ISSN: 1401-5617

Two-colored noncommutative Gerstenhaber formality and infinity Duflo isomorphism

Johan Alm

Research Reports in Mathematics
Number 2, 2011

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2011/2

Date of publication: May 18, 2011
2000 Mathematics Subject Classification: Primary 18G55.
Keywords: deformation quantization, operads, configuration spaces.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Filosofie Licentiatavhandling

Two-colored noncommmutative Gerstenhaber formality and infinity Duflo isomorphism

Johan Alm

Avhandlingen kommer att presenteras tisdagen den 30/5 2011,
kl. 13.15, rum 306, hus 6, Matematiska institutionen,
Stockholms universitet, Kräftriket.

Abstract

Using new configuration spaces, we give an explicit construction that extends Kontsevich's Lie-infinity quasi-isomorphism from polyvector fields to Hochschild cochains to a quasi-isomorphism of A-infinity algebras equipped with actions by homotopy derivations of the Lie algebra of polyvector fields. One may term this formality a formality of two-colored noncommutative Gerstenhaber homotopy algebras. In our result the action of polyvector fields by homotopy derivations of the wedge product on polyvector fields is not the adjoint action by the Schouten bracket, but a homotopy nontrivial and, in a sense, unique deformation of that action.

As an application we give an explicit Duflo-type construction for Lie-infinity algebras that generalizes the Duflo-Kontsevich isomorphism between the Chevalley-Eilenberg cohomology of the symmetric algebra on a Lie algebra and the Chevalley-Eilenberg cohomology of the universal enveloping algebra of the Lie algebra.

1

Introduction

Kontsevich's Formality Map is best understood as a morphism of two-colored operads

$$\mathcal{K}(\overline{C}(\mathbf{H})) \to \mathcal{E}nd(T_{\text{poly}}, \mathcal{O}),$$

where $\mathcal{K}(\overline{C}(\mathbf{H}))$ is the operad of fundamental chains of a certain cellular operad $\overline{C}(\mathbf{H})$ of compactified configuration spaces of points in the closed upper half-plane and $\mathcal{E}nd(T_{\text{poly}}, \mathcal{O})$ is the standard two-colored endomorphism operad on formal polyvector fields, T_{poly}, and formal smooth functions, $\mathcal{O}$, on some chosen graded vector space. The content of this map of operads is an L_∞ map from T_{poly} to the (differential) Hochschild cochain complex of $\mathcal{O}$. In this note we introduce a three-colored operad $\overline{CF}(\mathbf{H})$ of compactified configuration spaces of points in the closed upper half-plane equipped with a line parallel to the real axis, and, using the same techniques as Kontsevich, a representation

$$\mathcal{K}(\overline{CF}(\mathbf{H})) \to \mathcal{E}nd(T_{\text{poly}}, T_{\text{poly}}, \mathcal{O})$$

of its fundamental chains. This representation implies

- Kontsevich's L_∞ map $T_{\text{poly}} \to C(\mathcal{O}, \mathcal{O})$ to the Hochschild cochain complex of the associative algebra of functions,
- an L_∞ map $T_{\text{poly}} \to C^{\geq 1}(T_{\text{poly}}, T_{\text{poly}})$ to the Hochschild cochain complex of the associative algebra of polyvector fields, extending the canonical adjoint action of T_{poly} on itself,
- and a morphism $T_{\text{poly}} \to C(\mathcal{O}, \mathcal{O})$ of A_∞ algebras equipped with actions of the Lie algebra T_{poly} by homotopy derivations.

These data can be concisely encoded as a quasi-isomorphism of two-colored noncommutative G_∞ algebras.

The three-colored operad $\overline{CF}(\mathbf{H})$ is closely related to the moduli spaces of quilted holomorphic disks introduced in the context of Floer homology by Mau and Woodward in [**14**]. The moduli spaces of quilted holomorphic disks form a two-colored operad that can be embedded as a suboperad of our three-colored operad.

As an application we give an explicit strong homotopy version of the Duflo isomorphism. This generalizes earlier work by Calaque, Kontsevich, Manchon, Pevzner, Rossi, Torossian and others; see [**13, 17, 4, 18, 11**]. More specifically, we construct a universal and generically homotopy nontrivial A_∞ deformation $C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}}$ of the Chevalley-Eilenberg cochain algebra $C(\mathbf{g}, S(\mathbf{g}))$ and an A_∞ quasi-isomorphism $C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}} \to C(\mathbf{g}, U(\mathbf{g}))$ that on the cohomology level reproduces the Duflo-Kontsevich isomorphism of Chevalley-Eilenberg cohomologies. This implies that the Duflo-Kontsevich isomorphism can *not* be universally lifted to an A_∞ quasi-isomorphism $C(\mathbf{g}, S(\mathbf{g})) \to C(\mathbf{g}, U(\mathbf{g}))$ of the Chevalley-Eilenberg cochain algebras.

Acknowledgement. *It is a pleasure to thank Johan Granåker, Bruno Vallette and Thomas Willwacher for discussions and helpful criticism. Thanks to Malin Göteman for the TeX template. Special thanks to my supervisor, Sergei Merkulov for valuable ideas and encouragement, and to Carlo A. Rossi for his lucid critique of the first version of the article.*

Contents

2

Preliminaries

1. Finite sets

Given a natural number n, $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

The cardinality of a finite set A is written $|A|$, e.g. $|[n]| = n$.

Given finite sets A and B, we write $A + B$ for their disjoint union. We customarily write 0 for the empty set. If S is a subset of a finite set A, we customarily write $A - S$ for the complement of S in A. We write A/S for the set $A - S + \{S\}$. (So the cardinality of A/S is $|A| - |S| + 1$.) If A is an ordered finite set, then we say $S \subset A$ is a connected subset, and write $S < A$, if $s, s'' \in S$ and $s < s' < s'' \in A$ implies also $s' \in S$.

The group of permutations of a finite set T is denoted Σ_T, and $\Sigma_{[n]}$ is denoted Σ_n.

2. Differential graded vector spaces

In this section we state our conventions regarding differential graded (hencefort abbreviated dg) vector spaces.

A **dg vector space** is an indexed collection $V = \{V^p\}_{p \in \mathbf{Z}}$ of real vector spaces together with a collection $d_V = \{d_V^p\}_{p \in \mathbf{Z}}$ of linear maps $d_V^p : V^p \to V^{p+1}$ such that $d_V^{p+1} \circ d_V^p = 0$ for all p. If $v \in V^p$, then we define $|v| := p$ and say v is homogeneous of degree p. A **graded vector space** is a dg vector space (V, d_V) with $d_V^p = 0$ for all p. A **morphism of dg vector spaces** $f : (V, d_V) \to (W, d_W)$ is a collection of linear maps $\{f^p : V^p \to W^p\}_{p \in \mathbf{Z}}$ such that $f^{p+1} \circ d_V^p = d_W^p \circ f^p$ for all p. The evident composition rules for morphisms give us a category $\mathsf{Ch}(\mathbf{R})$ with dg vector spaces as objects and morphisms of dg vector spaces as arrows. We shall usually omit d_V from the notation and simply write V for $\{V^p, d_V^p\}_p$. An element v of a dg vector space V, written $v \in V$, is a vector v in the vector space $\bigoplus_d V^d$.

The **cohomology** of a dg vector space V is the graded vector space $H(V)$ with $H^p(V) := H(V)^p$ given as the quotient $\ker(d_V^p)/\mathrm{im}(d_V^{p-1})$. Elements of $\ker(d_V^p)$ are called **cocycles** of degree p and elements of $\mathrm{im}(d_V^{p-1})$ are called **coboundaries** of degree p.

Let r be some integer. The r-**fold suspension** of a dg vector space V is the dg vector space $V[r]$ with $V[r]^p := V^{p+r}$ and $d_{V[r]}^p := -s^r \circ d_V^{p+r} \circ s^{-r}$, where $s^r : V[r]^p \to V^{p+r}$ and $s^{-r} : V^{p+r+1} \to V[r]^{p+1}$ are the canonical isomorphisms of vector spaces.

Given dg vector spaces V and W their **tensor product** is the dg vector space $V \otimes W$ with $(V \otimes W)^n := \bigoplus_{p+q=n} V^p \otimes_{\mathbf{R}} W^q$ and $d_{V \otimes W} := d_V \otimes id_W + id_V \otimes d_W$. The **Koszul symmetry** for $V \otimes W$ is the morphism

$$S_{V \otimes W} : V \otimes W \to W \otimes V$$

given on vectors of homogeneous degree by

$$S_{V \otimes W}(v \otimes w) := (-1)^{|v| \cdot |w|} w \otimes v.$$

The tensor product, the Koszul symmetry and the tensor unit $\mathbf{R}$ give $\mathsf{Ch}(\mathbf{R})$ the structure of a symmetric monoidal category.

The **space of maps** from V to W is the dg vector space $Map(V, W)$ with

$$Map^n(V, W) := Map(V, W)^n := \prod_p Hom_{\mathbf{R}}(V^{p-n}, W^p),$$

where $Hom_{\mathbf{R}}(V^{p-n}, W^p)$ denotes the vector space of all linear maps from V^{p-n} to W^p, and differential is given on $\phi \in Map(V,W)^n)$ by $d^n_{Map(V,W)}\phi := d_W \circ \phi - (-1)^n d_V \circ \phi$. A vector ϕ of $Map(V,W)^n$ is called a **map of dg vector spaces** of degree n. Note that a morphism from V to W is the same thing as a cocycle of degree 0 of $Map(V,W)$. There is an adjunction formula

$$Hom_{\mathsf{Ch}(\mathbf{R})}(U \otimes V, W) \cong Hom_{\mathsf{Ch}(\mathbf{R})}(U, Map(V,W)).$$

3. Operads

Let $(\mathsf{V}, \otimes, I, S)$ be a cocomplete symmetric monoidal category and let C be a nonemtpy set. A **C-colored symmetric collection** $\mathcal{P}$ in V is the data of

- an object $\mathcal{P}(c_1, \ldots, c_n; c)$ of V, for each $n \geq 0$ and $(n+1)$-tuple $(c_1, ..., c_n; c)$ of elements of C,
- together with, for each $\sigma \in \Sigma_n$, a morphism $\sigma^* : \mathcal{P}(c_1, \ldots, c_n; c) \to \mathcal{P}(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)$ satisfying $\sigma_1^* \sigma_2^* = (\sigma_1\sigma_2)^*$ for all $\sigma_1, \sigma_2 \in \Sigma_n$.

One refers to elements of C as **colors**.

A **C-colored symmetric operad** in V is a C-colored collection $\mathcal{P}$ in V together with, for each $n \geq 1$, $(n+1)$-tuple $(c_1, \ldots_n; c)$ and tuples $\{(c^j_1, \ldots c^j_{k_j})\}_{1 \leq j \leq n}$ of lengths $k_j \geq 0$, a **composition morphism**

$$\mathcal{P}(c_1, \ldots, c_n; c) \otimes \bigotimes_{j=1}^{n} \mathcal{P}(c^j_1, \ldots, c^j_{k_j}; c_j) \to \mathcal{P}(c^1_1, \ldots, c^n_{k_n}; c).$$

The composition morphism are required to be associative in the obvious sense. A C-colored symmetric operad is said to be **unital** if for each color $c \in C$ there exists a $1_c : I \to \mathcal{P}(c; c)$ which is a (two-sided) unit for the composition morphisms.

If C has cardinality N and $\mathcal{P}$ is a C-colored operad, then we also say $\mathcal{P}$ is an N-colored operad.

Most operads used in this paper will have $\mathcal{P}(\emptyset; c) = \emptyset$ for all colors c. We shall also use simplified notation of the form

$$\mathcal{P}(m_1, \ldots, m_n; c) := \mathcal{P}(\underbrace{c_1, \ldots, c_1}_{m_1 \times}, \ldots, \underbrace{c_n, \ldots, c_n}_{m_n \times}; c).$$

If the color c is clear from the context we shall employ the further simplification $\mathcal{P}(m_1, \ldots, m_n) := \mathcal{P}(m_1, \ldots, m_n; c)$.

A **C-colored cooperad** in V is an operad in the opposite category V^{op}, i.e. it is a symmetric collection with cocomposition morphisms that are coassociative in a suitable sense.

A **dg (co)operad** refers to a (co)operad in the symmetric monoidal category of dg vector spaces.

4. Semialgebraic geometry

For a thorough treatment of the material in this section, see [**6**].

A **semialgebraic set** (in $\mathbf{R}^n$) is a finite union of finite intersections of solution sets out of polynomial equations or polynomial inequalities, for real polynomials in n variables. Semialgebraic sets are topologized as subsets. A **semialgebraic map** is a continuous map of semialgebraic sets whose graph is itself a semialgebraic set. The closure or interior of a semialgebraic set is again semialgebraic, and the inverse image of a semialgebraic map is also semialgebraic. A **semialgebraic manifold** of dimension k is, for our purposes, a semialgebraic set in $\mathbf{R}^k$ such that each point has a semialgebraic neighbourhood semialgebraically homeomorphic to $\mathbf{R}^k$ or $\mathbf{R}_{\geq 0} \times \mathbf{R}^{k-1}$. The boundary of a semialgebraic manifold is again semialgebraic. A **smooth semialgebraic submanifold** of a semialgebraic manifold is a semialgebraic subset that is also a smooth submanifold of the ambient euclidean space.

4.1. Semialgebraic chains. Let $\Omega^p_c(\mathbf{R}^k)$ denote the vector space of smooth differential p-forms on $\mathbf{R}^k$ with compact support. This vector space can be topologized in a natural way, and we let $\mathcal{C}^{-p}(\mathbf{R}^k)$ be the topological dual of $\Omega^p_c(\mathbf{R}^k)$. The adjoint of the de Rham differential yields a differential graded vector space $(\mathcal{C}^{-p}(\mathbf{R}^k), \partial)$, the **complex of smooth currents on $\mathbf{R}^k$**.

Let X be an oriented semialgebraic manifold in $\mathbf{R}^k$ and define $\mathcal{C}(X) \subset \mathcal{C}(\mathbf{R}^k)$ to be the subspace of currents that have support contained in X. For $V_1, \ldots, V_r$ p-dimensional disjoint smooth semialgebraic submanifolds of $\mathbf{R}^k$ with each closure $\overline{V}_i$ compact and contained in X and integers $n_1, \ldots, n_r$, there is a a current $\sum_i n_i[V_i]$ in $\mathcal{C}^{-p}(X)$ (defined by integration). The **complex of semialgebraic currents** on X, denoted $\mathrm{C}_{SA}(X)$, is the subcomplex of the complex of currents spanned by all currents of that form.

The association $X \mapsto \mathrm{C}_{SA}(X)$ is a symmetric monoidal functor from semialgebraic manifolds to differential graded vector spaces.

4.2. PA forms. Let $X \subset \mathbf{R}^k$ be a semialgebraic set and let $f = (f_0, f_1, \dots, f_p) : X \to \mathbf{R}^{p+1}$ be a semialgebraic map. This map defines a functional on $\mathrm{C}_{SA}^{-p}(X)$ by

$$\gamma \mapsto f_*(\gamma)(\rho \cdot x^0 dx^1 \wedge \cdots \wedge dx^p),$$

for $x^0, \dots, x^p$ the coordinates on $\mathbf{R}^{p+1}$ and ρ a smooth bump function which takes the value 1 on the support of $f_*(\gamma)$. Such a functional on semialgebraic currents on X is called a **minimal form** on X.

Let $\varphi : Y \to X$ be a semialgebraic map. Briefly, a **strongly continuous family of chains of dimension** p **along** φ is a function $\Phi : X \to \mathrm{C}_{SA}^{-p}(Y)$ such that there exists

- a finite semialgebraic stratification $\{S_i\}_{i \in I}$ of X together with, for each i, a compact p-dimensional semialgebraic manifold F_i and a semialgebraic map $g_i : \overline{S}_i \times F_i \to Y$,
- such that the composition $\phi \circ g_i$ is a trivial fibration $\overline{S}_i \times F_i \to \overline{S}_i$
- and $\Phi(x) = (g_i)_*([\{x\} \times F_i])$ for each $x \in \overline{S}_i$.

If Φ is a strongly continuous chain of dimension p along φ and γ is a semialgebraic q-current on X, then there is a trivialization of Φ (in above sense) that is adapted to γ in the sense that

$$\gamma = \sum_i n_i [\overline{S}_i].$$

Define a $p+q$-chain $\gamma \ltimes \Phi$ by

$$\gamma \ltimes \Phi := \sum_i n_i \cdot (g_i)_*([\overline{S}_i \times F_i]).$$

Take a minimal $(p+q)$-form μ on Y. Define a functional $\int_\Phi \mu$ on $\mathrm{C}_{SA}^{q}(X)$ by the formula

$$\langle \int_\Phi \mu, \gamma \rangle := \langle \mu, \gamma \ltimes \Phi \rangle.$$

The complex of PA forms on X is the subcomplex of the linearly dual complex of $\mathrm{C}_{SA}(X)$ spanned by all functionals of the form $\int_\Phi \mu$ (for some μ and some Φ). It is denoted $\Omega_{PA}(X)$ and it is a differential graded vetor space.

A semialgebraic bundle $\pi : E \to X$ admits local trivializations $\overline{S_i} \times F_i \cong E|_{\overline{S}_i}$. Hence the association $x \mapsto [\overline{\pi^{-1}(x)}]$ defines a strongly continuous chain F_π along π. We use the notation $\pi_*(\mu) := \int_{F_\pi} \mu$ for a minimal form μ on E and refer to the map π_* as the **fiber integration** along π.

3

Configuration space models for various homotopy algebras

In this section we define four different operads in the category of cellular compact semialgebraic manifolds. Two of the operads are our invention.

1. A configuration space model for L_∞

For an integer $\ell \geq 2$, let $\mathrm{Conf}_\ell(\mathbf{C})$ be the manifold of all injective maps of $[\ell] := \{1, \ldots, \ell\}$ into $\mathbf{C}$. The group of translations and positive dilations of the plane, $\mathbf{C} \rtimes \mathbf{R}_{>0}$, acts on the plane and hence (by postcomposition) on $\mathrm{Conf}_\ell(\mathbf{C})$. Define $C_\ell(\mathbf{C}) := \mathrm{Conf}_\ell(\mathbf{C})/\mathbf{C} \rtimes \mathbf{R}_{>0}$. Let $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$ be the real Fulton-MacPherson compactification (in the literature also called the Axelrod-Singer compactification) of $\mathrm{Conf}_\ell(\mathbf{C})$, i.e. the real oriented blow-up of $\mathbf{C}^\ell$ along all diagonals. The action by translations and positive dilations is smooth; hence extends uniquely to a smooth action on $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$. Define $\overline{C}_\ell(\mathbf{C})$ to be the quotient of $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$ by this action. It is a compact semialgebraic manifold with codimension one boundary

$$\bigsqcup_S C_{\ell-|S|+1}(\mathbf{C}) \times C_S(\mathbf{C})$$

given by products labelled by subsets $S \subset [\ell]$ (of cardinality $2 \leq |S| < \ell$). Moreover, the closure of $C_{\ell-|S|+1}(\mathbf{C}) \times C_S(\mathbf{C})$ in $\overline{C}_\ell(\mathbf{C})$ is the product $\overline{C}_{\ell-|S|+1}(\mathbf{C}) \times \overline{C}_S(\mathbf{C})$. This means that the family of spaces $\overline{C}(\mathbf{C}) = \{\overline{C}_\ell(\mathbf{C})\}$ together with the inclusions of boundary components and permutation actions by permutation of points assemble into the structure of an operad. We promote it to an operad of oriented semialgebraic manifolds as follows. Let $C_\ell^{\mathrm{std}}(\mathbf{C})$ be the submanifold of $\mathrm{Conf}_\ell(\mathbf{C})$ consisting of configurations x satisfying $\sum_{i=1}^\ell x_i = 0$ and $\sum_{i=1}^\ell |x_i|^2 = 1$. The manifolds $C_\ell(\mathbf{C})$ and $C_\ell^{\mathrm{std}}(\mathbf{C})$ are isomorphic. The manifold $\mathrm{Conf}_\ell(\mathbf{C})$ is canonically oriented; hence so is $C_\ell^{\mathrm{std}}(\mathbf{C})$. We orient $C_\ell(\mathbf{C})$ by pulling back the orientation on $C_\ell^{\mathrm{std}}(\mathbf{C})$. Requiring Stokes' formula (without a sign) to hold defines an orientation of the compactification $\overline{C}_\ell(\mathbf{C})$. It is easy to see that all permutations of $[\ell]$ preserve the orienation.

The boundary description describes a canonical stratification and the face complexes of the stratification of each component form an operad $\mathcal{K}(\overline{C}(\mathbf{C}))$ that this is freely generated as a graded operad by the set $\{[C_\ell(\mathbf{C})] \mid \ell \geq 2\}$ of "fundamental chains". We shall regard chains in the components as semialgebraic chains. It is well-known that representations of $\mathcal{K}(\overline{C}(\mathbf{C}))$ in a dg vector space V are in one-to-one correspondence with L_∞ structures on the suspension $V[1]$ of V; see e.g. [**5**].

2. A configuration space model for OCHA

Set $\mathbf{H} := \mathbf{R} \times \mathbf{R}_{\geq 0}$. For integers $m, n > 0$, with $2m + n \geq 2$, let $\mathrm{Conf}_{m,n}(\mathbf{H})$ be the manifold of injections of $[m] + [n]$ into $\mathbf{H}$ that map $[n]$ into the boundary $\mathbf{R} \times \{0\}$ of the half-plane and $[m]$ into the interior. The group of translations along the boundary and positive dilations, $\mathbf{R} \times \mathbf{R}_{>0}$, acts (by postcomposition) on $\mathrm{Conf}_{m,n}(\mathbf{H})$ and we let $C_{m,n}(\mathbf{H})$ be the quotient of this action. The embedding

$$\mathrm{Conf}_{m,n}(\mathbf{H}) \to \mathrm{Conf}_{2m+n}(\mathbf{C})$$

defined by sending a configuration in $[m] + [n] \hookrightarrow \mathbf{H}$ to its orbit under complex conjugation induces an embedding

$$C_{m,n}(\mathbf{H}) \to C_{2m+n}(\mathbf{C}) \subset \overline{C}_{2m+n}(\mathbf{C}).$$

The compactification $\overline{C}_{m,n}(\mathbf{H})$ of $C_{m,n}(\mathbf{H})$ was in [**11**] defined as the closure under this embedding. It is a semialgebraic manifold with $n!$ connected components. Let $\overline{C}^+_{m,n}(\mathbf{H})$ be the connected component that has the boundary points "compatibly ordered", by which we mean that if $i < j \in [n] = \{1 < \cdots < n\}$, then the point labelled by i is before the point labelled by j on the boundary for the orientation of the boundary induced by the orientation of the half-plane. This gives us a permutation-equivariant identification $\overline{C}_{m,n}(\mathbf{H}) \cong \overline{C}^+_{m,n}(\mathbf{H}) \times \Sigma_n$. The codimension one boundary of $\overline{C}^+_{m,n}(\mathbf{H})$ is

$$\bigsqcup_I \big(C^+_{m-|I|+1,n}(\mathbf{H}) \times C_I(\mathbf{C})\big) \sqcup \bigsqcup_{S,T} \big(C^+_{m-|S|,n-|T|+1}(\mathbf{H}) \times C^+_{S,T}(\mathbf{H})\big).$$

Here $C^+_{m-|I|+1,n}(\mathbf{H})$ is the interior of $\overline{C}^+_{m-|I|+1,n}(\mathbf{H})$, etc. The union is over all subsets $I \subset [m]$ and subsets $S \subset [m]$, $T < [n]$ such that all involved spaces are defined. This description of the boundary extends, via the identification $\overline{C}_{m,n}(\mathbf{H}) \cong \overline{C}^+_{m,n}(\mathbf{H}) \times \Sigma_n$, to boundary descriptions for all connected components, and defines the structure of a two-coloured operad on the collection $\overline{C}(\mathbf{H}) := \{\overline{C}_\ell(\mathbf{C}), \overline{C}_{m,n}(\mathbf{H})\}$, the points in the interior being inputs of one color and the points on the boundary being inputs of another color. The spaces $\overline{C}_{m,n}(\mathbf{H})$ are defined using embeddings into spaces fo the form $\overline{C}_\ell(\mathbf{C})$, for which we have chosen orientations. We orient the spaces $\overline{C}_{m,n}(\mathbf{C})$ by the pullback orientations of these embeddings.

The dg operad of face complexes of the stratification defined by the boundary decomposition is again generated by the fundamental chains. We denote this operad of fundamental chains $\mathcal{K}(\overline{C}(\mathbf{H}))$. A representation of it is referred to as an **open-closed homotopy algebra**, see [**7, 9**], henceforth abbreviated as an OCHA. An OCHA consists of a pair of dg vector spaces V and W, an L_∞ structure on $V[1]$, an A_∞ structure on W, and an L_∞ morphism from V to the Hochschild cochain complex of W.

We now define flag versions of the operads $\overline{C}(\mathbf{C})$ and $\overline{C}(\mathbf{H})$.

3. Flag version of $\overline{C}(\mathbf{C})$, a model for NCG_∞

Since the affine group preserves collinearity and parallel lines it makes sense to say that some points in a configuration $x \in C_\ell(\mathbf{C})$ are collinear on a line parallel to the real axis. For integers $p \geq 0$ and $q \geq 1$ with $p + q \geq 2$, define $CF_{p,q}(\mathbf{C}) \subset C_{[p]+[q]}(\mathbf{C})$ to be the subset of configurations for which the points labelled by $[q]$ are collinear on a line parallel to the real axis. Define $\overline{CF}_{p,q}(\mathbf{C})$ to be its closure inside $\overline{C}_{p+q}(\mathbf{C})$. It has $q!$ connected components. Let $CF^+_{p,q}(\mathbf{C})$ denote the interior of the connected component that has the collinear points compatibly ordered, by which we mean that if $i < j \in [q] = \{1 < \cdots < q\}$, then the point labelled by i is before the point labelled by j on their common line for the orientation of the line induced by the orientation of the plane. Then $CF_{p,q}(\mathbf{C}) \cong CF^+_{p,q}(\mathbf{C}) \times \Sigma_q$. We deduce that the codimension one boundary of the corresponding compact connected component, $\overline{CF}^+_{p,q}(\mathbf{C})$, is

$$\bigsqcup_I \big(CF^+_{p-|I|+1,q}(\mathbf{C}) \times C_I(\mathbf{C})\big) \sqcup \bigsqcup_{S,T} \big(CF^+_{p-|S|,q-|T|+1}(\mathbf{C}) \times CF^+_{S,T}(\mathbf{C})\big).$$

The union is over all subsets $I \subset [p]$, $S \subset [p]$, $T < [q]$ for which all involved spaces are defined. One can use the inclusions of boundary components to define a two-colored operad structure on the collection

$$\overline{CF}(\mathbf{C}) := \{\overline{C}_\ell(\mathbf{C}), \overline{CF}_{p,q}(\mathbf{C})\},$$

in a way completely analogous the previously discussed operadic structure on $\overline{C}(\mathbf{H})$.

Definition 3.0.1. We call $\overline{CF}(\mathbf{C})$ the operad of configurations on flags in the plane.

We orient the spaces of the form $\overline{CF}_{p,q}(\mathbf{C})$ by the pullback orientations of the defining embeddings into $\overline{C}_{p+q}(\mathbf{C})$. As before one then obtains a dg operad $\mathcal{K}(\overline{CF}(\mathbf{C}))$ of fundamental chains. It is almost identical to the operad $\mathcal{K}(\overline{C}(\mathbf{H}))$ of OCHAs: its representations also consist of an L_∞ algebra $V[1]$, an A_∞ algebra W and an L_∞ morphism from V to the Hochschild cochain complex of W. The difference lies in that the latter operad contains chains $[C_{m,n}(\mathbf{H})]$ with $n = 0$ while the former operad does not contain any chain of the form $[CF_{p,q}(\mathbf{C})]$ with $q = 0$. This means that the L_∞ map of an OCHA contains components $V^{\otimes p} \to W$, so called curvature terms, whilst the L_∞ map of a $\mathcal{K}(\overline{C}(\mathbf{H}))$-representation can not, i.e. it maps into the truncated Hochschild cochain complex $C^{\geq 1}(W, W)$.

Definition 3.0.2. We call $\mathcal{K}(\overline{CF}(\mathbf{C}))$ the operad of two-colored noncommutative G_∞ algebras.

Remark 3.0.1. Define a **two-colored noncommutative Gerstenhaber algebra** to be a pair (L, A), where $L[1]$ is a dg Lie algebra and A is a dg associative algebra, together with a dg Lie algebra morphism $L[1] \to \mathrm{Der}(A)$. Such algebras are representations of an operad $\mathcal{NCG}$ and $\mathcal{K}(\overline{CF}(\mathbf{C}))$ is the cobar construction on the Koszul dual cooperad of $\mathcal{NCG}$. We prove in an appendix that $\mathcal{NCG}$ is Koszul. Thus $\mathcal{K}(\overline{CF}(\mathbf{C}))$ indeed deserves to be called the operad of two-colored noncommutative G_∞ algebras.

We shall abbreviate "two-colored noncommutative G_∞ algebra" as NCG_∞ algebra.

4. Flag version of $\overline{C}(\mathbf{H})$, a model for flag OCHAs

There is also a flag version of the operad $\overline{C}(\mathbf{H})$, defined as follows. Let $k, m, n \geq 0$ be integers with $2k+m+n \geq 1$ if $m \geq 1$ and $k+n \geq 2$ if $m = 0$. Let $CF_{k,m,n}(\mathbf{H})$ be the subspace of $C_{k+m,n}(\mathbf{H})$ consisting of all configurations wherein the points labelled by $[m]$ are collinear on a line parallel to the boundary. Denote by $\overline{CF}_{k,m,n}(\mathbf{H})$ the closure inside $\overline{C}_{k+m,n}(\mathbf{H})$. Let $CF^+_{k,m,n}(\mathbf{H})$ be the connected component of $CF_{k,m,n}(\mathbf{H})$ that has both the collinear points and the boundary points compatibly ordered, i.e. if $i < j$ in $[m]$, then $x_i < x_j$ on their common line of collinearity, and if $r < s$ in $[n]$, then $x_r < x_s$ on the boundary. The codimension one boundary of its compactification, $\overline{CF}^+_{k,m,n}(\mathbf{H})$, has the form

$$\bigsqcup_I \big(CF^+_{k-|I|+1,m,n}(\mathbf{H}) \times C_I(\mathbf{C})\big) \sqcup \bigsqcup_{P,Q} \big(CF^+_{k-|P|,m-|Q|+1,n}(\mathbf{H}) \times CF^+_{P,Q}(\mathbf{C})\big)$$

$$\sqcup \bigsqcup_{S,T,U} \big(CF^+_{k-|S|,m-|T|,n-|U|+1}(\mathbf{H}) \times CF^+_{S,T,U}(\mathbf{H})\big).$$

The union is over all subsets $I, P, S \subset [k]$, $Q, T < [m]$, $S < [n]$ for which all involved spaces are defined. These boundary factorizations define an operad structure, but now in three colors, on the collection

$$\overline{CF}(\mathbf{H}) := \{\overline{C}_\ell(\mathbf{C}), \overline{CF}_{p,q}(\mathbf{C}), \overline{CF}_{k,m,n}(\mathbf{H})\}.$$

Definition 4.0.3. We call $\overline{CF}(\mathbf{H})$ the operad of configurations on flags in the half-plane.

Orient the spaces $\overline{CF}_{k,m,n}(\mathbf{H})$ by the pullback orientations of the embeddings into $\overline{C}_{k+m,n}(\mathbf{H})$. There is an associated operad $\mathcal{K}(\overline{CF}(\mathbf{H}))$ of fundamental chains.

Definition 4.0.4. We call $\mathcal{K}(\overline{CF}(\mathbf{H}))$ the operad of flag open-closed homotopy algebras, abbreviated as the operad of flag OCHAs.

Lemma 4.0.1. *A representation of the operad of flag open closed homotopy algebras in a triple (L, A, B) of chain complexes is equivalent to*

- *an NCG_∞ algebra structure on (L, A);*
- *an OCHA structure on (L, B);*
- *and a morphism from A to $C(B, B)$ of A_∞ algebras with L_∞ actions of L by homotopy derivations, where the Hochschild cochain complex of B is considered with the L-action induced by the OCHA structure.*

The first two listed items are obvious. Let $\mathcal{M}or_*(\mathcal{NCG})_\infty$ be the Koszul resolution of the operad, $\mathcal{M}or_*(\mathcal{NCG})$, whose representations are NCGAs (L, A), (L, A'), with the same dg Lie algebra L appearing in both pairs, and a morphism between the two dg associative algebras respecting the actions by L. See the appendix for some comments on why $\mathcal{M}or_*(\mathcal{NCG})$ is Koszul. The third item in the list is a $\mathcal{M}or_*(\mathcal{NCG})_\infty$-representation on $(L, A, C(B, B))$. The key to this correspondence is to change from the operadic perspective that the chains $[CF_{k,m,n}(\mathbf{H})]$ are represented as maps $L^{\otimes k} \otimes A^{\otimes m} \otimes B^{\otimes n} \to B$ to the perspective that they define maps

$$L^{\otimes k} \otimes A^{\otimes m} \to Map(B^{\otimes n}, B).$$

(This hom-adjunction argument exactly parallels the argument used for interpreting an OCHA structure $\{[C_{p,q}(\mathbf{H})] : L^{\otimes p} \otimes B^{\otimes q} \to B\}$ as an L_∞ morphism $L \to C(B, B)$, compare with [**9, 7**].) After this reinterpretation of the chains the argument reduces to (i) recognizing the induced NCG_∞ algebra structure on $(L, C(B, B))$ and (ii) comparing the differential on the chains to the differential on $\mathcal{M}or_*(\mathcal{NCG})_\infty$. The details are left to the reader. We work out some more explicit details in the subsequent sections.

Remark 4.0.2. Consider the two-colored suboperad of $\overline{CF}(\mathbf{H})$ on the components

$$\{\overline{CF}_{0,q}(\mathbf{C}), \overline{CF}_{0,m,0}(\mathbf{H}), \overline{CF}_{0,0,n}(\mathbf{H})\}.$$

It is isomorphic as an operad of compact semialgebraic manifolds to the operad of quilted holomorphic disks introduced by Mau and Woodward in [**14**]. Its operad of cellular chains is the operad of morphisms of A_∞ algebras.

4

(Co)operads of graphs

Kontsevich's proof of his Formality Conjecture and construction of a universal deformation quantization formula can be regarded[15] as the construction of

- a map of cooperads $\omega : \mathfrak{G}^c_{\overline{C}(\mathbf{H})} \to \Omega(\overline{C}(\mathbf{H}))$, where $\mathfrak{G}^c_{\overline{C}(\mathbf{H})}$ is a cooperad of Feynman diagrams,
- and a map of operads $\Phi : \mathfrak{G}_{\overline{C}(\mathbf{H})} \to \mathcal{E}nd(T_{\mathrm{poly}}, \mathcal{O})$ from the dual operad of Feynman diagrams.

Dualizing the map of cooperads and composing, one gets a representation

$$\Phi \circ \omega^* : \mathcal{K}(\overline{C}(\mathbf{H})) \to \mathfrak{G}_{\overline{C}(\mathbf{H})} \to \mathcal{E}nd(T_{\mathrm{poly}}, \mathcal{O})$$

of the fundamental chains of half-plane configurations, i.e. an OCHA structure on $(T_{\mathrm{poly}}, \mathcal{O})$. We shall show that Kontsevich's construction can be extended, essentially without any changes, to a representation

$$\Phi \circ \omega^* : \mathcal{K}(\overline{CF}(\mathbf{H})) \to \mathfrak{G}_{\overline{CF}(\mathbf{H})} \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{O})$$

of the operad of flag OCHAs. This is our NCG_∞ Formality Theorem. The new data added by extending Kontsevich's OCHA to a flag OCHA is a quasi-isomorphism $T_{\mathrm{poly}} \to C(\mathcal{O}, \mathcal{O})$ of A_∞ algebras with homotopy actions by T_{poly}.

The first construction we need for our extension of the Kontsevich representation is a suitable operad $\mathfrak{G}_{\overline{CF}(\mathbf{H})}$.

1. Directed graphs

Choose a finite set S. Let $fdgra^d_S$ be the set of all injective functions Γ of the set $[d]$ into $(S \times S) - \Delta$, for Δ the diagonal of S. We refer to such a Γ as a directed graph with d edges on the set S and introduce the following terminology:

- $E_\Gamma := \mathrm{im}(\Gamma)$ is the set of edges of Γ. We consider it as ordered by the given isomorphism with $[d]$. The element $\Gamma(i) \in E_\Gamma$ is written e_i and referred to as the ith edge.
- The function $s_\Gamma : E_\Gamma \subset S \times S \to S$ given by projection onto the first factor S is called the source map of Γ. The projection $t_\Gamma : E_\Gamma \to S$ onto the second factor is called the target map of Γ. An edge e is said to be directed from $s_\Gamma(e)$ to $t_\Gamma(e)$.
- The set S is called the set of vertices of Γ.
- The valence of a vertex is the number of edges having that vertex as either source or target.
- A connected component of Γ is a maximal (with respect to inclusions) subset $E \subset E_\Gamma$ with the property that $s_\Gamma(E) \cup t_\Gamma(E)$ and $s_\Gamma(E_\Gamma - E) \cup t_\Gamma(E_\Gamma - E)$ are disjoint. A graph with a single connected component is said to be connected.

Let $dgra^d_S$ be the subset of $fdgra^d_S$ of connected graphs. There is a natural action of the permutation groups Σ_d and Σ_S on $dgra^d_S$ by, respectively, reordering edges and permuting the vertices. Let sgn_d be the one-dimensional sign representation of Σ_d. Define, for any finite set I, of cardinality at least 2, the graded Σ_I-module

$$\mathfrak{G}^c_{\overline{C}(\mathbf{C})}(I) := \bigoplus_{j \geq 0} (\mathbf{R}\langle dgra^d_I\rangle \otimes_{\Sigma_d} sgn_d)[-d].$$

Elements of $\mathfrak{G}^c_{\overline{C}(\mathbf{C})}(I)^d$ may be represented as (linear combinations of) connected graphs with d directed edges ordered up to an even permutation, $|I|$ vertices labelled by I, without double edges and without tadpoles (edges that begin and end at the same vertex).

For a finite set P and a nonempty finite set Q, with $|P|+|Q|\geq 2$, let $dgra^d_{P,Q}$ be a copy of the subset of $fdgra^d_{P+Q}$ consisting of those graphs which have no connected components $E\subset E_\Gamma$ with $s_\Gamma(E)\cup t_\Gamma(E)\subset P$, and put

$$\mathfrak{G}^c_{\overline{CF(\mathbf{C})}}(P,Q):=\bigoplus_{d\geq 0}(\mathbf{R}\langle dgra^d_{P,Q}\rangle\otimes_{\Sigma_d} sgn_d)[-d].$$

The vertices labelled by P of a graph in $\mathfrak{G}^c_{\overline{CF(\mathbf{C})}}(P,Q)$ are called **free vertices** and the vertices labelled by Q are called **collinear vertices**. Our restrictions informally say that there are no connected components with only free vertices.

Assume given a triple of finite sets (K,M,N), with $2|K|+|M|+|N|\geq 1$ if M is nonempty, and $2|K|+|N|\geq 2$ if M is empty. Let $dgra^d_{K,M,N}$ be a copy of the subset of $dgra^d_{K,M+N}$ consisting of graphs Γ having no edge with source a vertex labelled by N. Set

$$\mathfrak{G}^c_{\overline{CF(\mathbf{H})}}(K,M,N):=\bigoplus_{d\geq 0}(\mathbf{R}\langle dgra^d_{K,M,N}\rangle\otimes_{\Sigma_d} sgn_d)[-d].$$

The vertices labelled by K of a graph in $\mathfrak{G}^c_{\overline{CF(\mathbf{H})}}(K,M,N)$ are called **free vertices**, the vertices labelled by M are called **collinear vertices** and the vertices labelled by N are called **boundary vertices**.

2. (Co)operad structures

We shall now describe how the vector spaces of (equivalence classes of) graphs defined above assemble into cooperads.

Given $\Gamma_2\in dgra^{d_2}_{S_2}$ and $\Gamma\in dgra^d_S$, where $d_2\leq d$ and $S_2\subset S$, we define an **embedding of Γ_2 as a full subgraph of** Γ to be an order-preserving inclusion $f:[d_2]\hookrightarrow[d]$ which makes

$$[d_2]\hookrightarrow[d]\xrightarrow{\Gamma}S\times S\ \text{ equal }\ [d_2]\xrightarrow{\Gamma_2}S_2\times S_2\subset S\times S.$$

An embedding of Γ_2 as a full subgraph of Γ is written $f:\Gamma_2\hookrightarrow\Gamma$. Given an embedding f as above, we define $\Gamma/\Gamma_2\in dgra^{d-d_2}_{S/S_2}$ to be the graph which, as a function, is the composition

$$[d-d_2]\cong[d]-\mathrm{im}(f)\xrightarrow{\Gamma}S\times S\to(S/S_2)\times(S/S_2).$$

Here the leftmost bijection is the unique order-preserving bijection and the rightmost arrow is given by the canonical projection of S onto $S/S_2=S-S_2+\{S_2\}$ (sending elements of S_2 to the element $\{S_2\}$). If $\Gamma_1=\Gamma/\Gamma_2$, $\Gamma_1\in dgra^{d_1}_{S_1+\{v\}}$ (so $S_1=S-S_2$ and we identify the singleton sets $\{v\}$ and $\{S_2\}$), then the embedding and the quotient define a bijection $[d_1]+[d_2]\to[d]$. This defines an order on $[d_1]+[d_2]$, using the order on $[d]$. This order on $[d_1]+[d_2]$ is related to the lexicographic order given by $[d_1]<[d_2]$ using a unique bijection. Define $\epsilon(\Gamma_2,\Gamma,\Gamma_1)$ to be the sign of that bijection. We may now define a cooperadic cocomposition

$$\mathfrak{G}^c_{\overline{C(\mathbf{C})}}(I_1+I_2)\to\mathfrak{G}^c_{\overline{C(\mathbf{C})}}(I_1+\{v\})\otimes\mathfrak{G}^c_{\overline{C(\mathbf{C})}}(I_2)$$

by

$$\Gamma\mapsto\sum_{\Gamma_1=\Gamma/\Gamma_2}\epsilon(\Gamma_2,\Gamma,\Gamma_1)\Gamma_1\otimes\Gamma_2.$$

The sum is over all embeddings of some Γ_2 into Γ.

Conclusion 2.0.1. The collection

$$\mathfrak{G}^c_{\overline{C(\mathbf{C})}}:=\{\mathfrak{G}^c_{\overline{C(\mathbf{C})}}(\ell)\}$$

carries a cooperad structure. The componentwise linear dual, $\mathfrak{G}_{\overline{C(\mathbf{C})}}:=\{\mathfrak{G}^c_{\overline{C(\mathbf{C})}}(\ell)^*\}$, is an operad.

We define a full subgraph embedding of a graph $\Gamma_2\in dgra^{d_2}_{P_2,Q_2}$ into a graph $\Gamma\in dgra^d_{P,Q}$ exactly as before, except that we now require $P_2\subset P$ and $Q_2\subset Q$ (not just $P_2+Q_2\subset P+Q$). The quotient Γ/Γ_2 is defined as before and regarded as an element of $dgra^{d-d_2}_{P-P_2,Q/Q_2}$. The sign $\epsilon(\Gamma_2,\Gamma,\Gamma_1)$ is also defined as before. With these conventions for subgraphs and quotients, above definitions for the cocomposition maps can be copied verbatim to define cocompositions

$$\mathfrak{G}^c_{\overline{CF(\mathbf{C})}}(P_1+P_2,Q_1+Q_2)\to\mathfrak{G}^c_{\overline{CF(\mathbf{C})}}(P_1,Q_1+\{v\})\otimes\mathfrak{G}^c_{\overline{CF(\mathbf{C})}}(P_2,Q_2).$$

The definitions repeat word for word when $\Gamma_2 \in dgra_I^{d_2}$, $\Gamma \in dgra^d_{P,Q}$ and $I \subset P$, if we agree on the convention that now Γ/Γ_2 belongs to $dgra^{d-d_2}_{P/I,Q}$, defining cocompositions

$$\mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(P+I,Q) \to \mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(P+\{v\},Q) \otimes \mathfrak{G}^c_{\overline{C}(\mathbf{C})}(I).$$

Conclusion 2.0.2. The collection $\mathfrak{G}^c_{\overline{CF}(\mathbf{C})} := \{\mathfrak{G}^c_{\overline{C}(\mathbf{C})}(\ell), \mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(p,q)\}$ carries a cooperad structure. The componentwise linear dual, $\mathfrak{G}_{\overline{CF}(\mathbf{C})} := \{\mathfrak{G}^c_{\overline{C}(\mathbf{C})}(\ell)^*, \mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(p,q)^*\}$, is an operad.

With the evident conventions for how to color the new vertex obtained by collapsing an embedded subgraph the same formulas define a cooperad structure on the collection

$$\mathfrak{G}^c_{\overline{CF}(\mathbf{H})} := \{\mathfrak{G}^c_{\overline{C}(\mathbf{C})}(\ell), \mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(p,q), \mathfrak{G}^c_{\overline{CF}(\mathbf{H})}(k,m,n)\}.$$

Its linear dual, denoted $\mathfrak{G}_{\overline{CF}(\mathbf{H})}$, is an operad.

3. de Rham field theory

Given a pair of distinct indices $i,j \in [k]+[m]+[n]$ we follow Kontsevich and define a function

$$\phi^h_{i,j} : CF_{k,m,n}(\mathbf{H}) \to \mathbf{S}^1,\ x + \mathbf{R} \rtimes \mathbf{R}_{>0} \mapsto \mathrm{Arg}\left(\frac{x_j - x_i}{x_j - \overline{x}_i}\right).$$

Here a barred variable denotes the complex conjugate variable. The function is smooth and extends to a smooth function defined on the compactified configuration space. Let ϑ be the homogeneous normalized volume form on $\mathbf{S}^1$.

Given a graph $\Gamma \in dgra^d_{k,m,n}$, define

$$\omega_\Gamma := \wedge_{i=1}^d (\phi^h_{s_\Gamma(e_i),t_\Gamma(e_i)})^*\vartheta.$$

The form ω_Γ is a smooth closed differential form of degree d on $\overline{CF}_{k,m,n}(\mathbf{H})$. We extend ω to a map of dg vector spaces $\mathfrak{G}^c_{\overline{CF}(\mathbf{H})}(k,m,n) \to \Omega(\overline{CF}_{k,m,n}(\mathbf{H}))$.

Define similarly, for indices $i,j \in [\ell]$, $\phi_{i,j} : C_\ell(\mathbf{C}) \to \mathbf{S}^1$ by

$$\phi_{i,j} : x + \mathbf{C} \rtimes \mathbf{R}_{>0} \mapsto \mathrm{Arg}(x_j - x_i).$$

The function ϕ extends to the compactification. For a graph $\Gamma \in dgra^d_\ell$, let

$$\omega_\Gamma := \wedge_{i=1}^d (\phi_{s_\Gamma(e_i),t_\Gamma(e_i)})^*\vartheta.$$

This allows us to define maps of dg vector spaces $\omega : \mathfrak{G}^c_{\overline{C}(\mathbf{C})}(\ell) \to \Omega(\overline{C}_\ell(\mathbf{C}))$. By identifying $\overline{CF}_{p,q}(\mathbf{C})$ with a subset of $\overline{C}_{p+q}(\mathbf{C})$ and $dgra^d_{p,q}$ with a subset of $dgra^d_{p+q}$ we can use this to define maps of dg vector spaces $\omega : \mathfrak{G}^c_{\overline{CF}(\mathbf{C})}(p,q) \to \Omega(\overline{CF}_{p,q}(\mathbf{C}))$ as well.

In all cases we interpret the form associated to a graph without edges as the function identically equal to 1.

Claim 3.0.1. The de Rham complex functor Ω is only comonoidal up to quasi-isomorphism with respect to the usual tensor product of dg vector spaces. Hence $\Omega(\overline{CF}(\mathbf{H}))$ is only a cooperad up to quasi-isomorphisms. This inconvenience can be ignored by working with a completed tensor product, regarding it, say, as a cooperad in the category of chain complexes of nuclear Fréchet spaces. Our mapping $\omega : \mathfrak{G}^c_{\overline{CF}(\mathbf{H})} \to \Omega(\overline{CF}(\mathbf{H}))$ is a morphism of cooperads in this category of cooperads.

We shall not prove this statement as it is a consequence of similar statements in [**15**].

4. A representation of the operad of graphs

Fix for the remainder of this section a graded vector space V, assumed finite-dimensional in each degree.

Define the **formal smooth functions on** V, denoted $\mathcal{O}$, to be the completed symmetric algebra on V^*. Define the **formal polyvector fields on** V, to be denoted T_{poly}, as the completed symmetric algebra on $V^* \oplus V[-1]$. Note that $\mathcal{O}$ is a subalgebra of T_{poly}.

Let τ be the image of id_V under $V \otimes V^* \to V \otimes V^*[1] \cong (V^* \otimes V[-1])^*$ and regard it as a map $V^* \otimes V[-1] \to \mathbf{R}$. It extends uniquely to a derivation of T_{poly}. This derivation defines an endomorphism (of degree -1) of $T_{\mathrm{poly}} \otimes T_{\mathrm{poly}}$ which we again denote τ. The **Schouten bracket** on T_{poly} is the map

$$[\,,\,]_S := m \circ \tau \circ (id + (21)),$$

where m denotes the product on T_{poly}. It is well-known that the Schouten bracket is a (degree -1) Lie bracket. Given a finite set S and distinct elements $s, t \in S$, define $\tau_{s,t}$ to be the endomorphism of $T_{\mathrm{poly}}^{\otimes S}$ acting as τ on the s-th factor times the t-th factor and as the identity on all others.

For a graph $\Gamma \in dgra^d_{k,m,n}$, let

$$\Phi_\Gamma := \varepsilon \circ m \circ \bigcirc_{i=1}^d \tau_{s_\gamma(e_i), t_\Gamma(e_i)} : T_{\mathrm{poly}}^{\otimes k} \otimes T_{\mathrm{poly}}^{\otimes m} \otimes \mathcal{O}^{\otimes n} \to \mathcal{O}.$$

Here ε is the projection of T_{poly} onto $\mathcal{O}$ defined by the projection $V^* \oplus V[-1] \to V^*$, we regard

$$T_{\mathrm{poly}}^{\otimes k} \otimes T_{\mathrm{poly}}^{\otimes m} \otimes \mathcal{O}^{\otimes n} \subset T_{\mathrm{poly}}^{\otimes k+m+n},$$

and $m : T_{\mathrm{poly}}^{k+m+n} \to T_{\mathrm{poly}}$ is the product. For a graph $\Gamma \in dgra^d_\ell$ we define

$$\Phi_\Gamma := m \circ \bigcirc_{i=1}^d \tau_{s_\gamma(e_i), t_\Gamma(e_i)} : T_{\mathrm{poly}}^{\otimes \ell} \to T_{\mathrm{poly}}.$$

For a graph $\Gamma \in dgra^d_{p,q}$ we use the same formula,

$$\Phi_\Gamma := m \circ \bigcirc_{i=1}^d \tau_{s_\gamma(e_i), t_\Gamma(e_i)} : T_{\mathrm{poly}}^{\otimes p} \otimes T_{\mathrm{poly}}^{\otimes q} \to T_{\mathrm{poly}}.$$

Claim 4.0.2. One verifies that these definitions define a morphism of dg operads

$$\Phi : \mathfrak{G}_{\overline{CF}(\mathbf{H})} \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{O}).$$

5

NCG_∞ formality

Combining the previous subsections, we have a representation

$$\Phi \circ \omega^* : \mathcal{K}(\overline{CF}(\mathbf{H})) \to \mathfrak{G}_{\overline{CF}(\mathbf{H})} \to \mathcal{E}nd(T_{\text{poly}}, T_{\text{poly}}, \mathcal{O}).$$

Since $\mathcal{K}(\overline{CF}(\mathbf{H}))$ is quasi-free the representation consists of a family of maps, one for each generator of $\mathcal{K}(\overline{CF}(\mathbf{H}))$, satisfying some quadratic identities coming from the boundary differential on $\mathcal{K}(\overline{CF}(\mathbf{H}))$. We shall denote the components as follows:

- $\lambda_\ell := \Phi \circ \omega^*([C_\ell(\mathbf{C})]) \in Map^{3-2\ell}(T_{\text{poly}}^{\otimes \ell}, T_{\text{poly}})$, for $\ell \geq 2$.
- $\nu_p := \Phi \circ \omega^*([CF_{0,q}^+(\mathbf{C})]) \in Map^{2-q}(T_{\text{poly}}^{\otimes q}, T_{\text{poly}})$ for $q \geq 2$.
- $\mu_n := \Phi \circ \omega^*([CF_{0,0,n}^+(\mathbf{H})]) \in Map^{2-n}(\mathcal{O}^{\otimes n}, \mathcal{O})$ for $n \geq 2$.
- $\mathcal{V}_{p,q} := \Phi \circ \omega^*([CF_{p,q}^+(\mathbf{C})]) \in Map^{2-2p-q}(T_{\text{poly}}^{\otimes p} \otimes T_{\text{poly}}^{\otimes q}, T_{\text{poly}})$ for $p, q \geq 1$.
- $\mathcal{U}_{k,n} := \Phi \circ \omega^*([CF_{k,0,n}^+(\mathbf{H})]) \in Map^{2-2k-n}(T_{\text{poly}}^{\otimes k} \otimes \mathcal{O}^{\otimes n}, \mathcal{O})$ for $k \geq 1$, $n \geq 0$.
- $\mathcal{Z}_{k,m,n} := \Phi \circ \omega^*([CF_{k,m,n}^+(\mathbf{H})]) \in Map^{1-2k-m-n}(T_{\text{poly}}^{\otimes k} \otimes T_{\text{poly}}^{\otimes m} \otimes \mathcal{O}^{\otimes n}, \mathcal{O})$ for $k \geq 0$, $m \geq 1$, $n \geq 0$.

Recall that the Hochschild cochain complex $C(A, A)$ of an A_∞ algebra A is

$$Map(T(A[1]), A), \text{ where } T(A[1]) = \bigoplus_{r \geq 0} A[1]^{\otimes r}$$

The brace operations on the Hochschild cochains complex are maps

$$(\,)\{\dots\}_p : C(A, A) \otimes \bigotimes_{i=1}^{p} C(A, A) \to C(A, A),\ p \geq 1,$$

defined for $x \in Map(A[1]^{\otimes r}, A)$, $x_i \in Map(A[1]^{\otimes r_i}, A)$, $1 \leq i \leq p \leq r$, $n = r + r_1 + \cdots + r_p - p$, by

$$x\{x_1, \dots, x_p\}_p(a_1, \dots, a_n) = \sum_{1 \leq i_1 < \cdots < i_p < r} \pm x(a_1, \dots, a_{i_1 - 1}, x_1(a_{i_1}, \dots), \dots, a_{i_p - 1}, x_p(a_{i_p}, \dots), \dots, a_n).$$

The Gerstenhaber bracket on the Hochschild cochain complex is the operation

$$[x, y]_G := x\{y\}_1 \pm y\{x\}_1.$$

It is a graded Lie bracket of degree -1 in our grading on the Hochschild cochain complex. Denote by $C^{\geq 1}(A, A)$ the subspace $Map(\bigoplus_{r \geq 1} A[1]^{\otimes r}, A)$. It is a graded Lie subalgebra. Set $(\,)\{\dots\} := \sum_{p \geq 1} (\,)\{\dots\}_p$ and define

$$br : C(A, A) \to C^{\geq 1}(C(A, A), C(A, A)), x \mapsto ()\{x\}_1 + x\{\dots\}.$$

One verifies that this is a map of graded Lie algebras.

An A_∞ structure on A is a Maurer-Cartan element $m = d + m_2 + \dots$ in $C^{\geq 1}(A, A)$. The differential $[m, \,]_G$ makes the Hochschild cochain complex a dg Lie algebra. It is also an A_∞ algebra with A_∞ structure the Maurer-Cartan element $\cup^m := br(m)$ of $C^{\geq 1}(C(A, A), C(A, A))$. When A has a given A_∞ structure m we shall usually write $C(m)$ for $C(A, A)$ with differential $[m, \,]_G$.

The interpretation of the components of our representation of $\mathcal{K}(\overline{CF}(\mathbf{H}))$ is that

- $\lambda = \{\lambda_\ell\}$ is an L_∞ structure on T_{poly}.
- $\nu = \{\nu_p\}$ is an A_∞ structure on T_{poly}.
- $\mu = \{\mu_n\}$ is an A_∞ structure on $\mathcal{O}$.
- $\mathcal{V} = \{\mathcal{V}_{p,q}\}$ is an L_∞ map $(T_{\mathrm{poly}}, \lambda) \to C^{\geq 1}(\nu)$.
- $\mathcal{U} = \{\mathcal{U}_{k,n}\}$ is an L_∞ map $(T_{\mathrm{poly}}, \lambda) \to C(\mu)$.
- $\mathcal{Z} = \{\mathcal{Z}_{k,m,n}\}$ is a morphism of A_∞ algebras
$$(T_{\mathrm{poly}}, \nu, \mathcal{V}) \to (C(\mu), \cup^\mu, br \circ \mathcal{U})$$
equipped with homotopy actions by $(T_{\mathrm{poly}}, \lambda)$.

This description is a result of the interpretation of the operad of flag open-closed homotopy algebras. All the component maps have an explicit description as sums over graphs, e.g.

$$\mathcal{V}_{p,q} = \sum_{[\Gamma]\in[dgra^{2p+q-2}_{p,q}]} \int_{\overline{CF}^+_{p,q}(\mathbf{C})} \omega_\Gamma \Phi_\Gamma,$$

with $[dgra^{2p+q-2}_{p,q}]$ the set of equivalence classes of graphs under the Σ_{2p+q-2}-action by permutation of edges. We shall use this description to give a more detailed description of the component maps. The main tool is "Kontsevich's vanishing lemma":

Lemma 0.0.2. [**11**] *Let X be a complex algebraic variety of dimension $N \geq 1$ and $Z_1, ..., Z_{2N}$ be rational functions on X, not equal identically to 0. Let U be any Zariski open subset of X such that each function Z_α is well-defined and nowhere vanishing on U, and that U consists of smooth points. Then the integral*

$$\int_{U(\mathbf{C})} \wedge_{\alpha=1}^{2N} d(\mathrm{Arg}(Z_\alpha))$$

is absolutely convergent and is equal to zero.

1. Descriptions of the involved structures

1.1. The L_∞ structure λ.

We have

$$\lambda_\ell = \sum_{[\Gamma]\in[dgra^{2\ell-3}_\ell]} \int_{\overline{C}_\ell(\mathbf{C})} \omega_\Gamma \Phi_\Gamma.$$

For $\ell \geq 3$, $C_\ell(\mathbf{C}) \cong \mathbf{S}^1 \times U$, with $U = (\mathbf{C} \setminus \{0,1\})^{\ell-2} \setminus$ *diagonals*. This identification can be obtained by using the translation freedom to fix the point labelled by 1, say, at the origin of $\mathbf{C}$ and using the dilation freedom to put the point labelled by 2, say, on the unit circle $\mathbf{S}^1$. Multiplying the remaining points by the inverse of the phase of the point labelled by 2 gives a point in U. Using this description we can reduce every integral

$$\int_{\overline{C}_\ell(\mathbf{C})} \omega_\Gamma$$

to an integral over a circle times an integral of the type appearing in Kontsevich's vanishing lemma. Hence all weights vanish for $\ell \geq 3$. The configuration space $\overline{C}_2(\mathbf{C})$ is a circle. The set of graphs $dgra^1_2$ contains two elements; the graph with an edge from 1 to 2 and the graph with an edge from 2 to 1. Both graphs have weight 1. It follows that λ_2 is the Schouten bracket. As all higher homotopies $\lambda_{\geq 3}$ vanish, this means λ is the usual graded Schouten Lie algebra structure on T_{poly}.

1.2. The A_∞ structure ν.

The A_∞ structure ν has components

$$\nu_p = \sum_{[\Gamma]\in[dgra^{p-2}_{0,p}]} \int_{\overline{CF}^+_{0,p}(\mathbf{C})} \omega_\Gamma \Phi_\Gamma.$$

The angle between collinear points is constant, so the differential form associated to a graph containing an edge connecting collinear vertices will be zero; hence no such graphs can contribute. It follows that the only graph which contributes is the graph with two vertices and no edge. The associated differential form is identically equal to one and we evaluate it on the one-point space $\overline{CF}_{0,2}(\mathbf{C})$. It follows that $\nu = \nu_2$ is the usual (wedge) product on T_{poly}.

1.3. The A_∞ structure μ. The operation μ_n is given by a sum over graphs in $dgra^{n-2}_{0,0,n}$. The set $dgra^{n-2}_{0,0,n}$ is empty if n is not equal to 2 since the condition that no edge begins at a boundary vertex forces a graph with only boundary vertices to have no edges. The space $\overline{CF}^+_{0,0,2}(\mathbf{H})$ is a point and the differential form associated to the graph with two vertices and no edge is the function identically equal to 1. The associated operator Φ_Γ is the wedge product of polyvector fields, restricted to a product on functions. It follows that $\mu = \mu_2$ is the usual associative (and commutative) product on $\mathcal{O}$.

1.4. The L_∞ map $\mathcal{V}$. Since

$$\mathcal{V}_{p,q} = \sum_{[\Gamma]\in[dgra^{2p+q-2}_{p,q}]} \int_{\overline{CF}^+_{p,q}(\mathbf{C})} \omega_\Gamma \Phi_\Gamma$$

and $\overline{CF}^+_{p,1}(\mathbf{C}) \cong \overline{C}_{p+1}(\mathbf{C})$, the argument regarding the L_∞ structure λ can be repeated to conclude that $\mathcal{V}_{p,1} = 0$ for $p \geq 2$, while

$$\mathcal{V}_{1,1} : T_{\text{poly}} \otimes T_{\text{poly}} \to T_{\text{poly}}, X \otimes \xi \mapsto [X, \xi]_S.$$

In other words, $\mathcal{V}_{1,1}$ is the adjoint action $T_{\text{poly}} \to \text{Der}(T_{\text{poly}})$ of T_{poly} on itself by derivations of the wedge product.

Using the translation freedom to put the collinear point labelled by 1 at the origin and the dilation freedom to put the collinear point labelled by 2 at 1 identifies $CF^+_{p,2}(\mathbf{C})$ with $(\mathbf{C} \setminus \{0,1\})^p \setminus diagonals$, so that one may again use Kontsevich's vanishing lemma and conclude that $\mathcal{V}_{p,2} = 0$ for all $p \geq 1$.

Reflection of the plane in the line of collinearity induces an involution f of $\overline{CF}^+_{p,q}(\mathbf{C})$. (Choosing representative configurations with the collinear points on the real axis identifies f with complex conjugation.) The map f preserves orientation if p is even and reverses it if p is odd. For $\Gamma \in dgra^{2p+q-2}_{p,q}$, $f^*\omega_\Gamma = (-1)^{2p+q-2}\omega_\Gamma = (-1)^q\omega_\Gamma$. Thus

$$(-1)^p \int_{\overline{CF}^+_{p,q}(\mathbf{C})} \omega_\Gamma = (-1)^q \int_{\overline{CF}^+_{p,q}(\mathbf{C})} \omega_\Gamma,$$

implying the integral is 0 whenever p and q have different parity, i.e. whenever $p+q$ is odd. This means that the first homotopy to $\mathcal{V}_{1,1}$ is given by $\mathcal{V}_{1,3}$. The angle between collinear points is constant, so the differential form associated to a graph containing an edge connecting collinear vertices will be zero. The set $dgra^3_{1,3}$ contains a unique graph without edges connecting collinear vertices, up to direction and ordering of edges, namely the graph with a free vertex of valence three and three collinear vertices of valence one. Hence there are eight (equivalence classes of) graphs (corresponding to the 2^3 ways to direct the three edges) contributing to $\mathcal{V}_{1,3}$. Each of these eight equivalence classes has a representative with the edges ordered so that e_i connects the free vertex with the collinear vertex labelled by i, $1 \leq i \leq 3$. These representatives all have weight $1/24$. To see this one may argue as follows.

Assume given a configuration in $CF^+_{1,3}(\mathbf{C})$. Use the freedom to translate along the imaginary axis to put the line of collinearity on the real axis. Use the freedom to translate along the real axis to put the free point on the imaginary axis. We are then left with a positive dilation that can be used to put the free point either at $+i$ or at $-i$, depending on wether it lies above or below the line of collinearity, respectively. These two types of configurations are mapped to each other by the involution f in the line of collinearity, discussed above. Denote the space of configurations of the first type, i.e. the subspace of $CF^+_{1,3}(\mathbf{C})$ where the free point lies above the line of collinearity, by C. It follows from the remarks on the involution f that the weight

$$\int_{\overline{CF}^+_{1,3}(\mathbf{C})} \omega_\Gamma$$

of a graph Γ entering the operation $\mathcal{V}_{1,3}$ may be calculated as

$$\int_{\overline{CF}^+_{1,3}(\mathbf{C})} \omega_\Gamma = 2 \int_C \omega_\Gamma.$$

We can identify C with the infinite open simplex $\{-\infty < x_1 < x_2 < x_3 < \infty\}$ and, for Γ the graph with the i-th edge directed from the free vertex to the i-th collinear vertex, we may then calculate

$$\begin{aligned}\int_C \omega_\Gamma &= \frac{1}{(2\pi)^3}\int_{-\infty<x_1<x_2<x_3<\infty} d\mathrm{Arg}(i-x_1)\wedge d\mathrm{Arg}(i-x_2)\wedge d\mathrm{Arg}(i-x_3)\\ &= \frac{1}{(2\pi)^3}\int_{-\infty<x_1<x_2<x_3<\infty} d\arctan(x_1)\wedge d\arctan(x_2)\wedge d\arctan(x_3)\\ &= \frac{1}{48}.\end{aligned}$$

The total weight is $2/48 = 1/24$.

It follows that

$$\mathcal{V}_{1,3} = \frac{1}{24} m\circ\Big(\tau_{1,4}\circ\tau_{1,3}\circ\tau_{1,2}+\tau_{1,4}\circ\tau_{1,3}\circ\tau_{2,1}+\tau_{1,4}\circ\tau_{3,1}\circ\tau_{1,2}+\tau_{4,1}\circ\tau_{1,3}\circ\tau_{1,2}$$
$$+\tau_{4,1}\circ\tau_{3,1}\circ\tau_{1,2}+\tau_{4,1}\circ\tau_{1,3}\circ\tau_{2,1}+\tau_{1,4}\circ\tau_{3,1}\circ\tau_{2,1}+\tau_{4,1}\circ\tau_{3,1}\circ\tau_{2,1}\Big)$$

as a map $T_{\mathrm{poly}}^{\otimes 1+3}\to T_{\mathrm{poly}}$. (The first of the four copies of T_{poly} acts on the last three.)

1.5. The L_∞ map $\mathcal{U}$. The map $\mathcal{U}$ is, by construction, Kontsevich's Formality Map. Recall that it's first Taylor component $\mathcal{U}_1 = \sum_{n\geq 0}\mathcal{U}_{1,n}$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

1.6. The map $\mathcal{Z}$ of NCG_∞ algebras. Since $\overline{CF}^{+}_{0,1,n}(\mathbf{H})$ is isomorphic to $\overline{CF}^{+}_{1,0,n}(\mathbf{H})$ and $dgra^n_{0,1,n}$ is isomorphic to $dgra^n_{1,0,n}$, for all n, the maps $\mathcal{Z}_{0,1,n}$ coincide with the maps $\mathcal{U}_{1,n}$. Hence the first Taylor component of $\mathcal{Z}$,

$$\sum_{n\geq 0}\mathcal{Z}_{0,1,n} : T_{\mathrm{poly}}\to C(\mu),$$

is the Hochschild-Kostant-Rosenberg (HKR) quasi-isomorphism. The higher components of Kontsevich's Formality Map $\mathcal{U}$ are homotopies measuring the failure of the HKR map to respect the Lie brackets. In the same way, the higher components of $\mathcal{Z}$ are homotopies that keep track of the failure of the HKR map to respect the associative products and the respective actions of T_{poly} by homotopy derivations of said associative products. Since the first component is the HKR morphism, we get the following theorem:

Theorem 1.6.1 (Main Theorem). *The map $Z = \{Z_{k,m} = \sum_{n\geq 0}\mathcal{Z}_{k,m,n}\}_{k\geq 0, m\geq 1}$ is an explicit NCG_∞ quasi-isomorphism from $((T_{\mathrm{poly}}, [\,,\,]_S), (T_{\mathrm{poly}}, \wedge, \mathcal{V}))$ to $((T_{\mathrm{poly}}, [\,,\,]_S), (C(\mathcal{O},\mathcal{O}), d_H + \cup, br\circ\mathcal{U}))$*

This statement implies the following A_∞ formality theorem:

Corollary 1.6.1. *The map $A = \{A_m := \sum_{n\geq 0}\mathcal{Z}_{0,m,n}\}_{m\geq 1}$ is an explicit A_∞ quasi-isomorphism from $(T_{\mathrm{poly}}, \wedge)$ to $(C(\mathcal{O},\mathcal{O}), d_H + \cup)$.*

This result has already been demonstrated, but in a different way, by Shoikhet; see [**18**].

6

The induced A_∞ structure

An NCG_∞ algebra consists in an L_∞ algebra (L,λ), an A_∞ algebra (A,ν) and an L_∞ morphism $\mathcal{V}: L \to C^{\geq 1}(\nu)$. Let $\hbar$ be a formal parameter. The map $\mathcal{V}$ induces a map on the sets of Maurer-Cartan elements,

$$\mathrm{MC}(L[[\hbar]]) \to \mathrm{MC}(C(\nu)[[\hbar]]), \pi \mapsto \sum_{p\geq 1} \frac{1}{p!}\mathcal{V}_{p,q}((\hbar\pi)^{\otimes p},\ ).$$

This gives us, for each Maurer-Cartan element π of L, an A_∞ structure

$$\nu_q^{\mathcal{V}(\pi)} := \nu_q + \sum_{p\geq 1} \frac{1}{p!}\mathcal{V}_{p,q}((\hbar\pi)^{\otimes p},\ ),\ q \geq 1,$$

on $A[[\hbar]]$.

If $\mathcal{Z}: (L, A, \lambda, \mathcal{V}, \nu) \to (L, B, \lambda, \mathcal{U}, \mu)$ is a morphism of NCG_∞ algebras (the same L_∞ algebra acting on both and we assume the NCG_∞ algebra morphism is the identity on the Lie-color), then, for any Maurer-Cartan element π of $L[[\hbar]]$, we get an induced map of A_∞ algebras

$$\mathcal{Z}^\pi : (A[[\hbar]], \nu^{\mathcal{V}(\pi)}) \to (B[[\hbar]], \mu^{\mathcal{U}(\pi)})$$

by $\mathcal{Z}^\pi_m := \mathcal{Z}_{0,m} + \sum_{k\geq 0} \frac{1}{k!}\mathcal{Z}_{k,m}((\hbar\pi)^{\otimes k},\ )$. See the appendix for the argument. If $\mathcal{Z}$ is a quasi-isomorphism, then $\mathcal{Z}^\pi$ is as well.

Applying this general construction to our representation $\Phi \circ \omega^*$ produces, for any Maurer-Cartan element $\pi \in T_{\mathrm{poly}}$ (i.e. a possibly graded Poisson structure),

- an A_∞ structure $\nu^{\mathcal{V}(\pi)}$ on $T_{\mathrm{poly}}[[\hbar]]$ with $\nu_1^{\mathcal{V}(\pi)} + \nu_2^{\mathcal{V}(\pi)} = \hbar[\pi,\ ]_S + \wedge$ as its first two Taylor components,
- the A_∞ cup product on the Hochschild cochains of $\mathcal{O}[[\hbar]]$ corresponding to the Kontsevich star product $\mu^{\mathcal{U}(\pi)}$ on $\mathcal{O}[[\hbar]]$ defined by π,
- and an A_∞ quasi-isomorphism $\mathcal{Z}^\pi : (T_{\mathrm{poly}}[[\hbar]], \nu^{\mathcal{V}(\pi)}) \to C(\mu^{\mathcal{U}(\pi)})[[\hbar]]$.

We record this fact as a corollary.

Corollary 0.6.2. *Let $\pi \in T_{\mathrm{poly}}$ be a Poisson structure. Then the A_∞ algebra $(T_{\mathrm{poly}}[[\hbar]], \nu^{\mathcal{V}(\pi)})$ is quasi-isomorphic as an A_∞ algebra to the algebra of Hochschild cochains on $\mathcal{O}[[\hbar]]$ equipped with the cup product corresponding to the Kontsevich star product defined by π. The map $\mathcal{Z}^\pi$ is an explicit such quasi-isomorphism.*

1. Homological properties of the exotic NCG_∞ algebra structure $\mathcal{V}$

Let $\mathcal{NCG}$ be the two-colored operad of noncommutative Gerstenhaber algebras and let $f: \mathcal{NCG} \to \mathfrak{G}_{\overline{CF}(\mathbf{C})}$ be the map which sends the bracket to the (sum of) graph(s) $e_{12}+e_{21} \in \mathfrak{G}_{\overline{CF}(\mathbf{C})}(2) = \mathfrak{G}_{\overline{C}(\mathbf{C})}(2)$, for e_{12} (e_{21}) the graph with vertices $\{1,2\}$ and a single edge from 1 to 2 (from 2 to 1), sends the product to the graph in $\mathfrak{G}_{\overline{CF}(\mathbf{C})}(0,2)$ which has two vertices and no edge, and sends the action to the graph in $\mathfrak{G}_{\overline{CF}(\mathbf{C})}(1,1)$ which is $e_{12}+e_{21}$ with the vertices in different colors. The composition $\Phi \circ f: \mathcal{NCG} \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ is the usual structure of NCGA on polyvector fields in terms of the wedge product and the Schouten bracket. The deformation complex of f is the mapping cone

$$\mathscr{C} := \mathrm{Cone}(\mathrm{Def}(\mathcal{L}ie^1_\infty \to \mathfrak{G}_{\overline{C}(\mathbf{C})})[-1] \to \mathrm{Def}(\mathcal{A}ss_\infty \to \textstyle\int \mathfrak{G}_{\overline{CF}(\mathbf{C})})).$$

See the appendix for notation and further details. The complex $\mathrm{Def}(\mathcal{L}ie^1_\infty \to \mathfrak{G}_{\overline{C}(\mathbf{C})})$ is a directed version of Kontsevich's graph complex, GC, and quasi-isomorphic to it, as shown in [**19**]. The operad $\int \mathfrak{G}_{\overline{CF}(\mathbf{C})}$ is a directed version of the operad $\mathcal{G}raphs$ used by Kontsevich in his proof in [**10**] of the formality of the little disks operad, and it is quasi-isomorphic to it[**19**]. Thomas Willwacher has proved the following:

Theorem 1.0.2. [**19**]

- $H^0(GC) \cong \mathfrak{grt}$ *as a graded Lie algebra.*
- $H^1(\mathrm{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)) \cong \mathfrak{grt} \oplus \mathbf{R}[-1]$ *as a vector space, where* $\mathbf{R}[-1]$ *is spanned by the class of the sum of graphs contributing to* $\mathcal{V}_{1,3}$.
- *The map* $GC[-1] \to \mathrm{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs)$ *is injective on cohomology.*

This theorem, together with the long exact sequence for our mapping cone, implies that $H^{d+1}(\mathscr{C}) \cong H^{d+1}(\mathrm{Def}(\mathcal{A}ss_\infty \to \mathcal{G}raphs))/H^d(GC)$. In particular, $H^1(\mathscr{C})$ is one-dimensional, spanned by the sum of graphs entering $\mathcal{V}_{1,3}$.

Using the representation Φ we can push this statement to a universal (or, rather, generic) statement about structures on polyvector fields.

Corollary 1.0.3. *The exotic* NCG_∞ *algebra structure* $\mathcal{V}$ *on polyvector fields is generically not homotopic to the usual such structure. Moreover, it represents the unique infinitesimal deformation of the usual structure.*

Corollary 1.0.4. *The* A_∞ *structures* $\wedge + \hbar[\pi, \,]$ *and* $\nu^{\mathcal{V}(\pi)}$ *on* $T_{\mathrm{poly}}[[\hbar]]$ *are, generically, not homotopic.*

We have to say generically because for some dimensions of the $\mathcal{O}$-module T_{poly} and for some degenerate Maurer-Cartan elements the corollaries might not be true.

7

A Duflo-type theorem

Kontsevich's paper [**11**] contained a (somewhat sketchy) proof that the tangential morphism of his Formality map, applied to a finite dimensional Lie algebra, defined an isomorphism $H(\mathbf{g}, S(\mathbf{g})) \to H(\mathbf{g}, U(\mathbf{g}))$ of Chevalley-Eilenberg cohomology algebras. This result was later given a detailed proof and generalized to an arbitrary dg Lie algebra of finite type, see [**13, 17, 4**]. In this section we discuss a homotopy generalization of this theorem.

Let $\mathbf{g}$ be a graded vector space which is of finite type (i.e. finite dimensional in each degree) and is concentrated in non-negative degrees. Let T_{poly} be the polyvector fields on $\mathbf{g}[1]$, so $T_{\mathrm{poly}} = S_a(\mathbf{g}^*[-1]) \otimes S_a(\mathbf{g})$. Identify T_{poly} with $Map(S_c(\mathbf{g}[1]), S_a(\mathbf{g}))$. The graded Lie algebra

$$\mathrm{Def}(\mathcal{L}ie, \mathbf{g})[-1] := \mathrm{Def}(\mathcal{L}ie_\infty \xrightarrow{0} \mathcal{E}nd(\mathbf{g}))[-1] = Map(S^{\geq 1}(\mathbf{g}[1])), \mathbf{g})$$

embeds into T_{poly} as a Lie subalgebra. Denote by $\mathcal{O} = S_a(\mathbf{g}^*[-1])$ the algebra of functions on $\mathbf{g}[1]$. Its Hoschschild cochain complex is

$$C(\mathcal{O}, \mathcal{O}) = Map(\mathrm{B}(S_a(\mathbf{g}^*[-1])), S_a(\mathbf{g}^*[-1])) \cong Map(S_c(\mathbf{g}[1]), \Omega(S_c(\mathbf{g}[1]))).$$

Here $\mathrm{B}(\,)$ denotes the (coassociative) bar construction and $\Omega(\,)$ denotes the (associative) cobar construction. In the isomorphism we use that $\mathbf{g}$ is of finite type and concentrated in degrees ≥ 0. These two assumptions ensure that $S_a(\mathbf{g}^*[-1])^* \cong S_c(\mathbf{g}[1])$.

After the above identifications the following result is a straight-forward corollary to our Main Theorem, 1.6.1.

Theorem 0.0.3. *The representation $\Phi \circ \omega^* : \mathcal{K}(\overline{CF}(\mathbf{H})) \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{O})$ induces an explicit quasi-isomorphism $Map(S_c(\mathbf{g}[1]), S_a(\mathbf{g})) \to Map(S_c(\mathbf{g}[1]), \Omega(S_c(\mathbf{g}[1])))$ of A_∞ algebras equipped with L_∞ actions by the graded Lie algebra $\mathrm{Def}(\mathcal{L}ie, \mathbf{g})$.*

As before, given a Maurer-Cartan element of $\mathrm{Def}(\mathcal{L}ie, \mathbf{g})$, we can push this to a quasi-isomorphism of the induced A-infintity structures. The formal parameter $\hbar$ may in the present case be discarded (set to 1). It's purpose is only to define filtrations that ensure we never encounter diverging sums, but in the present case one may use weight grading by tensor lengths to define such filtrations. It is a standard argument and we omit the details.

A Maurer-Cartan element Q of $\mathrm{Def}(\mathcal{L}ie, \mathbf{g})$ is precisely an L_∞ structure on $\mathbf{g}$. Assume Q given and interpret it as a coderivation of $S_c(\mathbf{g}[1])$, and denote the dg coalgebra $(S_c(\mathbf{g}[1]), Q)$ by $C(\mathbf{g})$. The cobar construction

$$\Omega(C(\mathbf{g})) =: U_\infty(\mathbf{g})$$

is the derived universal enveloping algebra of the L_∞ algebra $(\mathbf{g}, Q)$ introduced by V. Baranovsky in [**2**]. (Baranovsky has shown that it is quasi-isomorphic to the usual universal enveloping algebra in the special case that $\mathbf{g}$ is a dg Lie algebra.) Kontsevich's formality map $\mathcal{U}$ quantizes Q to a differential on $S_a(\mathbf{g}^*[-1])$. Denote $S_a(\mathbf{g}^*[-1])$ equipped with this differential by $C(\mathbf{g}, \mathbf{R})$. We have an isomorphism of algebras

$$C(C(\mathbf{g}, \mathbf{R}), C(\mathbf{g}, \mathbf{R})) \cong Map(C(\mathbf{g}), U_\infty(\mathbf{g})) =: C(\mathbf{g}, U_\infty(\mathbf{g})).$$

However, the induced A_∞ structure on $Map(S_c(\mathbf{g}[1]), S_a(\mathbf{g}))$ is not simply

$$C(\mathbf{g}, S(\mathbf{g})) = Map(C(\mathbf{g}), S_a(\mathbf{g})).$$

Instead, we obtain an A_∞ algebra $C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}}$, which is a (generically) homotopy nontrivial deformation of $C(\mathbf{g}, S(\mathbf{g}))$. The induced A_∞ quasi-isomorphism is

$$\mathcal{Z}^Q : C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}} \to C(\mathbf{g}, U_\infty(\mathbf{g})).$$

REMARK 0.0.1.

- The cohomologies $H(C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}})$ and $H(\mathbf{g}, S(\mathbf{g}))$ are isomorphic as associative algebras and the map on cohomology induced by $\mathcal{Z}^Q$ coincides, by construction, with the Duflo-Kontsevich isomorphism. Thus our theorem generalizes the Duflo-Kontsevich statement.
- Since $C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}}$ is, generically, not quasi-isomorphic to $C(\mathbf{g}, S(\mathbf{g}))$, but–by our theorem–is quasi-isomorphic to $C(\mathbf{g}, U_\infty(\mathbf{g}))$, it follows that there does not, generically, exist a quasi-isomorphism of A_∞ algebras

  $$C(\mathbf{g}, S(\mathbf{g})) \to C(\mathbf{g}, U_\infty(\mathbf{g})).$$

  In other words, it is impossible to find a universal A_∞ lift of the Duflo-Kontsevich isomorphism on Chevalley-Eilenberg cohomologies to the Chevalley-Eilenberg cochain algebras.

There is a canonical isomorphism between T_{poly} on $\mathbf{g}[1]$ and T_{poly} on $\mathbf{g}^*$. Above we used the first graded vector space, for which $\mathcal{O} = S_a(\mathbf{g}^*[-1])$. Application of Kontsevich's formality to the second case, for which $\mathcal{O} = S_a(\mathbf{g})$, quantizes an L_∞ structure $Q \in T_{\text{poly}}$ to a (flat) A_∞ structure $\star$ on $S(\mathbf{g})[[\hbar]]$. Calaque, Felder, Ferrario and Rossi constructed in [**3**] a nontrivial but explicit A_∞ $(S(\mathbf{g})[[\hbar]], \star) - C(\mathbf{g}, \mathbf{R})[[\hbar]]$-bimodule structure $K_\hbar$ on $\mathbf{R}[[\hbar]]$ and they proved that the derived left action

$$L : (S(\mathbf{g})[[\hbar]], \star) \to Map_\hbar(K_\hbar[1] \otimes \mathrm{B}(C(\mathbf{g}, \mathbf{R}))[[\hbar]], K_\hbar[1])$$

is a quasi-isomorphism of A_∞ algebras. Here $Map_\hbar$ denotes the mapping space of maps which are linear in $\hbar$. One may formally set $\hbar = 1$ in this quasi-isomorphism, for essentially the same reasons as those which allowed us to do so above, and then identify the term on the right (above) with the cobar construction $\Omega(C(\mathbf{g}))$. Thus the result of [**3**] implies that the quantization of the symmetric algebra on the L_∞ algebra $\mathbf{g}$, $(S(\mathbf{g}), \star)$, is quasi-isomorphic to Baranovsky's derived universal enveloping algebra of $\mathbf{g}$. A detailed proof of this will be contained in [**1**]. Together with our result this quasi-isomorphism implies that the A_∞ algebras $C(\mathbf{g}, S(\mathbf{g}))_{\text{exotic}}$ and $C(\mathbf{g}, (S(\mathbf{g}), \star))$ are quasi-isomorphic, though the quasi-isomorphism is presently not explicit.

8
Future work

This chapter is devoted to sketching a possible generalization of our results. We begin by recalling a construction due to Merkulov[**15**].

1. A configuration space model for morphisms of L_∞ algebras

The content of this section was invented by Merkulov; see [**15**].

Given $x = (x_1, \ldots, x_\ell)$ in $\mathrm{Conf}_\ell(\mathbf{C})$, define

$$x_c := \frac{1}{\ell}\sum_{i=1}^{\ell} x_i, \quad \|x\| := |x - x_c|.$$

The norm $|x - x_c|$ is the usual norm on vectors in $\mathbf{C}^\ell$. Recall from section 1 the manifold

$$C_\ell^{\mathrm{std}}(\mathbf{C}) := \{x \in \mathrm{Conf}_\ell(\mathbf{C}) \mid x_c = 0, \|x\| = 1\}.$$

Define for $\ell \geq 1$, $C'_\ell(\mathbf{C}) := \mathrm{Conf}_\ell(\mathbf{C})/\mathbf{C}$, the quotient by translations. There is an isomorphism

$$\psi_\ell : C'_\ell(\mathbf{C}) \to C_\ell^{\mathrm{std}}(\mathbf{C}) \times (0,\infty),\ [x] \mapsto \left(\frac{x - x_c}{\|x\|}, \|x\|\right).$$

Given a subset $I \subset [\ell]$, of cardinality at least 2, there is a canonical forgetful projection $\pi_A : C'_\ell(\mathbf{C}) \to C'_I(\mathbf{C})$. Define a compactification $\widehat{C}'_\ell(\mathbf{C})$ of $C'_\ell(\mathbf{C})$ as follows. If $\ell = 1$, then $C'_\ell(\mathbf{C})$ is already compact (it is a singleton) and we set $\widehat{C}'_1(\mathbf{C}) := C'_1(\mathbf{C})$. If $\ell \geq 2$ then we define $\widehat{C}'_\ell(\mathbf{C})$ as the closure of the embedding

$$C'_\ell(\mathbf{C}) \xrightarrow{(\pi_I)} \prod_{\substack{I\subset[\ell]\\|I|\geq 2}} C'_I(\mathbf{C}) \xrightarrow{(\psi_I)} \prod_{\substack{I\subset[\ell]\\|I|\geq 2}} C_I^{\mathrm{std}}(\mathbf{C}) \times (0,\infty) \longrightarrow \prod_{\substack{I\subset[\ell]\\|I|\geq 2}} \mathbf{C}^I \times [0,\infty].$$

This differs from the Fulton-MacPherson compactification in that not only do we add limit configurations of collapsing points, but also limit configurations where the points coalesce in clusters with finite distances within each cluster and infinite distances between all clusters. The codimension one boundary of $\widehat{C}'_\ell(\mathbf{C})$ is

$$\bigsqcup_{I\subset[\ell],|I|\geq 2} \left(C'_{\ell-|I|+1}(\mathbf{C}) \times C_I(\mathbf{C})\right) \sqcup \bigsqcup_{\substack{k\geq 2,|J_i|\geq 1\\ [\ell]=J_1+\ldots J_k}} \left(C_k(\mathbf{C}) \times C'_{J_1}(\mathbf{C}) \times \ldots C'_{J_k}(\mathbf{C})\right).$$

To interpret this boundary factorization as an operad structure one has to regard the configuration spaces $\overline{C}_I(\mathbf{C})$ and $\overline{C}_k(\mathbf{C})$ (the compacifications of $C_I(\mathbf{C})$ and $C_k(\mathbf{C})$ above) as being of different colors, say the colors “in” and “out”, respectively, while $\widehat{C}'_\ell(\mathbf{C})$ interpolates the colors, representing operations with ℓ in-inputs and an out-output. To distinguish the differently colored copies of $\overline{C}(\mathbf{C})$ we explicitly denote them $\overline{C}^{\mathrm{in}}(\mathbf{C})$ and $\overline{C}^{\mathrm{out}}(\mathbf{C})$. Then boundary inclusions give the collection

$$\widehat{C}'(\mathbf{C}) := \{\overline{C}_k^{\mathrm{in}}(\mathbf{C}), \widehat{C}'_r(\mathbf{C}), \overline{C}_\ell^{\mathrm{out}}(\mathbf{C})\}$$

a structure of two-colored operad in the category of compact semialgebraic manifolds. The components of the form $\overline{C}_p(\mathbf{C})$ have already been oriented. We orient components $\widehat{C}'_r(\mathbf{C})$ by orienting the interior by the isomorphism $C'_r(\mathbf{C}) \cong C_r^{\mathrm{std}}(\mathbf{C}) \times (0,\infty)$ and extending the orientation to the boundary by requiring that Stokes' formula holds (without a sign).

The operad $\widehat{C}'(\mathbf{C})$ is cellular and we denote the associated operad of fundamental (cellular) chains $\mathcal{K}(\widehat{C}(\mathbf{C}))$.

Lemma 1.0.1. [**15**] *The operad $\mathcal{K}(\widehat{C}'(\mathbf{C}))$ is isomorphic to the Koszul resolution of the operadic suspension of the two-colored operad of morphisms of Lie algebras. That is to say, a representation of $\mathcal{K}(\widehat{C}(\mathbf{C}))$ in a pair of dg vector spaces L_{in}, L_{out} is the same thing as L_∞ structures λ^{in}, λ^{out} on $L_{\mathrm{in}}[1]$, $L_{\mathrm{out}}[2]$, respectively, and an L_∞ morphism $(L_{\mathrm{in}}[1], \lambda^{\mathrm{in}}) \to (L_{\mathrm{out}}[1], \lambda^{\mathrm{out}})$.*

Another way to phrase the lemma is that $\mathcal{K}(\widehat{C}'(\mathbf{C}))$ is the operad of morphisms of $\mathcal{K}(\overline{C}(\mathbf{C}))$-algebras. The lemma is almost obvious; the suboperads $\mathcal{K}(\overline{C}^{\mathrm{in}}(\mathbf{C}))$ and $\mathcal{K}(\overline{C}^{\mathrm{out}}(\mathbf{C}))$ are both resolutions of the (operadic suspension) of the operad of Lie algebras. The boundary factorizations for $\{\widehat{C}'_r(\mathbf{C})\}$ are exactly the equiations for an L_∞ morphism.

2. Flag version of $\widehat{C}'(\mathbf{C})$, a model for morphisms of NCG_∞ algebras

For integers $r \geq 0, q \geq 1$, define $CF'_{r,s}(\mathbf{C})$ to be the submanifold of $C'_{r+s}(\mathbf{C})$ where the points labelled by $[s]$ are collinear on a line parallel to the real axis. This definition makes sense since the group of translations of the plane preserve collinearity. Define the compacification $\widehat{CF}'_{r,s}(\mathbf{C})$ to be the closure of $C'_{r+s}(\mathbf{C})$ inside $\widehat{C}'_{r+s}(\mathbf{C})$. Let $CF'^{+}_{r,s}(\mathbf{C})$ be the connected component of $CF'_{r,s}(\mathbf{C})$ that has the collinear points compatibly ordered, i.e. if $i<j$ in $[s]$, then $x_i < x_j$ on their common line of collinearity and let $\widehat{CF}'^{+}_{r,s}(\mathbf{C})$ be the associated connected component of the compactification. The codimension one boundary of $\widehat{CF}'^{+}_{r,s}(\mathbf{C})$ is

$$\bigsqcup_{I\subset[r]} \left(CF'^{+}_{r-|I|+1,s}(\mathbf{C}) \times C_I(\mathbf{C})\right) \sqcup \bigsqcup_{P\subset[r],Q\subset[s]} \left(CF'^{+}_{r-|P|,s-|P|+1}(\mathbf{C}) \times CF^{+}_{P,Q}(\mathbf{C})\right)$$

$$\sqcup \bigsqcup_{\substack{m,n \\ J_1+\dots J_m+A_1+\dots A_n=[r] \\ B_1+\dots+B_n=[r]}} \left(CF^{+}_{m,n}(\mathbf{C}) \times C'_{J_1}(\mathbf{C}) \times \cdots \times C'_{J_m}(\mathbf{C}) \times CF'^{+}_{A_1,B_1}(\mathbf{C}) \times \cdots \times CF'^{+}_{A_n,B_n}(\mathbf{C})\right).$$

The unions are over all (sub)sets for which all involved spaces are defined. This boundary factorization defines boundary factorizations of all connected components. To interpret it as an operad we need four colors; colors we shall refer to by Ass-in, Lie-in, Ass-out and Lie-out. The component $C_I(\mathbf{C})$ above is of the Lie-in-color. The component $CF^{+}_{P,Q}(\mathbf{C})$ has (Lie-in,Ass-in)-inputs and output Ass-in, and the component $CF^{+}_{m,n}(\mathbf{C})$ has (Lie-out,Ass-out)-inputs and output Ass-out. To distinguish these two incarnations of the operad $\overline{CF}(\mathbf{C})$ we denote them $\overline{CF}^{\mathrm{in}}(\mathbf{C})$ and $\overline{CF}^{\mathrm{out}}(\mathbf{C})$. The spaces $C'_{J_k}(\mathbf{C})$ have inputs in the color Lie-in and output in the color Lie-out. The spaces $CF'_{r,s}(\mathbf{C})$ have (Lie-in,Ass-in)-inputs and output Ass-out. We denote underlying collection of this four-colored operad as

$$\widehat{CF}'(\mathbf{C}) := \{\overline{C}^{\mathrm{in}}_r(\mathbf{C}), \overline{CF}^{\mathrm{in}}_{p,q}(\mathbf{C}), \widehat{C}'_\ell(\mathbf{C}), \widehat{CF}'_{s,t}(\mathbf{C}), \overline{C}^{\mathrm{out}}_k(\mathbf{C}), \overline{CF}^{\mathrm{out}}_{m,n}(\mathbf{C})\}.$$

Orient the manifolds $\widehat{CF}'_{s,t}(\mathbf{C})$ by the pullback orientations of the defining embeddings into $\widehat{C}'_{r+s}(\mathbf{C})$. We shall denote the associated operad of fundamental chains by $\mathcal{K}(\widehat{CF}'(\mathbf{C}))$. The following lemma is a simple extension of lemma 1.0.1.

Lemma 2.0.2. *A representation of the operad $\mathcal{K}(\widehat{CF}'(\mathbf{C}))$ is equivalent to two NCG_∞ algebras and an NCG_∞ morphism from one to the other.*

3. Cocycles from de Rham field theories

It is straight forward to extend the definitions employed in this paper to define a two-colored cooperad of graphs $\mathfrak{G}^c_{\widehat{C}'(\mathbf{C})}$, whose cocompositions mimick the compositions of $\widehat{C}'(\mathbf{C})$, together with a representation $\Phi : \mathfrak{G}_{\widehat{C}'(\mathbf{C})} \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}})$ of its linearly dual operad. In this paper we have constructed a map

$$\omega : \mathfrak{G}^c_{\overline{CF}(\mathbf{H})} \to \Omega(\overline{CF}(\mathbf{H})).$$

We did this by decorating edges of graphs with suitable pullbacks of the differential form $d\mathrm{Arg}(z-w)$ or its hyperbolic cousin

$$d \log \frac{(z-w)(z-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})}.$$

It was claimed in [**10**] that it is possible to instead use the “log propagator”

$$d \log \frac{z-w}{\bar{z}-w},$$

a claim which is highly nontrivial since this form is singular on the compactified configuration spaces. This claim still has no rigorous proof.

Say that a **propagator de Rham field theory** on $\widehat{C}'(\mathbf{C})$ is a map of dg cooperads

$$\xi : \mathfrak{G}^c_{\widehat{C}'(\mathbf{C})} \to \Omega(\widehat{C}'(\mathbf{C})).$$

which is defined by decorating edges of graphs with pullbacks of a fixed differential form. Any propagator de Rham field theory defines a morphism of dg operads

$$\Phi \circ \xi^* : \mathcal{K}(\widehat{C}'(\mathbf{C})) \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}}),$$

i.e. a pair of L_∞ structures λ^{in} and λ^{out} on T_{poly} and an L_∞ morphism from λ^{in} to λ^{out}. Merkulov showed that if ξ is a propagator de Rham field theory ξ with the property that λ^{in} and λ^{out} both equal to Schouten bracket, then the expression

$$\phi_\ell(\xi) = \sum_{\Gamma \in [(dgra_\ell^{2\ell-2})_{1\mathrm{vi}}]} \int_{\widehat{C}'_\ell(\mathbf{C})} \xi_\Gamma \Gamma,$$

where the subscript 1vi denotes the subset of 1-vertex irreducible graphs, is a cocycle of degree 0 in the graph complex GC. The results of Willwacher from [**19**] then imply that $\phi_\ell(\xi)$ represents an element of the Grothendieck-Teichmüller Lie algebra. Unfortunately, there are no known non-trivial such propagator de Rham field theories! The only good candidate is based on Kontsevich’s log propagator (defined above), and this propagator has some unresolved convergence issues.

Assume that there is a nontrivial propagator de Rham field theory ξ such that λ^{in} and λ^{out} both equal the Schouten bracket. One may introduce a four-colored graph cooperad $\mathfrak{G}^c_{\widehat{CF}'(\mathbf{C})}$ in the obvious way together with a representation

$$\Phi : \mathfrak{G}_{\widehat{CF}'(\mathbf{C})} \to \mathcal{E}nd(T_{\mathrm{poly}}, T_{\mathrm{poly}}, T_{\mathrm{poly}}, T_{\mathrm{poly}})$$

of its linearly dual operad. Since $\widehat{CF}'_{s,t}(\mathbf{C})$ embeds in $\widehat{C}'_{r+s}(\mathbf{C})$, it has to be possible to extend ξ to

$$\xi : \mathfrak{G}^c_{\widehat{CF}'(\mathbf{C})} \to \Omega(\widehat{CF}'(\mathbf{C})).$$

The composition $\Phi \circ \xi^*$ then defines a universal NCG_∞ automorphism $F+G$ of $((T_{\mathrm{poly}}, [\,,\,]_S), (T_{\mathrm{poly}}, \nu), \mathcal{V}))$, extending the L_∞ automorphism F of $(T_{\mathrm{poly}}, [\,,\,]_S)$ defined by Merkulov’s formula. One may then, given a Maurer-Cartan element π of $(T_{\mathrm{poly}}, [\,,\,]_S)$, push this to an A_∞ morphism

$$G^\pi : (T_{\mathrm{poly}}[[\hbar]], \nu^{\mathcal{V}(\pi)}) \to (T_{\mathrm{poly}}[[\hbar]], \nu^{\mathcal{V} \circ F(\pi)}).$$

This construction, if successful, would hopefully elucidate the relation suggested in [**10**] between the Grothendieck-Teichmüller Lie algebra and the theory of Duflo automorphisms.

$\mathcal{A}$

NCG_∞ algebras

Let $\mathcal{NCG}$ be the two-colored operad generated by a degree -1 Lie bracket $[x_1, x_2]$ in one color, call it $\mathbf{x}$, an associative degree 0 product $a_1 \cdot a_2$ in another color, call it $\mathbf{a}$, and an operation which we denote $x_1 \bullet a_1$, of the type $(\mathbf{x}, \mathbf{a}) \to \mathbf{a}$, which represents the bracket in derivations of the product. This is the operad of $NCGA$s.

Proposition 0.0.1. *The operad $\mathcal{NCG}$ is Koszul.*

Remark 0.0.2. After the publication of the first version of the present article, Hoefel and Livernet published a preprint[**8**] containing a different and detailed proof of this proposition.

Proof. We shall use the rewriting systems method of [**12**]. The rewriting rules are

$$\begin{aligned}
(a_1 \cdot a_2) \cdot a_3 &\mapsto a_1 \cdot (a_2 \cdot a_3) \\
[[x_1, x_2], x_3] &\mapsto -[[x_2, x_3], x_1] - [[x_3, x_1], x_2] \\
x_1 \bullet (a_1 \cdot a_2) &\mapsto (x_1 \bullet a_1) \cdot a_2 + a_1 \cdot (x_1 \bullet a_2) \\
[x_1, x_2] \bullet a_1 &\mapsto x_1 \bullet (x_2 \bullet a_1) - x_2 \bullet (x_1 \bullet a_1).
\end{aligned}$$

The critical monomials are $((a_1 \cdot a_2) \cdot a_3) \cdot a_4$, $[[[x_1, x_2], x_3], x_4]$, $x_1 \bullet ((a_1 \cdot a_2) \cdot a_3)$, $[x_1, x_2] \bullet (a_1 \cdot a_2)$ and $[[x_1, x_2], x_3] \bullet a_1$. The first two are known to be confluent as the operads $\mathcal{L}ie^1$ and $\mathcal{A}ss$ are known to be Koszul. The third critical monomial can be rewritten either as

$$\begin{aligned}
x_1 \bullet ((a_1 a_2) a_3) &\mapsto (x_1 \bullet (a_1 a_2)) a_3 + (a_1 a_2)(x_1 \bullet a_3) \\
&\mapsto ((x_1 \bullet a_1) a_2) a_3 + (a_1 (x_1 \bullet a_2)) a_3 + a_1 (a_2 (x_1 \bullet a_3)) \\
&\mapsto (x_1 \bullet a_1)(a_2 a_3) + a_1 ((x_1 \bullet a_2) a_3) + a_1 (a_2 (x_1 \bullet a_3))
\end{aligned}$$

or as

$$\begin{aligned}
x_1 \bullet ((a_1 a_2) a_3) &\mapsto x_1 \bullet (a_1 (a_2 a_3)) \\
&\mapsto (x_1 \bullet a)(a_2 a_3) + a_1 (x_1 \bullet (a_2 a_3)) \\
&\mapsto (x_1 \bullet a_1)(a_2 a_3) + a_1 ((x_1 \bullet a_2) a_3) + a_1 (a_2 (x_1 \bullet a_3)).
\end{aligned}$$

Since both ways give the same end result, $x_1 \bullet ((a_1 a_2) a_3)$ is confluent.

The critical monomial $[x_1, x_2] \bullet (a_1 \cdot a_2)$ can be rewritten either as

$$\begin{aligned}
[x_1, x_2] \bullet (a_1 a_2) &\mapsto ([x_1, x_2] \bullet a_1) a_2 + a_1 ([x_1, x_2] \bullet a_2) \\
&\mapsto (x_1 \bullet (x_2 \bullet a_1)) a_2 - (x_2 \bullet (x_1 \bullet a_1)) a_2 + a_1 (x_1 \bullet (x_2 \bullet a_2)) - a_1 (x_2 \bullet (x_1 \bullet a_2))
\end{aligned}$$

or

$$\begin{aligned}
[x_1, x_2] \bullet (a_1 a_2) &\mapsto x_1 \bullet (x_2 \bullet (a_1 a_2)) - x_2 \bullet (x_1 \bullet (a_1 a_2)) \\
&\mapsto x_1 \bullet ((x_2 \bullet a_1) a_2) + x_1 \bullet (a_1 (x_2 \bullet a_2)) - x_2 \bullet ((x_1 \bullet a_1) a_2) - x_2 \bullet (a_1 (x_1 \bullet a_2)) \\
&\mapsto (x_1 \bullet (x_2 \bullet a_1)) a_2 + (x_2 \bullet a_1)(x_1 \bullet a_2) + (x_1 \bullet a_1)(x_2 \bullet a_2) + a_1 (x_1 \bullet (x_2 \bullet a_2)) \\
&\quad - (x_2 \bullet (x_1 \bullet a_1)) a_2 - (x_1 \bullet a_1)(x_2 \bullet a_2) - (x_2 \bullet a_1)(x_1 \bullet a_2) - a_1 (x_2 \bullet (x_1 \bullet a_2)).
\end{aligned}$$

These two ways to rewrite the monomial agree, so it is also confluent. Confluence of the last critical monomial, $[[x_1, x_2], x_3] \bullet a_1$, is a similar straightforward manipulation and we omit it. $\square$

For a Koszul operad $\mathcal{P}$ the operad $\mathcal{M}or(\mathcal{P})$, whose representations are pairs of $\mathcal{P}$-algebras together with a morphism of $\mathcal{P}$-algebras between them, is (homotopy) Koszul by the results of [**16**]. An algebra for the resolution $\Omega(\mathcal{M}or(\mathcal{P})^{\text{i}})$ consists in two strong homotopy $\mathcal{P}$-algebras and a strong homotopy morphism between them. This general machinery produces a four-colored operad $\Omega(\mathcal{M}or(\mathcal{NCG})^{\text{i}})$ of morphisms of NCG_∞ algebras. It has two "Lie-colors" and two "Ass-colors". We can make it into a 3-colored operad by identifying the two Lie colors. (A representation of that new operad will be a morphism of NCG_∞ algebras having the same L_∞ algebra acting on both A_∞ algebras.) This operad includes generators describing an L_∞ endomorphism of the Lie-color. Quotient out these generators and get a new three-colored operad $\mathcal{M}or_*(\mathcal{NCG})_\infty$. Its representations are morphisms of NCG_∞ algebras that have the same L_∞ algebra acting on both A_∞ algebras, and for which the L_∞ endomorphism is the identity. It is easy to see, knowing that $\Omega(\mathcal{M}or(\mathcal{NCG})^{\text{i}}) \to \mathcal{M}or(\mathcal{NCG})$ is a quasi-isomorphism, that $\mathcal{M}or_*(\mathcal{NCG})_\infty$ is quasi-isomorphic to the operad $\mathcal{M}or_*(\mathcal{NCG})$ which has as representations two NCGAs with the same Lie algebra acting on both associative algebras and a morphism between the NCGAs which is the identity on the Lie algebra. Finally one can note that, in fact, $\mathcal{M}or_*(\mathcal{NCG})_\infty = \Omega(\mathcal{M}or_*(\mathcal{NCG})^{\text{i}})$.

REMARK 0.0.3. Consider the operad $\mathcal{NCG}^{(1)}$ of (one-colored) noncommutative Gerstenhaber algebras (chain complexes that are simultaneously a dg Lie algebra, with the bracket of degree -1, and an associative algebra, and the Lie bracket acts by derivations of the associative product). Our method to prove Koszulity of $\mathcal{NCG}$ does not repeat mutatis mutandum for $\mathcal{NCG}^{(1)}$. The problem is that one gets a new critical monomial, $[x_1x_2, x_3x_4]$, which is not confluent. This suggests (but does not prove) that $\mathcal{NCG}^{(1)}$ is not Koszul.

0.1. The deformation complex of $\mathcal{NCG}_\infty \to \mathcal{P}$. Recall that we denote the two colors of $\mathcal{NCG}$ by $\mathbf{x}$ and $\mathbf{a}$. Here $\mathbf{x}$ is the "Lie" color and $\mathbf{a}$ is the "Ass" color. Define $\mathcal{NCG}_\infty := \Omega(\mathcal{NCG}^{\text{i}})$.

Let $\mathcal{P}$ be a dg operad with colors $\mathbf{x}$ and $\mathbf{a}$ and assume given a morphism of operads $f : \mathcal{NCG}_\infty \to \mathcal{P}$. We shall describe the deformation complex $\mathrm{Def}(f)$.

We shall simplify notation and write $\mathcal{P}(k)$ for $\mathcal{P}(k, 0; \mathbf{x})$ and $\mathcal{P}(p, q)$ for $\mathcal{P}(p, q; \mathbf{a})$. As a chain complex,

$$\mathrm{Def}(f) = \prod_{k\geq 2} \mathcal{P}(k)_{\Sigma_k}[2-2k] \oplus \prod_{\substack{p\geq 0, q\geq 1\\ p+q\geq 2}} \mathcal{P}(p,q)_{\Sigma_p} \otimes sgn_q[1-2p-q].$$

This chain complex has a degree zero graded Lie bracket defined by taking the commutator of operadic composition. The map $f = (f_k) + (f_{p,q})$ is a Maurer-Cartan-element and the differential on the deformation complex is the internal differential on $\mathcal{P}$ plus the bracket $[f, \,]$. We can give a more suggestive formulation of the deformation complex as follows. The components (f_k) define a morphism $\lambda^f : \mathcal{L}ie^1_\infty \to \mathcal{P}$ and

$$\mathrm{Def}(\lambda^f) = \prod_{k\geq 2} \mathcal{P}(k)_{\Sigma_k}[2-2k]$$

with differential (the internal differential on $\mathcal{P}$ plus) $[(f_k), \,]$. Set

$$\textstyle\int \mathcal{P}(q) := \prod_p \mathcal{P}(p,q)_{\Sigma_p}[-2p].$$

The collection $\int \mathcal{P} = \{\int \mathcal{P}(q)\}$ has a structure of dg operad. (This can actually be interpreted as a categorical end: a Σ-bimodule can be regarded as a bifunctor and we take the limit over one argument.) The compositions of the Lie-color in $\mathcal{P}$ define a right action $\bullet$ of $\mathrm{Def}(\lambda^f)$ on $\int \mathcal{P}$ by operadic derivations. Add to the differential the term $[(f_{p,1}), \,] + (\,) \bullet (f_k)$. The remaining mixed components of f, i.e. $(f_{p,q})$ with $q \geq 2$, define a morphism $\mu^f : \mathcal{A}ss_\infty \to \int \mathcal{P}$ with $\mu^f_q = (f_{p,q})_{p\geq 0}$. We have

$$\mathrm{Def}(\mu^f) = \prod_{\substack{p\geq 0, q\geq 1\\ p+q\geq 2}} \mathcal{P}(p,q)_{\Sigma_p} \otimes sgn_q[1-2p-q].$$

The components $(f_{p,q})_{p\geq 1}$ define a map of complexes $\rho^f : \mathrm{Def}(\lambda^f)[-1] \to \mathrm{Def}(\mu^f)$ by $\gamma \mapsto (f_{p,q}) \circ \gamma$.

REMARK 0.1.1. The deformation complex $\mathrm{Def}(f)$ is isomorphic as a chain complex to the mapping cone of ρ^f and as a graded Lie algebra to $\mathrm{Def}(\lambda^f) \ltimes \mathrm{Def}(\mu^f)$.

0.2. Coalgebra description of NCG_∞ algebras. Algebras and morphisms for Koszul operads can be encoded in terms of coalgebras. This section is a remark on the coalgebra description of NCG_∞ algebras.

Let L and A be dg vector spaces. Denote by T_c the cofree coassociative counital algebra functor and by S_c the symmetric, i.e. cocommutative, version. Denote the weight n homogeneous parts with respect to the weight grading by tensor length using T_c^n and S_c^n. Introduce the coalgebra

$$C(L,A) := \bigoplus_{\substack{p\geq 0, q\geq 1\\ p+q\geq 2}} S_c^p(L[2])\otimes T_c^q(A[1]).$$

Let

$$\mathrm{Coder}_{T_c(A[1])}(C(L,A),T_c(A[1]))$$

be the dg Lie subalgebra of $\mathrm{Coder}(S_c(L[2])\otimes T_c(A[1]))$ obtained as the image of

$$Map(C(L,A),A[1]) \to \mathrm{Coder}(S_c^{\geq 1}(L[2])\otimes T_c(A[1])).$$

It follows from the description of the deformation complex of a morphism $\mathcal{NCG}_\infty \to \mathcal{P}$ that there is an isomorphism of dg Lie algebras;

$$\mathrm{Def}(\mathcal{NCG}_\infty \xrightarrow{0} \mathcal{E}\mathrm{nd}(L,A)) \cong \mathrm{Coder}(S_c(L[2])) \rtimes \mathrm{Coder}_{T_c(A[1])}(C(L,A),A[1]).$$

From this it follows that a structure of NCG_∞ algebra on (L,A) is equivalent to a degree $+1$

$$D = Q+V \in \mathrm{Coder}(S_c(L[2])) \rtimes \mathrm{Coder}_{T_c(A[1])}(C(L,A),T_c(A[1]))$$

such that $[D,D]=0$. Any such coderivation D defines a degree $+1$ coderivation of $S_c(L[2])\otimes T_c(A[1])$, also satisfying $[D,D]=0$, but the opposite is not true. (A general MC element in the dg Lie algebra of coderivations of $S_c(L[2])\otimes T_c(A[1])$ is a (weak) $OCHA$ structure on (L,A).) A morphism of NCG_∞ algebras from (L,A,D) to (L',A',D') consists of a pair

$$F+G \in \mathrm{Hom}(S_c^{\geq 1}(L[2]),L'[2]) \oplus \mathrm{Hom}(\bigoplus_{p\geq 0,q\geq 1} S_c^p(L[2])\otimes T_c^q(A[1]),A'[1])$$

such that $F+G$ lifts to a morphism of dg coalgebras

$$(S_c(L[2])\otimes T_c(A[1]),D) \to (S_c(L'[2])\otimes T_c(A'[1]),D').$$

0.3. Induced A_∞ morphisms. Let $F+G : (L,A,D) \to (L',A',D')$ be a morphism of NCG_∞ algebras, as in above section. Write $D=Q+V$ and split $V=\sum_{p\geq 0,q\geq 1,p+q\geq 2}$ as $\mu + U = \sum_q \mu_q + \sum_{p,q\geq 1} U_{p,q}$ (so $mu_q = V'_{0,q}$). Similarly split D' to $Q'+\mu'+U'$. This gives an alternative description of an NCG_∞ algebra as

- an L_∞ algebra $(L[1],Q)$,
- an A_∞ algebra $A,\mu)$,
- and an L_∞ morphism $U : L[1]\to C(\mu)[1]$ from $(L[1],Q)$ to the Hochschild cochain complex of $(A.\mu)$.

If $F+G$ is an NCG_∞ morphism, then the composite L_∞ morphism $U'\circ F : L[1]\to L'[1]\to C(\mu')[1]$ makes $(L,A',Q'+\mu'+U'\circ F)$ an NCG_∞ algebra, and we obtain an NCG_∞ morphism

$$id+G : (L,A,Q+\mu+U) \to (L,A',Q'+\mu'+U'\circ F).$$

Hence there is no essential loss of generality in assuming that $F=id$. With this assumption the equation satisfied by the components $G_{m,n} : L[2]^{\otimes m}\otimes A[1]^{\otimes n}\to A'[1]$ $(m\geq 0, n\geq 1)$ of G is

$$\partial_{Map(L[2]^{\otimes m}\otimes A[1]^{\otimes n},A'[1])} G_{m,n}$$

$$= \sum \pm G_{m-\ell+1,n}\circ Q_\ell + \sum \pm G_{m-p,n-q+1}\circ V_{p,q} + \sum \pm V'_{s,t}\circ (G_{m_1,n_1},\dots,G_{m_t,n_t}).$$

Let π be an MC element of L and let $(\dots)^\pi : Map(L[2]^{\otimes m}\otimes A[1]^{\otimes n},A'[1]) \to Map(A[1]^{\otimes n},A'[1])$ be the map $\Psi\mapsto \Psi^\pi := \Psi(\pi^{\otimes m},\dots)$. It is immediate from the MC equation for π that

$$(\partial_{Map(L[2]^{\otimes m}\otimes A[1]^{\otimes n},A'[1])})^\pi = \sum \pm(\Psi\circ Q_\ell) + \partial_{Map(A[1]^{\otimes n},A'[1])}(\Psi^\pi).$$

Hence $G^\pi_{m,n}$ satisfies

$$\partial_{Map(A[1]^{\otimes n},A'[1])}(G^\pi_{m,n}) = \sum \pm G^\pi_{m-p,n-q+1}\circ V^\pi_{p,q} + \sum \pm V'^\pi_{s,t}\circ (G^\pi_{m_1,n_1},\dots,G^\pi_{m_t,n_t}).$$

This means that $G_n = \sum_m G^{\pi}_{m,n}$ is an A_∞ morphism from A with the A_∞ whose q-th component is $\sum_p V^{\pi}_{p,q}$ to A' with the A_∞ structure whose t-th component is $\sum_s V'^{\pi}_{s,t}$. In general one should add a formal parameter $\hbar$ to get well-defined expressions but this technicality does not affect above argument.

Bibliography

1. J. Alm and C.A. Rossi, *The universal enveloping algebra of a L_∞ algebra and deformation quantization*, In preparation.
2. V. Baranovsky, *A universal enveloping for L-infinity algebras*, Mathematical Research Letters **6** (2008), 1073–1089.
3. D. Calaque, G. Felder, A. Ferrario, and C.A. Rossi, *Bimodules and branes in deformation quantization*, Compositio Mathematica **147** (2011), 105–160.
4. D. Calaque and C.A. Rossi, *Lectures on Duflo isomorphisms in Lie algebra and complex geometry*, `people.mpim-bonn.mpg.de/crossi/LectETHbook.pdf`.
5. E. Getzler and J.D.S Jones, *Operads, homotopy algebra, and iterated integrals for double loop spaces*, `arxiv:abs/hep-th/9403055`.
6. R.M. Hardt, P. Lambrechts, V. Turchin, and I. Volić, *Real homotopy theory of semi-algebraic sets*, `arxiv:0806.0476v2`.
7. E. Hoefel, *OCHA and the swiss-cheese operad*, Journal of Homotopy and Related Structures **4** (2009), no. 1, 123–151.
8. E. Hoefel and M. Livernet, *OCHA and Leibniz pairs, towards a Koszul duality*, `arXiv:1104.3607v1`.
9. T. Kajiura and J. Stasheff, *Homotopy algebras inspired by classical open-closed string field theory*, Communications in Mathematical Physics **263** (2006), 553–581.
10. M. Kontsevich, *Operads and Motives in Deformation Quantization*, Letters in Mathematical Physics **48** (1999), no. 1, 35–72.
11. ———, *Deformation Quantization of Poisson Manifolds*, Letters in Mathematical Physics **66** (2003), 157–216.
12. J.-L. Loday and B. Vallette, *Algebraic Operads*, `math.unice.fr/ brunov/Operads.pdf`.
13. D. Manchon and C. Torossian, *Cohomologie tangente et cup-produit pour la quantification de kontsevich*, Annales mathématiques Blaise Pascal **10** (2003), no. 1, 75–106.
14. S. Mau and C. Woodward, *Geometric realizations of the multiplihedron and its complexification*, `arXiv:0802.2120v4`.
15. S.A. Merkulov, *Operads, configuration spaces and quantization*, `arxiv:math/1005.3381`.
16. S.A. Merkulov and B. Vallette, *Deformation theory of representations of prop(erad)s*, Journal für die reine und angewandte Mathematik (Crelles Journal) **636** (2009), 125–174.
17. M. Pevzner and C. Torossian, *Isomorphisme de Duflo et cohomologie tangentielle*, Journal of Geometry and Physics **51** (2004), no. 4, 486–505.
18. B. Shoikhet, *On the A_∞ Formality Conjecture*, `arXiv:math/9809117v1`.
19. T. Willwacher, *M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra*, `arXiv:abs/1009.1654`.