give a proof for $\mathcal{X}_{\mathcal{Z}} \to \mathcal{Z}$ and $\mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ separately. Indeed, axiom **GS1** gives us the relations $\{\mathcal{X}\} = \{\mathcal{X}_{\mathcal{Z}}\} + \{\mathcal{X}_{\mathcal{U}}\}$ and $\{\mathcal{S}\} = \{\mathcal{Z}\} + \{\mathcal{U}\}$ which then would give us the desired result. It follows by noetherian induction that it is enough to prove that the proposition holds over a non-empty open subset of $\mathcal{S}$. Hence, by Proposition 2.1, we may assume that $\mathcal{S}$ is a global quotient $[S/\mathrm{GL}_n]$ with S a scheme. Consider the 2-Cartesian square

$$\begin{array}{ccc} \mathcal{X}_S & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{S}. \end{array}$$

The horizontal arrows are GL_n-torsors, so $\{S\} = \{\mathrm{GL}_n\}\{\mathcal{S}\}$ and $\{\mathcal{X}_S\} = \{\mathrm{GL}_n\}\{\mathcal{X}\}$ by Lemma 3.2. Combining these relations with the hypothesis about pullbacks to schemes gives $\{\mathrm{GL}_n\}\{\mathcal{X}\} = C \cdot \{\mathrm{GL}_n\}\{\mathcal{S}\}$. Since the factor $\{\mathrm{GL}_n\}$ is invertible, we may cancel it to get the desired result. □

Remark. By using a similar kind of argument, one can prove that the homomorphism $\mathrm{K}_0(\mathrm{Space}_k) \to \mathrm{K}_0(\mathrm{Stack}_k)$ induced by the inclusion of Space_k in Stack_k is a localisation map with respect to inversion of the classes $\{\mathrm{GL}_n\}$ for all n. For a complete proof, see [Eke09a].

Corollary 3.4. *Let G be a special group and let $\mathcal{T} \to \mathcal{S}$ be a G-torsor of algebraic stacks. Then we have the relation $\{\mathcal{T}\} = \{G\}\{\mathcal{S}\}$ in $K_0(\mathrm{Stack}_k)$. In particular, since $* \to \mathrm{B}G$ is a G-torsor, we have that $1 = \{G\}\{\mathrm{B}G\}$, so $\{G\}$ is invertible in $\mathrm{K}_0(\mathrm{Stack}_k)$. Furthermore, if F is an algebraic G-space and $\mathcal{E} \to \mathcal{S}$ is an F-fibration associated to a torsor as above, then $\{\mathcal{E}\} = \{F\}\{\mathcal{S}\}$.*

Proof. This is a direct application of Proposition 3.3 to Proposition 3.1. □

Corollary 3.5. *Let $G \to H$ be a homomorphism of algebraic groups with H special, and let F be an algebraic G-space with its G-action factoring through H. Assume that $\mathcal{X} \to \mathcal{S}$ is an F-fibration of stacks associated to a G-torsor. Then $\{\mathcal{X}\} = \{F\}\{\mathcal{S}\}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. Denote the G-torsor by $\mathcal{T} \to \mathcal{S}$. Since the action of G on F factors through H, we may view F as a H-space, which we denote ${}_HF$. Then we get a natural identification $F \cong {}_GH_H \times^H {}_HF$, where ${}_GH_H$ is just the group H regarded a (G, H)-space. The fibration $\mathcal{X}$ is obtained by taking the contraction product $\mathcal{T} \times^G ({}_GH_H \times^H {}_HF)$. Associativity of the contraction product gives that $\mathcal{X}$ is equivalent to the ${}_HF$-fibration associated to the H-torsor $\mathcal{T} \times^G {}_GH_H$. Since H is special, the result follows from Corollary 3.4. □

3.3 Computing the class of a classifying stack

In the actual computations, we shall use the following special cases of the multiplicativity results of the last section.

Proposition 3.6. *Let G be an affine group over a field k acting linearly on an n-dimensional k-vector space V. Then we have the relations*

$$\{\mathrm{B}G\} = \{[V/G]\} \cdot \mathbb{L}^{-n} = \{[\mathbb{P}(V)/G]\} \cdot \frac{\mathbb{L} - 1}{\mathbb{L}^n - 1}$$

in $\mathrm{K}_0(\mathrm{Stack}_k)$.

Proof. For the first equality, we apply Corollary 3.5, with GL_n as our special group, the space V as our fibre and the 1-morphism $[V/G] \to \mathrm{B}G$ as our V-fibration associated to the G-torsor $* \to \mathrm{B}G$. For the second equality, we instead have the fibre $\mathbb{P}(V)$ and the $\mathbb{P}(V)$-fibration $[\mathbb{P}(V)/G] \to \mathrm{B}G$ associated to the same torsor as above. □

Proposition 3.7. *Let $1 \to G \to E \to K \to 1$ be an exact sequence of algebraic space groups, flat over an algebraic stack $\mathcal{S}$, with E special. Then we have the relation*

$$\{\mathrm{B}_{\mathcal{S}}G\} = \{K\}/\{E\}$$

in $\mathrm{K}_0(\mathrm{Stack}_{\mathcal{S}})$.

Proof. The action of E on K by left translation gives an E-torsor $K \to [K/E]$, so $\{K\} = \{E\}\{[K/E]\}$ by Corollary 3.4. By the same corollary, we know that the class $\{E\}$ is invertible. Hence the result follows follows from the fact that the stack $[K/E]$ is equivalent to $\mathrm{B}_{\mathcal{S}}G$. □

Remark. Note that since E is special, it has affine fibres. This property is stable under taking closed subgroups, so the same is true for G. Since G is assumed to be flat, and also of finite presentation by our default assumption, over $\mathcal{S}$, it follows that its classifying stack is algebraic with affine stabilisers. Hence the statement above makes sense.

As a direct application of Proposition 3.7, we compute the class of the classifying stack of the group of n-th roots of unity.

Proposition 3.8. *The class of classifying stack* $\mathrm{B}\mu_n$ *for the group of* n*-th roots of unity is trivial in* $\mathrm{K}_0(\mathrm{Stack}_k)$ *for any field* k*.*

Proof. Consider the Kummer sequence $1 \to \mu_n \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to 1$. Since $\mathbb{G}_m$ is special, the statement follows from Proposition 3.7. □

For more complicated groups G, it is harder to compute the class of the classifying stack $\mathrm{B}G$. Our strategy will be to find a linear representation V and invoke Corollary 3.6. This reduces the problem to compute the class of the stack $[V/G]$. Stratifying V into locally closed G-invariant subschemes allows us to reduce the problem further. Before we illustrate how this can be done by computing some examples, we prove some propositions which will allow us to think of the action in a purely topological way for smooth groups G.

Assume that G acts on a space X. Then we have two maps $\sigma, q \colon G \times X \to X$, which are the action map and the projection on the second factor respectively. Recall that a subsheaf $Z \subset X$ is said to be *invariant* under the the action provided that the pullback of Z along σ is equal to the pullback along q when viewed as subsheaves of $G \times X$. If Z is G-invariant, then we get a 2-Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ [Z/G] & \longrightarrow & [X/G] \end{array}$$

allowing us to descend the properties of $Z \to X$ to the induced 1-morphism $[Z/G] \to [X/G]$.

Lemma 3.9. *Let* G *be a smooth group acting on a scheme* X *of finite type over a field* k*, and let* Y *be a reduced locally closed subscheme of* X*. Then* Y *is invariant under* G *if its closed points are in the set theoretic sense.*

Proof. Let σ and q denote the morphisms from $G \times X \to X$ given by the action and the second projection respectively. If we assume that $Y \to X$ is an open immersion, then the pullbacks along σ and q will be as well. Since, under our finiteness assumptions, an open subscheme is uniquely determined by the underlying set of closed points, the proposition holds in this case.

If we instead assume that Y is closed and reduced, the same will hold for both pullbacks since the group is smooth. Now the same argument as in the open case apply, since the reduced scheme structure on a closed set is unique.

In the general case, we have a factorisation $Y \to \overline{Y} \to X$ of immersions with the first being open and the second being closed and with $\overline{Y}$ reduced. Since the action is continuous, the set of closed points in $\overline{Y}$ will be G-invariant, and the lemma follows from the previous two cases. □

Proposition 3.10. *Let G be an algebraic group acting on a reduced scheme X over a field k containing a rational point x. Assume that G acts transitively on closed points. Then $[X/G]$ is equivalent to $\mathrm{B}G_x$, where G_x denotes the stabiliser of x under the action of G.*

Proof. The fact that the induced 1-morphism $\mathrm{B}G_x \to [X/G]$ is fully faithful is formal. Denote the pullback of x along σ by R. In order to verify that $\mathrm{B}G_x \to [X/G]$ is also essentially surjective, we prove that the composition

$$r\colon R \to G \times X \xrightarrow{q} X$$

is faithfully flat. Note that R admits a natural G-space structure, making the morphism r G-equivariant. On generalised points, this action is given by $h \cdot (g, e) := (gh^{-1}, he)$. Since X is reduced, it follows from generic flatness that there is an open subscheme $U \subset X$ over which r is flat. Using the G-equivariance and transitivity on points, we see that r must be flat everywhere. The transitivity also implies that r is surjective, so r is indeed faithfully flat. □

We end the section by working out some examples using the techniques described in this section. The results will be used in the next section.

Proposition 3.11. *Consider the group $G = \mathbb{G}_\mathrm{m} \rtimes \Sigma_2$, with Σ_2 acting as the automorphism group of $\mathbb{G}_\mathrm{m}$. If 2 is invertible in the field k, then $\{\mathrm{B}G\} = \mathbb{L}(\mathbb{L}^2 - 1)^{-1}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. To see this, consider the following action of G on $\mathbb{P}^1$. The subgroup $\mathbb{G}_\mathrm{m}$ acts by multiplication on the first homogeneous coordinate and by multiplication with the inverse on the second. The subgroup Σ_2 acts by permuting the homogeneous coordinates. This action obviously comes from a linear action, so we may apply Proposition 3.6 and get $\{\mathrm{B}G\} = \{[\mathbb{P}^1/G]\}(\mathbb{L}+1)^{-1}$.

The G-space $\mathbb{P}^1$ has two orbits. A closed orbit containing the point $(1:0)$ and an open orbit containing $(1:1)$. The stabilisers of these points are $\mathbb{G}_m$ and $\mu_2 \times \Sigma_2$ respectively. Stratifying the stack $[\mathbb{P}^1/G]$ according to these orbits and applying Proposition 3.10 gives the relation $\{[\mathbb{P}^1/G]\} = \{\mathrm{B}\mathbb{G}_\mathrm{m}\} + \{\mathrm{B}(\mu_2 \times \Sigma_2)\}$. Since $\mathbb{G}_\mathrm{m}$ is special, we have $\{\mathrm{B}\mathbb{G}_\mathrm{m}\} = (\mathbb{L}-1)^{-1}$. Furthermore $\mu_2 \times \Sigma_2 = \mu_2 \times \mu_2$ under our assumptions on the base field, so $\mathrm{B}(\mu_2 \times \Sigma_2)$ is isomorphic to $\mathrm{B}\mu_2 \times \mathrm{B}\mu_2$ which has class 1 according to Proposition 3.8. Combining these relations gives the desired result. □

Proposition 3.12. *Consider the subgroup $G = \mu_n \rtimes \Sigma_2$ of the group $\mathbb{G}_\mathrm{m} \rtimes \Sigma_2$ from the previous proposition. Assume that 2 is invertible in the field k and that 4 does not divide n. Then $\{\mathrm{B}G\} = 1$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. If $n = 2q$, with q odd, we have an isomorphism $G \simeq \mu_2 \times (\mu_q \rtimes \Sigma_2)$ which gives $\mathrm{B}G \simeq \mathrm{B}\mu_2 \times \mathrm{B}(\mu_q \rtimes \Sigma_2)$. This reduces the problem to the case when n is odd.

By using the same representation of G on $\mathbb{P}^1$ as in the proof of the previous proposition, we get the relation $\{\mathrm{B}G\} = \{[\mathbb{P}^1/G]\}(\mathbb{L}+1)^{-1}$. As before, we may also isolate the closed orbit $\{0, \infty\}$ to get the relation $\{[\mathbb{P}^1/G]\} = \{\mathrm{B}\mu_n\} + \{[U/G]\} = 1 + \{[U/G]\}$, where U denotes the complement. Since the subgroup μ_n acts freely on U, we have an isomorphism $[U/G] \cong [(U/\mu_n)/\Sigma_2]$. Here $U/\mu_n \cong \mathbb{G}_m$ on which Σ_2 acts as the automorphism group of $\mathbb{G}_m$. This action has two fixed points, namely ± 1. The quotient $\mathbb{G}_\mathrm{m}/\Sigma_2$ is isomorphic to $\mathbb{A}^1$. Therefore, the quotient $(\mathbb{G}_\mathrm{m} \setminus \{\pm 1\})/\Sigma_2$ is isomorphic to $\mathbb{A}^1$ minus two points. Hence the usual stratification argument gives $\{[\mathbb{G}_\mathrm{m}/\Sigma_2]\} = 2\{\mathrm{B}\Sigma_2\} + \mathbb{L} - 2 = \mathbb{L}$. The result follows by combining the relations. □

As a corollary, we get a very special case of Ekedahl's result that $\{\mathrm{B}\Sigma_n\} = 1$ for all n without any assumptions on the base field k [Eke09b].

Corollary 3.13. *Assume that 6 is invertible in k and that k contains the third roots of unity. Then $\{\mathrm{B}\Sigma_3\} = 1$.*

Remark. Note that the proof above does not work without modification for n even, since then the subgroup $\mu_2 < \mu_n$ acts trivially on $\mathbb{P}^1$.

4 The classes of $\mathrm{B}\,\mathrm{PGL}_2$ and $\mathrm{B}\,\mathrm{PGL}_3$

In this section, we start our investigation of the classes of $\mathrm{B}\,\mathrm{PGL}_2$ and $\mathrm{B}\,\mathrm{PGL}_3$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.

The group PGL_n, being the automorphism group of $\mathbb{P}^{n-1}$, has a natural action on the space H of degree d hypersurfaces in $\mathbb{P}^{n-1}$. In the case when $d = n$, this action is induced by a *linear* representation of PGL_n. Indeed, assume that $d = n$ and let V be an n-dimensional vector space. Consider the action of $\mathrm{GL}(V)$ on the space $(\mathrm{S}^d V)^\vee$ of d-forms given by

$$\alpha \cdot f = v \mapsto (\det \alpha) f(\alpha^{-1}(v)), \qquad \alpha \in \mathrm{GL}(V), \qquad f \in (\mathrm{S}^d V)^\vee.$$

Since $d = n$, the centre of $\mathrm{GL}(V)$ acts trivially on $(\mathrm{S}^d V)^\vee$, which gives us a linear PGL_n-representation. The space H of degree d-hypersurfaces in $\mathbb{P}^{n-1}$ is the projectivisation $\mathbb{P}((\mathrm{S}^d V)^\vee)$. For a detailed proof of this, see [Ser06, 4.3.2].

The fact that the group action is induced by a linear representation allows us to apply Proposition 3.6. This reduces the problem of computing the class of $\mathrm{B}\,\mathrm{PGL}_n$ to computing the class of the stack quotient $[H/\mathrm{PGL}_n]$. The space H admits a stratification into the open subspace H_ns and the closed subspace H_sing, which denote the spaces of non-singular and singular hypersurfaces respectively. Since this stratification is PGL_n-invariant, we get a corresponding stratification of $[H/\mathrm{PGL}_n]$ into $[H_\mathrm{ns}/\mathrm{PGL}_n]$ and $[H_\mathrm{sing}/\mathrm{PGL}_n]$.

When computing the stabilisers, it is convenient to use coordinates. When doing this, we will use the same conventions as described in the appendix. In order to avoid non-reduced stabiliser groups, we shall assume that $n!$ is invertible in the base field k. We let K denote an arbitrary field extension of k.

4.1 The class of $\mathrm{B}\,\mathrm{PGL}_2$

In the case $n = 2$, the spaces H_{ns} and H_{sing} consist of one orbit each. Let xy and x^2 be representatives for these orbits and let G_{xy} and G_{x^2} denote the corresponding stabilisers.

We prove that the group G_{x^2} is isomorphic to $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$. This follows if we prove that the stabiliser of the corresponding action of GL_2 is the subgroup of lower triangular matrices. This is easily seen to be true on K-points. Furthermore, a general element $I + \varepsilon(a_{ij})$ of $\mathrm{Lie}_K \mathrm{GL}_2$ takes the form x^2 to $x^2 + 2x\varepsilon(a_{11}x + a_{12}y)$. Since we are in characteristic $\neq 2$, this forces $a_{12} = 0$ for an element of the stabiliser of x^2. Hence the dimensions of the Lie algebra and the group coincides. This proves that the stabiliser is smooth and so is determined by its points.

The stabiliser of xy in GL_2 is the subgroup of monomial matrices. As in the previous case, this is first verified on points. A similar Lie-algebra computation as above, gives that the stabiliser is smooth regardless of the characteristic of the field. Taking the quotient with the scalar matrices gives $G_{xy} = \mathbb{G}_{\mathrm{m}} \rtimes \Sigma_2$.

The group $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$ is special, so the class of its classifying stack may be computed directly as the inverse $(\mathbb{L}(\mathbb{L}-1))^{-1}$ of the class of the group by Corollary 3.4. The class of $\mathrm{B}(\mathbb{G}_{\mathrm{m}} \rtimes \Sigma_2)$ was shown to be $\mathbb{L}(\mathbb{L}^2-1)^{-1}$ in Proposition 3.11. Combining these results gives the expression

$$\left(\frac{1}{\mathbb{L}(\mathbb{L}-1)} + \frac{\mathbb{L}}{\mathbb{L}^2-1}\right)\frac{\mathbb{L}-1}{\mathbb{L}^3-1} = \frac{1}{\mathbb{L}(\mathbb{L}^2-1)}$$

for the class of $\mathrm{B}\,\mathrm{PGL}_2$. This is the inverse of the class of PGL_2, which proves Theorem A.

4.2 The classes corresponding to singular plane cubics

There are eight singular cubic curves in $\mathbb{P}^2$ up to projective equivalence. These are listed in the appendix. This gives a stratification of H_{sing} into eight orbits, each containing rational points. Hence we get a corresponding stratification of the stack $[H_{\mathrm{sing}}/\,\mathrm{PGL}_3]$ into eight locally closed substacks, each equivalent to the classifying stack of a stabiliser. This allows us to write the class of $[H_{\mathrm{sing}}/\,\mathrm{PGL}_3]$ as the sum of the classes of these classifying stacks. The stabiliser groups are

a)	$\mathbb{G}_{\mathrm{a}}^2 \rtimes \mathrm{GL}_2$,	*b)*	$\mathbb{G}_{\mathrm{a}}^2 \rtimes \mathbb{G}_{\mathrm{m}}^2$,	*c)*	$\mathbb{G}_{\mathrm{a}}^2 \rtimes G$,	*d)*	N_3,
e)	$\mathbb{G}_{\mathrm{m}} \rtimes \Sigma_2$,	*f)*	$\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$,	*g)*	$\mathbb{G}_{\mathrm{m}}$,	*h)*	$\mu_3 \rtimes \Sigma_2$

and we will compute the classes of their classifying stacks to

a)	$\mathbb{L}^{-3}(\mathbb{L}+1)^{-1}(\mathbb{L}-1)^{-2}$	*b)*	$\mathbb{L}^{-2}(\mathbb{L}-1)^{-2}$	*c)*	$\mathbb{L}^{-2}(\mathbb{L}-1)^{-1}$	*d)*	$\{\mathrm{B}N_3\}$
e)	$\mathbb{L}(\mathbb{L}+1)^{-1}(\mathbb{L}-1)^{-1}$	*f)*	$\mathbb{L}^{-1}(\mathbb{L}-1)^{-1}$	*g)*	$(\mathbb{L}-1)^{-1}$	*h)*	1.

Most of these are easy to compute. Indeed, the groups in the cases a, b, f and g are special, so the classes of their classifying stacks are inverses to the classes of the groups themselves. The classes in the cases e end h are given by Proposition 3.11 and 3.12 respectively.

In case c, we have the group $\mathbb{G}_{\mathrm{a}}^2 \rtimes G$, where G is the subgroup $\mathbb{G}_{\mathrm{m}} \rtimes \Sigma_3$ of GL_2 generated by its centre and the embedding of Σ_3 induced by its irreducible 2-dimensional representation. The inclusions $G \hookrightarrow \mathrm{GL}_2$ and $\mathbb{G}_a^2 \rtimes G \hookrightarrow \mathbb{G}_a^2 \rtimes \mathrm{GL}_2$ both give rise to the same quotient space. Since both groups on the right hand side of these arrows are special, we get the relation

$$\{\mathrm{B}G\}\{\mathrm{GL}_2\} = \{\mathrm{B}(\mathbb{G}_a^2 \rtimes G)\}\{\mathbb{G}_a^2 \rtimes \mathrm{GL}_2\}$$

by Proposition 3.7. This reduces the problem of computing $\{\mathrm{B}(\mathbb{G}_a^2 \rtimes G)\}$ to computing $\{\mathrm{B}G\}$. To do this, consider the representation V given by the embedding of G in GL_2 just mentioned. This gives us $\mathbb{L}^2\{\mathrm{B}G\} = \{[V/G]\}$. The stack $\{[V/G]\}$ may be stratified in the substacks $\mathrm{B}G$ and $[V_0/G]$, where V_0 denotes the subspace of V where the origin is removed. Using the scissors relations and solving for $\{\mathrm{B}G\}$ gives $\{\mathrm{B}G\} = [V_0/G]/(\mathbb{L}^2-1)$. Note that we have an equivalence $[V_0/G] \cong [\mathbb{P}^1/\Sigma_3]$. The action of Σ_3 on $\mathbb{P}^1$ in the latter stack quotient factors through GL_2. Hence $\{[\mathbb{P}^1/\Sigma_3]\} = \{\mathbb{P}^1\}\{\mathrm{B}\Sigma_3\} = \mathbb{L}+1$. This allows us to conclude that the class of $\mathrm{B}(\mathbb{G}_a^2 \rtimes G)$ is $\mathbb{L}^{-2}(\mathbb{L}-1)^{-1}$.

5 Some equivalences of moduli stacks

Recall that H_{ns} denotes the space of smooth degree 3 hypersurfaces in $\mathbb{P}^2$. In the last section, we saw how the class of $\mathrm{B}\,\mathrm{PGL}_3$ was related to the class of the stack $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$. We shall now study the stack quotient $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ more closely. It may be worth noting that in this section we will not need any restrictions on the base we are working over. The results hold over $\operatorname{Spec}\mathbb{Z}$.

Since all degree 3 hypersurfaces in $\mathbb{P}^2$ are smooth genus 1 curves, it seems natural to assume that $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ is somehow related to the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves. The main result of this section is that $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ is equivalent to the neutral gerbe $\mathrm{B}_{\mathcal{M}_{1,1}}\mathcal{E}[3]$ over $\mathcal{M}_{1,1}$ associated to the 3-torsion subgroup $\mathcal{E}[3]$ of the universal curve $\mathcal{E}$. We do this by first establishing the equivalence to $\mathcal{M}_{1,(3)}$, the moduli stack of genus 1 curves polarised in degree 3.

5.1 Moduli of polarised genus 1 curves

Consider a smooth genus 1 curve $C \to S$ over a scheme. By a *polarisation* of C in degree d, we mean a global section of the sheaf $\underline{\mathrm{Pic}}^d_{C/S}$. Since for any morphism $S' \to S$ there is a natural identification of $\underline{\mathrm{Pic}}_{C/S} \times_S S'$ with $\underline{\mathrm{Pic}}_{C_{S'}/S'}$, we may pull back polarisations on $C \to S$ to $C_{S'} \to S'$. This allows us to define the fibred category $\mathcal{M}_{1,(d)}$ of genus one curves polarised in degree d. The objects are genus 1 curves together with degree d polarisations, and the morphisms are Cartesian squares respecting these polarisations. That $\mathcal{M}_{1,(d)}$ is a stack follows from the sheaf property of $\underline{\mathrm{Pic}}^d_{C/S}$.

We want to establish an equivalence between the stack quotient $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ and $\mathcal{M}_{1,(3)}$. First we give an explicit description of the pre-stack quotient $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]^{\mathrm{pre}}$ as a category fibred in groupoids over the category of schemes. Its object are the same as the objects of H_{ns}, i.e. smooth genus 1 curves embedded in $\mathbb{P}^2_T$ over some scheme T. Now let $f\colon T' \to T$ be a morphism of schemes and let $\iota'\colon C' \hookrightarrow \mathbb{P}^2_{T'}$ and $\iota\colon C \hookrightarrow \mathbb{P}^2_T$ be objects over T' and T respectively. A morphism from ι' to ι over f is then given by a pair (σ, α), where α is an automorphism of $\mathbb{P}^2_{T'}$ and $\sigma\colon C' \to C$ is a morphism such that the diagram

$$\begin{array}{ccc} C' & \xhookrightarrow{\alpha \circ \iota'} & \mathbb{P}^2_{T'} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mathbb{P}^2(f)} \\ C & \xhookrightarrow{\iota} & \mathbb{P}^2_T \end{array}$$

is Cartesian.

Now we define a 1-morphism $f\colon [H_{\mathrm{ns}}/\operatorname{PGL}_3] \to \mathcal{M}_{1,(3)}$ of stacks. By the universal property of stackification, it is enough to define it on the pre-stack quotient, which we denote by $[H_{\mathrm{ns}}/\operatorname{PGL}_3]^{\mathrm{pre}}$. It takes objects $\iota\colon C \hookrightarrow \mathbb{P}^2_T$ to pairs $(C \to T, [\iota^*\mathcal{O}(1)])$ and morphisms (σ, α) to σ. Note that f is well-defined on objects since smooth degree 3 hyper surfaces of $\mathbb{P}^2$ are smooth genus 1 curves and well-defined on morphisms since the automorphism α does not affect the isomorphism class of the pulled back line bundle.

Proposition 5.1. *The 1-morphisms $f\colon [H_{\mathrm{ns}}/\operatorname{PGL}_3] \to \mathcal{M}_{1,(3)}$ defined in the paragraph above is an equivalence of stacks.*

Proof. Let $\iota\colon C \to \mathbb{P}^2_T$ be an object of $[H_{\mathrm{ns}}/\operatorname{PGL}_3]^{\mathrm{pre}}$ over a scheme T, and denote the structure maps to T by $q\colon C \to T$ and $p\colon \mathbb{P}^2 \to T$ respectively. We also use the shorthand notation $\mathscr{L}$ for the invertible sheaf $\iota^*\mathscr{O}(1)$. In order to prove that f is fully faithful, it is enough to prove that it induces an isomorphism between the automorphism group of $\iota\colon C \to \mathbb{P}^2_T$ in $[H_{\mathrm{ns}}/\operatorname{PGL}_3]$ and the automorphism group of $(q\colon C \to T, [\mathscr{L}])$ in $\mathcal{M}_{1,(3)}$.

First note that the $\mathscr{O}_T$-module homomorphism $p_*\mathscr{O}(1) \to q_*\mathscr{L}$ corresponding to the embedding as described in [DG61, §4.2] is an isomorphism. Since this may be verified locally, we may assume that we have a short exact sequence of quasi-coherent $\mathscr{O}_{\mathbb{P}^2}$-modules

$$0 \to \mathscr{O}(-3) \to \mathscr{O} \to \iota_*\mathscr{O}_C \to 0. \tag{1}$$

Tensoring with the fundamental sheaf $\mathscr{O}(1)$ and using the projection formula on the last term gives a new short exact sequence

$$0 \to \mathscr{O}(-2) \to \mathscr{O}(1) \to \iota_*\mathscr{L} \to 0.$$

Pushing this forward to T gives rise to the exact sequence

$$0 \to p_*\mathscr{O}(-2) \to p_*\mathscr{O}(1) \to q_*\mathscr{L} \to R^1p_*\mathscr{O}(-2).$$

The map in the middle is the canonical map mentioned above, and it is an isomorphism since both the first and last terms vanish [Har77, Thm. III.5.1]. This allows us to assume that $\mathbb{P}^2_T = \mathbb{P}(q_*\mathscr{L})$ and that the embedding ι corresponds to the canonical map $\varepsilon\colon q^*q_*\mathscr{L} \to \mathscr{L}$.

The functor f is faithful. To prove this, it is enough to show that for any automorphism of $\iota\colon C \to \mathbb{P}(q_*\mathscr{L})$ of the form (id_C, α), the $\mathbb{P}(q_*\mathscr{L})$-automorphism α is the identity. This may be verified locally. Hence we may assume that the automorphism α is of the form $\mathbb{P}(\beta)$, where β is an $\mathscr{O}_T$-module automorphism of $q_*\mathscr{L}$. The criterion that α fixes the embedding ι is that there exists an $\mathscr{O}_C$-module automorphism γ of $\mathscr{L}$ such that the diagram

$$\begin{array}{ccc} q^*q_*\mathscr{L} & \longrightarrow & \mathscr{L} \\ {\scriptstyle q^*\beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ q^*q_*\mathscr{L} & \longrightarrow & \mathscr{L} \end{array}$$

commutes. By using the adjunction property of the pair (q^*, q_*), we see that β must be of the form $q_*\gamma$. The automorphism γ may be viewed as a global section of $\mathscr{O}_C^\times$. If we apply p_* to the exact sequence (1), we get the exact sequence

$$0 \to p_*\mathscr{O}(-3) \to p_*\mathscr{O} \to q_*\mathscr{O}_C \to R^1p_*\mathscr{O}(-3).$$

Since both $p_*\mathscr{O}(-3)$ and $R^1p_*\mathscr{O}(-3)$ vanish, we see that $q_*\mathscr{O}_C \cong p_*\mathscr{O}$, with the latter sheaf being isomorphic to $\mathscr{O}_T$. Hence $q_*\gamma$ is a global section of $\mathscr{O}_T^\times$. It follows that the automorphism α is the identity.

The functor f is full. To prove this, we need to verify that the map on automorphisms is surjective. Let σ be a T-automorphism of C such that $[\sigma^*\mathscr{L}] = [\mathscr{L}]$ in $\underline{\mathrm{Pic}}_{C/T}(T)$. It is enough to show that σ locally is given by an automorphism of $\mathbb{P}(q_*\mathscr{L})$, so we may assume that $\sigma^*\mathscr{L} \simeq \mathscr{L}$. The new embedding $\iota\circ\sigma$ then corresponds to the automorphism $\alpha\colon q_*\mathscr{L} \to q_*\mathscr{L}$ given by $s \mapsto \sigma^*(s)$. It follows that $\mathbb{P}(\alpha)\colon \mathbb{P}(q_*\mathscr{L}) \to \mathbb{P}(q_*\mathscr{L})$ is our sought automorphism.

The functor f is essentially surjective. This may also be checked fppf-locally. Hence, given an object $(q\colon C \to T, \lambda)$ of $\mathcal{M}_{1,(3)}$, we may assume that λ comes from a line bundle $\mathscr{L}$ of degree 3 on C. By Proposition 2.3, the push forward $q_*\mathscr{L}$ is locally free of rank 3, and we get an embedding of C into the projective bundle $\mathbb{P}(q_*\mathscr{L})$. By extending the base further if necessary, we may assume that this bundle is $\mathbb{P}^2_T$, so our object $(q\colon C \to T, \lambda)$ comes from an object of $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]^{\mathrm{pre}}$. □

5.2 An interlude on torsors

In order to describe $\mathcal{M}_{1,(3)}$ as a classifying stack, we would like to reinterpret polarisations in terms of torsors. It turns out that much of this may be worked out in the general theory for torsors for abelian sheaves over an arbitrary site $\mathcal{C}$. Hence we make a short interlude, working in this generality.

Let A be a fixed sheaf of abelian groups on $\mathcal{C}$. Given an A-torsor T and a positive integer n, we have a map $n_T\colon T \to T^n$ taking a local section t of T to its n-fold contraction power $(t, \dots, t)$. In particular, $n_A\colon A \to A$ is the map taking a generalised point a to its n-fold product a^n using the group law. The kernel of this map is the n-torsion subgroup of A, which we denote by $A[n]$. If A is an n-torsion group, there is a canonical identification of T^n with A for each A-torsor T.

Lemma 5.2. *Let A be a sheaf of abelian groups on a site $\mathcal{C}$ and T an A-torsor. If A is an n-torsion group, then the torsor T^n has a canonical global section κ.*

Proof. Fix an object $S \in \mathcal{C}$ such that $T(S)$ is non-empty and let $x, y \in T(S)$. Then $y = a \cdot x$ for some group element $a \in A(S)$. We have $y^n = (a \cdot x)^n = a^n \cdot x^n = x^n$, since A is an n-torsion group. It follows that T^n has a canonical S-point $\kappa_S = x^n$. Taking a covering S_i such that $T(S_i)$ has sections, the canonical local sections κ_{S_i} glue together to the global section κ. ☐

We define the category $\mathrm{B}_n A$, fibred over $\mathcal{C}$, as the category of pairs $(T \to S, \lambda \colon S \to T^n)$, where $T \to S$ is an A-torsor over some object S in $\mathcal{C}$ and λ is a global section of T^n. Morphisms are pullbacks of sheaves respecting the global sections. Recall that the inclusion $A[n] \to A$ induces a morphism $\mathrm{B}A[n] \to \mathrm{B}A$ taking an $A[n]$-torsor T to the A-torsor $T' = A \times^{A[n]} T$. The canonical global section κ of T^n allows us to define a canonical global section $(1, \kappa)$ of $(T')^n \cong A \times^{A[n]} T^n$. Hence we get a natural map

$$\mathrm{B}A[n] \to \mathrm{B}_n A$$

through which $\mathrm{B}A[n] \to \mathrm{B}A$ factors. This is not an equivalence in general, but we have the following result.

Proposition 5.3. *Let A be a sheaf of abelian groups such that the map $n_A \colon A \to A$ is surjective. Then the natural map $\mathrm{B}A[n] \to \mathrm{B}_n A$ is an equivalence of stacks.*

Proof. We prove the equivalence by constructing a 2-inverse explicitly. Given an object $(T \to S, \lambda)$, we may define the subsheaf $T_\lambda \subset T$ over S as the pullback of $n_T \colon T \to T^n$ along the map $\lambda \colon S \to T^n$. On S'-points, this may be described as

$$T_\lambda(S') := \{x \in T(S') \mid x^n = \lambda \text{ in } T^n(S')\}.$$

From this description it is straightforward to verify that the A-action on T restricts to a well-defined $A[n]$-action on T_λ. This is free and transitive, making T_λ a pseudo-torsor for $A[n]$. Locally, the morphism n_T is just n_A, so n_T is surjective. Hence the same holds for the structure map $T_\lambda \to S$, which proves that T_λ actually is a torsor.

Given objects (T, λ) and T' in $\mathrm{B}_n A$ and $\mathrm{B}A[n]$ respectively, we have natural maps

$$\eta_{(T,\lambda)} \colon A \times^{A[n]} T_\lambda \to T, \qquad \varepsilon_{T'} \colon T' \to (A \times^{A[n]} T')_{(1,\kappa)}$$

given on generalised points by $(a, t) \mapsto at$ and $t \mapsto (1, t)$ respectively. The reader may verify that these are isomorphisms in the categories $\mathrm{B}_n A$ and $\mathrm{B}A[n]$ respectively. ☐

5.3 The stack of polarised genus 1 curves as a gerbe

Now we apply the results from the previous subsection to our situation with the stack $\mathcal{M}_{1,(n)}$ to show that it is a gerbe over $\mathcal{M}_{1,1}$. However, we cannot use the result directly, since the base $\mathcal{M}_{1,1}$ is a stack rather than a scheme. Proposition 5.3 could be generalised to this situation, but we shall instead just give the explicit description in this special case. Since it should be easy to fill in the details, we shall allow ourselves to be somewhat sketchy.

The fibred category $\mathrm{B}_{\mathcal{M}_{1,1}}\mathcal{E}[n]$ over schemes has pairs

$$(E \to S, T \to S),$$

as objects, where $E \to S$ is an elliptic curve and $T \to S$ is an $E[n]$-torsor. We will use the rest of the section to prove the following proposition:

Proposition 5.4. *The stack $\mathcal{M}_{1,(n)}$ is equivalent to $\mathrm{B}_{\mathcal{M}_{1,1}}\mathcal{E}[n]$.*

Consider the fibred category $(\mathrm{B}_n)_{\mathcal{M}_{1,1}}\mathcal{E}$ over schemes whose objects are triples

$$(E \to S, T \to S, \lambda\colon S \to T^n),$$

where $E \to S$ is an elliptic curve and $T \to S$ is an E-torsor. Now let $(C \to S, \lambda\colon S \to \underline{\mathrm{Pic}}^n_{C/S})$ be an object of $\mathcal{M}_{1,(n)}$. Since the Picard sheaf $\underline{\mathrm{Pic}}_{C/S}$ is an extension of $\underline{\mathrm{Pic}}^0_{C/S}$ by $\mathbb{Z}_S$, the component $\underline{\mathrm{Pic}}^1_{C/S}$ is an $\underline{\mathrm{Pic}}^0_{C/S}$-torsor and $\underline{\mathrm{Pic}}^n_{C/S}$ is canonically isomorphic to its n-th contraction power. The group $\underline{\mathrm{Pic}}^0_{C/S}$ is an elliptic curve, being the Jacobian of a genus 1 curve. Hence we get a well-defined 1-morphism $\mathcal{M}_{1,(n)} \to (\mathrm{B}_n)_{\mathcal{M}_{1,1}}\mathcal{E}$ over $\mathcal{M}_{1,1}$ taking the object to

$$(\underline{\mathrm{Pic}}^0_{C/S}, \underline{\mathrm{Pic}}^1_{C/S}, \lambda\colon S \to \underline{\mathrm{Pic}}^n_{C/S}).$$

Note that since $C \to S$ is a smooth genus 1 curve, there is a canonical isomorphism $C \to \underline{\mathrm{Pic}}^1_{C/S}$, so this 1-morphism has an obvious 2-inverse.

Now we consider the functor $f\colon \mathrm{B}_{\mathcal{M}_{1,1}}\mathcal{E}[n] \to (\mathrm{B}_n)_{\mathcal{M}_{1,1}}\mathcal{E}$. This is defined analogously with the equivalence in the previous section by taking $(E \to S, T \to S)$ to $(E \to S, E \times^{E[n]} T, (1, \kappa))$. For an arbitrary scheme S and a morphism $S \to \mathcal{M}_{1,1}$, corresponding to an elliptic curve $E \to S$, the functor above pulls back to the functor $f_S\colon \mathrm{B}_S E[n] \to (\mathrm{B}_n)_S E$. Since $n_E : E \to E$ is an isogeny, and in particular surjective on the underlying sheaves, we are now in the situation where we can apply Proposition 5.3. Therefore f_S, and hence also f, is an equivalence, and we are done.

6 Finishing the computation for the class of $\mathrm{B}\,\mathrm{PGL}_3$

In this final section, we show that the class of $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ equals $\mathbb{L}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$ under our hypothesis on k. This is the last piece of information we need in order to prove Theorem B. This is done by considering the description of $[H_{\mathrm{ns}}/\,\mathrm{PGL}_3]$ as a classifying stack for the 3-torsion subgroup $\mathcal{E}[3]$ of the universal curve over $\mathcal{M}_{1,1}$ established in the previous section. For primes ℓ which are invertible in the structure sheaf of the base, the Weil pairing gives the ℓ-torsion subgroup of an elliptic curve the structure of a *symplectic local system*. We start by recalling this notion.

Let ℓ be an arbitrary prime. By a rank n *local system* over a scheme S, we mean a sheaf V which is locally isomorphic to a rank n $\mathbb{F}_\ell$-vector space considered as a constant sheaf.

A *symplectic local system* is a pair (V, ω), where V is a local system and $\omega\colon V \times V \to \mathbb{F}_\ell$ is a symplectic form. Assume that S is connected and that $\Gamma = \pi_1(S, \overline{\xi})$ is the étale fundamental group of S with respect to some geometric point $\overline{\xi} \in S$. Then the pair (V, ω) corresponds to a pair consisting of an n-dimensional Γ-representation over $\mathbb{F}_\ell$ and a Γ-invariant symplectic form. By abuse of notation, we denote this pair by (V, ω) as well.

Now we fix an odd prime ℓ and a 2-dimensional symplectic local system (V, ω) over a connected scheme S with fundamental group Γ. Let $V_0 \subset V$ denote the Γ-invariant subset where the origin in V has been removed. We get a surjection from the free Γ-module $\mathbb{Z}[V_0]$ on the Γ-set V_0 to V taking a formal sum to an actual sum. This gives rise to an exact sequence

$$0 \to K \to \mathbb{Z}[V_0] \to V \to 0$$

of Γ-modules.

Denote the set of lines through the origin in V by $\mathbb{P}(V)$. Then we have a surjection $V_0 \to \mathbb{P}(V)$ of Γ-sets inducing a surjection $\mathbb{Z}[V_0] \to \mathbb{Z}[\mathbb{P}(V)]$ of Γ-modules. The map $K \to \mathbb{Z}[\mathbb{P}(V)]$ given by the obvious composition is also a surjection. Indeed, each standard basis element $(\mu : \lambda)$ of $\mathbb{Z}[\mathbb{P}(V)]$ lifts to $2(\mu, \lambda) - (2\mu, 2\lambda)$ in K.

The symplectic form ω allows us to define an endomorphism φ on $\mathbb{Z}[V_0]$ by

$$v \mapsto \sum_{\omega(v,u)=1} u, \qquad v, u \in V_0.$$

This is Γ-equivariant since ω is Γ-invariant. The image of φ lies in K. This can be seen by choosing v' such that $\omega(v, v') = 1$ and letting W be the subspace of vectors u such that $\omega(v, u) = 0$. Then v maps to $\#W \cdot v' + \sum_{u \in W} u$ in V, which indeed is zero.

The endomorphism φ descends to a corresponding endomorphism φ' on $\mathbb{Z}[\mathbb{P}(V)]$ given by

$$P \mapsto \sum_{\omega(P,Q)\neq 0} Q, \qquad P, Q \in \mathbb{P}(V).$$

Since $\mathbb{P}(V)$ has $\ell + 1$ points, this endomorphism is described by an $\ell + 1$ by $\ell + 1$ matrix with respect to the standard basis. All the elements of this matrix are one, except for the elements on the main diagonal which are zero. Since such a matrix has determinant $-\ell$, it follows that φ' is injective with cokernel of order ℓ. Hence we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'' & \longrightarrow & K' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}[V_0] & \longrightarrow & K & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}[\mathbb{P}(V)] & \longrightarrow & \mathbb{Z}[\mathbb{P}(V)] & \longrightarrow & \mathbb{Z}/\ell\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proposition 6.1. *Let S be a scheme and let (V, ω) be an $\mathbb{F}_3$-symplectic local system of rank 2 over S. Then the class $\{\mathrm{B}_S V^\vee\} = 1$ in $\mathrm{K}_0(\mathrm{Stack}_S)$.*

Proof. First one needs to check that $A = 0$ in the diagram above. A straightforward computation gives $\det \varphi = -3^3$. Here we view φ as an endomorphism of $\mathbb{Z}[V_0]$. Since K has index ℓ^2 in $\mathbb{Z}[V_0]$, it follows that B has order ℓ. Therefore the map $B \to \mathbb{Z}/\ell\mathbb{Z}$ must be an isomorphism, which indeed gives $A = 0$ by exactness of the last column.

Next we take the Cartier dual of the diagram. The maps $\mathbb{Z}[V_0]^\vee \to K''^\vee$ and $K^\vee \to K'^\vee$ are both $\mathbb{Z}[\mathbb{P}(V)]^\vee$-torsors. Since $\mathbb{Z}[\mathbb{P}(V)]^\vee$ is quasi-split, and therefore special, we get $\{\mathbb{Z}[\mathbb{P}(V)]^\vee\}\{K''^\vee\} = \{\mathbb{Z}[V_0]^\vee\}$ and $\{\mathbb{Z}[\mathbb{P}(V)]^\vee\}\{K'^\vee\} = \{K^\vee\}$. Since we have seen that $K' \simeq K''$, it follows that $\{\mathbb{Z}[V_0]^\vee\} = \{K^\vee\}$.

But $\mathbb{Z}[V_0]^\vee$ is also a quasi-split torus. Hence the result follows by applying Proposition 3.7 to the exact sequence

$$0 \to V^\vee \to \mathbb{Z}[V_0]^\vee \to K^\vee \to 0.$$

□

Remark. The map φ is defined also when we have a symplectic local system of higher rank. It does, however, not induce an isomorphism between K'' and K' in general. Indeed this is not even true for rank 2 symplectic local systems. In this case experiments suggest that the determinant of φ is given by $(-1)^{\frac{\ell-1}{2}}\ell^{\binom{\ell}{2}}$. In other words, the fact that we get an isomorphism in the case we are interested in seems to be a coincidence.

Corollary 6.2. *Let k be a field of characteristic not equal to 3 containing all third roots of unity. Then the class of $\mathcal{M}_{1,(3)}$ equals the class of $\mathcal{M}_{1,1}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. Let $E \to S$ be an elliptic curve over a scheme S. Since 3 is invertible, the 3-torsion subgroup $E[3]$ is étale over S and locally isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Since also k contains all third roots of unity, the Weil pairing gives a symplectic form on $E[3]$ and the same holds for the dual $E[3]^\vee$. Hence, by the previous proposition, we have $\{\mathrm{B}_S E[3]\} = 1$. Since we have established the equivalence $\mathcal{M}_{1,(3)} \simeq \mathrm{B}_{\mathcal{M}_{1,1}}\mathcal{E}[3]$ the result now follows by applying Proposition 3.3 with $C = 1$. □

Proposition 6.3. *Let k be a field in which 6 is invertible. Then $\{\mathcal{M}_{1,1}\} = \mathbb{L}$ in $\mathrm{K}_0(\mathrm{Stack}_k)$.*

Proof. For ease of notation, we denote $\mathcal{M}_{1,1}$ by $\mathcal{M}$. There is a map $j\colon \mathcal{M} \to \mathbb{A}^1 = \operatorname{Spec} k[t]$ to the coarse moduli space induced by the classical j-invariant. Consider the closed points $\{0\}$ and $\{1728\}$ in $\mathbb{A}^1$, and denote their complement by U. This induces a stratification of $\mathcal{M}$ into the closed substacks $\mathcal{M}_0$ and $\mathcal{M}_{1728}$ and the open complement $\mathcal{M}_U$.

The stack $\mathcal{M}_U$ is equivalent to $\mathrm{B}_U\Sigma_2$ over U. Indeed, the inertia of $\mathcal{M}_U \to U$ is the automorphism group of the universal elliptic curve $\mathcal{E}_U \to \mathcal{M}_U$, which is Σ_2 since we removed the curves with j-invariants 0 or 1728. In particular, the inertia stack is faithfully flat of finite presentation over $\mathcal{M}_U$, so $\mathcal{M}_U \to U$ is a gerbe. Moreover, we see that it is the neutral gerbe since $\mathcal{M}_U \to U$ has a section. This section is induced by the elliptic curve E defined by the equation

$$y^2z + xyz = x^3 - \frac{36}{t-1728}xz^2 - \frac{1}{t-1728}z^3$$

over U. It is a straightforward computation to check that $E \to U$ is an elliptic curve whose fibres E_t have j-invariant t over closed points $t \in U$.

It is of course easy to construct elliptic curves with j-invariants 0 and 1728 over k, so both the stacks $\mathcal{M}_0$ and $\mathcal{M}_{1728}$ are neutral gerbes over k. Since we assume that 6 is invertible in the base field, the automorphism groups of elliptic curves with j-invariants 0 and 1728 are μ_6 and μ_4 respectively [Hus04, 3.4]. It follows that $\mathcal{M}_0 \simeq \mathrm{B}\mu_6$ and $\mathcal{M}_{1728} \simeq \mathrm{B}\mu_4$.

Now it follows by Proposition 3.8 and the scissors relations that the class of $\mathcal{M}$ equals

$$\{\mathrm{B}_U\Sigma_2\} + \{\mathrm{B}\mu_6\} + \{\mathrm{B}\mu_4\} = \mathbb{L} - 2 + 1 + 1 = \mathbb{L}$$

in $\mathrm{K}_0(\mathrm{Stack}_k)$. □

We are now in position to prove Theorem B.

Proof of Theorem B. Using the notation introduced in Section 4, we let H denote the space of plane cubics, H_{sing} the subspace of singular cubics and H_{ns} the space of non-singular cubics. Since $H \simeq \mathbb{P}^9$, we get

$$\{\mathrm{B}\,\mathrm{PGL}_3\} = \{[H/\mathrm{PGL}_3]\}\frac{\mathbb{L}-1}{\mathbb{L}^{10}-1}$$

by Proposition 3.6. Since H_{sing} is a closed PGL_3-invariant subspace of H, we get the identity $\{[H/\mathrm{PGL}_3]\} = \{[H_{\mathrm{sing}}/\mathrm{PGL}_3]\} + \{[H_{\mathrm{ns}}/\mathrm{PGL}_3]\}$ by the scissors relations. Combining Corollary 6.2 and Proposition 6.3, we get $\{[H_{\mathrm{ns}}/\mathrm{PGL}_3]\} = \mathbb{L}$ under the hypothesis on the base field. For the readers convenience, we again list the classes of the classifying stacks for the stabilisers of the singular curves as described in Section 4.2.

a)	$\mathbb{L}^{-3}(\mathbb{L}+1)^{-1}(\mathbb{L}-1)^{-2}$	*b)*	$\mathbb{L}^{-2}(\mathbb{L}-1)^{-2}$	*c)*	$\mathbb{L}^{-2}(\mathbb{L}-1)^{-1}$	*d)*	$\{\mathrm{B}N_3\}$
e)	$\mathbb{L}(\mathbb{L}+1)^{-1}(\mathbb{L}-1)^{-1}$	*f)*	$\mathbb{L}^{-1}(\mathbb{L}-1)^{-1}$	*g)*	$(\mathbb{L}-1)^{-1}$	*h)*	1.

Since $\{[H_{\mathrm{sing}}/\mathrm{PGL}_3]\}$ is simply the sum of these classes, we get the desired result by elementary algebraic manipulations. □

Remark. Denote by $\mathrm{K}_0^{\mathrm{PGL}_3}(\mathrm{Stack}_k)$ the ring where we formally add the relations $\{T\} = \{\mathrm{PGL}_3\}\{S\}$ for all PGL_3-torsors $T \to S$ in $\mathrm{K}_0(\mathrm{Stack}_k)$. As a corollary of Theorem B, we get that the class of $\mathrm{B}N_3$ is

$$\frac{\mathbb{L}^3}{(\mathbb{L}-1)^2(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)}$$

in $\mathrm{K}_0^{\mathrm{PGL}_3}(\mathrm{Stack}_k)$. It is possible to check this more directly by considering the natural action of N_3 on $\mathbb{P}^2$. This action has three orbits represented by the points $(1:0:0)$, $(1:1:0)$ and $(1:1:1)$ respectively. The classes of the classifying stacks of the stabilisers for these points are quite easily computed, even in $\mathrm{K}_0(\mathrm{Stack}_k)$. The reader may do this and verify that the formula for $\{N_3\}$ indeed is correct. Note that since the action we considered is not induced by a linear action, we cannot apply Proposition 3.6 and get the class in $\mathrm{K}_0(\mathrm{Stack}_k)$ by using this representation.

A Singular plane cubics and stabilisers

Throughout the appendix, we let k be a field in which 6 is invertible. It is a classical result that there exist eight singular cubic curves in $\mathbb{P}^2_k$ up to projective equivalence. These correspond to orbits in the space of singular cubics in $\mathbb{P}^2$ under the natural action of PGL_3 by change of coordinates. In this appendix, we will determine the stabiliser groups corresponding to these orbits up to isomorphism. The result is described in the table below.

	Description	Standard Form	Components	Stabiliser
a)	Triple line	x^3	$3 \cdot 1$	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathrm{GL}_2$
b)	Double and single line	x^2y	$2 \cdot 1 + 1$	$\mathbb{G}_\mathrm{a}^2 \rtimes \mathbb{G}_\mathrm{m}^2$
c)	Three lines through a point	$x^2y + xy^2$	$1 + 1 + 1$	$\mathbb{G}_\mathrm{a}^2 \rtimes G$
d)	Three general lines	xyz	$1 + 1 + 1$	N_3
e)	Int. conic and general line	$xyz + z^3$	$2 + 1$	$\mathbb{G}_\mathrm{m} \rtimes \Sigma_2$
f)	Int. conic and tangent line	$y^2z + x^2y$	$2 + 1$	$\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$
g)	Cuspidal cubic	$x^2z + y^3$	3	$\mathbb{G}_\mathrm{m}$
h)	Nodal cubic	$xyz + x^3 + y^3$	3	$\mu_3 \rtimes \Sigma_2$

The table lists the type of singular curve, the equation for a prototypical curve, and the degrees of the components of the curve as well as the stabiliser group up to isomorphism. The symbol N_3 denotes the normaliser of the maximal torus in PGL_3. Explicitly, this group may be described as the group of monomial 3×3 matrices up to multiplication by a scalar. The group denoted by G is the subgroup of GL_2 generated by the scalar matrices and the embedding of Σ_3 in GL_2 induced by the 2-dimensional irreducible representation.

In characteristic zero, the stabiliser groups are determined by the points of the underlying topological space. In positive characteristic however, we must also account for the possibility of the stabilisers not being reduced. Our assumption on the base field asserts that this situation does not occur. This may be verified by determining the dimension of the Lie algebra for the stabiliser. We will go through these arguments in detail for the first computations only and leave the rest for the reader to verify.

When using coordinates in our arguments, we use the convention that PGL_3 acts by standard transformation of coordinates on $\mathbb{P}^2$ from the left. This means that the action on forms is dual and hence is a right action. We will frequently represent elements in PGL_3 as 3×3-matrices. When doing so, taking the quotient by the scalar matrices is implicit. The corresponding convention applies when we discuss the Lie algebra of PGL_3.

A.1 Three Lines

First we treat the case when the form defining the curve is a product of three linear forms. There are four different configurations to consider.

(a) A triple line. We choose our prototypical curve such that it is defined by the form x^3. On points, this is the same as the stabiliser of the line $x = 0$. This consists of

the matrices (a_{ij}) such that $a_{12} = a_{13} = 0$. By normalising the coordinates by setting $a_{11} = 1$, we get that this group is isomorphic to $\mathbb{G}_a^2 \rtimes \mathrm{GL}_2$.

Now let $A = I + \varepsilon(a_{ij})$ be a general element of the Lie algebra of PGL_3, i.e. a $\overline{k}[\varepsilon]$-point mapping to the identity. Then $x^3 \cdot A$ is

$$x^3 + 3\varepsilon x^2(a_{11}x + a_{12}y + a_{13}z).$$

This gives the condition $3a_{12} = 3a_{13} = 0$. Since we assume that 3 is invertible, we get $a_{12} = a_{13} = 0$. We conclude that both the Lie-algebra and the group have the same dimension, so the stabiliser is smooth and therefore reduced. Therefore $\mathrm{PGL}^3_{x^3} \simeq \mathbb{G}_a^2 \rtimes \mathrm{GL}_2$.

(b) A double and a single line. This time we choose x^2y as our standard representative. An element (a_{ij}) of the stabiliser must preserve both the line $x = 0$ as well as the line $y = 0$. This forces $a_{12} = a_{13} = a_{21} = a_{23} = 0$. By normalising $a_{33} = 1$, we see that the reduced stabiliser is $\mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$.

Now we consider a general element $A = I + \varepsilon(a_{ij})$ of the Lie-algebra in the same way as in the previous case. Then we get that $x^2y \cdot A$ equals

$$x^2y + \varepsilon(2xy(a_{11}x + a_{12}y + a_{13}z) + x^2(a_{21}x + a_{22}y + a_{23}z)).$$

Since 2 is invertible, this gives the conditions $a_{12} = a_{13} = a_{21} = a_{23} = 0$. Again we see that the dimension is right, so we get $\mathrm{PGL}^3_{x^2y} \simeq \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$.

(c) Three lines intersecting at a single point. Let $x^2y + xy^2$ be our standard form. An element (a_{ij}) of the stabiliser must preserve the intersection point $(0:0:1)$ of the three lines. This forces $a_{13} = a_{23} = 0$. By normalising $a_{33} = 1$, we see that the stabiliser is a subgroup of $\mathbb{G}_\mathrm{a}^2 \rtimes \mathrm{GL}_2$.

Next we determine the stabiliser of our standard form under the action of the subgroup GL_2. This corresponds to the problem of finding the stabiliser of an unordered triple of distinct points in $\mathbb{P}^1$ under the standard action of GL_2. Since the corresponding action of PGL_2 is simply 3-transitive, the stabiliser is an extension of Σ_3 by $\mathbb{G}_m$. One verifies that this is the subgroup G as described in the introduction of this section. Since the subgroup $\mathbb{G}_\mathrm{a}^2$ clearly stabilises our standard from, we get that the reduced stabiliser is $\mathbb{G}_\mathrm{a}^2 \rtimes G$.

The corresponding calculation for the Lie-algebra for $\mathrm{PGL}^3_{x^2y+xy^2}$ as in the previous cases leads to the relations $a_{11} = a_{22}$ and $a_{12} = a_{13} = a_{21} = a_{23} = 0$. This shows that the dimension is right regardless of characteristic, so $\mathrm{PGL}^3_{x^2y+xy^2} \simeq \mathbb{G}_\mathrm{a}^2 \rtimes G$.

(d) Three lines in general position. Choose xyz as standard form. The stabiliser has to preserve the lines x, y and z up to permutation. If we impose an ordering on the lines, the stabiliser consists of the diagonal matrices. Since we may reorder the lines by using permutation matrices, the group PGL^3_{xyz} is generated by the diagonal and the permutation matrices. One verifies that also in this case the stabiliser is smooth regardless of characteristic. Thus the stabiliser PGL^3_{xyz} is the group N_3 described in the introduction of the appendix.

A.2 An integral conic and a line

There are two types of cubic curves consisting of an integral conic and a line. The line is either tangent to the cubic or intersects it at two distinct points.

(e) Integral conic and non-tangent line. The intersection between the curves determine an unordered pair of points $P1$, $P2$. The tangents to the conic at these points are distinct by Bézout's Theorem. Hence they intersect in a third point Q.

Chose $xyz + z^3$ as the standard form defining our cubic. For this curve, the points $P1$, $P2$ and Q as defined above have coordinates $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ respectively. Any element of the stabiliser $\mathrm{PGL}^3_{xyz+z^3}$ must preserve these points, so the stabiliser is contained in the group generated by the diagonal matrices and the permutation matrix switching the first two coordinates.

The subgroup of the diagonal matrices $\mathrm{diag}(a : b : c)$ stabilising the form $xyz + z^3$ is defined by the equation $abc = c^3$. Hence it is isomorphic to $\mathbb{G}_m$, as seen by the parametrisation $t \mapsto \mathrm{diag}(t : t^{-1} : 1)$. It follows that the stabiliser is isomorphic to $\mathbb{G}_\mathrm{m} \rtimes \Sigma_2$, where Σ_2 acts non-trivially on $\mathbb{G}_\mathrm{m}$.

(f) Integral conic and tangent line. In this case, we let $y^2z + x^2y$ be the standard form defining our curve. The stabiliser $\mathrm{PGL}^3_{y^2z+x^2y}$ must preserve the intersection point $(0:0:1)$ between the conic and the tangent line, as well as the tangent line $y = 0$ itself. Hence it must be a subgroup of the group of projective matrices of the following form:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The additional requirement that it also should preserve the conic $yz + x^2$ gives the equations

$$a_{11}^2 = a_{22}a_{33}, \qquad a_{12}^2 + a_{22}a_{32} = 0, \qquad a_{22}a_{31} + 2a_{11}a_{12} = 0.$$

This resulting subgroup is isomorphic to $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$, which may be seen by using the parametrisation described below.

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & -ab & 0 \\ 0 & a^2 & 0 \\ 2b & -b^2 & 1 \end{pmatrix}$$

A.3 An integral cubic

There are two types of integral singular cubics, both having exactly one singularity. The singularity is either a node or a cusp.

(g) Cuspidal cubic. A cuspidal cubic has exactly one singularity and one inflection point. We denote these points by $P1$ and $P2$ respectively. Consider the reduced line associated to the tangent cone at $P1$ and the tangent line at $P2$. As a consequence of Bézout's Theorem, these lines are distinct. Hence they have a unique intersection point $P3$, which does not lie on the line between $P1$ and $P2$.

We choose the standard cuspidal cubic $x^2z + y^3$. In this case, the coordinates of the points $P1$, $P2$ and $P3$ as described above are $(0:0:1)$, $(1:0:0)$ and $(0:1:0)$ respectively. Since the stabiliser $\mathrm{PGL}^3_{x^2z+y^3}$ preserves these points, it must be a subgroup of the group of diagonal matrices. Introducing coordinates $\mathrm{diag}(a : b : c)$ for these matrices, we see that the stabiliser is the subgroup defined by the equations $a^2c = b^3$. This group is isomorphic to $\mathbb{G}_m$ via the parametrisation $t \mapsto \mathrm{diag}(t^3 : t^2 : 1)$.

(h) Nodal cubic. The standard nodal cubic $xyz + x^3 + y^3$ has the tangent cone $xy = 0$ at the singularity. All its three inflection points are distinct and lie at the line $z = 0$ at infinity. Hence the stabiliser group $\mathrm{PGL}^3_{xyz+x^3+y^3}$ must preserve the forms xy and z. It is therefore a subgroup of the group generated by the diagonal matrices and the permutation matrix exchanging the x- and y-coordinates. The diagonal matrices $\mathrm{diag}(a:b:c)$ which preserve the form $xyz+x^3+y^3$ are those satisfying the equations $abc = a^3 = b^3$. These matrices form a group isomorphic to μ_3 through the parametrisation $\zeta \mapsto \mathrm{diag}(\zeta : \zeta^2 : 1)$, where $\zeta^3 = 1$. We conclude that $\mathrm{PGL}^3_{xyz+x^3+y^3}$ is isomorphic to $\mu_3 \rtimes \Sigma_2$.

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