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EXTENSION AND EMBEDDING OF PLURISUBHARMONIC CURRENTS

AHMAD K. AL ABDULAALI

Abstract. In this paper we study the extension of currents across small obstacles. Our main results are:

- Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT \geq -S$ on $\Omega\setminus A$ for some positive closed currents S on Ω. Assume that the Hausdorff measure $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is closed and negative supported in A.
- Let u be a positive exhaustion strictly 0-convex function on an open subset Ω of $\mathbb{C}^n$ and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a positive current of bidimension (p,p) on $\Omega\setminus A$ such that $dd^cT \leq S$ on $\Omega\setminus A$ for some positive currents S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists. If $p \geq 2$, dd^cS is of locally finite mass and $u \in \mathcal{C}^2$, then $\widetilde{dd^cT}$ exists and $\widetilde{dd^cT} = dd^c\widetilde{T}$.

Furthermore, we generalize some results on the intersection and the extension of currents.

1. Introduction

According to our nature, there is always a motivation behind any behavior. So you may ask yourself now "What is the motivation which drove us to write this paper?". This question will be clarified in the next few lines.

In this paper we are interested with the currents and its extension. In 2003, Dabbek, Elkhadhra and El Mir [8] proved one of the strongest results in this field which asserts:

Let A be a closed complete pluripolar set of an open subset Ω of C^n and T a positive current of bidimension (p,p) on $\Omega\setminus A$. Suppose that $\widetilde{T}$ and $\widetilde{dd^cT}$ exist (or $dd^cT \leq 0$), then there exists a positive current R supported in A such that $\widetilde{dd^cT} - dd^c\widetilde{T} = R$.

The authors proved the existence of the residual current R without requiring anything from dT and this is the strength of this result. In that article they studied the extension of plurisubharmonic currents across pluripolar sets and across zero sets of strictly k-convex functions. Three years later Dinh and Sibony [11] generalized the above result and found the residual

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current R when $dd^cT \leq S$ for some positive currents S on Ω. Now a question occurs, can we show the results in [8] if we replace $dd^cT \leq 0$ by $dd^cT \leq S$, specially in the case when A is a zero set of strictly k-convex function? Actually, this is the main motivation behind this paper. In addition, the present paper is inspired by the articles [5], [6] and [7].

1.1. **Survey of results in this paper.** The theme in this paper is to improve the results in [8]. More precisely, we study the extendability of positive currents T after a little squeeze for dd^cT into the positive side. So as shown in the figure below, this work can be considered as an embedding of the plurisubharmonic currents.

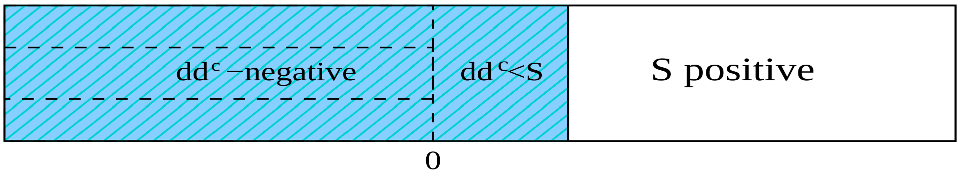

FIGURE 1. Embedding of dd^c-negative currents into our study

The paper consists of four sections. In the first section we gave some definitions, basic properties and some facts about the currents.

The section following that one is concerned with the intersection and the extension of currents. So, we considered the case where A is a closed complete pluripolar subset and started with the following result.

Let T be a positive dd^c-negative current of bidimension (p, p) on a complex manifold Ω of dimension n and A be a closed complete pluripolar subset of Ω such that the Hausdorff measure $\mathcal{H}_{2p-1}(A) = 0$. Let S be a positive and closed current of bidimension $(1, 1)$ on Ω and smooth on $\Omega \setminus A$. If g is a solution of $dd^cg = S$ on an open set $U \subset \Omega$ and g_j a sequence of smooth plurisubharmonic functions such that (g_j) converges to g in $\mathcal{C}^2(U \setminus A)$ then the sequence $(dd^cg_j \wedge T)$ is locally bounded in mass in Ω.

By this result we can find a subsequence g_{j_s} such that the sequence $dd^cg_{j_s} \wedge T$ converges weakly* to a current denoted by $S \wedge T$. But what about the uniqueness of $S \wedge T$? We ask this question since this intersection between S and T is well defined in some cases. Infact, For T is pluriharmonic Dinh and Sibony [10] in 2004 defined the current $S\wedge T$ but when S is smooth on Ω where Ω is a compact Khäler manifold. In 2008, Alessandrini and Bassanelli [3] generalized the result in [10] by proving the existence of $S \wedge T$ when A is a proper analytic subset set with $(n-p) + dimA \leq dim\Omega$ and S is smooth on $\Omega\setminus A$. In particular, $S\wedge T$ is defined as a limit of $dd^cg_j\wedge T$.

Also in this section we obtained another result which generalizes a convergence theorem of closed positive currents.

Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a closed and positive current of bidimension (p,p) on $\Omega \setminus A$. Suppose that $\mathcal{H}_{2p-1}(A) = 0$. Let g be a locally bounded plurisubharmonic function on Ω and (g_j) be a decreasing sequence of smooth plurisubharmonic functions on $\Omega \setminus A$ converging pointwise to g on $\Omega \setminus A$. Then for all j the extension $\widetilde{g_jT}$ exists and the sequence $(\widetilde{g_jT})$ converges.

This result improves a very well known result when A is empty (cf. [9]). We ended this section by proving the following theorem

Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive closed currents S on Ω. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is closed and negative supported in A.

This theorem extends the case when $S = 0$ which was proved in [8]. We used this extension to give a version of Chern-Levine-Nirenberg inequality.

Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive closed currents S on Ω. Let K and L compact set in Ω with $L \subset\subset K$. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$, then there exists a constant $C_{K,L} > 0$ such that for all u smooth bounded plurisubharmonic on Ω we have the following estimate

$$\int_{L\setminus A} T \wedge du \wedge d^cu \wedge \beta^{p-1} \leq C_{K,L} \sup_{z\in K} |u(z)|^2 (\|\widetilde{T}\|_K + \|\widetilde{dd^cT}\|_K)$$

In the third section we considered A as a zero set of plurisubharmonic function u with the very special choice that u is positive exhaustion strictly 0-convex. In this section we included our second main theorem.

Let u be a positive exhaustion strictly 0-convex function on an open subset Ω of $\mathbb{C}^n$ and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive currents S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists. If $p \geq 2$, dd^cS is of order zero and $u \in \mathcal{C}^2$, then $\widetilde{dd^cT}$ exists and

$\widetilde{dd^cT} = dd^c\widetilde{T}$.

The case when u is positive strictly k-convex and $S = 0$ was proved in [8]. In 2005 Dabbek and Elkhadhra [6] assumed the case when T is a positive current such that $\widetilde{dT}$ and u a positive strictly k-convex. Recently, the result has been declared by Dabbek and Noureddine [7] in the case of quasi-plurisubharmonic currents.

The study in this paper led us to many intersting open problems in this field. So we dedicated the final section to exposing these problems that might be a start point for the next work.

2. Preliminaries and notations

Let Ω be an open subset of $\mathbb{C}^n$. Let $\mathcal{D}_{p,q}(\Omega, k)$ be the space $\mathcal{C}^k$ compactly supported differential forms of bidegree (p, q) on Ω. A form $\varphi \in \mathcal{D}_{p,p}(\Omega, k)$ is said to be strongly positive form if φ can be written as

$$\varphi(z) = \sum_{j=1}^{N} \gamma_j(z)\ i\alpha_{1,j} \wedge \overline{\alpha}_{1,j} \wedge ... \wedge i\alpha_{p,j} \wedge \overline{\alpha}_{p,j}$$

where $\gamma_j \geq 0$ and $\alpha_{s,j} \in \mathcal{D}_{0,1}(\Omega, k)$. Then $\mathcal{D}_{p,p}(\Omega, k)$ admits a basis consisting of strongly positive forms (see [9], CH III, (1.4) Lemma). We denote by $SP_{p,p}(\Omega)$ the space of strongly positive forms on Ω. The dual space $\mathcal{D}'_{p,q}(\Omega, k)$ is the space of currents of bidimension (p, q) or bidegree $(n-p, n-q)$ and of order k. A current $T \in \mathcal{D}'_{p,p}(\Omega, k)$ is said to be positive if $\langle T, \varphi\rangle \geq 0$ for all forms $\varphi \in \mathcal{D}_{p,p}(\Omega, k)$ that are strongly positive. If $T \in \mathcal{D}'_{p,p}(\Omega, k)$ then it can be written as

$$T = i^{(n-p)^2} \sum_{|I|=|J|=n-p} T_{I,J} dz_I \wedge d\overline{z}_J$$

where $T_{I,J}$ are distributions on Ω. For the positive current $T \in \mathcal{D}'_{p,p}(\Omega, k)$ the mass of T is denoted by $\|T\|$ and defined by $\sum |T_{I,J}|$ where $|T_{I,J}|$ are the total variations of the measures $T_{I,J}$. Let $\beta = dd^c|z|^2$ be the Khäler form on $\mathbb{C}^n$ (where $d =$ and $d^c = i(-\partial + \overline{\partial})$ thus $dd^c = 2i\partial\overline{\partial}$), then there exists a constant $C > 0$ depends only on n and p such that

$$T \wedge \frac{\beta^p}{2^p p!} \leq \|T\| \leq C\ T \wedge \beta^p$$

A current T is said to be $\mathbb{C}$-normal if T and dd^cT are of locally finite mass. We recall that T is $\mathbb{C}$-flat current if $T = F + \partial H + \overline{\partial} S + \partial\overline{\partial} R$, where R, H, S and R are currents with locally integrable coefficients. On this class of currents the theorem says that for $\mathbb{C}$-flat current T of bidimension (p, p) if $\mathcal{H}_{2p}(SuppT) = 0$, then $T = 0$.

Along the way a current T is said to be plurisubharmonic if dd^cT is positive current. Let (χ_n) be a smooth bounded sequence which vanishes on a neighborhood of closed subset $A \subset \Omega$ and χ_n converges to $1_{\Omega\setminus A}$, and T be a current defined on $\Omega \setminus A$. If $\chi_n T$ has a limit which does not depend on (χ_n), this limit is the trivial extension of T by zero across A noted by $\widetilde{T}$. Thus, $\widetilde{T}$ exists if and only if $\|T\|$ has locally finite mass on Ω.

We end this section by giving the following two theorems. The first one is called Chern-Levine-Nirenberg inequality and the second is a modification for that inequality proved by Dinh and Sibony [11].

Theorem 2.1. *Let Ω be an open subset of $\mathbb{C}^n$ and T be a closed positive current of bidimension (p,p). Let $u_1, ..., u_p$ are locally bounded plurisubharmonic functions on Ω. For all compact subsets K, L of Ω with $L \subset\subset K$, there exists s constant $C_{K,L} \geq 0$ such that*

$$\|T \wedge dd^c u_1 \wedge ... \wedge dd^c u_p\|_L \leq C_{K,L}\|T\|_K \sup_{z\in K}|u_1(z)|... \sup_{z\in K}|u_p(z)|$$

Theorem 2.2. *Let Ω be an open subset of $\mathbb{C}^n$. Let K and L compact set in Ω with $L \subset\subset K$. Assume that $T \in \mathcal{D}'_{p,p}(\Omega)$ is positive and dd^cT is of order zero, then there exists a constant $C_{K,L} > 0$ such that for all u smooth bounded plurisubharmonic on Ω we have the following estimate*

$$\int_L T \wedge du \wedge d^c u \wedge \beta^{p-1} \leq C_{K,L} \sup_{z\in K}|u(z)|^2(\|T\|_K + \|dd^cT\|_K)$$

3. Extension over pluripolar sets and intersection of currents

In this section we give continuation and extension results across pluripolar sets. So let us start with a theorem which deals with the intersection of currents.

Theorem 3.1. *Let T be a positive dd^c-negative current of bidimension (p,p) on a complex manifold Ω of dimension n and A be a closed complete pluripolar subset of Ω such that the Hausdorff measure $\mathcal{H}_{2p-1}(A) = 0$. Let S be a positive and closed current of bidimension $(1,1)$ on Ω and smooth on $\Omega \setminus A$. If g is a solution of $dd^c g = S$ on an open set $U \subset \Omega$ and g_j a sequence of smooth plurisubharmonic functions such that (g_j) converges to g in $\mathcal{C}^2(U\setminus A)$ then the sequence $(dd^c g_j \wedge T)$ is locally bounded in mass in Ω.*

Proof. The problem is local so we can consider that U is an open ball centered at 0 and contained in Ω. Since $dd^c g = S$, $S \geq 0$ and S smooth on $\Omega \setminus A$ then we can assume that g is plurisubharmonic on U and smooth on $U \setminus A$. Put $s = p - 1$ then $\mathcal{H}_{2s+1}(A) = 0$. So for almost all choices of unitary coordinates $(z_1, ..., z_n) = (z', z''), z' = (z_1, ..., z_s), z'' = (z_{s+1}, ..., z_n)$ and almost all radii of balls $B'' = B(0, r'') \subset \mathbb{C}^{n-s}$, the set $\partial B'' \times \{0\} \cap A = \phi$. Therefore there exist B' an open set of $\mathbb{C}^s$ and $0 < t < 1$ such that $B' \times \{|z''| \geq t\} \cap A = \phi$. We may assume that g_j is positive function on $B' \times B''$.

Now, Let $\rho \in \mathcal{C}_0^\infty(B')$ with $\rho(z') \geq 0$ and $\rho = 1$ on $\frac{1}{2}B'$ and let $\psi(z'') \in \mathcal{C}_0^\infty(B'')$ such that $\psi(z'') = 1$ on $\{z'' : |z''| < t\}$. We have

$$\int_{\frac{1}{2}B' \times \{z'':|z''|<t\}} dd^c g_j \wedge T \wedge (dd^c|z'|^2)^s \\ \leq \int_{B' \times B''} \psi(z'') dd^c g_j \wedge T \wedge \rho(z') \wedge (dd^c|z'|^2)^s$$

But

$$\int_{B' \times B''} \psi(z'') dd^c g_j \wedge T \wedge \rho(z') \wedge (dd^c|z'|^2)^s \\ = \int_{B' \times B''} g_j dd^c(\psi(z'')T) \wedge \rho(z') \wedge (dd^c|z'|^2)^s \tag{3.1}$$

Let $\varepsilon > 0$ small enough so that $t_\varepsilon := t - \varepsilon$ satisfies the separation axiom above and take $\varphi \in \mathcal{C}^\infty(\mathbb{C}^{n-s})$, with support in $|z''| > t_\varepsilon$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\{z'' \in \mathbb{C}^{n-s}, |z''| > t\}$. Since $dd^cT \leq 0$, then by (3.1) we get.

$$\int_{B' \times B''} \psi(z'') dd^c g_j \wedge T \wedge \rho(z') \wedge (dd^c|z'|^2)^s \\ = \int_{B' \times B'' \setminus \{z'':|z''|<t_\varepsilon\}} \varphi(z'') g_j dd^c(\psi(z'')T) \wedge \rho(z') \wedge (dd^c|z'|^2)^s \\ + \int_{B' \times B'' \cap \{z'':|z''|<t\}} (1 - \varphi(z'')) g_j dd^c T \wedge \rho(z') \wedge (dd^c|z'|^2)^s \\ \leq \int_{B' \times B''} dd^c(\varphi(z'') g_j) \wedge \psi(z'') T \wedge \rho(z') \wedge (dd^c|z'|^2)^s \tag{3.2}$$

As the current T is positive and $dd^c g_j$ converging to $dd^c g$ in $Supp\ \varphi$, the last right hand side integral in (3.2) is bounded independently of j.

Notice that what we have shown is true for almost all choices of unitary coordinates (z', z''). Hence, for all compact K subset of U we have $\sup_j \|dd^c g_j \wedge T\|_K < \infty$ and from Banach-Alaoglu the sequence $(dd^c g_j \wedge T)$ is contained in a compact set, then there exist subsequence (g_{j_k}) of (g_j) such that $dd^c g_{j_k} \wedge T$ converges weakly* to a current denoted by $S \wedge T$. $\square$

The case when T is pluriharmonic was studied before. Actually, Alessandrini and Bassanelli in [3] proved the uniqueness of $S \wedge T$ as a limit of $dd^c g_j \wedge T$ when A is proper analytic subset set with $(n-p) + dimA \leq dim\Omega$. In [10] Dinh and Sibony defined the intersection of currents $S \wedge T$ but when S is smooth on Ω where Ω is a compact Khäler manifold. So, we generalized the conditions of Alessandrini and Bassanelli but we didn't obtain the uniqueness.

Theorem 3.2. *Let T be a positive and closed current of bidimension (p, p) on a complex manifold Ω of dimension n and A be a closed complete pluripolar subset of Ω such that the Hausdorff measure $\mathcal{H}_{2p-1}(A) = 0$. Let S be a positive and closed current of bidimension $(1, 1)$ on Ω with locally bounded dd^c-solutions on $\Omega \setminus A$. If g is a solution of $dd^c g = S$ on an open set $U \subset \Omega$ and (g_j) is a sequence of decreasing plurisubharmonic smooth functions on*

U which is converging pointwise to g in $U \setminus A$. Then the sequence (g_jT) converges.

Before we prove this theorem let us show the following result

Proposition 3.3. *Let A be a closed complete pluripolar subset of an open subset $\Omega \subset \mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive and closed currents S on Ω. Let v be a plurisubharmonic function of class $\mathcal{C}^2$, $v \geq -1$ on Ω such that*

$$\Omega' = \{z \in \Omega : v(z) < 0\}$$

is relatively compact in Ω. Let $K \subset \Omega'$ be a compact subset and let us set

$$c_K = -\sup_{z\in K} v(z)$$

Then there exists a constant $\eta \geq 0$ such that for all integer $1 \leq s \leq p$ and for every plurisubharmonic function u of class $\mathcal{C}^2$ on Ω' satisfying $-1 \leq u < 0$ we have,

$$\int_{K\setminus A} T \wedge (dd^cu)^p \leq c_K^{-s} \int_{\Omega'\setminus A} T \wedge (dd^cv)^s \wedge (dd^cu)^{p-s} + \eta\|S\|_{\Omega'}$$

This proposition generalizes a result in [8] where the authors considered the case of positive and dd^c-negative current.

Proof. We follow the same techniques in [8]. By ([12], Proposition II.2), there exists a negative plurisubharmonic function f on Ω' which is smooth on $\Omega' \setminus A$ such that

$$A \cap \Omega' = \{z \in \Omega' : f(z) = -\infty\}$$

We choose λ, μ such that $0 < \mu < \lambda < c_K$. For $m \in \mathbb{N}$ and ε small enough we set

$$\varphi_m(z) = \mu\, u(z) + \tfrac{f(z)+m}{m+1} \text{ and } \varphi_{m,\varepsilon}(z) = \max{}_\varepsilon\,(v(z)+1, \varphi_m(z))$$

where $\max_\varepsilon$ is the convolution of the function $(x_1,x_2) \mapsto \max(x_1,x_2)$ by a positive regularization kernel on $\mathbb{R}^2$ depending only on $\|(x_1,x_2)\|$. Thus we have $\varphi_{m,\varepsilon}(z) \in psh(\Omega') \cap \mathcal{C}^\infty(\Omega')$. Furthermore, $\varphi_{m,\varepsilon}(z) = v(z)+1$ in a neighborhood of $\partial\Omega' \cup (\Omega' \cap \{f \leq -m\})$. Consider the open subset

$$\Omega'_m = \Omega' \cap \{f > -m\}$$

Then by Stokes formula we have

$$\int_{\Omega'_m} T \wedge (dd^cu)^{p-s} \wedge (dd^c\varphi_{m,\varepsilon})^{s-1} \wedge dd^c(\varphi_{m,\varepsilon} - v - 1)$$
$$\leq \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (dd^cu)^{p-s} \wedge (dd^c\varphi_{m,\varepsilon})^{s-1}$$

Hence

$$\begin{aligned} &\int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^s \\ &\quad \le \int_{\Omega'_m} (\varphi_{m,\varepsilon}-v-1) S\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^{s-1} \\ &\quad\quad + \int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^{s-1}\wedge dd^c v \end{aligned} \tag{3.3}$$

Let us set

$$S_{k,\varepsilon} := \int_{\Omega'_m} (\varphi_{m,\varepsilon}-v-1) S\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^{s-1-k}\wedge (dd^c v)^k$$

By iterating the operation in (3.3), we deduce that

$$\int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^s \le \int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon}$$

Let $R>0$ and $K_R=\{z\in K: f(z)\ge -R\}$. For m sufficiently large, $K_R\subset\Omega'_m$ and for any $z\in K_R$,

$$\varphi_m(z)\ge -\mu+\frac{m-R}{m+1} > 1-\lambda$$

Moreover, $v\le -c_K$ on K_R so we get

$$v+1\le 1-c_K\le 1-\lambda$$

then $\varphi_{m,\varepsilon}=\varphi_m$ in a neighborhood of K_R. Therefore, by the above inequality we obtain

$$\int_{K_R} T\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_m)^s \le \int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon}$$

Notice that $(dd^c\varphi_m)^s\ge \mu^s\,(dd^c u)^s$ because $dd^c f\ge 0$. So

$$\mu^s\int_{K_R} T\wedge (dd^c u)^p \le \int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c v)^s + \sum_{k=0}^{s-1} S_{k,\varepsilon} \tag{3.4}$$

Now we will show that $S_{k,\varepsilon}$ is bounded for each k. Since S is positive and closed, then using ([9], ChIII, (3.6) Corollary) and Chern-Levine-Nirenberg inequality there exists $\eta_k\ge 0$ such that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega'_m} S\wedge (dd^c u)^{p-s}\wedge (dd^c\varphi_{m,\varepsilon})^{s-1-k}\wedge (dd^c v)^k \\ &= \int_{\Omega'_m} S\wedge (dd^c u)^{p-s}\wedge (dd^c\max(\varphi_m, v+1))^{s-1-k}\wedge (dd^c v)^k \\ &\le \eta_k \sup_{z\in\Omega'} |\max(\varphi_m(z), v(z)+1)|^{s-k-1}\sup_{z\in\Omega'}|u(z)|^{p-s}\sup_{z\in\Omega'}|v(z)|^k\|S\|_{\Omega'} \end{aligned}$$

Therefore there exists $\eta\ge 0$ making (3.2) as follows

$$\mu^s\int_{K_R} T\wedge (dd^c u)^p \le \int_{\Omega'_m} T\wedge (dd^c u)^{p-s}\wedge (dd^c v)^s + \eta\|S\|_{\Omega'}$$

We finished the proof by letting first $m\to\infty$ and secondly $R\to\infty$. □

In our proof of Theorem 3.2. we will use the previous proposition and also the following lemma which is considered one of the most significant results in this field.

Lemma 3.4. *([8], Theorem 1) Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative plurisubharmonic current of bidimension (p,p) on $\Omega \setminus A$ such that $\mathcal{H}_{2p}(A\cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists and is negative plurisubharmonic on Ω. Furthermore the current $\widetilde{dd^cT} - dd^c\widetilde{T}$ is closed and negative supported in A.*

Proof of theorem 3.2. We keep the notations of Theorem 3.1. By the same argument in Theorem 3.1 we can consider that g is plurisubharmonic in U and locally bounded on $U \setminus A$. Since $g_j \in \mathcal{C}^\infty(U) \cap psh(U)$ we may assume that $g_j \le 0$ on $B' \times B'' \subset a\triangle^n$ for the unit polydisk $\triangle^n = \triangle^s \times \triangle^{n-s}$ contained in Ω and $s = p-1$. Take $0 < t < a$ and define,

$$\rho_\varepsilon = \max{}_\varepsilon\left(\tfrac{|z'|^2-a}{a}, \tfrac{|z''|^2-a^2}{a^2-t^2}\right)$$

We have

$$-1 \le \rho_\varepsilon < 0 \text{ in } a\triangle^n \text{ and } \rho_\varepsilon = \tfrac{|z'|^2-a}{a} \text{ on } |z''| < t$$

Now

$$\int_{a\triangle^n} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s = \int_{a\triangle^s\times\{z'':|z''|<t\}} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s + \int_{a\triangle^s\times\{z'':t\le|z''|\le a\}} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s$$

Since $\rho_\varepsilon = \frac{|z'|^2-a}{a}$ on $|z''| < t$, then we obtain that

$$\int_{a\triangle^n} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s = \int_{a\triangle^s\times\{z'':t\le|z''|\le a\}} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s$$

The sequence (g_j) converges pointwise to g on $U \setminus A$ so we get

$$\lim_{j\to\infty}\int_{a\triangle^n} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s = \int_{a\triangle^s\times\{z'':t\le|z''|\le a\}} gT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s$$

As g is locally bounded on $\Omega \setminus A$, the current gT is well defined on $\Omega \setminus A$ and locally this current is negative and plurisubharmonic. So by Lemma 3.4 and ([8], Theorem 1), $\widetilde{gT}$ and $\widetilde{dd^cg\wedge T}$ exist. By applying Theorem 2.2. on $gT \wedge dd^c\rho_\varepsilon$, we obtain that the right hand side in the above equality is bounded independently of ε. Therefore we have

$$\lim_{j\to\infty}\int_{a\triangle^n} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s < \infty$$

The current $g_jT \le 0$, $dd^c(g_jT) = dd^cg_j \wedge T$ which is positive and ρ_ε satisfies all conditions in Proposition 3.3, so for $B' \times B''$ we have

$$\lim_{j\to\infty}\int_{B'\times B''} g_jT\wedge \beta \wedge \left(dd^c|z'|^2\right)^s \le \lim_{j\to\infty}\int_{a\triangle^n} g_jT\wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s < \infty \tag{3.5}$$

And as in the proof of Theorem 3.1, one can find a subsequence $g_{j_s}T$ which is convergent in weak* topology. But we still need to show the convergence of (g_jT). In order to do this we first remember that any differential form can be written as a linear combination of strongly positive forms. So it is enough to verify the convergence for each strongly positive form $\varphi \in SP_{p,p}(\Omega)$. But for $\varphi \in SP_{p,p}(\Omega)$ the sequence $\langle g_jT, \varphi\rangle$ is decreasing in a compact set so it is converging. Hence g_jT converges weakly* in Ω. Proving Theorem 3.2.

Remark 3.5. If we start with a function g which is plurisubharmonic on an open subset Ω of $\mathbb{C}^n$ and locally bounded on $\Omega \setminus A$ where A as above. We find that dd^cg is a positive current. So by Theorem 3.2, if (g_j) is a decreasing sequence of smooth plurisubharmonic functions converging pointwise to g on $\Omega \setminus A$, then we guarantee the convergence of g_jT. This shows that this result generalizes a very well known convergence theorem when g is locally bounded plurisubharmonic function on Ω and the sequence (g_j) is decreasing and converging pointwise to g on the whole of Ω. (cf. [9], Ch.III, (3.7) Theorem).

In view of the proof of Theorem 3.2, we saw how is it nice sometime to play with a combination of plurisubharmonic function g and positive closed current T. Since in this case we have that locally the current gT is negative and plurisubharmonic, allowing us a good area to use the results of closed currents and plurisubharmonic currents. The following corollary one of the aspects of this combination.

Corollary 3.6. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T be a closed and positive current of bidimension (p,p) on $\Omega \setminus A$. Suppose that $\mathcal{H}_{2p-1}(A) = 0$. Let g be a locally bounded plurisubharmonic function on Ω and (g_j) be a decreasing sequence of smooth plurisubharmonic functions on $\Omega \setminus A$ converging pointwise to g on $\Omega \setminus A$. Then for all j the extension $\widetilde{g_jT}$ exists and the sequence $(\widetilde{g_jT})$ converges to $\widetilde{gT}$.*

Proof. Our problem is local so we may assume that $g_j \leq 0$. Now locally the current g_jT is negative and plurisubharmonic. Since $\mathcal{H}_{2p-1}(A) = 0$, then $\mathcal{H}_{2p}(A) = 0$ and by Lemma 3.4, the extension $\widetilde{g_jT}$ exists. Under the same notation of Theorem 3.2 we have

$$\begin{aligned}\lim_{j\to\infty} \int_{a\triangle^n \setminus A} g_jT \wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s \\ = \int_{a\triangle^s \times \{z'':t\leq|z''|\leq a\}} gT \wedge dd^c\rho_\varepsilon \wedge \left(dd^c|z'|^2\right)^s\end{aligned}$$

By similar argument as in the proof of Theorem 3.2, we find that the sequence $(\widetilde{g_jT})$ is weakly* convergent. □

We end this section with a result which can be considered as a version of Chern-Levine-Nirenberg inequality. But before this let us give the following theorem.

Theorem 3.7. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive closed currents S on Ω. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is closed and negative supported in A.*

Proof. Let us first assume that $\widetilde{T}$ exists. Then by [11], the extension $\widetilde{dd^cT}$ exists and R is negative current. Since S is closed, then $dd^cT + S$ is closed positive current on $\Omega \setminus A$, and by El Mir-Skoda result [12] $\widetilde{dd^cT}$ is closed. Therefore R is closed negative current.

In order to show the existence of $\widetilde{T}$ we will proceed as in [8]. Since the problem is local, we will show that T is of lacally finite mass near every point z_0 in A. Without loss of generality, one can assume that z_0 is the origin. Since $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$, there exists a system of coordinates and a polydisk $\triangle^p \times \triangle^{n-p} \subset \mathbb{C}^p \times \mathbb{C}^{n-p}$ such that $(A \cap \overline{SuppT}) \cap (\triangle^p \times \partial\triangle^{n-p}) = \phi$. Moreover the projection map $\pi : (A \cap \overline{SuppT}) \cap (\triangle^p \times \triangle^{n-p}) \to \triangle^p$ is proper, and as $\pi(A \cap \overline{SuppT})$ is closed with a zero Lebesgue measure in $\triangle^p$ one can find an open subset $O \subset \triangle^p \setminus \pi(A \cap \overline{SuppT})$. Therefore the current has locally finite on $O \times \triangle^{n-p}$. Let $0 < \delta < 1$ such that $(A \cap \overline{SuppT}) \cap (\triangle^p \times \triangle^{n-p}(1-\delta, 1+\delta)) = \phi$ and fix a and t two reals in $(\delta, 1)$ such that $a < t$. Set

$$\rho_\varepsilon = \max\nolimits_\varepsilon \left(\pi^*\rho, \tfrac{1}{t^2-a^2}(|z''|^2 - t^2)\right)$$

where ρ is a plurisubharmonic function on $\triangle^p$ such that $(dd^c\rho)^p$ supported in O. We have $-1 \leq \rho_\varepsilon < 0$ in $t\triangle^n$ and $\rho_\varepsilon = \pi^*\rho$ on $|z''| \leq a$, and we obtain

$$\int_{(t\triangle^n)\setminus A} T \wedge (dd^c\rho_\varepsilon)^p = \int_{(t\triangle^p)\times\{|z''|<a\}\setminus A} T \wedge (dd^c(\pi^*\rho))^p + \int_{(t\triangle^p)\times\{a\leq|z''|<t\}} T \wedge (dd^c\rho_\varepsilon)^p$$

since $(dd^c\pi^*\rho)^p$ supported in $O \times \triangle^{n-p}$ then both integrals of the right hand side are finite. By applying Proposition 3.3. on $-T$ we deduce that $\widetilde{T}$ exists. □

Notice that Theorem 3.7 generalizes the case when $S = 0$ which is done in [8].

Corollary 3.8. *Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive closed currents S on Ω. Assume that $\mathcal{H}_{2p-2}(A \cap \overline{SuppT}) = 0$. Then $\widetilde{T}$ exists. Furthermore the current $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

The case when $\widetilde{dd^cT}$ exists and $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$ done by Dabbek in [5]. Dabbek proved that in this case the residual current is positive and closed by using the same technique in [8] with the local potential of a positive closed current given in [4].

Proof. Applying El Mir and Feki result [13] for the current $dd^cT + S$, the extension $\widetilde{dd^cT}$ exists. Now, the result follows from Theorem 5 in [8]. □

Theorem 3.9. *Let A be a closed complete pluripolar subset of an open set Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive closed currents S on Ω. Let K and L compact set in Ω with $L \subset\subset K$. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$, then there exists a constant $C_{K,L} > 0$ such that for all u smooth bounded plurisubharmonic on Ω we have the following estimate*

$$\int_{L\setminus A} T \wedge du \wedge d^cu \wedge \beta^{p-1} \leq C_{K,L} \sup_{z\in K} |u(z)|^2 (\|\widetilde{T}\|_K + \|\widetilde{dd^cT}\|_K)$$

Proof. From Theorem 3.7. the extensions $\widetilde{T}$ and $\widetilde{dd^cT}$ exist. Moreover, $\widetilde{T}$ is positive and $\widetilde{dd^cT}$ is of order zero. So the result follows from Theorem 2.2. □

4. The extension across a zero set of 0-convex function

In this section, we will show a continuation result for positive current when $dd^cT \leq S$ for positive current S, across a zero set of a strictly 0-convex function with some special properties.

Definition 4.1. *Let u be a continuous real function defined on an open subset Ω of $\mathbb{C}^n$. we say that u is strictly k-convex if there exists a continuous $(1,1)$-form γ defined on Ω which admits $(n-k)$-positive eigenvalues at each point, and such that the current $dd^cu - \gamma$ is positive on Ω.*

In what remains we consider u to be a positive exhaustion strictly 0-convex function on an open subset Ω of $\mathbb{C}^n$, and set $A = u^{-1}\{0\}$. In [8], the authors discussed the extension of the positive plurisubharmonic currents across the zero of strictly k-convex functions. With our distinguished choice of u we generalize their result in the case of strictly 0-convex.

Proposition 4.2. *Let Ω be an open subset of $\mathbb{C}^n$ and $\Omega_c = \{u(z) \leq c\}$. Let T be a positive current of bidimension (p,p) on $\Omega \setminus \Omega_c$ such that $dd^cT \leq S$ on $\Omega \setminus \Omega_c$ for some positive currents S on Ω. If $p \geq 1$, then T is of finite mass near Ω_c.*

Proof. We can assume that $u \in C^\infty(\Omega \setminus A)$. Indeed, by Richberg's theorem there exists a strictly plurisubharmonic function v on Ω satisfies that,

$$u(z) \leq v(z) \leq u(z) + d(z,A), \forall z \in \Omega$$

implies that $A = v^{-1}\{0\}$. So, from now on in this proof u is smooth on $\Omega \setminus A$.

Since u is exhaustion, then for each c we can find an open set M_c contains Ω_c such that $\overline{M_c}$ contained in Ω and ∂M_c is oriented by outer normal. Now our aim is to prove that

$$\int_{M_c\setminus A} T\wedge\beta^p<\infty$$

Let φ_ε is the regularization kernel on $\mathbb{C}^n$ depending only on $|z|$. Since u is strictly then there exists $\alpha>0$ so that $dd^cu\geq\alpha\beta$ on Ω, therefore we can choose ε_n small enough to make $u_n=u*\varphi_{\varepsilon_n}$ satisfies

$$\tfrac{1}{2^n}\leq u-u_n\leq\tfrac{1}{n} \text{ and } dd^cu_n\geq\tfrac{\alpha}{2}\beta$$

Let us define

$$\rho_{\varepsilon'_n}=\max\nolimits_{\varepsilon'_n}\{u_n-c-\tfrac{1}{2^n},0\}$$

where $\max_\varepsilon$ is the convolution of the function $(x_1,x_2)\mapsto\max(x_1,x_2)$ by a positive regularization kernel on $\mathbb{R}^2$ depending only on $\|(x_1,x_2)\|$. Notice that

$$\max\nolimits_\varepsilon(x_1,x_2)=\max(x_1,x_2)$$

outside a neighborhood of $x_1=x_2$. This definition implies that $\rho_{\varepsilon'_n}=0$ in a neighborhood O_{c+n} of Ω_c which is depending on n, and allows us to speak about $\rho_{\varepsilon'_n}T$ on the whole of Ω. Now, set

$$\Gamma_{c+n}=\{z\in M_c:u\leq c+\frac{1}{n}+\frac{1}{2^n}\}$$

On $M_c\setminus\Gamma_{c+n}$ we have $u_n-c-\frac{1}{2^n}>0$, implies that $dd^c\rho_{\varepsilon'_n}\geq\frac{\alpha}{2}\beta$ on $M_c\setminus\Gamma_{c+n}$. Let us take a function g with support in $\Omega\setminus\Omega_c$ such that $g=1$ in a neighborhood of ∂M_c, $0\leq g\leq 1$ and vanishes in a neighborhood of Ω_c. Let $T_{\varepsilon_k}=T*\varphi_{\varepsilon_k}$ be a smoothing of T which is of course convergent weakly* to T, hence

$$\int_{M_c\setminus\Gamma_{c+n}} T\wedge\beta^p\leq\frac{\alpha}{2}\lim_{\varepsilon_k\to0}\int_{M_c}T_{\varepsilon_k}\wedge dd^c\rho_{\varepsilon'_n}\wedge\beta^{p-1} \tag{4.1}$$

On the other hand we have

$$\begin{aligned}\int_{M_c}T_{\varepsilon_k}\wedge dd^c\rho_{\varepsilon'_n}\wedge\beta^{p-1}&=\int_{M_c}T_{\varepsilon_k}\wedge dd^c(\rho_{\varepsilon'_n}g+\rho_{\varepsilon'_n}(1-g))\wedge\beta^{p-1}\\&=\int_{M_c}T_{\varepsilon_k}\wedge dd^c(\rho_{\varepsilon'_n}g)\wedge\beta^{p-1}\\&\quad+\int_{M_c}\rho_{\varepsilon'_n}(1-g)dd^cT_{\varepsilon_k}\wedge\beta^{p-1}\\&\leq\int_{M_c}T_{\varepsilon_k}\wedge dd^c(\rho_{\varepsilon'_n}g)\wedge\beta^{p-1}\\&\quad+\int_{M_c}\rho_{\varepsilon'_n}(1-g)S_{\varepsilon_k}\wedge\beta^{p-1}\end{aligned} \tag{4.2}$$

The nice choice of g makes the sequence $(\rho_{\varepsilon'_n}g)$ converges uniformly to $(u-c)g$. Moreover, on $Supp\ g\cap Supp\ \rho_{\varepsilon'_n}$ the positive current T has locally finite mass. Hence the last right hand side integrals in (4.2) are bounded independently of ε_k and n. In virtue of (4.1) we deduce that T is of finite mass on $M_c \setminus \Omega_c$ □

Remark 4.3. In view of the previous proof we used the exhaustion just to make $\Omega_c \subset\subset \Omega$. So, the proof is valid if Ω_c is relatively compact in Ω.

Theorem 4.4. *Let Ω be an open subset of $\mathbb{C}^n$ and set $A = u^{-1}\{0\}$. Let T be a positive current of bidimension (p,p), $p \geq 1$ on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive current S on Ω, then $\widetilde{T}$ exists. If $log(u)$ is plurisubharmonic then*

1. *$\widetilde{dT} = d\widetilde{T}$ while $\widetilde{dT}$ exists.*
2. *$\widetilde{dd^cT}$ exists and $\widetilde{dd^cT} - dd^c\widetilde{T}$ is a positive current.*

Proof. In the previous proposition we have already proved the existence of $\widetilde{T}$. Since the problem near A, then we can assume that $\Omega = M_0$ where M_0 as in Proposition 2.4. To prove (1) we use that $log(u) = \varphi$ is plurisubharmonic and $A = \{z, log(u(z)) = -\infty\}$. Then there exist $u_n \in psh(\Omega)$, $0 \leq u_n \leq 1$ such that the sequence u_n is converging locally uniformly to 1 on $\Omega \setminus A$ and vanishing near A. Hence $\lim_{n\to\infty} u_nT = \widetilde{T}$. We first show that $du_n \wedge T \to 0$.

Let ϕ be a smooth $(0,1)$-form with support in a compact subset K of Ω. We have to show

$$\int T \wedge du_n \wedge \phi \wedge \beta^{p-1} \to 0$$

Fix $\varepsilon > 0$, let U_ε and $U_{\frac{\varepsilon}{2}}$ are neighborhoods of A with $U_{\frac{\varepsilon}{2}} \subset U_\varepsilon$ such that

$$\int_{U_\varepsilon} i\widetilde{T} \wedge \phi \wedge \overline{\phi} \wedge \beta^{p-1} \leq \varepsilon^2$$

Since dd^cT of finite mass on $\Omega \setminus A$ and u_n converging locally uniformly to 1 on $\Omega \setminus A$, then applying Theorem 2.2 for $u_n - 1$, gives us

$$\begin{aligned}
&\lim_{n\to\infty} \left| \int_{\Omega\setminus U_\varepsilon} T \wedge d(u_n - 1) \wedge \phi \wedge \beta^{p-1} \right|^2 \\
&\leq \lim_{n\to\infty} \left(\int_{\Omega\setminus U_\varepsilon} T \wedge d(u_n) \wedge d^c(u_n) \wedge \beta^{p-1} \right) \left(\int_{\Omega\setminus U_\varepsilon} iT \wedge \phi \wedge \overline{\phi} \wedge \beta^{p-1} \right) \\
&\leq \lim_{n\to\infty} C \sup_{\Omega\setminus U_{\frac{\varepsilon}{2}}} |u_n - 1|^2 \left(\|T\|_{\Omega\setminus U_{\frac{\varepsilon}{2}}} + \|dd^cT\|_{\Omega\setminus U_{\frac{\varepsilon}{2}}} \right) = 0
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \int_{\Omega\setminus U_\varepsilon} iT \wedge du_n \wedge \phi \wedge \beta^{p-1} = 0$$

On the other hand, Cauchy Schwartz inequality gives

$$\begin{aligned}
&\left| \int_{U_\varepsilon} T \wedge du_n \wedge \phi \wedge \beta^{p-1} \right| \\
&\leq \left(\int_{U_\varepsilon} T \wedge du_n \wedge d^cu_n \wedge \beta^{p-1} \right)^{\frac{1}{2}} \left(\int_{U_\varepsilon} iT \wedge \phi \wedge \overline{\phi} \wedge \beta^{p-1} \right)^{\frac{1}{2}}
\end{aligned}$$

The second factor in the right hand side from the above inequality is bounded by ε. So it is enough to show that the second integral is uniformly bounded with respect to n. we can not proceed as the previous part since we don't know that dd^cT has finite mass near A.

Let χ be a smooth compactly supported function on Ω such that $0 \leq \chi \leq 1$ and $\chi = 1$ on K. Set

$$I_n := \int T\chi^2 \wedge du_n \wedge d^c u_n \wedge \beta^{p-1}$$

By simple computation we get

$$T \wedge du_n \wedge d^c u_n = \frac{1}{2} T \wedge dd^c u_n^2 - u_n T \wedge dd^c u_n \tag{4.3}$$

Since u_n is positive smooth plurisubharmonic function, then the current $u_n T \wedge dd^c u_n \wedge \beta^{p-1}$ is positive. So by (4.3), we have

$$\begin{aligned}
I_n &\leq \frac{1}{2} \int \chi^2 T \wedge dd^c u_n^2 \wedge \beta^{p-1} \\
&= \frac{1}{2} \int dd^c \left(\chi^2 T\right) u_n^2 \\
&= \frac{1}{2} \int \chi^2 dd^c T \wedge u_n^2 \wedge \beta^{p-1} + \int dd^c \chi^2 \wedge T u_n^2 \wedge \beta^{p-1} \\
&\quad - 4 \int \chi u_n T \wedge du_n \wedge d^c \chi \wedge \beta^{p-1} \\
&\leq \frac{1}{2} \left[\int \chi^2 dd^c T \wedge u_n^2 \wedge \beta^{p-1} + \left| \int dd^c \chi^2 \wedge T u_n^2 \wedge \beta^{p-1} \right| \right] \\
&\quad + 4 \left| \int \chi u_n T \wedge du_n \wedge d^c \chi \wedge \beta^{p-1} \right| \\
&\leq \frac{1}{2} \left[\int \chi^2 S \wedge u_n^2 \wedge \beta^{p-1} + \left| \int dd^c \chi^2 \wedge T u_n \wedge \beta^{p-1} \right| \right] \\
&\quad + 4 \left| \int \chi u_n T \wedge du_n \wedge d^c \chi \wedge \beta^{p-1} \right|
\end{aligned} \tag{4.4}$$

Since $\widetilde{T}$ exists, S is positive and $0 \leq u_n \leq 1$, then

$$\frac{1}{2} \left[\int \chi^2 S \wedge u_n^2 \wedge \beta^{p-1} + \left| \int dd^c \chi^2 \wedge T u_n \wedge \beta^{p-1} \right| \right] < \infty$$

We still need to estimate the last integral in (4.4). To do so we use again the nice Cauchy Schwartz inequality, and find

$$\begin{aligned}
&4 \left| \int \chi u_n T \wedge du_n \wedge d^c \chi \wedge \beta^{p-1} \right| \\
&\leq 4 \left(\int \chi^2 T \wedge du_n \wedge d^c u_n \wedge \beta^{p-1} \right)^{\frac{1}{2}} \left(\int u_n^2 T \wedge d\chi \wedge d^c \chi \wedge \beta^{p-1} \right)^{\frac{1}{2}}
\end{aligned}$$

But $\left(\int u_n^2 T \wedge d\chi \wedge d^c \chi \wedge \beta^{p-1}\right)^{\frac{1}{2}}$ is bounded. Therefore there exist positive real numbers M and C such that $I_n \leq M + C I_n^{\frac{1}{2}}$, and I_n is bounded. So we have proved that $T \wedge du_n \to 0$. Since T is real, then by taking the conjugate we get $T \wedge d^c u_n \to 0$ as well. So, if $\widetilde{dT}$ exists, then $\widetilde{dT} = \lim_{n\to\infty} u_n dT$. But

$$d\widetilde{T} = \lim_{n\to\infty} d(u_n T) = \lim_{n\to\infty} u_n dT = \widetilde{dT}$$

Proving our first claim. To prove the second one, we first note that

$$u_n dd^c T - dd^c u_n \wedge T = dd^c(u_n T) - d(T \wedge d^c u_n) + d^c(T \wedge du_n) \tag{4.5}$$

From this computation we obtain

$$u_n(dd^c T - S) - dd^c u_n \wedge T = -u_n S + dd^c(u_n T) - d(T \wedge d^c u_n) + d^c(T \wedge du_n)$$

The right hand side converges to $-\widetilde{S} + dd^c\widetilde{T}$ where $\widetilde{S}$ is the trivial extension of S on Ω. Since $dd^c T \leq S$ and T is positive, then $u_n(dd^c T - S) - dd^c u_n \wedge T$ is negative. We then deduce that $-\widetilde{S} + dd^c\widetilde{T}$ is negative. Thus there exists a positive current F on Ω such that $-\widetilde{S} + dd^c\widetilde{T} = -F$. Therefore, $dd^c\widetilde{T}$ has order zero because S and F has locally finite mass near A. But we have

$$u_n dd^c T = u_n dd^c\widetilde{T} = u_n(-F + S)$$

Therefore $dd^c T$ has locally finite mass near A.

The right hand side in (4.5) converges to $\widetilde{dd^c T} - dd^c\widetilde{T}$ and the left hand side converges to $\lim_{n\to\infty} dd^c u_n \wedge T$, proving that $\widetilde{dd^c T} - dd^c\widetilde{T}$ is positive. □

In the proof of (1) and (2) in Theorem 4.4, we followed Dinh and Sibony in their proof of Theorem 1.3 in [11]. In that result the authors studied the existence of $\widetilde{dd^c T}$ when $T \geq 0$, $dd^c T \leq S$ on $\Omega \setminus A$ and $\widetilde{T}$ exists. The case when $S = 0$ was proved by Dabbek, Elkhadhra and El Mir in [8].

Remark 4.5. In some cases the conditions in Theorem 4.4. can be stronger than we assumed as we will see in the following remarks.

(1) As we saw in Remark 4.3. we don't need the exhaustion if A is relatively compact in Ω.

(2) If we take u any positive exhaustion strictly 0-convex function of class $\mathcal{C}^2$ and don't request any thing from $log(u)$, in this case we can control the mass of $dd^c T$ outside A. Indeed, let $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth positive increasing function such that $\rho(t) = 0$ if $t \leq \frac{1}{2}$ and $\rho(t) = 1$ if $t \geq 1$. Put

$$\rho_r = \rho(\frac{u^2}{r})$$

We have ρ_r is of class $\mathcal{C}^2$, $0 \leq \rho_r \leq 1$ and satisfying that

$$\lim_{r\to 0} \rho_r = 1_{\Omega\setminus A}$$

and by Dini's lemma the convergence is uniformly on each compact in $\Omega \setminus A$. Then $\widetilde{T} = \lim_{r\to 0} \rho_r T$. We want to show that

$$\lim_{r\to 0} T \wedge d\rho_r = 0. \tag{4.6}$$

outside A. Let ϕ smooth $(0,1)$-form with support in a compact subset K of Ω. We have to show

$$\int T \wedge d\rho_r \wedge \phi \wedge \beta^{p-1} \longrightarrow 0$$

Fix $\varepsilon > 0$ and let $U_\varepsilon = \{z \in \Omega, u < \varepsilon\}$ be a neighborhood of A which is relatively compact in Ω, such that

$$\int_{U_\varepsilon} i\widetilde{T} \wedge \phi \wedge \overline{\phi} \leq \varepsilon^2 \tag{4.7}$$

For $r < \varepsilon^2$ the function ρ_r is equal to 1 on $\Omega \setminus U_\varepsilon$. Hence

$$d\rho_r(z) = 2\frac{u(z)}{r}\rho'(\frac{u^2(z)}{r})du(z) = 0, \forall z \in \Omega \setminus U_\varepsilon$$

Therefore

$$\lim_{n\to\infty} \left| \int_{\Omega\setminus U_\varepsilon} T \wedge d\rho_r \wedge \phi \wedge \beta^{p-1} \right| = 0$$

This doesn't solve the problem since we need also to show (4.6) in U_ε. Actually we are not quite sure about the existence of $\widetilde{dd^cT}$ when $p \geq 1$. We have this feeling since the result have been discussed only when $p \geq 2$, but at the same time nobody has shown -at least in our knowledge- the sharpness of this condition. However, we can relax the conditions of Theorem 4.4 and prove our second main result.

Theorem 4.6. *Let u be a positive exhaustion strictly 0-convex function on an open subset Ω of $\mathbb{C}^n$ and set $A = \{z \in \Omega : u(z) = 0\}$. Let T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive currents S on Ω. If $p \geq 1$, then $\widetilde{T}$ exists. If $p \geq 2$, dd^cS is of locally finite mass and $u \in \mathcal{C}^2$, then $\widetilde{dd^cT}$ exists and $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

Proof. We keep the notations of Proposition 4.2, and set $F = S - dd^cT$. Then F is a $\mathbb{C}$-normal current on $\Omega \setminus A$, and as we did in Proposition 4.2, we define $F_{\varepsilon_k} = F * \varphi_{\varepsilon_k}$. Hence

$$\int_{M\setminus\Gamma_n} F \wedge \beta^{p-1} \leq \frac{\alpha}{2} \lim_{\varepsilon_k \to 0} \int_M F_{\varepsilon_k} \wedge dd^c \rho_{\varepsilon'_n} \wedge \beta^{p-2} \tag{4.8}$$

On the other hand we have

$$\begin{aligned} \int_M F_{\varepsilon_k} \wedge dd^c \rho_{\varepsilon'_n} \wedge \beta^{p-2} &= \int_M F_{\varepsilon_k} \wedge dd^c(\rho_{\varepsilon'_n} g + \rho_{\varepsilon'_n}(1-g)) \wedge \beta^{p-2} \\ &= \int_M F_{\varepsilon_k} \wedge dd^c(\rho_{\varepsilon'_n} g) \wedge \beta^{p-2} \\ &\quad + \int_M \rho_{\varepsilon'_n}(1-g) dd^c F_{\varepsilon_k} \wedge \beta^{p-2} \\ &= \int_M F_{\varepsilon_k} \wedge dd^c(\rho_{\varepsilon'_n} g) \wedge \beta^{p-2} \\ &\quad + \int_M \rho_{\varepsilon'_n}(1-g) dd^c S_{\varepsilon_k} \wedge \beta^{p-2} \end{aligned} \tag{4.9}$$

The sequence $(\rho_{\varepsilon'_n} g)$ converges uniformly to $(u-c)g$. Moreover, on $Supp\ g \cap Supp\ \rho_{\varepsilon'_n}$ the positive current F has locally finite mass. So by the nice property of dd^cS, we deduce that the last right hand side integrals in (4.9) are bounded independently of ε_k and n, and from (4.8) the extension $\widetilde{F}$ exists. Implies that $\widetilde{dd^cT}$ exists, and the theorem follows by applying ([8], Theorem 4). □

Corollary 4.7. *Under the same hypotheses of Theorem 4.4., if $log(u)$ is plurisubharmonic, S is closed and $\mathcal{H}_{2p}(A) = 0$ then $R = \widetilde{dd^cT} - dd^c\widetilde{T}$ is closed. Moreover $R = 0$ as soon as $\mathcal{H}_{2p-2}(A) = 0$.*

Proof. The case when $\mathcal{H}_{2p}(A) = 0$ follows from Theorem 3.7, since $log(u)$ is plurisubharmonic. To show the second part we note first that the current $dd^c\widetilde{T}$ is closed, so we need to show that $\widetilde{dd^cT}$ is so. But $S - dd^cT$ is closed positive current on $\Omega \setminus A$. So by [13], the extension $\widetilde{S - dd^cT}$ exists and is closed positive, hence $\widetilde{dd^cT}$ is closed. Therefore in this case the current R is $\mathbb{C}$-flat, and since our set A is thin so that $\mathcal{H}_{2p-2}(A) = 0$ then by the support theorem $R = 0$. □

In the end of this section we will return again to the case when S is closed positive current but this time we will prove the extension over closed set.

Theorem 4.8. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for closed positive currents S on Ω. Assume that $\mathcal{H}_{2p-3}(\overline{SuppT} \cap A) = 0$, then $\widetilde{T}$ exists and $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

Proof. Let $F = dd^cT + S$, then F is closed positive current on $\Omega \setminus A$ of bidimension $(p-1, p-1)$. Since $\mathcal{H}_{2(p-1)-1}(\overline{SuppT} \cap A) = 0$, by ([15], Theorem 6), $\widetilde{F}$ exists and cosed positive current on Ω. Implies that $\widetilde{dd^cT}$ exists, and by ([8], Theorem 5) we obtain our result. □

If $S = 0$, Theorem 4.8 due to Dabbek , Elkhadhra and El Mir [8].

5. Open problems

(1) Let T be a positive pluriharmonic current of bidimension (p,p) on a complex manifold Ω of dimension n and A be a closed complete pluripolar subset of Ω such that the Hausdorff measure $\mathcal{H}_{2p-1}(A) = 0$. Let S be a positive and closed current of bidimension $(1,1)$ on Ω and smooth on $\Omega \setminus A$. If g is a solution of $dd^cg = S$ on an open set $U \subset \Omega$ and $g_j \in \mathcal{C}^\infty(U) \cap psh(U)$ such that the sequence (g_j) converges to g in $\mathcal{C}^2(U \setminus A)$
   - Is $S \wedge T$ well defined? i.e. does $dd^cg_j \wedge T$ converge?
   - Under the same hypotheses, does g_jT converge?

(2) Is the following assertion true?

Let A be a closed complete pluripolar subset of an open subset $\Omega \subset \mathbb{C}^n$ and T be a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive currents S on Ω. Let v be a plurisubharmonic function of class $\mathcal{C}^2$, $v \geq -1$ on Ω such that

$$\Omega^{'} = \{z \in \Omega : v(z) < 0\}$$

is relatively compact in Ω. Let $K \subset \Omega'$ be a compact subset and let us set

$$c_K = -\sup_{z\in K} v(z)$$

Then there exists a constant $\eta \geq 0$ such that for all integer $1 \leq s \leq p$ and for every plurisubharmonic function u of class $\mathcal{C}^2$ on Ω' satisfying $-1 \leq u < 0$ we have,

$$\int_{K\setminus A} T \wedge (dd^cu)^p \leq c_K^{-s} \int_{\Omega'\setminus A} T \wedge (dd^cv)^s \wedge (dd^cu)^{p-s} + \eta\|S\|_{\Omega'}$$

(3) In [8] the authors proved the previous problem when $S = 0$ and used it in the proof of Lemma 3.4. So one can arise the following problem.

Let A be a closed complete pluripolar subset of an open subset Ω of $\mathbb{C}^n$ and T a negative current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \geq -S$ on $\Omega \setminus A$ for some positive currents S on Ω. Assume that $\mathcal{H}_{2p}(A \cap \overline{SuppT}) = 0$. Does $\widetilde{T}$ exist?

(4) Let u be a positive exhaustion strictly k-convex function on an open subset Ω of $\mathbb{C}^n$ and $A = u^{-1}\{0\}$. Let T be a positive current of bidimension (p,p), $p \geq 1$ on $\Omega \setminus A$ such that $dd^cT \leq S$ on $\Omega \setminus A$ for some positive currents S on Ω. can we prove the existence of $\widetilde{T}$ and $\widetilde{dd^cT}$?

(5) Can we show Theorem 4.8., when S positive and plurisubharmonic (resp. dd^c-negative) current?

As we saw in Theorem 4.8., the key of the proof was Harvey's result for closed positive currents. So once we obtain Harvey's result for positive plurisubharmonic currents, we answare this question.

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