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POLYNOMIALITY

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POLYNOMIALITY

Qimh Xantcha

Thesis submitted for the degree of Licentiate of Philosophy, to be presented at the Department of Mathematics at Stockholm University on the 4th of June, 2009.

Jag vill icke säga Licentiatens namn, men initial-bokstafven var X.

— Carl Jonas Love Almqvist, *Svensk Rättstafnings-Lära*

Acknowledgements

Our gratitude towards Professor Torsten Ekedahl, the most brilliant mathematician south of the North Pole, is perhaps best expressed by the phrase with which Cathos greeted Mascarille in Molière's play *Les Précieuses ridicules*:

Pour voir chez nous le mérite, il a fallu que vous l'y ayez amené.

Contents

CHAPTER 0

Preliminaries

En tycktes vara hvass, och grep mig an för stöld,
Att jag ur böcker tog, med andras tankar jäste;
Men huru vet hon det, som aldrig nånsin läste?

— Hedvig Charlotta Nordenflycht,
Satir emot afundsjuka fruntimmer

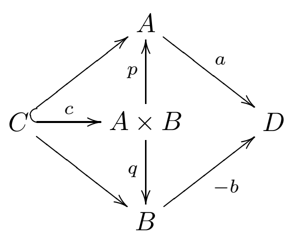
1. Ring Theory

The following proposition is known to mathematicians as Delsarte's Lemma, but there seems to be no tangible way to attach his name unto it. As it is a *very* general theorem, we choose to place it here among the preliminaries, rather than in Chapter 1, where it is applied. It holds not only for rings, but also for groups, linear spaces, modules,..., with virtually identical proofs (but it is false for monoids). Unfortunately we need to prove it twice, as we will have use for both a ring-theoretical and an abstract nonsense version for abelian categories. (Both are probably special cases of some as yet undiscovered Universal Delsarte's Lemma, which we leave as an exercise for the interested reader to find.)

Theorem 1: The Ring-Theoretical Delsarte's Lemma. *In the diagram below, A, B and C are (commutative, unital) rings, such that $C \subseteq A \times B$ and the projections*

$$pc\colon C \to A, \qquad qc\colon C \to B$$

are both onto. Then A and B have a common factor ring D which completes the diagram into a pullback square:



Equivalently,

$$C = \operatorname{Ker}(a + b).$$

Proof. Note that

$$A \cap C = \operatorname{Ker} pc, \qquad B \cap C = \operatorname{Ker} qc,$$

from which the Fundamental Homomorphism Theorem gives

$$A = \operatorname{Im} pc \cong C/\operatorname{Ker} pc = C/(A \cap C)$$
$$B = \operatorname{Im} qc \cong C/\operatorname{Ker} qc = C/(B \cap C),$$

and hence

$$A/(A \cap C) \cong C/\big((A \cap C) + (B \cap C)\big) \cong B/(B \cap C).$$

We may therefore define

$$D = C/\big((A \cap C) + (B \cap C)\big),$$

and let $a\colon A \to D$ and $-b\colon B \to D$ be the natural quotient maps.

To find the kernel of $a + b$, suppose $x \in A$ and $y \in B$ satisfy

$$0 = a(x + A \cap C) + b(y + B \cap C) = x - y + \big((A \cap C) + (B \cap C)\big).$$

This means

$$x + x' = y + y',$$

for some $x' \in A \cap C$ and $y' \in B \cap C$. But then

$$z = x + x' = y + y' \in A \cap B = 0,$$

so in fact $x = -x'$ and $y = -y'$ are both in C. Consequently, $C = \operatorname{Ker}(a+b)$. □

2. Commutative Algebra

Theorem 2: Chevalley's Dimension Argument. *When R is a finitely generated (non-trivial) ring, the (in)equality*

$$\dim R/pR = \dim \mathbf{Q} \otimes_{\mathbf{Z}} R \leq \dim R - 1$$

holds for all but finitely many prime numbers p. When R is an integral domain of characteristic 0, there is in fact equality for all but finitely many primes p.

Proof. In the case of positive characteristic n, the formula will hold trivially, for then

$$\mathbf{Q} \otimes_{\mathbf{Z}} R = 0 = R/pR,$$

except when $p \mid n$.

Consider now the case when R is an integral domain of characteristic 0. We have an embedding $\varphi\colon \mathbf{Z} \to R$, and a corresponding dominant morphism

$\operatorname{Spec}\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} \mathbf{Z}$ of integral schemes, which is of finite type. Letting $\operatorname{Frac} P$ denote the fraction field of R/P, we may define

$$\begin{aligned} C_n &= \{P \in \operatorname{Spec} \mathbf{Z} \mid \dim(\operatorname{Spec}\varphi)^{-1}(P) = n\} \\ &= \{P \in \operatorname{Spec} \mathbf{Z} \mid \dim R \otimes_{\mathbf{Z}} \operatorname{Frac} P = n\} \\ &= \{(p) \mid \dim R/pR = n\} \cup \{(0) \mid \dim R \otimes_{\mathbf{Z}} \mathbf{Q} = n\}, \end{aligned}$$

and this set, by Chevalley's Constructibility Theorem[1], will contain a dense, open set in $\operatorname{Spec} \mathbf{Z}$ if $n = \dim R - \dim \mathbf{Z}$. Such a set must contain (0) and (p) for all but finitely many primes p, so for those primes,

$$\dim \mathbf{Q} \otimes_{\mathbf{Z}} R = \dim R/pR = \dim R - 1.$$

Now let R be an arbitrary ring of characteristic 0. For any prime ideal Q, R/Q will be an integral domain (but not necessarily of characteristic 0!), and so we can apply the preceding to obtain

$$\dim \mathbf{Q} \otimes_{\mathbf{Z}} R/Q = \dim R/(Q + pR) = \dim R/Q - 1,$$

for all but finitely many primes p. The prime ideals of $\mathbf{Q} \otimes_{\mathbf{Z}} R$ are all of the form $\mathbf{Q} \otimes_{\mathbf{Z}} Q$, where Q is a prime ideal in R. Moreover,

$$(\mathbf{Q} \otimes_{\mathbf{Z}} R)\big/(\mathbf{Q} \otimes_{\mathbf{Z}} Q) = \mathbf{Q} \otimes_{\mathbf{Z}} R/Q.$$

It follows that

$$\begin{aligned} \dim \mathbf{Q} \otimes_{\mathbf{Z}} R &= \max_{Q \in \operatorname{Spec} R} \dim(\mathbf{Q} \otimes_{\mathbf{Z}} R)\big/(\mathbf{Q} \otimes Q) \\ &= \max_{Q \in \operatorname{Spec} R} \dim \mathbf{Q} \otimes_{\mathbf{Z}} R/Q \\ &= \max_{Q \in \operatorname{Spec} R} \dim R/(Q + pR) \\ &= \max_{\overline{Q} \in \operatorname{Spec} R/pR} (R/pR)\big/\overline{Q} = \dim R/pR \end{aligned}$$

for all but finitely many p, because the maxima are taken over the finitely many *minimal* prime ideals only. In a similar fashion,

$$\dim \mathbf{Q} \otimes_{\mathbf{Z}} R \leq \dim R - 1,$$

and the theorem is proved. □

An immediate corollary is that $\mathbf{Q}$-algebras are never finitely generated, which is of course fun to know.

---

[1] This proposition appears to belong to the folklore of algebraic geometry. An explicit reference is Théorème 2.3 of [11].

3. Category Theory

The following is a (not exhaustive) list of the categories we will use. Those which are not standard will of course be defined somewhere in the text.

$\mathfrak{CRing}$ Commutative, unital rings.

$\mathfrak{CAlg}$ Commutative, unital algebras.

$\mathfrak{NRing}$ Numerical rings.

$\mathfrak{NAlg}$ Numerical algebras.

$\mathfrak{Mod}$ Modules.

$\mathfrak{FMod}$ Free modules.

$\mathfrak{XMod}$ Finitely generated, free modules.

$\mathfrak{Num}$ Numerical functors.

$\mathfrak{SPol}$ Strictly polynomial functors.

$\mathfrak{HPol}$ Homogeneous polynomial functors.

$\mathfrak{Set}$ Sets.

$\mathfrak{MSet}$ Multisets.

$\mathfrak{Laby}$ The labyrinth category.

When C is a category, we let

$$C^\circ$$

denote the opposite category. Given two objects $X, Y \in C$, the arrow set of X and Y will in general be denoted by

$$C(X, Y).$$

There are two exceptions to this rule. When inside a module category, the homomorphisms between the R-modules M and N will be denoted by

$$\mathrm{Hom}_R(M, N)$$

(and the letter R will be omitted if the ring is clear from the context (which it generally is)). Also, when inside a functor category, the natural transformations between the functors F and G will be denoted by

$$\mathrm{Nat}(F, G)$$

(or just $\mathrm{Nat}(F)$ if $F = G$).

Given two categories A and B, we let

$$\mathrm{Fun}(A, B)$$

denote the category of functors from A to B.

We now describe the abstract version of Delsarte's Lemma.

Theorem 3: The Abstract Delsarte's Lemma. *We work inside an abelian category.*

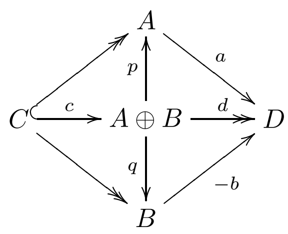
In the diagram below, let A, B and C be such that $C \subseteq A \oplus B$ and the arrows

$$pc\colon C \to A, \qquad qc\colon C \to B$$

are epic. Then A and B have a common quotient object

$$A \xrightarrow{a} D \xleftarrow{b} B\ ,$$

which completes the diagram into a pullback square:



Equivalently,

$$C = \operatorname{Ker}(a+b).$$

In fact, we may take $D = (A \oplus B)/C$.

Conversely, let a common quotient object

$$A \xrightarrow{a} D \xleftarrow{b} B$$

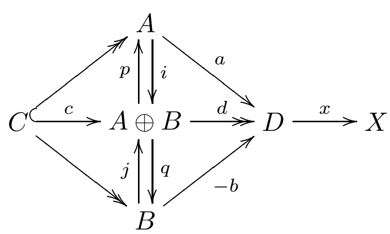
of A and B be given. Then the projections of

$$C = \operatorname{Ker}(a+b\colon A \oplus B \to D)$$

on A and B are epimorphisms.

Proof. Quick and easy way out: diagram-chasing and an off-hand reference to Mitchell's Embedding Theorem.

That would be cheating, though. We prefer to do it by abstract nonsense. Consider the following tangle, where we have defined $d = \operatorname{Coker} c$:



To show $a = di$ is epic, let x be any arrow such that $xdi = xa = 0$. Then

$$xd = xd \circ 1_{A\oplus B} = xd(ip + jq) = xdip + xdjq = xdjq,$$

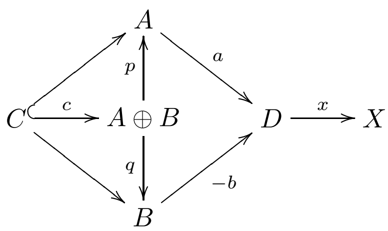
from which $0 = xdc = xdjqc$, but since qc is epic, it must be that $xdj = 0$. Hence $xd = xdjq = 0$, from which $x = 0$, using that d is epic. Similarly, $-b$ is epic.

Since we defined $d = \operatorname{Coker} c$, the sequence

$$0 \longrightarrow C \xrightarrow{c} A \oplus B \xrightarrow{d} D \longrightarrow 0$$

is exact, and it now follows from Proposition 2.53 of [8] that the above square is in fact not only a pullback square, but a Doolittle square[2].

For the converse, suppose the vee given, and define $c = \operatorname{Ker}(ap+bq)$. Inspect the following diagram:



c is the equalizer of ap and $-bq$ and hence the square is a pullback square. By the Pullback Theorem (Theorem 2.54 of [8]), qc is an epimorphism since a is, and similarly for pc. □

4. Set Theory

We will everywhere use the standard notation

$$[n] = \{1, \dots, n\}.$$

The text is pervaded by the use of multisets. They are formally introduced in Chapter 6, and the reader may want to skip ahead when need arises.

[2] A *Doolittle square* is a square which is both a pullback and a pushout square.

CHAPTER 1

Numerical Rings

At the age of twenty-one he wrote a treatise upon the Binomial Theorem, which has had a European vogue.

— Sherlock Holmes's description of Professor Moriarty;
Arthur Conan Doyle, *The Final Problem*

Numerical rings were (presumably) first discovered[1] in 2002 by Torsten Ekedahl, see [7], who used them as a natural setting for integral homotopy theory. However, once defined, these remarkable rings were immediately put to use, and no detailed study was ever made of their elementary properties. This is unfortunate, as the numerical rings turn out to present an array of rather pleasant properties, some of which may come somewhat as a surprise.

1. Numerical Rings

The original definition, in [7], of a numerical ring was quite a non-explicit one. It was stated in terms of three mysterious polynomials, the exact nature of which was never made precise. Our definition intends to remedy this.

Definition 1. A **numerical ring** is a commutative ring with unity equipped with unary operations $r \mapsto \binom{r}{n}$, $n \in \mathbf{N}$, called **binomial coefficients**, satisfying the following axioms:

I. $$\binom{a+b}{n} = \sum_{p+q=n} \binom{a}{p}\binom{b}{q}.$$

II. $$\binom{ab}{n} = \sum_{m=0}^{n} \binom{a}{m} \sum_{\substack{q_1+\cdots+q_m=n \\ q_i \geq 1}} \binom{b}{q_1}\cdots\binom{b}{q_m}.$$

III. $$\binom{a}{m}\binom{a}{n} = \sum_{k=0}^{n} \binom{a}{m+k}\binom{m+k}{n}\binom{n}{k}.$$

[1] He uses himself the word "introduced", but humility has always been among his chief virtues.

IV. $\binom{1}{n} = 0$ when $n \geq 2$.

V. $\binom{a}{0} = 1$ and $\binom{a}{1} = a$.

◇

The original definition also included a (non-explicit) formula for reducing the *composition* $\binom{\binom{a}{m}}{n}$ of binomial coefficients to simple ones. Surprisingly enough, this formula will be a consequence of the five axioms we have listed.

It follows easily from axioms I, IV and V, that when these functions are evaluated on multiples of unity, we retrieve the ordinary binomial coefficients, namely

$$\binom{m \cdot 1}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!} \cdot 1, \quad m \in \mathbf{N}$$

Since $\binom{n \cdot 1}{n} = 1$, but $\binom{0}{n} = 0$ unless $n = 0$, we see that a numerical ring has necessarily characteristic 0.

The numerical structure on a given ring is always unique. This will be proved shortly.

Example 1. Every $\mathbf{Q}$-algebra is numerical with the usual definition of binomial coefficients:

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

The numerical axioms may be proved either directly, or by manipulating formal power series. △

Example 2. For any integer m, the ring $\mathbf{Z}[m^{-1}]$ is numerical. Since it inherits the binomial coefficients from $\mathbf{Q}$, it is just a matter of verifying that this ring is closed under binomial coefficients. Because

$$\binom{\frac{a}{f}}{n} = \frac{\frac{a}{f}(\frac{a}{f}-1)\cdots(\frac{a}{f}-(n-1))}{n!} = \frac{a(a-f)\cdots(a-(n-1)f)}{n!f^n},$$

it will suffice to prove that whenever $p^i \mid n!$, but $p \nmid b$, then

$$p^i \mid (a+b)(a+2b)\cdots(a+nb).$$

To this end, let

$$n = c_m p^m + \cdots + c_1 p + c_0, \qquad 0 \leq c_i \leq p-1,$$

be the base p representation of n. For fixed k and $0 \leq d < c_k$, the numbers

$$a + (c_m p^m + \cdots + c_{k+1} p^{k+1} + dp^k + i)b, \qquad 1 \leq i \leq p^k, \tag{1.1}$$

will form a set of representatives for the congruence classes modulo p^k, as do of course the numbers

$$c_m p^m + \cdots + c_{k+1} p^{k+1} + dp^k + i, \qquad 1 \leq i \leq p^k. \tag{1.2}$$

Note that if $x \equiv y \mod p^k$ and $j \leq k$, then $p^j \mid x$ iff $p^j \mid y$. Hence there are at least as many factors p among the numbers (1.1) as among the numbers (1.2). The claim now follows. △

Example 3. As a special case $m = 1$ of the preceding example, $\mathbf{Z}$ is a numerical ring. For this ring there is actually another way of proving the numerical axioms. We shall indiate how they may be arrived at as solutions to problems of enumerative combinatorics:

> *Axiom I.* We have balls of two types: round balls, square balls. If we have a round balls and b square balls, in how many ways may we choose n balls? Let p be the number of round balls chosen, and q the number of square balls.
>
> *Axiom II.* We have a chocolate box containing a rectangular $a \times b$ array of pralines, and we wish to eat n of these. In how many ways can this be done? Suppose the pralines we choose to feast upon are located in m of the a rows, and let q_i be the number of chosen pralines in row number i of these m.
>
> *Axiom III.* We are given a mathematicians, of which m do analysis and n algebra. Naturally there exist people who do both. How many possible distributions of skills are possible? Let k be the number of mathematicians who do only algebra.
>
> *Axiom IV – V.* Clear.

△

Example 4. The set

$$S = \{f \in \mathbf{Q}[x] \mid f(\mathbf{Z}) \subseteq \mathbf{Z}\}$$

of **numerical maps** on $\mathbf{Z}$ is numerical. Addition and multiplication of functions are evaluated pointwise, as are binomial coefficients:

$$\binom{f}{n}(x) = \binom{f(x)}{n} = \frac{f(x)(f(x)-1)\cdots(f(x)-n+1)}{n!}.$$

We will see later that S is also the free numerical ring (on the singleton set $\{x\}$).

Seizing the opportunity, we recall that any numerical map may be written uniquely as a **numerical polynomial**

$$f(x) = \sum c_n \binom{x}{n}, \quad c_n \in \mathbf{Z}.$$

This example may be generalized to any set of variables, and will later be seen to constitute the free numerical ring. △

Example 5. The operations $r \mapsto \binom{r}{n}$, being given by rational polynomials, are continuous as maps $\mathbf{Q}_p \to \mathbf{Q}_p$ in the p-adic topology. It should be well known that $\mathbf{Z}$ is dense in the ring $\mathbf{Z}_p$, and that $\mathbf{Z}_p$ is closed in $\mathbf{Q}_p$. Since the binomial coefficients leave $\mathbf{Z}$ invariant, the same must then be true of $\mathbf{Z}_p$, which is thus a numerical ring.

This provides an alternative proof of the fact that $\mathbf{Z}[m^{-1}]$ is closed under biomial coefficients. For this is evidently true of the localizations $\mathbf{Z}_{(p)} = \mathbf{Q} \cap \mathbf{Z}_p$, and therefore also for

$$\mathbf{Z}[m^{-1}] = \bigcap_{p \nmid m} \mathbf{Z}_{(p)}.$$

△

Example 6. Products of numerical rings are numerical. More generally, projective limits of numerical rings are numerical. △

Example 7. It is shown in [7] that the tensor product of two numerical rings over $\mathbf{Z}$ is numerical. In fact, a more general statement is proved, namely that if $R \to S$ and $R \to T$ are homomorphisms of numerical rings, then $S \otimes_R T$ is numerical and is the categorical pushout. △

2. Elementary Identities

Theorem 1. *The following formulæ are valid in any numerical ring:*

1. $\displaystyle\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$ *when* $r \in \mathbf{Z}$.
2. $\displaystyle n!\binom{r}{n} = r(r-1)\cdots(r-n+1)$.
3. $\displaystyle n\binom{r}{n} = (r-n+1)\binom{r}{n-1}$.

Proof. The map

$$\varphi\colon (R,+) \to (1+tR[[t]],\cdot), \qquad r \mapsto \sum_{n=0}^{\infty} \binom{r}{n} t^n$$

is by axioms I and V a group homomorphism. Therefore, when $r \in \mathbf{Z}$,

$$\varphi(r) = \varphi(1)^r = (1+t)^r,$$

which expands as usual (with ordinary binomial coefficients) by the Binomial Theorem. This proves equation 1. An inductive proof will also work.

To prove equations 2 and 3, we proceed differently. By axiom III,

$$r\binom{r}{n-1} = \binom{r}{n-1}\binom{r}{1} = \sum_{k=0}^{1} \binom{r}{n-1+k}\binom{n-1+k}{1}\binom{1}{k}$$

$$= \binom{r}{n-1}\binom{n-1}{1}\binom{1}{0} + \binom{r}{n}\binom{n}{1}\binom{1}{1}$$
$$= (n-1)\binom{r}{n-1} + n\binom{r}{n},$$

which reduces to equation 3.

Equation 2 then follows inductively from equation 3. □

It may be noted, that axiom II has so far not been needed. Consequently, whenever a **Q**-algebra comes equipped with unary operations $r \mapsto \binom{r}{n}$, satisfying the axioms I, III, IV and V, it follows that in fact

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

3. Torsion

In this section we shall prove that numerical rings lack torsion, referring of course to **Z**-torsion.

First some lemmata concerning binomial and multinomial coefficients:

Lemma 1. *Let m be an integer. If p is prime and $p^l \mid m$, but $p \nmid k$, then $p^l \mid \binom{m}{k}$.*

Proof. p^l divides the right-hand side of

$$k\binom{m}{k} = m\binom{m-1}{k-1},$$

and therefore also the left-hand side. But p^l is relatively prime to k, so in fact $p^l \mid \binom{m}{k}$. □

Lemma 2. *Let $m_1, m_2, \ldots$ be integers. If $n = \sum_{i=1}^{\infty} m_i i$ is prime and $m = \sum m_i$, then*

$$m \mid \binom{m}{\{m_i\}},$$

unless $m_1 = m = n$, and all other $m_i = 0$.

Proof. Let a prime power $p^l \mid m$. Because of the relation $n = \sum m_i i$, not all m_i can be divisible by p, unless we are in the exceptional case $m_1 = m = p = n$ given above. Say $p \nmid m_j$; then

$$\binom{m}{\{m_i\}_i} = \binom{m}{m_j}\binom{m-m_j}{\{m_i\}_{i\neq j}}$$

is divisible by p^l according to Lemma 1. The claim follows. □

Lemma 3. *Let R be a numerical ring, $r \in R$, and $m, n \in \mathbf{N}$. If $nr = 0$, also $mn\binom{r}{m} = 0$.*

Proof. Follows inductively, since if $nr = 0$, then

$$mn\binom{r}{m} = n(r-m+1)\binom{r}{m-1} = -n(m-1)\binom{r}{m-1}.$$

□

Theorem 2. *Numerical rings are torsionfree.*

Proof. Suppose $nr = 0$ in R and, without any loss of generality, that n is prime.

$$0 = \binom{0}{n} = \binom{nr}{n} = \sum_{m=0}^{n}\binom{r}{m}\sum_{\substack{q_1+\cdots+q_m=n\\ q_i\geq 1}}\binom{n}{q_1}\cdots\binom{n}{q_m}$$
$$= \sum_{m=0}^{n}\binom{r}{m}\sum_{\substack{\sum m_i=m\\ \sum m_i i=n}}\binom{m}{\{m_i\}}\prod_i\binom{n}{i}^{m_i},$$

where, for given numbers q_i, we let m_i denote the number of these that are equal to i (of course $i \geq 1$ and $m_i \geq 0$). Given the numbers m_i, values may be distributed to the numbers q_i in $\binom{m}{\{m_i\}}$ ways, which accounts for the multinomial coefficient above.

We claim the inner sum is divisible by mn when $m \geq 2$. For when $2 \leq m \leq n-1$, $m \mid \binom{m}{\{m_i\}}$ by Lemma 2; also, there must exist some $0 < j < n$ such that $m_j > 0$, and for this j, Lemma 1 says $n \mid \binom{n}{j}^{m_j}$. In the case $m = n$, obviously all $m_i = 0$ for $i \geq 2$, and $m_1 = n$, so the inner sum equals $\binom{n}{1}^n$, which is divisible by $n^2 = mn$.

We can now employ Lemma 3 to kill all terms except $m = 1$. But this term is simply $\binom{r}{1} = r$, which is then equal to 0. □

This theorem is surprising indeed. We know of no other example of a variety of algebras, of which the axioms imply absense of torsion in a non-trivial way; that is, without implying a $\mathbf{Q}$-algebra structure. Not only that, the theorem is also a most crucial result in the theory of numerical rings. Over the course of the following sections, we will deduce several corollaries, seemingly without effort.

4. Uniqueness

Theorem 3. *There is at most one numerical ring structure on a given ring.*

Proof. We know that $n!\binom{r}{n} = r(r-1)\cdots(r-n+1)$, and that $n!$ is not a zero divisor. □

5. Embedding into Q-Algebras

Theorem 4. *Every numerical ring may be embedded in a* $\mathbf{Q}$*-algebra, where the binomial coefficients are given by the usual formula*

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

Proof. If R is torsionfree, the map $R \to \mathbf{Q} \otimes_{\mathbf{Z}} R$ is an embedding. □

We point out that this gives an alternative characterization of numerical rings, namely as torsionfree rings R which are closed in $\mathbf{Q} \otimes_{\mathbf{Z}} R$ under the operations

$$r \mapsto \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

6. Iterated Binomial Coefficients

In $\mathbf{Z}$, there "exists" a formula for iterated binomial coefficients:

$$\binom{\binom{r}{m}}{n} = \sum_{k=1}^{mn} g_k \binom{r}{k}, \tag{1.3}$$

in the sense that there are unique integers g_k making the formula valid for every $r \in \mathbf{Z}$. There seems to be no closed formula for them, however; confer [9]. Note, however, that (1.3) is a polynomial identity with rational coefficients, which means it must hold in any $\mathbf{Q}$-algebra, and therefore in any numerical ring:

Theorem 5. *The formula*

$$\binom{\binom{r}{m}}{n} = \sum_{k=1}^{mn} g_k \binom{r}{k}$$

for iterated binomial coefficients that is valid in $\mathbf{Z}$*, is valid in every numerical ring.*

7. Numerical Ring Homomorphisms

Definition 2. A **numerical ring homomorphism** $\varphi\colon R \to S$ between numerical rings is a ring homomorphism preserving binomial coefficients:

$$\varphi\left(\binom{r}{n}\right) = \binom{\varphi(r)}{n}.$$

S is then a **numerical algebra** over R. ◇

We denote by $\mathfrak{NRing}$ the category of numerical rings, and by ${}_R\mathfrak{NAlg}$, or simply $\mathfrak{NAlg}$, the category of numerical algebras over some fixed numerical base ring R.

Theorem 6. *Every ring homomorphism of numerical rings is numerical, so that* $\mathfrak{NRing}$ *is a full subcategory of* $\mathfrak{CRing}$.

Proof. Let a ring homomorphism $\varphi\colon R \to S$ of numerical rings be given. Because of the lack of torsion, the equation

$$\begin{aligned} n!\varphi\left(\binom{r}{n}\right) &= \varphi\left(n!\binom{r}{n}\right) = \varphi(r(r-1)\cdots(r-n+1)) \\ &= \varphi(r)(\varphi(r)-1)\cdots(\varphi(r)-n+1) = n!\binom{\varphi(r)}{n} \end{aligned}$$

implies $\varphi\left(\binom{r}{n}\right) = \binom{\varphi(r)}{n}$, so that φ is numerical. □

8. Free Numerical Rings

Definition 3. Given a set X, the **free numerical ring** on X is the numerical ring $\mathbf{Z}\binom{X}{-}$ satisfying

$$\mathfrak{NRing}\left(\mathbf{Z}\binom{X}{-}, R\right) \cong \mathfrak{Set}(X, R)$$

functorially in the numerical ring R. ◇

This is the usual construction of a free object. We now provide an explicit description.

Recall from Example 4, that a *numerical polynomial* in the variables $x_1, \ldots,$ x_k is a formal (finite) linear combination

$$f(x) = \sum c_{n_1,\ldots,n_k}\binom{x_1}{n_1}\cdots\binom{x_k}{n_k}, \qquad c_{n_1,\ldots,n_k} \in \mathbf{Z},$$

that a *numerical map* is a rational polynomial mapping $\mathbf{Z}$ to itself, and that these two concepts are essentially one and the same.

Let $\tilde{E}(X)$ be the set of all finite words that can be formed from the alphabet

$$X \cup \left\{+, -, \cdot, 0, 1, \binom{-}{n} \;\middle|\; n \in \mathbf{N}\right\},$$

where $+$ and $\cdot$ are binary, $-$ and $\binom{-}{n}$ are unary, and 0 and 1 are nullary (this is the so-called term algebra of universal algebra; confer Definition II.10.4 of [3]). Impose (divide away) the axioms of a commutative ring with unity, as well as the numerical axioms, to create a numerical ring $E(X)$.

Theorem 7. *We have the following isomorphisms:*

$$\mathbf{Z}\binom{X}{-} \cong E(X) \cong \{f \in \mathbf{Q}[X] \mid f(\mathbf{Z}^X) \subseteq \mathbf{Z}\},$$

so that every element of $\mathbf{Z}\binom{X}{-}$ *may be uniquely expressed as a numerical polynomial (or viewed as a numerical map).*

Proof. The numerical axioms, together with the formula for iterated binomial coefficients, can be used to reduce any element of $E(X)$ to a numerical polynomial. The fact that the ring of numerical maps exists and is numerical, proves that the numerical polynomials are also linearly independent, so that the expression of an element as a numerical polynomial is also unique.

From this it is evident that $E(X)$ is free on X, for any set map $\varphi\colon X \to R$ may be uniquely extended to $E(X)$ by setting

$$\varphi\left(\sum c_{n_1,\ldots,n_k}\binom{x_1}{n_1}\cdots\binom{x_k}{n_k}\right) = \sum c_{n_1,\ldots,n_k}\binom{\varphi(x_1)}{n_1}\cdots\binom{\varphi(x_k)}{n_k}.$$

□

9. Numerical Universality

Theorem 8: The Numerical Universality Principle. *A numerical polynomial identity* $p(x_1,\ldots,x_k) = 0$ *universally valid in* $\mathbf{Z}$ *is valid in every numerical ring.*

Proof. View p as an element of $\mathbf{Z}\binom{x_1,\ldots,x_k}{-}$. It is the zero numerical map, and therefore also the zero numerical polynomial. □

We thus have a canonical embedding

$$\mathbf{Z}\binom{x_1,\ldots,x_k}{-} \to \mathbf{Z}^{\mathbf{Z}^k}$$
$$p(x_1,\ldots,x_k) \mapsto (p(n_1,\ldots,n_k))_{(n_1,\ldots,n_k)\in\mathbf{Z}^k}.$$

Example 8. Numerical rings are special λ-rings in the sense of [10]. (A more readable account is [12].) First recall that a *λ-ring* (called *pre-λ-ring* by some) is a commutative ring with unity, equipped with unary operations λ^n, $n \in \mathbf{N}$, satisfying the following axioms:

1. $\lambda^0(a) = 1$.
2. $\lambda^1(a) = a$.
3. $\lambda^n(a+b) = \sum_{p+q=n} \lambda^p(a)\lambda^q(b)$.

For a numerical ring we can clearly put $\lambda^n(a) = \binom{a}{n}$.

The definition of a special λ-*ring* (called just *special* λ-*ring* by others) involves three more axioms, which are quite cumbersome, and will not be stated here. They are, however, of a polynomial nature, so their verification in a numerical ring will simply consist in verifying a number of numerical polynomial identities. As these are valid in $\mathbf{Z}$ (for $\mathbf{Z}$ itself is well known to be a λ-ring), they will hold in every numerical ring by Numerical Universality. △

10. The Nilradical

Yet another pleasant property of numerical rings is the following.

Theorem 9: Fermat's Little Theorem. *In numerical rings,*

$$a^p - a \equiv 0 \mod p$$

for any prime p.

Proof. Since $f(x) = \frac{x^p - x}{p}$ is a numerical map, it may be written as a numerical polynomial $f(x) \in \mathbf{Z}\binom{x}{-}$. But then evidently $a^p - a = pf(a) \in pR$. □

Example 9. The polynomial f may in fact be given explicitly. For when $a \in \mathbf{N}$, we may calculate the number of maps $[p] \to [a]$ as

$$a^p = \sum_{k=1}^{p} S(p,k) \binom{a}{k},$$

where $S(p,k)$ denotes the number of onto functions $[p] \to [k]$. By enumerative combinatorics, the numbers $S(p,k)$, except for $S(p,1) = 1$, are all divisible by p, and so

$$\frac{a^p - a}{p} = \sum_{k=2}^{p} \frac{S(p,k)}{p} \binom{a}{k}.$$

It follows from the Numerical Universality Principle that this formula is valid in every numerical ring. △

Theorem 10. *The nilradical of a numerical ring is divisible, and hence a vector space over* $\mathbf{Q}$.

Proof. Let p be a prime and suppose r lies in the nilradical of R. Fermat's Little Theorem states $p \mid r(r^{p-1} - 1)$, from which it inductively follows that

$$p \mid r(r^{2^m(p-1)} - 1)$$

for all $m \in \mathbf{N}$. A large enough m will kill r, and we conclude that $p \mid r$. □

11. Numerical Ideals and Factor Rings

We shall now make a (very) short survey of numerical ideals and factor rings.

Theorem 11. *Let I be an ideal of the numerical ring R. Defining*

$$\binom{r+I}{n} = \binom{r}{n} + I$$

will yield a well-defined numerical structure on R/I iff

$$\binom{e}{n} \in I$$

for every $e \in I$ and $n \neq 0$.

Proof. The condition is clearly necessary. To show sufficiency, note that, when $r \in R$ and $e \in I$,

$$\binom{r+e}{n} = \sum_{p+q=n} \binom{r}{p}\binom{e}{q} \equiv \binom{r}{n}\binom{e}{0} = \binom{r}{n} \mod I,$$

when $\binom{e}{j} \in I$ for $j > 0$. The numerical axioms in R/I then follow immediately from those in R. $\square$

Definition 4. An ideal I of a numerical ring satisfying the condition of the previous theorem will be called a **numerical ideal**. $\diamond$

Example 10. $\mathbf{Z}$ does not possess any non-trivial numerical ideals, because all its non-trivial factor rings have torsion. Neither do the rings $\mathbf{Z}[m^{-1}]$. $\triangle$

Theorem 12. *Suppose R is a (commutative, unital) ring, having an ideal I which is a vector space over $\mathbf{Q}$, and for which R/I is numerical. Then R itself is numerical.*

Proof. Since I and R/I are both torsionfree, so is R, and there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} I = I & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} R & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} R/I & \longrightarrow & 0
\end{array}$$

It suffices to show that R is closed under the formation of binomial coefficients in $\mathbf{Q} \otimes_{\mathbf{Z}} R$. Let $r \in R$.

$$\frac{r(r-1)\cdots(r-n+1)}{n!} + I = \binom{r+I}{n}$$

when calculated in the ring $\mathbf{Q} \otimes_{\mathbf{Z}} R/I$. Since $\binom{r+I}{n}$ in fact lies in R/I, it must be that $\frac{r(r-1)\cdots(r-n+1)}{n!} \in R$, and we are finished. $\square$

Note that the quotient map $R \to R/I$ will automatically be a numerical ring homomorphism.

12. Finitely Generated Numerical Rings

Lemma 4. *If a ring R is torsionfree and finitely generated as an abelian group, its fraction ring is $\mathbf{Q} \otimes_{\mathbf{Z}} R$.*

Proof. By the Structure Theorem for Finitely Generated Abelian Groups, $R \cong \mathbf{Z}^n$ for some n, considered as a group. Let $a \in \mathbf{Z}^n$. Multiplication by a is a linear transformation on $\mathbf{Z}^n$, and so may be considered an integer matrix A. The condition that a not be a zero divisor corresponds to A being non-singular. It then has an inverse, with *rational* entries, and the inverse of a is given by

$$a^{-1} = A^{-1}1 \in \mathbf{Q}^n = \mathbf{Q} \otimes_Z R$$

where 1 denotes the column vector which is the multiplicative identity of R. □

Lemma 5. *Let A be the algebraic integers in the field $K \supseteq \mathbf{Q}$. If K is finitely generated over $\mathbf{Q}$, A is finitely generated over $\mathbf{Z}$.*

The following theorem (with proof) is due to Torsten Ekedahl. It classifies completely those numerical rings which are finitely generated as *rings* (forgetting the numerical structure). Recall from Example 2 that $\mathbf{Z}[m^{-1}]$ inherits a numerical structure from $\mathbf{Q}$, and that products of numerical rings are numerical, with componentwise evaluation of binomial coefficients. Recall also the infamous Delsarte's Lemma. We proved it for rings, but the same proof goes through for numerical rings.

Theorem 13: The Structure Theorem for Finitely Generated Numerical Rings. *Let R be a numerical ring which is finitely generated as a ring. Then there exist unique positive integers $m_1, \ldots, m_k$ such that*

$$R \cong \mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}].$$

Proof. We first impose the stronger hypothesis that R be finitely generated as an abelian group, so that $R \cong \mathbf{Z}^n$ as groups.

If $r^n = 0$, then r is divisible by p for all primes $p > n$ because of Fermat's Little Theorem. But in $\mathbf{Z}^n$ this can only be if $r = 0$, so R is reduced. By the lemma above, the fraction ring of R is $\mathbf{Q} \otimes_{\mathbf{Z}} R$. As this is reduced and artinian, being finite-dimensional over $\mathbf{Q}$, it is a product $\prod K_j$ of fields. The projections of R on the factors K_j will then each be numerical.

Hence, we first consider the special case when R is included in a field, in which we let A be the algebraic integers. Let us examine the subgroup $A \cap R$ of A. Since $A \subseteq \mathbf{Q} \otimes_{\mathbf{Z}} R$, an arbitrary element of A will have an integer multiple lying in R. This means $A/(A \cap R)$ is a torsion group. Also, the fraction ring $\mathbf{Q} \otimes_{\mathbf{Z}} R$ is finitely generated over $\mathbf{Q}$, so from the lemma above, we deduce that A is finitely generated over $\mathbf{Z}$. Because the factor group $A/(A \cap R)$ is both finitely generated and torsion, it is killed by a single integer N, so that $N(A/(A \cap R)) = 0$, and as a consequence

$$(A \cap R)[N^{-1}] = A[N^{-1}].$$

Now let $z \in A$ and let p be a prime. The element $z \in A[N^{-1}] = (A \cap R)[N^{-1}]$ can be written $z = \frac{a}{N^k}$, where $a \in A \cap R$ and $k \in \mathbf{N}$. Using Fermat's Little Theorem(s),

$$(N^k)^p = N^k + pn$$
$$a^p = a + pb$$

for some $n \in \mathbf{Z}$ and $b \in R$. Observe that pb belongs to $A \cap R$, hence to $A[N^{-1}]$, so that $b \in A$, as long as p does not divide N.

We then have

$$\begin{aligned} z^p - z &= \frac{a^p}{N^{kp}} - \frac{a}{N^k} = \frac{a + pb}{N^k + pn} - \frac{a}{N^k} \\ &= \frac{(a + pb)N^k - a(N^k + pn)}{(N^k + pn)N^k} = p\frac{N^k b - na}{(N^k + pn)N^k} = p\frac{N^k b - na}{N^{(p+1)k}}, \end{aligned}$$

so that $pu = z^p - z \in A$ for some $u \in A[N^{-1}]$, assuming $p \nmid N$. But then in fact $u \in A$.

Consequently, for all $z \in A$ and all sufficiently large primes p, $z^p - z \in pA$, so that $z^p = z$ in A/pA. Being reduced and artinian, A/pA may be written as a product of fields, and because of the equation $z^p = z$, these fields must all equal $\mathbf{Z}/p$, which means all sufficiently large primes split completely in A. It is then a consequence of Tchebotarev's Density Theorem[2] that $\mathbf{Q} \otimes_{\mathbf{Z}} R = \mathbf{Q}$, and consequently that $R = \mathbf{Z}$ (recall that R was assumed finitely generated as an abelian group). This concludes the proof in this special case.

In the general case, recall that $R = \prod R_j$ was included in product of numerical rings, each of which is isomorphic to $\mathbf{Z}[m^{-1}]$ according to the above argument. But these rings have no non-trivial (numerical) ideals, so by Delsarte's Lemma, R must be the whole product.

Finally, we abandon the assumption that R be finitely generated as a group, and assume it finitely generated as a ring only. Because of the relation $p \mid r^p - r$, R/pR will be a finitely generated torsion group, and hence zero-dimensional, for each prime p. It then follows from Chevalley's Dimension Argument that $\dim \mathbf{Q} \otimes_{\mathbf{Z}} R = 0$, so that $\mathbf{Q} \otimes_{\mathbf{Z}} R$ is a finite-dimensional vector space over $\mathbf{Q}$. Only finitely many denominators are employed in a basis, so there exists an integer M for which $R[M^{-1}]$ is finitely generated over $\mathbf{Z}[M^{-1}]$.

We can now more or less repeat the previous argument. $R[M^{-1}]$ will still be reduced, and as before, $\mathbf{Q} \otimes_{\mathbf{Z}} R[M^{-1}]$ will be finite-dimensional, hence a product of fields, and we may reduce to the case when $\mathbf{Q} \otimes_{\mathbf{Z}} R[M^{-1}]$ is a field. Letting

[2](A special case of) Tchebotarev's Density Theorem states the following: The density of the primes that split completely in a number field K equals $\frac{1}{|\mathrm{Gal}(K/\mathbf{Q})|}$. In our case, this set has density 1.

A denote the algebraic integers in $\mathbf{Q} \otimes_{\mathbf{Z}} R[M^{-1}]$, the factor group $A/R[M^{-1}]$ will be finitely generated and torsion, and hence killed by some integer, so that again we are lead to $R[N^{-1}] = A[N^{-1}]$. As before, we may draw the conclusion that $\mathbf{Q} \otimes_{\mathbf{Z}} R = \mathbf{Q}$, and consequently that $R = \mathbf{Z}[N^{-1}]$. This concludes the proof in the general case. □

13. Modules

A most elegant application of the Structure Theorem for Finitely Generated Numerical Rings is to classify torsionfree modules.

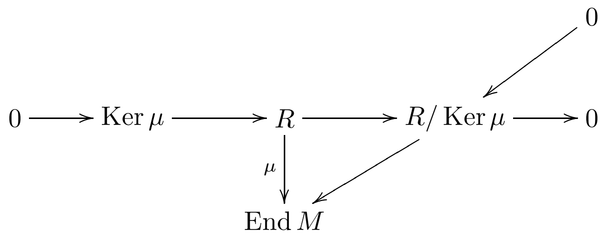
Lemma 6. *For a ring homomorphism $\varphi\colon R \to S$, where R is numerical and S is torsionfree,* $\operatorname{Ker}\varphi$ *will be a numerical ideal.*

Proof.

$$n!\varphi\left(\binom{r}{n}\right) = \varphi\left(n!\binom{r}{n}\right) = \varphi(r(r-1)\cdots(r-n+1)) = 0,$$

if $r \in \operatorname{Ker}\varphi$ and $n > 0$. Thus $\binom{r}{n} \in \operatorname{Ker}\varphi$, which is then numerical. □

Let M be a torsionfree module over the numerical ring R, with module structure given by the group homomorphism $\mu\colon R \to \operatorname{End} M$. We have the following commutative diagram:



$\operatorname{End} M$ is torsionfree, so by the lemma $\operatorname{Ker}\mu$ is a numerical ideal. Therefore $R/\operatorname{Ker}\mu$ will be a numerical ring, over which M is also a module.

Assume now also that $\operatorname{End} M$ is finitely generated as a module over $\mathbf{Z}[N^{-1}]$ for some integer N. Because $\mathbf{Z}[N^{-1}]$ is a noetherian ring, $\operatorname{End} M$ is a noetherian module. Hence its submodule $R/\operatorname{Ker}\mu$ is finitely generated as a module over $\mathbf{Z}[N^{-1}]$, and therefore also as a ring. Not only that, but $R/\operatorname{Ker}\mu$ is in fact numerical, so by the Structure Theorem,

$$R/\operatorname{Ker}\mu \cong \mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}],$$

for unique numbers $m_1, \ldots, m_k$. The module M itself will split up as a direct sum

$$M = M_1 \oplus \cdots \oplus M_k,$$

with each M_j a module over $\mathbf{Z}[m_j^{-1}]$. M_j is torsionfree, and therefore in fact free over $\mathbf{Z}[m_j^{-1}]$, because of these rings being principal. We have thus proved:

Theorem 14. *Over a numerical ring, let M a torsionfree module, which is finitely generated over $\mathbf{Z}[N^{-1}]$ for some integer N. Then there exist positive integers m_j, r_j such that*

$$M \cong \mathbf{Z}[m_1^{-1}]^{r_1} \oplus \cdots \oplus \mathbf{Z}[m_k^{-1}]^{r_k}$$

as a module over

$$\mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}].$$

14. The Binomial Theorem

Given a numerical ring and a (commutative, unital) algebra A over R, we have an induced *exponentiation* on $1+\sqrt{0}$, given by the following binomial expansion:

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$

The sum is of course a finite one.

The numerical axioms imply the following properties for this exponentiation:

I. $(1+x)^r(1+x)^s = (1+x)^{r+s}$.

II. $\big((1+x)^r\big)^s = (1+x)^{rs}$.

III. $(1+x)^r(1+y)^r = \big((1+x)(1+y)\big)^r$.

IV. $(1+x)^1 = 1+x$.

V. $(1+x)^r \equiv 1+rx \mod (\sqrt{0})^2$.

Exponentiation will thus make the abelian group $(1+\sqrt{0}, \cdot)$ into an R-module. Indeed, property III shows that exponentiation by r gives an endomorphism $\epsilon(r)$ of the group, and properties I, II and IV show that

$$\epsilon_A \colon R \to \operatorname{End}(1+\sqrt[A]{0}, \cdot)$$

is a unital ring homomorphism.

This module structure is *natural* in the following sense. Given two algebras A and B and an algebra homomorphism $\varphi\colon A \to B$, the following diagram commutes for any $r \in R$:

$$\begin{array}{ccc} 1+\sqrt[A]{0} & \xrightarrow{\epsilon_A(r)} & 1+\sqrt[A]{0} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ 1+\sqrt[B]{0} & \xrightarrow[\epsilon_B(r)]{} & 1+\sqrt[B]{0} \end{array}$$

We now reverse the procedure:

Theorem 15: The Binomial Theorem.

Given a numerical ring R, the equation

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \tag{1.4}$$

defines a module structure on $(1+\sqrt[A]{0}, \cdot)$, which is natural in R-algebras A, and satisfies

$$(1+x)^r \equiv 1 + rx \mod (\sqrt{0})^2. \tag{1.5}$$

Conversely, given a ring R and a natural module structure on $(1+\sqrt[A]{0}, \cdot)$ (for all R-algebras A) satisfying (1.5), there is a (necessarily unique) numerical ring structure on R, fulfilling the equation (1.4).

Proof. There remains to establish the second part. So, let a natural module structure be given, and consider $\epsilon_A \colon R \to \operatorname{End}(1+\sqrt[A]{0}, \cdot)$, where $A = R[t]/(t^{N+1})$, and N is some large number. We have

$$\epsilon(r)(1+t) = (1+t)^r = a_0 + a_1 t + \cdots + a_N t^N,$$

and clearly the coefficients a_n are independent of N. Therefore, we may without ambiguity define $\binom{r}{n} = a_n$. This will make the binomial expansion identity hold in A, and then it will hold everywhere by naturality.

It is now immediate that the axioms for a numerical ring hold, as they are simply direct translations of the module axioms. For example, identification of the coefficients of t^n in

$$\sum_{i=0}^{\infty} \binom{r}{i} t^i \sum_{j=0}^{\infty} \binom{s}{j} t^j = (1+t)^r (1+t)^s = (1+t)^{r+s} = \sum_{n=0}^{\infty} \binom{r+s}{n} t^n$$

proves axiom I. (Proving III will of course involve the polynomial ring in two variables.) □

And this little "treatise on the Binomial Theorem" closes the chapter on numerical rings.

CHAPTER 2

Polynomial Maps

> [...] je donnerais bien cent sous au mathématicien qui me démontrerait par une équation algébrique l'existence de l'enfer.
>
> — Honoré de Balzac, *La Peau de chagrin*

In this and the succeeding chapters, we will consider a fixed base ring of scalars, commutative and unital and which, when referred to by name, will be called R. We adopt the following conventions:

I. All modules will be R-modules, and all algebras will be commutative and unital R-algebras.

II. We will use $\mathfrak{Mod}$ to denote the category of R-modules, and $\mathfrak{CAlg}$ for the category of commutative, unital R-algebras.

III. All tensor products will be computed over R, unless otherwise stated.

IV. "Homomorphism" with no further qualification will denote an R-module homomorphism (or R-linear map).

V. When discussing non-strict polynomiality, R will also be assumed numerical, and $\mathfrak{NAlg}$ will denote the category of numerical R-algebras.

At his leisure, the reader may put $R = \mathbf{Z}$, and anywhere substitute "abelian group" for "module".

We shall consider maps $f\colon M \to N$ between modules, and they shall almost never be homomorphisms. Indeed, they shall be generalizations of ordinary polynomial maps as defined on fields. The problem is how to form "polynomials" on general modules, where there is no multiplication in sight. We recall the following quibble[1]:

> And God said unto the animals: "Go out into the world and multiply!"
>
> But the snake answered: "How could I? I am an adder!"

[1] In some versions of this myth, it is said that God constructed a table made of wood for the snakes to crawl upon, since even adders can multiply on a log table. God does not seem to be familiar with tensor products.

Returning to the modules, two different approaches present themselves. We may choose to talk about (let us phrase it carefully) "polynomial-like[2]" maps as maps satisfying certain equations that are somehow thought to characterize polynomials. This road will indeed be explored; for these equations to be sensible, a numerical base ring is required.

A completely different method, with the advantage of producing entities that actually look like polynomials, is to use scalar extension. Quoting from [14], "[...] la généralisation en vue devrait conduire à associer, à «quelque chose» qui s'écrirait : $x_1T_1 + \cdots x_pT_p$, une «autre chose» qui s'écrirait

$$\sum_{i=1}^{q} y_i Q_i(T_1, \ldots, T_p),$$

les Q_i étant cette fois des polynomes. Manifestement s'introduisent ici les modules produits tensoriels [...]." This seems to be the most elegant solution, and is used to define strict polynomial maps (called polynomial laws) in [14].

Classically, (non-strict) polynomial maps were defined using the first method, but this was before numerical rings were discovered. With this new class of rings at our disposal, we shall be able to use the method of scalar extension also for non-strict maps, which will provide a beautiful unification of the two notions of polynomiality.

1. Polynomiality

We shall begin by making an extremely general discussion of polynomiality, and then identify the two definitions which will actually be used in the sequel.

Let D be a finitary algebraic category, so that it is an equational class in the sense of universal algebra (and hence a variety of algebras by the HSP Theorem; see for example [3]). We require D to be a subcategory of $\mathfrak{Mod}$, so that the objects of D are first of all R-modules.

For a set of variables V, we let $\langle V \rangle_D$ denote the free algebra on V in D.

Definition 1. A **D-polynomial** over a module M (not necessarily in D) in the variables $x_1, \ldots, x_k$, is an element of

$$M \otimes \langle x_1, \ldots, x_k \rangle_D.$$

A **linear form** over M in these same variables is a polynomial of the form

$$\sum u_j \otimes x_j,$$

for some $u_j \in M$. ◇

Theorem 1: Ekedahl's Esoteric Polynomiality Principle. *Let two modules M and N be given, and a family of maps*

$$f_A \colon M \otimes A \to N \otimes A, \qquad A \in D.$$

The following statements are equivalent:

[2] På svenska: *polynomaktiga*.

A. For every D-polynomial $p(x) = p(x_1, \ldots, x_k)$ over M there is a unique D-polynomial $q(x) = q(x_1, \ldots, x_k)$ over N, such that for all $A \in D$ and all $a_j \in A$,

$$f_A(p(a)) = q(a).$$

B. For every linear form $l(x)$ over M there is a unique D-polynomial $q(x)$ over N, such that for all $A \in D$ and all $a_j \in A$,

$$f_A\,(l(a)) = q(a).$$

C. The map

$$f\colon M \otimes - \to N \otimes -$$

is a natural transformation between functors $D \to \mathfrak{Set}$.

Proof. It is trivial that A implies B. Given statement B, and a homomorphism $\varphi\colon A \to B$ mapping a_j to b_j, the following commutative diagram proves the naturality of f:

$$\begin{array}{ccc} M \otimes A & \xrightarrow{f_A} & N \otimes A \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle 1\otimes\varphi} \\ M \otimes B & \xrightarrow[f_B]{} & N \otimes B \end{array} \qquad \begin{array}{ccc} \sum u_j \otimes a_j & \longrightarrow & q(a) \\ \downarrow & & \downarrow \\ \sum u_j \otimes b_j & \longrightarrow & q(b) \end{array}$$

Finally, suppose f natural. Given

$$p(x) \in M \otimes \langle x_1, \ldots, x_k \rangle_D,$$

define

$$q(x) = f_{\langle x_1, \ldots, x_k \rangle_D}\,(p(x))\,,$$

and for any $A \in D$ and $a_j \in A$, define the homomorphism

$$\varphi\colon \langle x_1, \ldots, x_k \rangle_D \to A, \qquad x_j \mapsto a_j.$$

Then by naturality of f, the following diagram commutes:

$$\begin{array}{ccc} M \otimes \langle x_1, \ldots, x_k \rangle & \xrightarrow{f_{\langle x_1, \ldots, x_k \rangle}} & N \otimes \langle x_1, \ldots, x_k \rangle \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle 1\otimes\varphi} \\ M \otimes A & \xrightarrow[f_A]{} & N \otimes A \end{array} \qquad \begin{array}{ccc} p(x) & \longrightarrow & q(x) \\ \downarrow & & \downarrow \\ p(a) & \dashrightarrow & q(a) \end{array}$$

q is evidently unique, which proves A. □

Definition 2. When the conditions of the theorem are fulfilled, we call f a ***D*-polynomial map** from M to N. ◇

When f is D-polynomial, part B of the theorem tells us that

$$\sum u_j \otimes a_j \mapsto q(a)$$

for some D-polynomial q. Naïvely, if we want the coefficients a_j of the elements u_j to transform as generalized polynomials, formed using some operations, the correct setting is the category of algebras using these same operations.

Example 1. A $\mathfrak{Mod}$-polynomial map $f\colon M \to N$ is just a linear transformation $M \to N$. This is because, by B above, f_R will map $\sum u_j \otimes r_j$ to $\sum v_j \otimes r_j$ for all $r_j \in R$, and such a map is easily seen to be linear. Conversely, any module homomorphism induces a natural transformation $M \otimes - \to N \otimes -$. △

Example 2. Let S be an R-algebra. An ${}_S\mathfrak{Mod}$-polynomial map $M \to N$ is a transformation

$$M \otimes A \to N \otimes A,$$

natural in the S-module A, which is the same as a natural transformation

$$(M \otimes S) \otimes_S - \to (N \otimes S) \otimes_S -.$$

This is simply an ${}_S\mathfrak{Mod}$-polynomial map $M \otimes S \to N \otimes S$, or, as we noted in the previous example, an S-linear map from $M \otimes S$ to $N \otimes S$. △

The last two examples will be the important ones:

Example 3. A $\mathfrak{CAlg}$-polynomial map $M \to N$ is a *strict polynomial map*, or *polynomial law* in the sense of [14]. For every linear form $\sum u_j \otimes x_j$ over M there are unique elements $v_\mu \in N$, μ running over all multi-indices, such that for all algebras A and all $a_j \in A$,

$$f_A\left(\sum u_j \otimes a_j\right) = \sum v_\mu \otimes a^\mu.$$

Intuitively, the coefficients of the elements u_j "transform as ordinary polynomials". △

Example 4. Suppose now that the base ring R is numerical, and consider the category $\mathfrak{NAlg}$ of numerical algebras over R. An $\mathfrak{NAlg}$-polynomial map $M \to N$ is what will be called a *polynomial map*. For every linear form $\sum u_j \otimes x_j$ over M there are unique elements $v_\mu \in N$, μ running over all multi-indices, such that for all algebras A and all $a_j \in A$,

$$f_A\left(\sum u_j \otimes a_j\right) = \sum v_\mu \otimes \binom{a}{\mu}.$$

Intuitively, the coefficients of the elements u_j "transform as numerical polynomials". △

2. Polynomial Maps

The key to understanding polynomials is the following property of ordinary polynomials f (over some field): If $f(x) = a$ is constant, clearly

$$f(x) - f(0) = 0$$

for all x. If $f(x) = a + bx$ is linear, then

$$f(x+y) - f(x) - f(y) + f(0) = 0$$

for all x, y. A generalization to arbitrary degrees is immediate and leads to the following definition, presumably first explicitly stated by Eilenberg and Mac Lane in [6]:

Definition 3. The nth **deviation** of a map $f\colon M \to N$ is the map

$$f(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} f\left(\sum_{i \in I} x_i\right)$$

of $n+1$ variables. $\diamond$

The idea here is that the nth deviation measures how much f deviates from being polynomial of degree n. We have for example

$$\begin{aligned} f(x \diamond y) &= f(x+y) - f(x) - f(y) + f(0) \\ f(\diamond x) &= f(x) - f(0), \end{aligned}$$

and, of course,

$$f(\diamond) = f(0).$$

We let

$$f\left(\Diamond_n x\right) = f(\underbrace{x \diamond \cdots \diamond x}_{n}).$$

Definition 4. The map $f\colon M \to N$ is **polynomial** of degree n if its nth deviation vanishes:

$$f(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

for any $x_i \in M$. $\diamond$

Let us, for clarity, point out, that the diamond sign itself does not work as an operator; the entity $x \diamond y$ does not have a life of its own, and cannot exist outside the scope of an argument of a map.

3. Numerical Maps

After defining the deviation, a polynomial map of degree n between *abelian groups* is classically[3] defined as a map of which the nth deviation vanishes. While this works well enough for modules over $\mathbf{Z}$, we would like to include modules over more general rings.

Recall that an extra condition

$$f(rx) = rf(x)$$

need be imposed on a group homomorphism to make it a module homomorphism (but that this is automatic when the base ring is $\mathbf{Z}$). Using binomial coefficients, we generalize to arbitrary numerical modules. The base ring R of scalars is now of course assumed numerical.

Definition 5. The map $f\colon M \to N$ is **numerical** of degree at most n if it satisfies the following two equations, for all $x_i, x \in M$ and all $r \in R$:

$$f(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$
$$f(rx) = \sum_{k=0}^{n} \binom{r}{k} f\left(\mathop{\Diamond}_{k} x\right).$$

$\diamond$

It is of course straightforward to define what it means for f to have degree *exactly* n, but this is never needed. Therefore, when we speak of a map as being of degree n, it is to be understood: degree n *or less*.

Example 5. A map is of degree 0 iff it is constant. It is of degree 1 iff it is a homomorphism translated by a constant. △

Example 6. The numerical maps $f\colon \mathbf{Z} \to \mathbf{Z}$ of degree n, are precisely the ones given by numerical polynomials of degree n:

$$f(x) = \sum_{k=0}^{n} c_k \binom{x}{k}.$$

△

Lemma 1. *For r in a numerical ring and natural numbers $m \geq n$, the following formula holds:*

$$\sum_{k=m}^{n} (-1)^k \binom{r}{k}\binom{k}{m} = (-1)^n \binom{r}{m}\binom{r-m-1}{n-m}.$$

Proof. Induction (and, optionally, a quick reference to the Numerical Universality Principle). □

[3] Of course, [6] itself never bothers to make this definition, but instead moves on to more important topics.

Theorem 2. *The map $f\colon M \to N$ is numerical of degree n iff its nth deviations vanish, and it satisfies the equation*

$$f(rx) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} f(mx),$$

for any $r \in R$ and $x \in M$.

Proof. This follows from the lemma:

$$\begin{aligned}\sum_{k=0}^{n} \binom{r}{k} f\left(\underset{k}{\Diamond}\, x\right) &= \sum_{k=0}^{n} \binom{r}{k} \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} f(mx) \\ &= \sum_{m=0}^{n} (-1)^{-m} \left(\sum_{k=m}^{n} (-1)^{k} \binom{r}{k} \binom{k}{m} \right) f(mx) \\ &= \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} f(mx).\end{aligned}$$

□

4. The Augmentation Algebra

We now wish to find an alternative way of describing these numerical maps. Recall that the free module on a set M is the set

$$R[M] = \left\{ \sum a_j [x_j] \;\middle|\; a_j \in R,\ x_j \in M \right\}$$

of formal (finite) linear combinations of elements of M. It obviously has a module structure, and if M is itself a module, it also carries a multiplication, namely the **sum multiplication**

$$[x][y] = [x+y],$$

extended by linearity. It makes $R[M]$ into a commutative, associative algebra with unity $[0]$, called the **augmentation algebra**.

When M additionally has an algebra structure, there is another canonical operation on the augmentation algebra, namely the **product multiplication**, defined by

$$[x] \star [y] = [xy].$$

This multiplication has identity element $[1]$, but is of course commutative only if M is. The latter operation will make an apparition later on, in the context of Morita equivalence.

In the present discussion, we assume M to be a module only, and hence use the sum multiplication. The map

$$M \to R[M], \qquad x \mapsto [x],$$

is a map between modules, and so we may form its nth deviation

$$(x_1, \ldots, x_{n+1}) \mapsto [x_1 \diamond \ldots \diamond x_{n+1}].$$

The following lemma then follows easily from the definition of deviation.

Lemma 2.

$$[x_1 \diamond \ldots \diamond x_{n+1}] = ([x_1] - [0]) \cdots ([x_{n+1}] - [0]) \,.$$

Defining a filtration in $R[M]$ (a decreasing sequence of ideals) by

$$I_n = ([x_1 \diamond \ldots \diamond x_{n+1}] \mid x_i \in M) + \left([rx] - \sum_{k=0}^{n} \binom{r}{k} \left[\mathop{\Diamond}_{k} x \right] \,\middle|\, r \in R,\ x \in M \right),$$

and then letting

$$R[M]_n = R[M]/I_n,$$

we have a canonical map

$$\begin{aligned} \delta_n \colon M &\to R[M]_n \\ x &\mapsto [x], \end{aligned}$$

which is numerical of degree n. And not only that:

Theorem 3. *The map δ_n is the* universal *numerical map of degree n, in that every numerical map $f\colon M \to N$ of degree n has a unique factorization through it.*

$$\begin{array}{ccc} M & \xrightarrow{\delta_n} & R[M]_n \\ & \searrow^{f} & \vdots \\ & & N \end{array}$$

Proof. Given a map $f\colon M \to N$, we extend it linearly to $f\colon R[M] \to N$, by which procedure it automatically becomes a homomorphism. The theorem then amounts to the trivial observation that f is numerical of degree n iff it kills I_n, so that it factors through $R[M]_n$. □

The augmentation quotients of a free module M are given by the next theorem.

Theorem 4. *In the polynomial algebra $R[t_1, \ldots, t_k]$, let J_n be the ideal generated by monomials of degree greater than n. Then*

$$R[R^k]_n \cong R[t_1, \ldots, t_k]/J_n$$

as algebras. In particular, $R[R^k]_n$ is a free module.

Proof. Each t_i is nilpotent in $R[t_1, \dots, t_k]/J_n$, and so we may define exponentiation $(1+t_i)^r$ for any $r \in R$. Accordingly define, for a tuple $(r_1, \dots, r_k) \in R^k$,

$$\begin{aligned} \varphi\colon R[R^k] &\to R[t_1, \dots, t_k] \\ [(r_1, \dots, r_k)] &\mapsto (1+t_1)^{r_1} \cdots (1+t_k)^{r_k}. \end{aligned}$$

Using multi-index notation $\varrho = (r_1, \dots, r_k)$, we may write this more succinctly as

$$[\varrho] \mapsto (1+t)^{\varrho}.$$

The map φ is linear by definition, and also multiplicative, since

$$\varphi([\varrho][\sigma]) = \varphi([\varrho + \sigma]) = (1+t)^{\varrho+\sigma} = (1+t)^{\varrho}(1+t)^{\sigma} = \varphi(\varrho)\varphi(\sigma).$$

It maps I_n into J_n, because, when $\varrho_1, \dots, \varrho_{n+1} \in R^k$,

$$\begin{aligned} \varphi([\varrho_1 \diamond \cdots \diamond \varrho_{n+1}]) &= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} \varphi\left(\left[\sum_{j \in J} \varrho_j\right]\right) \\ &= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} (1+t)^{\sum_{j \in J} \varrho_j} \\ &= \prod_{j=1}^{n+1} \big((1+t)^{\varrho_j} - 1\big) \in J_n. \end{aligned}$$

Also, for $s \in R$ and $\varrho \in R^k$,

$$\begin{aligned} \varphi\left([s\varrho] - \sum_{m=0}^{n} \binom{s}{m} \left[\mathop{\diamond}_{m} \varrho\right]\right) &= \varphi\left([s\varrho] - \sum_{m=0}^{n} \binom{s}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} [j\varrho]\right) \\ &= (1+t)^{s\varrho} - \sum_{m=0}^{n} \binom{s}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (1+t)^{j\varrho} \\ &= (1+t)^{s\varrho} - \sum_{m=0}^{n} \binom{s}{m} \big((1+t)^{\varrho} - 1\big)^m \\ &= (p(t)+1)^s - \sum_{m=0}^{n} \binom{s}{m} p(t)^m, \end{aligned}$$

where, in the last step, we let $p(t) = (1+t)^{\varrho} - 1$. By the Binomial Theorem, we have

$$(p(t)+1)^s = \sum_{m=0}^{\infty} \binom{s}{m} p(t)^m,$$

but since the terms of index $n+1$ and higher yield an $(n+1)$st degree polynomial, the above difference will belong to J_n. We therefore have an induced map

$$\varphi\colon R[R^k]_n \to R[t_1, \dots, t_k]/J_n.$$

We now define a map

$$\begin{aligned}\psi\colon R[t_1,\ldots,t_k] &\to R[R^k]_n\\ t_{p_1}\cdots t_{p_m} &\mapsto [e_{p_1}\diamond\cdots\diamond e_{p_m}]\end{aligned}$$

in the reverse direction. Again, ψ is additive by definition, and multiplicative because of Lemma 2. The vanishing of ψ on J_n induces a map

$$\psi\colon R[t_1,\ldots,t_k]/J_n \to R[R^k]_n.$$

It is easy to verify that φ and ψ are inverse to each other. □

For future reference, we also explore the grading of $R[M]$ induced by the filtration I_n.

Theorem 5. *Let M be free on k generators $e_1,\ldots,e_k$. The map*

$$\begin{aligned}\xi\colon S^n(M) &\to I_{n-1}/I_n\\ e_1^{n_1}\cdots e_k^{n_k} &\mapsto \left[\mathop{\Diamond}_{n_1} e_1\diamond\cdots\diamond\mathop{\Diamond}_{n_k} e_k\right]+I_n\end{aligned}$$

gives an isomorphism

$$S(M)\to I_{-1}/I_0\oplus I_0/I_1\oplus\cdots$$

of graded algebras.

Proof. Under the isomorphism $R[R^k]_n\cong R[t_1,\ldots,t_k]/J_n$, the ideal I_{n-1} will correspond to J_{n-1}, and consequently $I_{n-1}/I_n\cong J_{n-1}/J_n$. Under this correspondence, ξ simply takes $e^\nu\mapsto t^\nu$, and is of course an isomorphism. (An alternative is to use Theorem 2 directly.) □

5. Properties of Numerical Maps

We now elaborate on the behaviour of numerical maps. To begin with, we note that not only do the nth deviations of an nth degree map vanish, but its lower order deviations are also quite pleasant.

Theorem 6. *The map $f\colon M\to N$ is numerical of degree n iff for any $a_1,\ldots,$ $a_k\in R$ and $x_1,\ldots,x_k\in M$, the following equation holds:*

$$f(a_1x_1\diamond\cdots\diamond a_kx_k)=\sum_{\substack{\#S=[k]\\|S|\le n}}\prod_{j\in\#S}\binom{a_j}{\deg j}f\left(\mathop{\Diamond}_{j\in S}x_j\right),$$

where the sum is taken over multisets S.

Proof. If f is of degree n, calculate in the augmentation algebra $R[M]_n$:

$$\begin{aligned}
[a_1x_1 \diamond \ldots \diamond a_kx_k] &= ([a_1x_1] - [0]) \cdots ([a_kx_k] - [0]) \\
&= \sum_{q_1=1}^{\infty} \binom{a_1}{q_1} \Big[\mathop{\lozenge}_{q_1} x_1 \Big] \cdots \sum_{q_k=1}^{\infty} \binom{a_k}{q_k} \Big[\mathop{\lozenge}_{q_k} x_k \Big] \\
&= \sum_{q_1=1}^{\infty} \cdots \sum_{q_k=1}^{\infty} \binom{a_1}{q_1} \cdots \binom{a_k}{q_k} \Big[\mathop{\lozenge}_{q_1} x_1 \diamond \cdots \diamond \mathop{\lozenge}_{q_k} x_k \Big].
\end{aligned}$$

The theorem now follows after application of f. The converse is trivial. □

This proof is pure magic! It is absolutely vital that the calculation be carried out in the augmentation algebra, as there would have been no way to perform the above trick had the map f been applied directly.

We now turn our attention towards the binomial coefficients themselves and prove that, considered as maps $R \to R$, they are numerical. This is of course hardly surprising, as they are more or less given by polynomials (in the enveloping **Q**-algebra).

Theorem 7. *The binomial coefficient $x \mapsto \binom{x}{n}$ is numerical of degree n.*

Proof. It is numerical of degree n in **Z**, and therefore also in R by the Numerical Universality Principle. □

We now have the following description of numerical maps.

Theorem 8. *The map $f\colon M \to N$ is numerical of degree n iff for any $u_1, \ldots, u_k \in M$ there exist unique elements $v_\mu \in N$, μ varying over all multi-indices with $|\mu| \le n$, such that*

$$f(r_1u_1 + \cdots + r_ku_k) = \sum_{\mu} \binom{r}{\mu} v_\mu,$$

for any $r_1, \ldots, r_k \in R$.

Proof. We assume f is numerical of degree n, and suppose first that $M = R^k$ is free of rank k and $u_j = e_j$. By the preceding theorems, numerical maps $f\colon R^k \to N$ of degree n correspond to linear maps

$$\overline{f}\colon R[t_1, \ldots, t_k]/J_n \to N.$$

We have the following factorization:

$$\begin{array}{ccc}
R^k & \xrightarrow{\delta_n} & R[R^k]_n \cong R[t_1, \ldots, t_k]/J_n \\
& \searrow^{f} & \downarrow \overline{f} \\
& & N
\end{array}$$

Say the monomial t^μ is mapped to $v_\mu \in N$; then $[(r_1, \ldots, r_k)] \in R[R^k]_n$ corresponds in $R[t_1, \ldots, t_k]/J_n$ to

$$(1+t_1)^{r_1} \cdots (1+t_k)^{r_k} = \left(\sum_{m_1} \binom{r_1}{m_1} t_1^{m_1}\right) \cdots \left(\sum_{m_k} \binom{r_k}{m_k} t_k^{m_k}\right),$$

and so is mapped by f (or $\overline{f}$) to

$$\sum_{m_1,\ldots,m_k} \binom{r_1}{m_1} \cdots \binom{r_k}{m_k} v_{(m_1,\ldots,m_k)}.$$

Thus, the action of f is

$$f(r_1, \ldots, r_k) = \sum_\mu \binom{r}{\mu} v_\mu,$$

as desired. In this case the elements v_μ are clearly unique.

In the general case, when M is allowed to be any module, we study the composition

$$\begin{aligned} &R^k \to M \to N\colon \\ &(r_1, \ldots, r_k) \mapsto r_1 u_1 + \cdots + r_k u_k \mapsto f(r_1 u_1 + \cdots + r_k u_k). \end{aligned}$$

By the preceding argument, this map is of the desired form, and the v_μ will again be unique.

The converse is trivial. □

Finally, we make the promised connection with $\mathfrak{NAlg}$-polynomiality. Let $f\colon M \to N$ be an $\mathfrak{NAlg}$-polynomial map. From the Polynomiality Principle, we know that for every linear form $l(x)$ over M there is a unique $\mathfrak{NAlg}$-polynomial $q(x)$ over N, such that for all $A \in \mathfrak{NAlg}$ and all $a_j \in A$,

$$f_A\left(l(a)\right) = q(a).$$

We say that f is of **bounded degree** n if the degree of the polynomial q is uniformly bounded above ny n (independent of l).

The main theorem linking the two notions of polynomiality states:

Theorem 9. *$f\colon M \to N$ is numerical of degree n iff it may be extended to a (unique) $\mathfrak{NAlg}$-polynomial map of bounded degree n.*

Proof. Given a numerical map f, fix the elements v_μ from the preceding theorem. We then have a map

$$f_A\colon M \otimes A \to N \otimes A, \qquad \sum u_j \otimes x_j \mapsto \sum_\mu v_\mu \otimes \binom{x}{\mu},$$

By the Polynomiality Principle, f_A is a natural transformation. The converse is trivial. □

Example 7. Here is an example to show that requiring bounded degree is necessary. Let $U = \langle u_1, u_2, \ldots \rangle$ be free on an infinite basis. The map

$$f_A \colon U \otimes A \to U \otimes A, \qquad \sum u_k \otimes a_k \mapsto \sum u_k \otimes \binom{a_k}{k}$$

is $\mathfrak{NAlg}$-polynomial, but not numerical of any finite degree n. △

6. Strict Polynomial Maps

We no longer assume a numerical base ring R, as we turn our attention toward strict polynomial maps. Norbert Roby invented these (he called them polynomial laws), and all the facts stated in this section may be found in [14].

Definition 6. A **strict polynomial map** between modules $f \colon M \to N$ is a $\mathfrak{CAlg}$-polynomial map; that is, a natural transformation

$$M \otimes - \to N \otimes -$$

between functors $\mathfrak{CAlg} \to \mathfrak{Set}$. ◇

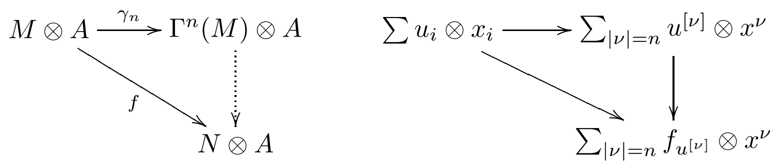
Some elementary facts we shall need about a strict polynomial map $f \colon M \to N$ are the following:

1. From the Polynomiality Principle, the following proposition is immediately deduced: For any $u_1, \ldots, u_k \in M$ there exist unique elements $v_\nu \in N$ (only finitely many of which are non-zero), with μ varying over all multi-indices, such that
$$f(u_1 \otimes x_1 + \cdots + u_k \otimes x_k) = \sum_\nu v_\nu \otimes x^\nu$$
for all x_j in all algebras. We shall write $f_{u^{[\nu]}} = v_\nu$.
2. f is said to have degree (at most) n, if $f_{u^{[\nu]}} = 0$ when $|\nu| > n$.
3. f is said to be **homogeneous** of degree n if $f(az) = a^n f(z)$ for all a in all algebras A and all $z \in M \otimes A$. This amounts to saying that $f_{u^{[\nu]}} \neq 0$ only when $|\nu| = n$.
4. When f is homogeneous of degree n, note that
$$f_{u^{[n]}} = f(u).$$
5. Any f has a unique decomposition into homogeneous components, namely:
$$f(u_1 \otimes x_1 + \cdots + u_k \otimes x_k) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} f_{u^{[\nu]}} \otimes x^\nu$$
(only a finite number of terms being non-zero).

6. Finally, there is a fundamental relationship between homogeneous maps and divided power algebras: For any module M there is a *universal* homogeneous map

$$\gamma_n \colon M \to \Gamma^n(M), \qquad \sum u_i \otimes x_i \mapsto \sum_{|\nu|=n} u^{[\nu]} \otimes x^\nu$$

of degree n, through which every map $f\colon M \to N$ of degree n factors uniquely:



In other words, there is a canonical isomorphism between the module of homogeneous polynomial maps of degree n from M to N and the module of homomorphisms from $\Gamma^n(M)$ to N.

7. Pray note that the map

$$\begin{aligned} \Gamma^n(M) &\to N \\ u^{[\nu]} &\mapsto f_{u^{[\nu]}} \end{aligned}$$

is a module homomorphism (for fixed f).

7. The Divided Power Algebra

The elementary theory of divided power modules (and ditto algebras) can be found in [14].

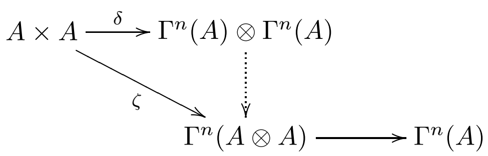
When A is an algebra, the nth divided power module $\Gamma^n(A)$ comes equipped with a natural multiplication. First of all, note that there is a canonical map

$$\delta \colon A \times A \to \Gamma^n(A) \otimes \Gamma^n(A), \qquad (x, y) \mapsto x^{[n]} \otimes y^{[n]},$$

which is universal for bihomogeneous maps of bidegree (n, n) out of $A \times A$. Because the map

$$\zeta \colon A \times A \to \Gamma^n(A \otimes A), \qquad (x, y) \mapsto (x \otimes y)^{[n]},$$

is bihomogeneous of degree (n, n), it will have a unique factorization through $\Gamma^n(A) \otimes \Gamma^n(A)$, as in the following diagram:



Composition with the canonical (linear) map

$$\Gamma^n(A \otimes A) \to \Gamma^n(A), \qquad (x \otimes y)^{[n]} \mapsto (xy)^{[n]},$$

results in the following multiplication on $\Gamma^n(A)$:

$$\Gamma^n(A) \otimes \Gamma^n(A) \to \Gamma^n(A), \qquad x^{[n]} \otimes y^{[n]} \mapsto (xy)^{[n]}.$$

It will be called the **product multiplication** on $\Gamma^n(A)$.

Contrast this with the **divided power multiplication**, defined on $\Gamma(M)$ for any module M, which is simply juxtaposition:

$$x^{[m]} \cdot y^{[n]} = x^{[m]} y^{[n]}.$$

The components $\Gamma^n(M)$ are not even closed under this operation.

It deserves to be pointed out, and emphasized strongly, that the nth divided power module $\Gamma^n(M)$ is *not* generated by the pure divided powers $z^{[n]}$, for $z \in M$, as the following example shows.

Example 8. Consider in $\Gamma^3(\mathbf{Z}^2)$ a pure power

$$(a_1 e_1 + a_2 e_2)^{[3]} = a_1^3 e_1^{[3]} + a_1^2 a_2 e_1^{[2]} e_2 + a_1 a_2^2 e_1 e_2^{[2]} + a_2^3 e_2^{[3]}.$$

Observe that the coefficients of $e_1^{[2]} e_2$ and $e_1 e_2^{[2]}$ have the same parity. Therefore it is impossible to write $e_1^{[2]} e_2$ as a linear combination of pure powers. △

$\Gamma^n(M)$ is, however, "universally" generated by pure powers over all algebras, in the following sense:

Theorem 10: The Divided Power Lemma.

A natural transformation

$$\zeta \colon \Gamma^n(M) \otimes - \to N \otimes -,$$

between functors $\mathfrak{CAlg} \to \mathfrak{Mod}$, *is uniquely determined by its effect on* pure *divided powers* $z^{[n]}$ *(when* $z \in M \otimes A$ *for some algebra* A*).*

More generally, a natural transformation

$$\zeta \colon \Gamma^m(M) \otimes \Gamma^n(M) \otimes - \to N \otimes -$$

is uniquely determined by its effect on tensor products $z^{[m]} \otimes w^{[n]}$ *of pure powers.*

Proof. It suffices to show that if ζ vanishes on pure powers, it is identically zero. Indeed, linear maps $\Gamma^n(M) \to N$ correspond to homogeneous maps $M \to N$:

$$\begin{array}{ccc} \Gamma^n(M) \otimes A & \xrightarrow{\zeta} & N \otimes A \\ {\scriptstyle \gamma_n}\uparrow & \nearrow_{\overline{\zeta}} & \\ M \otimes A & & \end{array} \qquad \begin{array}{ccc} \left(\sum u_i \otimes x_i\right)^{[n]} & \longrightarrow & 0 \\ \uparrow & \nearrow & \\ \sum u_i \otimes x_i & & \end{array}$$

Since $\overline{\zeta} = 0$, also $\zeta = 0$.

For the second part, proceed similarly, noting that linear maps $\Gamma^m(M) \otimes \Gamma^n(M) \to N$ correspond to bihomogeneous maps $M \oplus M \to N$. $\square$

CHAPTER 3

Polynomial Functors

Och när jag stod där gripen, kall av skräck
och fylld av ängslan inför hennes tillstånd
begynte plötsligt mimans fonoglob
att tala till mig på den dialekt
ur högre avancerad tensorlära
som hon och jag till vardags brukar mest.

— Harry Martinson, *Aniara*

In this chapter, we turn to interpreting our different notions of polynomiality in terms of functors. We are convinced that the two "correct" notions are numerical and strict polynomial functors. Non-strict polynomiality, the original concept, works well enough over $\mathbf{Z}$, but is much too weak a notion over a general base ring. But the price to pay for upgrading to the stronger notion is the restriction to numerical base rings.

We recall our convention of a fixed base ring R, over which all modules, algebras, tensor products, etc., are taken, and which is assumed numerical when discussing numericality.

1. Module Functors

By a **module functor**, we shall understand a functor $F\colon \mathfrak{Mod} \to \mathfrak{Mod}$ mapping modules to modules. We shall mostly be content to consider functors defined only on the most simple of modules, namely the free and finitely generated ones. They constitute a subcategory of $\mathfrak{Mod}$, which will be denoted by $\mathfrak{XMod}$ (the letter X intended to suggest "eXtra nice modules"!). We shall let $\mathfrak{FMod}$ be the category of free modules, be they finitely or infinitely generated.

As it turns out, a functor defined on the subcategory $\mathfrak{XMod}$ has a unique "well-behaved" extension to the whole module category. In this introductory section we shall describe this extension process, and thus convince ourselves that there is no serious imposition in considering only functors $\mathfrak{XMod} \to \mathfrak{Mod}$, as will be done hereafter.

First, let us recall what it means for a functor, not necessarily additive, to be right-exact:

Definition 1. A functor F between abelian categories is **right-exact** if for any exact sequence:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

the associated sequence:

$$F(A \oplus B) \xrightarrow[]{\substack{F(\alpha+1_B) \\ -F(1_B)}} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0$$

is also exact. ◇

This definition agrees with the usual one in the case of an additive functor. In fact, the usual definition actually *implies* additivity of the functor, which is why it is useless to us.

We now state the main result on module functors, along with an outline of the proof:

Theorem 1.

1. *Any functor* $\mathfrak{XMod} \to \mathfrak{Mod}$ *has a unique extension to a functor* $\mathfrak{FMod} \to \mathfrak{Mod}$*, which commutes with inductive limits.*

2. *Any functor* $\mathfrak{FMod} \to \mathfrak{Mod}$ *has a unique right-exact extension to a functor* $\mathfrak{Mod} \to \mathfrak{Mod}$*.*

The first part follows from Lazard's Theorem, stating that every flat module is an inductive limit of finitely generated free modules. Given a functor $G\colon \mathfrak{XMod} \to \mathfrak{Mod}$, we may hence define $\overline{G}\colon \mathfrak{FMod} \to \mathfrak{Mod}$ by

$$\overline{G}(\varinjlim M_\alpha) = \varinjlim G(M_\alpha),$$

for an inductive limit $\varinjlim M_\alpha$ of finitely generated free modules. This definition is probably independent of the inductive system.

The second part of the theorem is an immediate consequence of Theorem 2.14 in [2]. (The crucial properties are the closure of $\mathfrak{FMod}$ under direct sums, and that its objects are projective and generate $\mathfrak{Mod}$.) The extension procedure (which essentially uses parts of the Dold–Puppe construction originally presented in [5]) may be summarized thus: Given a module M, choose a resolution of free modules P and Q:

$$Q \xrightarrow{\psi} P \longrightarrow M \longrightarrow 0$$

Define the extension $\overline{F}\colon \mathfrak{Mod} \to \mathfrak{Mod}$ of $F\colon \mathfrak{FMod} \to \mathfrak{Mod}$ by the equation

$$\overline{F}(M) = F(P) \Big/ \Big[F(\pi)\big(\operatorname{Ker} F(\pi + \psi\xi)\big)\Big],$$

where π and ξ are the canonical projections:

$$P \xleftarrow{\pi} P \oplus Q \xrightarrow{\xi} Q$$

This definition extends F, because for free M, we may take the free resolution:

$$0 \xrightarrow{0} M \longrightarrow M \longrightarrow 0$$

with $\pi = 1_M$ and $\xi = 0$, so that

$$\begin{aligned}\overline{F}(M) &= F(M)\Big/\Big[F(\pi)\big(\operatorname{Ker} F(\pi + \psi\xi)\big)\Big] = F(M)\Big/\Big[F(1_M)(\operatorname{Ker} F(1_M))\Big] \\ &= F(M)\Big/\Big[1_{F(M)}(\operatorname{Ker} 1_{F(M)})\Big] = F(M)/0 \cong F(M).\end{aligned}$$

2. The Cross-Effects

An arbitrary module functor may be analysed in terms of its cross-effects. These may be defined as either of four modules, neither more canonical than the others. Given a direct sum $M = M_1 \oplus \cdots \oplus M_n$, let

$$\pi_j \colon M \to M$$

be projection on the jth summand,

$$\varrho_j \colon M \to M/M_j$$

retraction from the jth summand, and

$$\tau_j \colon M/M_j \to M$$

insertion of 0 into the jth summand. We then have:

Theorem 2. *For a module functor F, the following four modules are naturally isomorphic:*

A. $\operatorname{Im}\big[F(\pi_1 \diamond \cdots \diamond \pi_n) \colon F(M) \to F(M)\big]$.

B. $\operatorname{Ker}\big[(F(\varrho_1), \ldots, F(\varrho_n)) \colon F(M) \to \bigoplus F(M/M_j)\big]$.

C. $\operatorname{Coker}\big[F(\tau_1) + \cdots + F(\tau_n) \colon \bigoplus F(M/M_j) \to F(M)\big]$.

D. $\operatorname{Coim}\big[F(\pi_1 \diamond \cdots \diamond \pi_n) \colon F(M) \to F(M)\big]$.

Proof. We only show the modules in A and B to be equal, and leave the rest to the reader.

Suppose $z \in \operatorname{Ker}(F(\varrho_1), \dots, F(\varrho_n))$. Note that if $j \neq i$, then $\pi_i \varrho_j = \pi_i$, and consequently, if $j \notin I$, then

$$F\left(\sum_{i \in I} \pi_i\right)(z) = F\left(\sum_{i \in I} \pi_i\right) F(\varrho_j)(z) = 0.$$

It follows that

$$\begin{aligned} F(\pi_1 \diamond \cdots \diamond \pi_n)(z) &= \sum_{I \subseteq [n]} (-1)^{n-|I|} F\left(\sum_{i \in I} \pi_i\right)(z) \\ &= F\left(\pi_1 + \cdots + \pi_n\right)(z) = F(1)(z) = z. \end{aligned}$$

Conversely, assume $z \in \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n)$, so that $z = F(\pi_1 \diamond \cdots \diamond \pi_n)(y)$. Then, since

$$\varrho_j \pi_i = \begin{cases} \pi_i & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

we get

$$\begin{aligned} F(\varrho_j)(z) = F(\varrho_j) F(\pi_1 \diamond \cdots \diamond \pi_n)(y) &= \sum_{I \subseteq [n]} (-1)^{n-|I|} F\left(\varrho_j \sum_{i \in I} \pi_i\right)(y) \\ = \sum_{I \subseteq [n]} (-1)^{n-|I|} F\left(\sum_{i \in I \setminus \{j\}} \pi_i\right)(y) = 0, \end{aligned}$$

because sets I with and without j will cancel each other out. □

Definition 2. We define the nth **cross-effect** of F as the multifunctor

$$F^{\dagger}\left(M_1 | \dots | M_n\right) = \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n)$$

of n arguments (it could be defined as any of the four modules above). We shall use the short-hand notation

$$F^{\dagger}(M_i|_{i \in I})$$

for the $|I|$'th cross-effect of the modules M_i. ◇

In each of the four cases above, it is implicit how the resulting cross-effect functor will act on arrows. For example, if the cross-effect is viewed as $\operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n)$, then for given $\alpha_j \colon M_j \to M'_j$, the following diagram will commute:

$$\begin{array}{ccc} F(\bigoplus M_i) & \xrightarrow{F(\bigoplus \alpha_i)} & F(\bigoplus M'_i) \\ {\scriptstyle F(\diamond\, \iota_i \pi_i)} \downarrow & & \downarrow {\scriptstyle F(\diamond\, \iota'_i \pi'_i)} \\ F(\bigoplus M_i) & \xrightarrow[F(\bigoplus \alpha_i)]{} & F(\bigoplus M'_i) \end{array}$$

Therefore, there will be an induced map

$$F^{\dagger}\left(\alpha_1|\ldots|\alpha_n\right) :\ \operatorname{Im} F\left(\Diamond\, \iota_i \pi_i\right) \to \operatorname{Im} F\left(\Diamond\, \iota_i' \pi_i'\right).$$

Similar arguments may be constructed for the other three possibilities.

Theorem 3: The Cross-Effect Decomposition.

$$F(M_1 \oplus \cdots \oplus M_n) = \bigoplus_{I \subseteq [n]} F^{\dagger}(M_i|_{i \in I}).$$

Proof. See [6]. □

3. Polynomial Functors

We now turn to interpreting our three notions of polynomiality, in order from the weakest to the strongest. We begin with plain polynomiality, of which the defining property is classically taken as the vanishing of the cross-effects.

Definition 3. The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **polynomial** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is. ◇

Examples will be found later on, as all numerical and strict polynomial functors are also polynomial.

Theorem 4. *F is polynomial of degree n iff its $(n+1)$st cross-effect vanishes.*

Proof. Suppose the $(n+1)$st cross-effect vanishes and consider $n+1$ maps $\alpha_j\colon M \to N$. Create $n+1$ modules $M_j = M$ and $n+1$ modules $N_j = N$, let

$$\pi_j\colon \bigoplus N_i \to N_j, \qquad \iota_j\colon N_j \to \bigoplus N_i$$

denote the jth projection and inclusion, respectively, and define

$$\sigma\colon \bigoplus N_i \to N, \quad (y_1, \ldots, y_{n+1}) \mapsto \sum y_i.$$

The following equality is easily checked:

$$\begin{aligned} &F(N) \leftarrow F\left(\bigoplus N_i\right) \leftarrow F\left(\bigoplus N_i\right) \leftarrow F(M)\colon \\ &F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1}) = F(\sigma) \circ F(\iota_1\pi_1 \diamond \cdots \diamond \iota_{n+1}\pi_{n+1}) \circ F((\alpha_1, \ldots, \alpha_{n+1})) \end{aligned}$$

But the middle component is zero by assumption, and we are done.

The converse is trivial. □

4. Numerical Functors

We now assume a numerical base ring.

Definition 4. The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **numerical** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. $\diamond$

Note the inconspicuous assumption on uniformly bounded degree of the arrow maps. We shall presently see what happens when this assumption is dropped.

Also note that, over the base ring $\mathbf{Z}$, the notions of polynomial and numerical functor coincide.

Example 1. The numerical functors F of degree 0 are the constant ones:

$$F(M) = K.$$

The functors of degree 1 are those of the form

$$F(M) = K \oplus E(M),$$

where K is a fixed module and E is R-linear. $\triangle$

Example 2. The tensor power $T^n(M)$, the symmetric power $S^n(M)$, the exterior power $\Lambda^n(M)$, and the divided power $\Gamma^n(M)$ are all nth degree functors. $\triangle$

A **natural transformation** $\eta\colon F \to G$ of numerical functors is a family

$$\eta = (\eta_M : F(M) \to G(M) \mid M \in \mathfrak{XMod})$$

such that for any modules M and N, any numerical algebra A, and any $\omega \in A \otimes \operatorname{Hom}(M,N)$, the following diagram commutes:

$$\begin{array}{ccc} A \otimes F(M) & \xrightarrow{1\otimes\eta_M} & A \otimes G(M) \\ {\scriptstyle F(\omega)}\downarrow & & \downarrow{\scriptstyle G(\omega)} \\ A \otimes F(N) & \xrightarrow[1\otimes\eta_N]{} & A \otimes G(N) \end{array} \tag{3.1}$$

We let $\mathfrak{Num}_n$ be the category whose objects are numerical functors of degree (at most) n, with arrows natural transformations. It is easy to see that it is abelian (the case $R = \mathbf{Z}$ is well known). It is also closed under direct sums, and we will see in Chapter 4 that it possesses a compact progenerator. By Morita equivalence, it is equivalent to a module category.

5. Properties of Numerical Functors

Let us hasten to point out, that our definition of natural transformation is unnecessarily complicated. A consequence of Theorem 8 of Chapter 2 is that a polynomial functor is uniquely determined by its underlying functor. In view of this, the following theorem is hardly surprising. The reason for adopting the more complicated condition as definition, is to conform to the situation for strict polynomial functors. These, it may be recalled, are not determined by their underlying functors.

Theorem 5. *The diagram (3.1) commutes for any natural transformation $\eta\colon F \to G$.*

Proof. Consider homomorphisms $\alpha_1, \dots, \alpha_k\colon M \to N$. Assume

$$F(a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k) = \sum_{\mu} \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} \otimes \beta_\mu$$

$$G(a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k) = \sum_{\nu} \binom{a_1}{n_1} \cdots \binom{a_k}{n_k} \otimes \gamma_\nu,$$

for any $a_1, \dots, a_k$ in any numerical algebra A, where we denote $\mu = (m_1, \dots, m_k)$ and $\nu = (n_1, \dots, n_k)$. The naturality of η ensures that

$$\sum_{\mu} \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} \eta_N \beta_\mu = \sum_{\nu} \binom{a_1}{n_1} \cdots \binom{a_k}{n_k} \gamma_\nu \eta_M.$$

Specialize first to the case $a_2 = a_3 = \cdots = 0$, to obtain

$$\sum_{m_1} \binom{a_1}{m_1} \eta_N \beta_{m_1 0 \dots} = \sum_{n_1} \binom{a_1}{n_1} \gamma_{n_1 0 \dots} \eta_M.$$

By successively letting $a_1 = 0, 1, 2, \dots$, it will be seen that

$$\eta_N \beta_{(m_1, 0, \dots)} = \gamma_{(m_1, 0, \dots)} \eta_M$$

for all m_1. It is now easy to show inductively, that

$$\eta_N \beta_\mu = \gamma_\mu \eta_M$$

for all μ. The commutativity of the diagram (3.1), for

$$\omega = a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k,$$

is then demonstrated by the following instantiation:

$$\begin{array}{ccc}
b \otimes x & \longrightarrow & b \otimes \eta_M(x) \\
\downarrow & & \downarrow \\
\sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \beta_\mu(x) & \longrightarrow & \left[\begin{array}{l} \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \eta_N \beta_\mu(x) \\ = \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \gamma_\mu \eta_M(x) \end{array}\right]
\end{array}$$

□

Theorem 6. *The following conditions are equivalent on a polynomial functor F of degree n.*

A.

$$F(r\alpha) = \sum_{k=0}^{n} \binom{r}{k} F\left(\underset{k}{\Diamond}\, \alpha\right),$$

for any homomorphism α and $r \in R$ (numerical functor).

B.

$$F(r\alpha) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m\alpha),$$

for any homomorphism α and $r \in R$.

A′.

$$F(r \cdot 1_{R^n}) = \sum_{k=0}^{n} \binom{r}{k} F\left(\underset{k}{\Diamond}\, 1_{R^n}\right),$$

for $r \in R$.

B′.

$$F(r \cdot 1_{R^n}) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m \cdot 1_{R^n}),$$

for $r \in R$.

Proof. That A and B are equivalent follows from Theorem 2 of Chapter 2, as does the equivalence of A′ with B′. Clearly B implies B′, so there remains to establish that B′ implies B. Hence assume B′.

If $q \leq n$, the equation

$$F(r \cdot 1_{R^q}) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m \cdot 1_{R^q})$$

holds, because 1_{R^q} factors through 1_{R^n}. Putting

$$Z(m) = (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m},$$

we calculate for $q > n$

$$\begin{aligned} F(r \cdot 1_{R^q}) &= F(r\pi_1 + \cdots + r\pi_q) \\ &= -\sum_{I \subset [q]} (-1)^{q-|I|} F\left(\sum_{i \in I} r\pi_i\right) \\ &= -\sum_{I \subset [q]} (-1)^{q-|I|} \sum_{m=0}^{n} Z(m) F\left(\sum_{i \in I} m\pi_i\right) \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{m=0}^{n} Z(m) \sum_{I\subset[q]} (-1)^{q-|I|} F\left(\sum_{i\in I} m\pi_i\right) \\
&= \sum_{m=0}^{n} Z(m) F(m\pi_1 + \cdots + m\pi_q) = \sum_{m=0}^{n} Z(m) F(m \cdot 1_{R^q}).
\end{aligned}
$$

The third and sixth steps are because the qth deviation vanishes, and the fourth step is by induction (on q). Finally, the equation will also hold for an arbitrary homomorphism $\alpha\colon R^p \to R^q$, because

$$
\begin{aligned}
F(r\alpha) &= F(r \cdot 1_{R^q}) F(\alpha) \\
&= \sum_{m=0}^{n} Z(m) F(m \cdot 1_{R^q}) F(\alpha) \\
&= \sum_{m=0}^{n} Z(m) F(m\alpha).
\end{aligned}
$$

□

Theorem 7. *The module functor F is numerical of degree n iff for any r_1, $\ldots, r_k \in R$ and homomorphisms $\alpha_1, \ldots, \alpha_k$, the following equation holds:*

$$
F(r_1\alpha_1 \diamond \cdots \diamond r_k\alpha_k) = \sum_{\substack{\#S=[k] \\ |S|\le n}} \prod_{j\in\#S} \binom{r_j}{\deg j} F\left(\Diamond_{j\in S} \alpha_j\right),
$$

where the sum is taken over multisets S.

Proof. Theorem 6 of Chapter 2. □

6. The Hierarchy of Numerical Functors

We shall say that a map f, or a family of such, is **multiplicative** if

$$
f(z)f(w) = f(zw)
$$

whenever z and w are entities («quelques choses») such that the equation makes sense, and also

$$
f(1) = 1,
$$

where the symbol 1 is to be interpreted in a natural way. An ordinary functor is by definition multiplicative.

Also, we say that a family of maps is polynomial of **bounded degree**, if every map is numerical of some fixed degree n.

Theorem 8. *Consider the following constructs, where A ranges over all numerical algebras:*

A. A family of ordinary functors $E_A\colon {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$*, commuting with extension of scalars.*

B. A functor $J\colon \mathfrak{XMod} \to \mathfrak{Mod}$ *of which the arrow functions are multiplicative maps*

$$J_A\colon \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N)),$$

natural in A.

C. A functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ *of which the arrow functions are multiplicative maps*

$$F_A\colon A \otimes \operatorname{Hom}_R(M, N) \to A \otimes \operatorname{Hom}_R(F(M), F(N)),$$

natural in A (numerical maps).

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow functions are assumed to have uniformly bounded degree, all three are equivalent.

Proof. Given E, define J by

$$J(M) = E_R(M)$$

and the following diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_A(A \otimes M, A \otimes N) & \xrightarrow{E_A} & \operatorname{Hom}_A(E_A(A \otimes M), E_A(A \otimes N)) \\
 & \searrow^{J} \;(\text{dotted}) & \updownarrow \\
 & & \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N))
\end{array}$$

The properties required of J are immediate.

Conversely, given J, define the E by the equation

$$E_A(A \otimes M) = A \otimes J(M)$$

and the diagram:

$$\begin{array}{ccc}
 & & \operatorname{Hom}_A(E_A(A \otimes M), E_A(A \otimes N)) \\
 & \nearrow^{E_A} \;(\text{dotted}) & \Vert \\
\operatorname{Hom}_A(A \otimes M, A \otimes N) & \xrightarrow[J]{} & \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N))
\end{array}$$

Also, it is easy to define J from F, using the following diagram:

$$\begin{array}{ccc}
A \otimes \operatorname{Hom}_R(M, N) & \xrightarrow{F} & A \otimes \operatorname{Hom}_R(F(M), F(N)) \\
\updownarrow & & \downarrow \\
\operatorname{Hom}_A(A \otimes M, A \otimes N) & \xrightarrow[J]{\ \cdots\ } & \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N))
\end{array}$$

The left column in the diagram is an isomorphism as long as M and N are free.

So far the proofs have been completely straightforward, but we now turn to the more difficult procedure of defining F from J, modelled on the corresponding proof for strict polynomial functors in [15]. Given M and N, find a free resolution:

$$R^{(\lambda)} \longrightarrow R^{(\kappa)} \longrightarrow J(M) \longrightarrow 0$$

Apply the contravariant, left-exact functor $\mathrm{Hom}_A(A \otimes -, A \otimes J(N))$, where A is any numerical algebra:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N)) & \xrightarrow{\iota} & (A \otimes J(N))^{\kappa} & \xrightarrow{\sigma} & (A \otimes J(N))^{\lambda} \\
& \uparrow J & & \vdots\uparrow \zeta & & \\
& A \otimes \mathrm{Hom}(M, N) & \xrightarrow{\delta_n} & A \otimes R[\mathrm{Hom}(M, N)]_n & &
\end{array}$$

The homomorphism

$$\iota J \colon A \otimes \mathrm{Hom}(M, N) \to (A \otimes J(N))^{\kappa}$$

may be split up into components

$$(\iota J)_k \colon A \otimes \mathrm{Hom}(M, N) \to A \otimes J(N),$$

for each $k \in \kappa$. These are numerical of degree n, and will factor over δ_n via some linear ζ_k. Together these yield a linear map

$$\zeta \colon A \otimes R[\mathrm{Hom}(M, N)]_n \to (A \otimes J(N))^{\kappa}$$

making the above square commute.

Now, $\sigma\zeta\delta_n = \sigma\iota J = 0$, which gives $\sigma\zeta = 0$. Using the exactness of the upper row in the diagram, ζ factors via some

$$\xi \colon R[\mathrm{Hom}(M, N)]_n \to \mathrm{Hom}(J(M), J(N)),$$

and because of the injectivity of ι, also J will factor over δ_n. The following diagram will therefore commute:

$$\begin{array}{ccc}
\mathrm{Hom}(J(M), J(N)) & \xrightarrow{\iota} & J(N)^{(\kappa)} \\
\uparrow J & \nwarrow \xi & \uparrow \zeta \\
\mathrm{Hom}(M, N) & \xrightarrow{\delta_n} & R[\mathrm{Hom}(M, N)]_n
\end{array}$$

Because J factors over $R[\mathrm{Hom}(M, N)]_n$, it is numerical of degree n, and so may be used to construct the F above. □

We thus obtain the following hierarchy of functors:

> *Numerical functors* are required to satisfy all three conditions A, B and C, and to be of bounded degree.

A functor satisfying condition C, with no assumption on the degree, could rightly be called **locally numerical**, but this concept will not be used in the sequel.

A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **analytic**.

Example 3. The functors S, T, Γ and Λ are all analytic. Of these, only Λ is locally numerical. △

7. Strict Polynomial Functors

We now develop the theory for strict polynomial functors, to make it run in parallel with that of non-strict functors. The base ring R is no longer assumed numerical.

Definition 5. The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **strictly polynomial** of degree n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. ◇

Example 4. The functors T^n, S^n, Λ^n and Γ^n are in fact strict polynomial functors of degree n. △

By a **natural transformation** $\eta\colon F \to G$ of strict polynomial functors, we mean a family

$$\eta = (\eta_M : F(M) \to G(M) \mid M \in \mathfrak{XMod})$$

such that for any modules M and N, any algebra A, and any $\omega \in A \otimes \operatorname{Hom}(M,N)$, the following diagram commutes:

$$\begin{array}{ccc} A \otimes F(M) & \xrightarrow{1\otimes\eta_M} & A \otimes G(M) \\ \downarrow{\scriptstyle F(\omega)} & & \downarrow{\scriptstyle G(\omega)} \\ A \otimes F(N) & \xrightarrow[1\otimes\eta_N]{} & A \otimes G(N) \end{array}$$

We let $\mathfrak{SPol}_n$ be the category whose objects are strict polynomial functors of degree (at most) n, with arrows natural transformations. It is well known to be abelian.

It is clear that every strict polynomial functor is also numerical of the same degree.

8. The Hierarchy of Strict Polynomial Functors

As for numerical functors, we have the following three characterizations of strict polynomial functors.

Theorem 9. *Consider the following constructs, where A ranges over all algebras:*

A. *A family of ordinary functors $E_A\colon {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$, commuting with extension of scalars.*

B. *A functor $J\colon \mathfrak{XMod} \to \mathfrak{Mod}$ of which the arrow functions are multiplicative maps*

$$J_A\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N)),$$

natural in A.

C. *A functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ of which the arrow functions are multiplicative maps*

$$F_A\colon A \otimes \mathrm{Hom}_R(M, N) \to A \otimes \mathrm{Hom}_R(F(M), F(N)),$$

natural in A (strict polynomial maps).

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow functions are assumed to have uniformly bounded degree, all three are equivalent.

Proof. The proof is exactly analogous to the one given for polynomial functors, except that, in the proof that B implies C, the module

$$\bigoplus_{k=0}^{n} \Gamma^k \,\mathrm{Hom}(M, N)$$

is used in place of $R[\mathrm{Hom}(M, N)]_n$. The details are found in [15]. □

As in the numerical case, we obtain the following hierarchy:

Strict polynomial functors are required to satisfy all three conditions A, B and C, and to be of bounded degree.

A functor satisfying condition C, with no assumption on the degree, could be called **locally strict polynomial**, but this concept will not be used in the sequel.

A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **strictly analytic**.

9. Homogeneous Polynomial Functors

Rather than considering arbitrary strict polynomial functors, we shall from now on limit our attention to homogeneous ones.

Definition 6. The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **homogeneous** of degree n if every arrow map

$$F\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is. ◇

The subcategory of homogeneous functors will be denoted by $\mathfrak{HPol}_n$. It is abelian, and nothing essential will be lost by considering such functors only, as the following theorem shows. It is proved in [15].

Theorem 10. *A strict polynomial functor decomposes as a unique direct sum of homogeneous functors. The only possible natural transformation between homogeneous functors of different degrees is the zero transformation.*

Like $\mathfrak{Num}_n$, $\mathfrak{HPol}_n$ will shortly be proved to possess a compact progenerator and to be closed under direct sums, and hence be equivalent to a module category.

10. Analytic Functors

We here make a close examination of the analytic functors. We opt not to prove the first of these results, as it should be well known. The second result can likely be improved upon. It seems rather probable that the analytic functors are precisely the inductive limits of *numerical* functors.

Theorem 11. *The strict analytic functors are precisely the infinite direct sums (or, equivalently, inductive limits) of strict polynomial functors.*

Theorem 12. *Over a noetherian base ring, the analytic functors are precisely the inductive limits of locally numerical functors.*

Proof. Inductive limits of numerical, or even analytic, functors will clearly be analytic. For if the functors F_i, for $i \in I$, are analytic, then for any $\alpha \in \operatorname{Hom}_A(A \otimes M, A \otimes N)$, we have

$$F_i(\alpha)\colon A \otimes F_i(M) \to A \otimes F_i(N).$$

Therefore

$$\varinjlim F_i(\alpha)\colon A \otimes \varinjlim F_i(M) \to A \otimes \varinjlim F_i(N),$$

since tensor products commute with inductive limits, which yields a map

$$\varinjlim F_i\colon \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes \varinjlim F_i(M), A \otimes \varinjlim F_i(N)).$$

Suppose now conversely that F is analytic; the maps

$$F\colon \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes F(M), A \otimes F(N))$$

are then multiplicative and natural in A. To show F is the inductive limit of locally numerical functors, it is sufficient to construct, given a module P and an element $u \in F(P)$, a locally numerical subfunctor G of F such that u belongs to $G(P)$.

To this end, define the functor G by

$$G(M) = \langle F(\alpha)(u) \mid \alpha\colon P \to M \rangle\,,$$

and observe that the modules $G(M)$ are invariant under the action of F. Thus, G is indeed a subfunctor of F, and clearly $u \in G(P)$. To see that G is locally numerical, let $\{\epsilon_1, \dots, \epsilon_m\}$ be a basis for $\operatorname{Hom}(P, M)$. Let $A = R\binom{t_1,\dots,t_m}{-}$. Then

$$F\left(\sum t_k \otimes \epsilon_k\right) : A \otimes F(P) \to A \otimes F(M), \qquad 1 \otimes u \mapsto \sum \binom{t}{\mu} \otimes v_\mu,$$

for finitely many elements $v_\mu \in F(M)$. Specializing $t_k \mapsto a_k \in R$, we get

$$F\left(\sum a_k \epsilon_k\right) : F(P) \to F(M), \qquad u \mapsto \sum \binom{a}{\mu} v_\mu,$$

which shows

$$\begin{aligned} G(M) &= \langle F(\alpha)(u) \mid \alpha\colon P \to M \rangle \\ &= \left\langle F\left(\sum a_k \epsilon_k\right)(u) \;\middle|\; a_k \in R \right\rangle = \left\langle \sum \binom{a}{\mu} v_\mu \;\middle|\; a_k \in R \right\rangle \end{aligned}$$

is finitely generated. Since R is noetherian, $G(M)$ is also finitely presented. We have therefore the following commutative diagram, where the right column is an isomorphism, for any *flat* numerical algebra A:

$$\begin{array}{ccc} A \otimes \operatorname{Hom}(M, N) & \xrightarrow{\;G\;} & \operatorname{Hom}_A(A \otimes G(M), A \otimes G(N)) \\ & \searrow & \updownarrow \\ & & A \otimes \operatorname{Hom}(G(M), G(N)) \end{array}$$

The existence of the diagonal map for flat A is enough for G to be locally numerical. □

11. The Deviations

We shall here make a more detailed study of deviations in the context of functors. We introduce the notation

$$M \sqsubseteq X \times Y,$$

to denote that M is a subset of $X \times Y$, and that both the canonical projections are *onto*.

Lemma 1. *Let m and n be natural numbers, $L \subseteq [m] \times [n]$, and let $Y(m, n, k)$ denote the number of sets K of cardinality k satisfying*

$$L \subseteq K \sqsubseteq [m] \times [n].$$

Then

$$\sum_k (-1)^k Y(m, n, k) = 0,$$

unless L is of the form $P \times Q$, for $P \subseteq [m]$, $Q \subseteq [n]$.

Proof. If L is not of the given form, there exists an (a, b) which is not in L, but such that some (a, j) and some (i, b) are in L. Then, for any set $K \subseteq [m] \times [n]$ containing (a, b), K itself will satisfy the given set inclusions iff $K \setminus \{(a, b)\}$ does. Because the cardinalities of these sets differ by one, the corresponding terms in the above sum will have opposing signs, and hence cancel. □

Lemma 2. *Let m, n, p and q be natural numbers, and let $Y(m, n, k)$ denote the number of sets K of cardinality k satisfying*

$$[p] \times [q] \subseteq K \sqsubseteq [m] \times [n].$$

Then

$$\sum_k (-1)^k Y(m, n, k) = (-1)^{m+n+p+q+pq}.$$

Proof. The formula is evidently true for $m = p$ and $n = q$, for then $Y(p, q, pq) = 1$ and all other $Y(p, q, k) = 0$. We now do recursion. Consider the pair $(m, n) \in [m] \times [n]$. The sets K containing (m, n) will fall into two classes: those where (m, n) is *mandatory* in order to satisfy $K \sqsubseteq [m] \times [n]$, and those where it is not. For the latter class we may proceed as in the preceding proof: Taking such a K and removing (m, n) will yield another set counted in the sum above, but of cardinality decreased by one. Since these two types of sets exactly pair off, with opposing signs, their contribution to the given sum is zero.

Consider then those K of which (m, n) *is* a mandatory element. They fall into three categories:

- Some $(m, j) \in K$, for $1 \leq j \leq n - 1$, but no $(i, n) \in K$, for $1 \leq i \leq m - 1$. The number of such sets is $Y(m, n - 1, k - 1)$.
- No $(m, j) \in K$, for $1 \leq j \leq n - 1$, but some $(i, n) \in K$, for $1 \leq i \leq m - 1$. The number of such sets is $Y(m - 1, n, k - 1)$.
- No $(m, j) \in K$, for $1 \leq j \leq n - 1$, and no $(i, n) \in K$, for $1 \leq i \leq m - 1$. The number of such sets is $Y(m - 1, n - 1, k - 1)$.

Assuming the proposed formula is valid for lesser values of m and n, we calculate by induction:

$$\sum_k (-1)^k Y(m, n, k)$$

$$
\begin{aligned}
&= \sum_k (-1)^k \big(Y(m, n-1, k-1) + Y(m-1, n, k-1) + Y(m-1, n-1, k-1)\big) \\
&= -\big((-1)^{m+n-1+p+q+pq} + (-1)^{m-1+n+p+q+pq} + (-1)^{m-1+n-1+p+q+pq}\big) \\
&= (-1)^{m+n+p+q+pq},
\end{aligned}
$$

as desired. ☐

With these combinatorial prerequisites, we may state and prove our main result on deviations.

Theorem 13: The Deviation Formula. *For a module functor F, and homomorphisms $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$,*

$$
F(\alpha_1 \diamond \cdots \diamond \alpha_m) F(\beta_1 \diamond \cdots \diamond \beta_n) = \sum_{K \sqsubseteq [m]\times[n]} F\left(\mathop{\Diamond}_{(i,j)\in K} \alpha_i \beta_j \right).
$$

Proof. We have

$$
\begin{aligned}
\sum_{K \sqsubseteq [m]\times[n]} F\left(\mathop{\Diamond}_{(i,j)\in K} \alpha_i \beta_j \right) &= \sum_{K \sqsubseteq [m]\times[n]} \sum_{L \subseteq K} (-1)^{|K|-|L|} F\left(\sum_{(i,j)\in L} \alpha_i \beta_j \right) \\
&= \sum_{L \subseteq [m]\times[n]} \sum_{L \subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|-|L|} F\left(\sum_{(i,j)\in L} \alpha_i \beta_j \right) \\
&= \sum_{L \subseteq [m]\times[n]} (-1)^{|L|} F\left(\sum_{(i,j)\in L} \alpha_i \beta_j \right) \sum_{L \subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|} \\
&= \sum_{P\times Q \subseteq [m]\times[n]} (-1)^{|P||Q|} F\left(\sum_{(i,j)\in P\times Q} \alpha_i \beta_j \right) (-1)^{m+n+|P|+|Q|+|P||Q|} \\
&= \sum_{P \subseteq [m]} (-1)^{m-|P|} F\left(\sum_{i\in P} \alpha_i \right) \sum_{Q \subseteq [n]} (-1)^{n-|Q|} F\left(\sum_{j\in Q} \beta_j \right) \\
&= F(\alpha_1 \diamond \cdots \diamond \alpha_m) F(\beta_1 \diamond \cdots \diamond \beta_n),
\end{aligned}
$$

where in the fifth step the lemmata were used to evaluate the inner sum. ☐

12. The Multicross-Effects

Given a direct sum $M = M_1 \oplus \cdots \oplus M_n$, let $\pi_i \colon M \to M$ denote the ith projection. Recall that the cross-effects of a module functor F are given by the formula

$$
F^\dagger(M_i|_{i\in I}) = \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n),
$$

for $I \subseteq [n]$.

The cross-effects of a strict polynomial functor may in fact be dissected further into so called *multicross-effects*. These are described using the language of multisets, which are formally introduced in Chapter 6.

Let F be a strict polynomial functor and let $\alpha_i\colon M \to N$ be homomorphisms. We recall that the maps $F_{\alpha^{[\mu]}}\colon F(M) \to F(N)$, for multi-indices μ, are defined by the universal validity of the equation

$$F\left(\sum a_i \otimes \alpha_i\right) = \sum a^\mu \otimes F_{\alpha^{[\mu]}}.$$

Definition 7. Let A be a multiset with $|A| = n$ and $\#A = [n]$. We define the **multicross-effect** of F of type A to be the multifunctor

$$F^\dagger_A\left(M_1|\ldots|M_n\right) = \operatorname{Im} F_{\pi^{[A]}}$$

of n arguments. ◇

Theorem 14: The Multicross-Effect Decomposition. *For F a strict polynomial (or strict analytic) functor,*

$$F^\dagger\left(M_1|\ldots|M_n\right) = \bigoplus_{\substack{\#A=[n]\\|A|=n}} F^\dagger_A\left(M_1|\ldots|M_n\right),$$

and consequently,

$$F(M_1 \oplus \cdots \oplus M_n) = \bigoplus_{\substack{\#A\subseteq[n]\\|A|=n}} F^\dagger_A(M_a|_{a\in\#A}).$$

Proof. The equation defining the $F_{\pi^{[\mu]}}$ is

$$F\left(\sum a_i \otimes \pi_i\right) = \sum a^\mu \otimes F_{\pi^{[\mu]}},$$

from which it immediately follows that

$$1 = F(1) = F\left(\sum \pi_i\right) = \sum F_{\pi^{[\mu]}}.$$

Furthermore, the equation

$$\begin{aligned}\sum a^\mu b^\nu \otimes F_{\pi^{[\mu]}} F_{\pi^{[\nu]}} &= F\left(\sum a_i \otimes \pi_i\right) F\left(\sum b_j \otimes \pi_j\right)\\ &= F\left(\sum a_k b_k \otimes \pi_k\right) = \sum (ab)^\lambda \otimes F_{\pi^{[\lambda]}}\end{aligned}$$

shows that $F_{\pi^{[\mu]}} F_{\pi^{[\nu]}} = 0$ whenever $\mu \neq \nu$, and also that $F^2_{\pi^{[\mu]}} = F_{\pi^{[\mu]}}$. Consequently, the images of the maps $F_{\pi^{[\mu]}}$ form a direct sum decomposition. □

Note in particular the following special case:

$$F(R^n) = \bigoplus_{\substack{\#A \subseteq [n] \\ |A|=n}} F_A^\dagger(R|_{a \in \#A}),$$

which we choose to write more succinctly as

$$F(R^n) = \bigoplus_{\substack{\#A \subseteq [n] \\ |A|=n}} F_A^\dagger(R^n),$$

CHAPTER 4

Module Representations

Et la glace où se fige un réel mouvement
Reste froide malgré son détestable ouvrage.
La force du miroir trompa plus d'un amant
Qui crut aimer sa belle et n'aima qu'un mirage.

— Guillaume Apollinaire, *La Force du Miroir*

T. I. Pirashvili, [13], showed in 1988 how polynomial functors may be viewed as modules. Fifteen years later, Ekedahl and Salomonsson, [15], came to realize that also strict polynomial functors admit a module interpretation. In this chapter we describe these two module categories.

As before, we assume a fixed base ring R, which is assumed to be numerical when dealing with numerical functors.

1. The Fundamental Numerical Functor

Two numerical functors of supreme importance are the following.

Theorem 1. *The functor $R[-]_n$, given by*

$$M \mapsto R[M]_n$$
$$\big[\chi\colon M \to N\big] \mapsto \begin{bmatrix} [\chi]\colon R[M]_n \to R[N]_n \\ [x] \mapsto [\chi(x)] \end{bmatrix}$$

is numerical of degree n, as is the functor $R[\operatorname{Hom}(K,-)]_n$ for a fixed module K.

Proof. The first functor is immediately seen to be of degree n, for if $\chi_j\colon M \to N$, and $x \in M$, then

$$[\chi_1 \diamond \cdots \diamond \chi_{n+1}]([x]) = [\chi_1(x) \diamond \cdots \diamond \chi_{n+1}(x)] = 0;$$

and if $a \in R$ and $\chi\colon M \to N$, then

$$[a\chi]([x]) = [a\chi(x)] = \sum_{k=0}^{n} \binom{a}{k} \Big[\underset{k}{\Diamond}\, \chi(x)\Big] = \sum_{k=0}^{n} \binom{a}{k} \Big[\underset{k}{\Diamond}\, \chi\Big]([x]).$$

The second functor is the composition of an nth degree functor with a linear one and is therefore also of degree n. ◻

The latter functor $R[\mathrm{Hom}(K,-)]_n$ above takes

$$M \mapsto R[\mathrm{Hom}(K,M)]_n$$
$$\big[\chi\colon M \to N\big] \mapsto \begin{bmatrix} [\chi_*]\colon R[\mathrm{Hom}(K,M)]_n \to R[\mathrm{Hom}(K,N)]_n \\ [\alpha] \mapsto [\chi \circ \alpha] \end{bmatrix}$$

and certainly deserves to be called the **fundamental numerical functor** of degree n, for reasons that will presently be made clear.

Example 1. As an example of a functor which is polynomial, but not numerical, of degree 1, let the base ring be $\mathbf{R}$, and define for real vector spaces

$$F\colon {}_{\mathbf{R}}\mathfrak{Mod} \to {}_{\mathbf{R}}\mathfrak{Mod}, \quad V \mapsto \mathbf{R}[V]/\left\langle [x+y]-[x]-[y]\right\rangle.$$

Clearly F is additive, its first deviation vanishes, and therefore also its second cross-effect. But F is not numerical (of any degree), for

$$F(\sqrt{2}\colon \mathbf{R} \to \mathbf{R})\colon [1] \mapsto [\sqrt{2}]$$
$$\sqrt{2}F(1\colon \mathbf{R} \to \mathbf{R})\colon [1] \mapsto \sqrt{2}[1],$$

and these are not equal in

$$F(\mathbf{R}) = \mathbf{R}[\mathbf{R}]/\left\langle [x+y]-[x]-[y]\right\rangle.$$

In fact, F is not numerical of any degree, as it is impossible to express $F(\sqrt{2})$ as a linear combination of $F(0), F(1), \ldots$. △

2. Yoneda Correspondence for Numerical Functors

The functors $R[\mathrm{Hom}(K,-)]_n$ just introduced are to numerical functors what the Hom-functors are to ordinary functors, in that we have the following Yoneda Lemma for natural transformations between $R[\mathrm{Hom}(K,-)]_n$ and an arbitrary nth degree functor F:

Theorem 2: The Numerical Yoneda Lemma. *Let K be a fixed module and F a numerical functor of degree n. The map*

$$\Upsilon\colon \mathrm{Nat}\big(R[\mathrm{Hom}(K,-)]_n, F\big) \to F(K)$$
$$\eta \mapsto \eta_K([1_K])$$

is an isomorphism of modules.

Proof. The proof is the usual one. Consider the commutative diagram:

$$\begin{array}{ccccccc} K & & R[\mathrm{Hom}(K,K)]_n & \xrightarrow{\eta_K} & F(K) & \quad [1_K] & \longrightarrow & \eta_K([1_K]) \\ \downarrow \alpha & & \downarrow [\alpha_*] & & \downarrow F(\alpha) & \quad \downarrow & & \downarrow \\ M & & R[\mathrm{Hom}(K,M)]_n & \xrightarrow{\eta_M} & F(M) & \quad [\alpha] & \longrightarrow & \eta_M([\alpha]) = F(\alpha)(\eta_K([1_K])) \end{array}$$

Upon inspection, we find that Υ has the inverse

$$y \mapsto \begin{bmatrix} \eta_M \colon R[\mathrm{Hom}(K,M)]_n \to F(M) \\ [\alpha] \mapsto F(\alpha)(y) \end{bmatrix}.$$

Here we use the numericality of F to ensure that the map

$$\mathrm{Hom}(K,M) \to \mathrm{Hom}(F(K),F(M))$$

will factor through $R[\mathrm{Hom}(K,M)]_n$. ☐

In particular, we have a module isomorphism

$$\mathrm{Nat}(R[\mathrm{Hom}(K,-)]_n) \cong R[\mathrm{Hom}(K,K)]_n = R[\mathrm{End}\,K]_n,$$

given by the map

$$\eta \mapsto \eta_K([1_K])$$

with inverse

$$[\sigma] \mapsto \begin{bmatrix} \eta_M \colon R[\mathrm{Hom}(K,M)]_n \to R[\mathrm{Hom}(K,M)]_n \\ [\alpha] \mapsto [\alpha \circ \sigma]. \end{bmatrix}.$$

We recall that $R[\mathrm{End}\,K]$ and its quotients $R[\mathrm{End}\,K]_n$ feature two distinct multiplications, namely the sum multiplication $[\sigma][\tau] = [\sigma + \tau]$ and the product multiplication $[\sigma][\tau] = [\tau\sigma]$. The Yoneda map does not respect the former in any way, but it will reverse the latter.

Theorem 3. *Under the Yoneda correspondence, the rings*

$$\big(\mathrm{End}\,R[\mathrm{Hom}(K,-)]_n\big)^{\circ} \cong R[\mathrm{End}\,K]_n,$$

where the latter is equipped with the product multiplication.

3. Morita Equivalence for Numerical Functors

We shall now specialize the fundamental functor to the case $K = R^n$. But first, a preliminary lemma:

Lemma 1. *A polynomial nth degree functor that vanishes on R^n is identically zero.*

Proof. For $q \le n$, 1_{R^q} factors via R^n, so that $1_{F(R^q)} = F(1_{R^q}) = 0$ factors via $F(R^n) = 0$.

Now proceed by induction and suppose $F(R^{q-1}) = 0$ for some $q \ge n+1$. Decompose $1_{R^q} = \iota_1\pi_1 + \cdots + \iota_q\pi_q$, where $\pi_j \colon R^q \to R$ denotes the jth projection and $\iota_j \colon R \to R^q$ the jth inclusion. Since F is of degree $q-1$,

$$0 = F(\iota_1\pi_1 \diamond \cdots \diamond \iota_q\pi_q) = \sum_{X \subseteq \{\iota_1\pi_1,\ldots,\iota_q\pi_q\}} (-1)^{q-|X|} F\left(\sum_X \iota_j\pi_j\right).$$

If $|X| \le q-1$, $F\left(\sum_X \iota_j\pi_j\right) = 0$, since $\sum_X \iota_j\pi_j$ factors via R^{q-1} and we assumed $F(R^{q-1}) = 0$. The only remaining term in the sum above is then $0 = F(\iota_1\pi_1 + \cdots + \iota_q\pi_q) = F(1_{R^q}) = 1_{F(R^q)}$. ☐

Theorem 4. *$R[\mathrm{Hom}(R^n,-)]_n$ is a compact progenerator*[1] *for $\mathfrak{Num}_n$, through which there is a Morita equivalence:*

$$\mathfrak{Num}_n \sim {}_{R[R^{n\times n}]_n}\mathfrak{Mod},$$

where $R[R^{n\times n}]_n$ carries the product multiplication.

More precisely, the functor F corresponds to the abelian group $F(R^n)$, with module structure given by the equation

$$[s]x = F(s)(x).$$

Proof. To show $R[\mathrm{Hom}(R^n,-)]_n$ is projective, we must show that

$$P = \mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n,-)$$

is right-exact, or preserves epimorphisms. Hence let $\eta\colon F \to G$ be epic, so that each η_M is onto. From the following diagram, constructed by aid of the Yoneda Lemma, it follows that η_* is epic:

$$\begin{array}{ccc}
\mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n,F) & \xrightarrow{\eta_*} & \mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n,G) \\
\Upsilon\big\updownarrow & & \big\updownarrow\Upsilon \\
F(R^n) & \xrightarrow{\quad\eta_{R^n}\quad} & G(R^n)
\end{array}$$

To show $R[\mathrm{Hom}(R^n,-)]_n$ is a generator, we use the lemma.

$$0 = \mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n,F) \cong F(R^n)$$

implies $F=0$, so P fails to kill non-zero objects.

Compactness of $R[\mathrm{Hom}(R^n,-)]_n$ follows from the computation

$$\begin{aligned}
\mathrm{Nat}\left(R[\mathrm{Hom}(R^n,-)]_n,\bigoplus F_k\right) &\cong \left(\bigoplus F_k\right)(R^n) = \bigoplus F_k(R^n) \\
&\cong \bigoplus \mathrm{Nat}\left(R[\mathrm{Hom}(R^n,-)]_n,F_k\right),
\end{aligned}$$

again using the Yoneda Lemma (twice).

As $\mathfrak{Num}_n$ is closed under direct sums, we have a Morita equivalence:

$$\mathfrak{Num}_n \;\underset{R[\mathrm{Hom}(R^n,-)]_n\otimes-}{\overset{\mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n,-)}{\rightleftarrows}}\; {}_S\mathfrak{Mod}$$

where $S = \left(\mathrm{Nat}\, R[\mathrm{Hom}(R^n,-)]_n\right)^\circ \cong R[\mathrm{End}\, R^n]_n = R[R^{n\times n}]_n$.

[1] A *progenerator* of an abelian category is a projective generator. It is *compact* when the corresponding Hom-functor commutes with arbitrary direct sums.

To prove the last assertion of the theorem, we first note that F corresponds to

$$\mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n, F) \cong F(R^n),$$

again by the celebrated Yoneda Lemma. We now investigate on the module structure on $F(R^n)$. Under the Yoneda map, an element $x \in F(R^n)$ will correspond to the natural transformation

$$\begin{bmatrix} \eta\colon R[\mathrm{Hom}(R^n,-)]_n \to F \\ \eta_M\colon R[\mathrm{Hom}(R^n,M)]_n \to F(M) \\ [\alpha] \mapsto F(\alpha)(x) \end{bmatrix},$$

extended by linearity. Likewise, $[s] \in R[R^{n\times n}]_n$ will correspond to

$$\begin{bmatrix} \sigma\colon R[\mathrm{Hom}(R^n,-)]_n \to R[\mathrm{Hom}(R^n,-)]_n \\ \sigma_M\colon R[\mathrm{Hom}(R^n,M)]_n \to R[\mathrm{Hom}(R^n,M)]_n \\ [\alpha] \mapsto [\alpha \circ s] \end{bmatrix},$$

again extended by linearity. Multiplying (the scalar) σ with η in the module

$$\mathrm{Nat}(R[\mathrm{Hom}(R^n,-)]_n, F)$$

gives as product the transformation

$$\begin{bmatrix} \eta\circ\sigma\colon R[\mathrm{Hom}(R^n,-)]_n \to F \\ (\eta\circ\sigma)_M\colon R[\mathrm{Hom}(R^n,M)]_n \to F(M) \\ [\alpha] \mapsto F(\alpha\circ s)(x) \end{bmatrix},$$

which under the Yoneda map corresponds to

$$(\eta\circ\sigma)_{R^n}([1_{R^n}]) = F(1_{R^n} \circ s)(x) = F(s)(x)$$

in $F(R^n)$. The scalar multiplication on $F(R^n)$ is therefore given by the formula

$$[s]x = F(s)(x),$$

and the proof is finished. □

Example 2. Consider the constant functor $C\colon R^k \mapsto R$ and the identity functor $I\colon R^k \mapsto R^k$. They are both of the first degree (of course, C is in fact of degree zero), which means they will both under the Morita equivalence correspond to the abelian group $C(R) = I(R) = R$. Their module structures over $R[R]_1 = \langle [0_R], [1_R] \rangle$ will differ, however. For C, the scalar multiplication is given by

$$\begin{aligned}(a[0_R] + b[1_R])x &= C(a[0_R] + b[1_R])(x) = aC(0_R)(x) + bC(1_R)(x) \\ &= a\,1_R(x) + b\,1_R(x) = (a+b)x,\end{aligned}$$

whereas for I the action is

$$\begin{aligned}(a[0_R] + b[1_R])x &= I(a[0_R] + b[1_R])(x) = aI(0_R)(x) + bI(1_R)(x) \\ &= a0_R(x) + b\,1_R(x) = bx.\end{aligned}$$

△

4. The Fundamental Homogeneous Functor

With modifications, the above theory for numerical functors will have a strictly polynomial counterpart. The appropriate progenerator will of course no longer be $R[\mathrm{Hom}(R^n,-)]_n$, but will involve the divided power functor Γ^n.

The strict polynomiality of Γ^n is of course an immediate consequence of the fact that it commutes with extension of scalars:

$$\Gamma^n(A \otimes M) = A \otimes \Gamma^n(M).$$

We would, however, like to examine its behaviour a little more closely. In order to do so, we define a representation

$$\Gamma^n \operatorname{Hom}(M,N) \to \operatorname{Hom}(\Gamma^n(M), \Gamma^n(N)).$$

Given $a_i \in A$ (where is A is some algebra) and $\alpha_i \in \operatorname{Hom}(M,N)$ (where M and N are modules), let the equation

$$\Gamma^n\colon \begin{bmatrix} \sum_i a_i \otimes \alpha_i \\ \in A \otimes \operatorname{Hom}(M,N) \end{bmatrix} \mapsto \begin{bmatrix} \sum_{|\nu|=n} a^\nu \otimes \alpha^{[\nu]} \\ \in A \otimes \operatorname{Hom}(\Gamma^n(M), \Gamma^n(N)) \end{bmatrix},$$

define the homomorphisms

$$\alpha^{[\nu]}\colon \Gamma^n(M) \to \Gamma^n(N).$$

Thus, the symbol $\alpha^{[\nu]}$ may be interpreted either as an element of $\Gamma^n \operatorname{Hom}(M,N)$ or as a map $\Gamma^n(M) \to \Gamma^n(N)$ (and sometimes both).

We state the following theorem, which should be well known:

Theorem 5. *The functor Γ^n, given by*

$$M \mapsto \Gamma^n(M)$$

$$\begin{bmatrix} \sum_i a_i \otimes \alpha_i \\ \in A \otimes \operatorname{Hom}(M,N) \end{bmatrix} \mapsto \begin{bmatrix} (\sum_i a_i \otimes \alpha_i)^{[n]} = \sum_{|\nu|=n} a^\nu \otimes \alpha^{[\nu]} \\ \in A \otimes \operatorname{Hom}(\Gamma^n(M), \Gamma^n(N)) \end{bmatrix}$$

is strictly polynomial of homogeneous degree n, as is the functor $\Gamma^n \operatorname{Hom}(K,-)$ for a fixed module K.

We define the **fundamental homogeneous polynomial functor** to be $\Gamma^n \operatorname{Hom}(K,-)$, given by the following formula:

$$M \mapsto \Gamma^n \operatorname{Hom}(K,M)$$

$$\begin{bmatrix} \sum_i a_i \otimes \alpha_i \\ \in A \otimes \operatorname{Hom}(M,N) \end{bmatrix} \mapsto \begin{bmatrix} (\sum_i a_i \otimes (\alpha_i)_*)^{[n]} = \sum_{|\nu|=n} a^\nu \otimes (\alpha_*)^{[\nu]} \\ \in A \otimes \operatorname{Hom}(\Gamma^n \operatorname{Hom}(K,M), \Gamma^n \operatorname{Hom}(K,N)) \end{bmatrix}.$$

5. Yoneda Correspondence for Homogeneous Functors

Parallelling the development for polynomial functors, we state and prove the Yoneda Lemma for homogeneous polynomial functors. Note that the module K is here required to be *free*.

Theorem 6: The Homogeneous Yoneda Lemma. *Let K be a fixed, free module and F a homogeneous functor of degree n. The map*

$$\begin{aligned} \Upsilon\colon \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-), F) &\to F(K) \\ \eta &\mapsto \eta_K(1_K^{[n]}) \end{aligned}$$

is an isomorphism of modules.

Proof. Since all the modules involved are free, we may without difficulty define

$$\Xi\colon y \mapsto \begin{bmatrix} \zeta_M : \Gamma^n \operatorname{Hom}(K,M) \to F(M) \\ \beta^{[\mu]} \mapsto F_{\beta^{[\mu]}}(y) \end{bmatrix}.$$

Pray note that $F_{\beta^{[n]}} = F(\beta)$. ζ_M is evidently a well-defined group homomorphism, being the composite of $\beta^{[\mu]} \mapsto F_{\beta^{[\mu]}}$ with evaluation at y.

ζ is natural, because of the following commutative diagram:

$$\begin{array}{ccccccc} M & \quad & \Gamma^n \operatorname{Hom}(K,M) & \xrightarrow{\zeta_M} & F(M) & \quad & \beta^{[\mu]} \longrightarrow F_{\beta^{[\mu]}}(y) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle(\alpha_*)^{[n]}}\downarrow & & \downarrow{\scriptstyle F_{\alpha^{[n]}}} & & \downarrow \qquad\qquad \downarrow \\ N & & \Gamma^n \operatorname{Hom}(K,N) & \xrightarrow{\zeta_N} & F(N) & & (\alpha\beta)^{[\mu]} \longrightarrow F_{(\alpha\beta)^{[\mu]}}(y) = F_{\alpha^{[n]}\beta^{[\mu]}}(y) \end{array}$$

Now we show the above formula indeed gives the inverse of Υ. On the one hand, it is clear that

$$\Upsilon\Xi(y) = \Upsilon(\zeta) = \zeta_K(1_K^{[n]}) = F_{1_K^{[n]}}(y) = F(1_K)(y) = y.$$

On the other hand, starting from η and letting $y = \Upsilon(\eta) = \eta_K(1_K^{[n]})$ define $\zeta = \Xi(y)$, we see that

$$\zeta_M(\beta^{[n]}) = F_{\beta^{[n]}}(y) = F_{\beta^{[n]}}(\eta_K(1_K^{[n]})) = F(\beta)(\eta_K(1_K^{[n]})) = \eta_M(\beta^{[n]}),$$

where the last equality is due to the following commutative diagram:

$$\begin{array}{ccccccc} K & \quad & \Gamma^n \operatorname{Hom}(K,K) & \xrightarrow{\eta_K} & F(K) & \quad & 1_K^{[n]} \longrightarrow \eta_K(1_K^{[n]}) \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle(\beta_*)^{[n]}}\downarrow & & \downarrow{\scriptstyle F(\beta)} & & \downarrow \qquad\qquad \downarrow \\ M & & \Gamma^n \operatorname{Hom}(K,M) & \xrightarrow{\eta_M} & F(M) & & \beta^{[n]} \longrightarrow \eta_M(\beta^{[n]}) = F(\beta)(\eta_K(1_K^{[n]})) \end{array}$$

η and ζ then agree everywhere by the Divided Power Lemma. □

In particular, we have a module isomorphism

$$\mathrm{Nat}(\Gamma^n \operatorname{Hom}(K,-)) \cong \Gamma^n \operatorname{Hom}(K,K) = \Gamma^n(\operatorname{End} K),$$

given by the map

$$\eta \mapsto \eta_K(1_K^{[n]})$$

with inverse

$$\sigma^{[n]} \mapsto \begin{bmatrix} \eta_M : \Gamma^n \operatorname{Hom}(K,M) \to \Gamma^n \operatorname{Hom}(K,M) \\ \alpha^{[n]} \mapsto (\alpha \circ \sigma)^{[n]} \end{bmatrix}.$$

As in the numerical case, this is a ring isomorphism when $\Gamma^n(\operatorname{End} K)$ is equipped with the reverse product multiplication:

Theorem 7. *Let K be free. Under the Yoneda correspondence, the rings*

$$\big(\mathrm{Nat}(\Gamma^n \operatorname{Hom}(K,-))\big)^\circ \cong \Gamma^n(\operatorname{End} K),$$

where the latter is equipped with the product multiplication.

6. Morita Equivalence for Homogeneous Functors

Again, we specialize to the case $K = R^n$ to obtain a compact progenerator:

Theorem 8. *$\Gamma^n \operatorname{Hom}(R^n,-)$ is a compact progenerator for $\mathfrak{HPol}_n$, through which there is a Morita equivalence:*

$$\mathfrak{HPol}_n \sim {}_{\Gamma^n(R^{n\times n})}\mathfrak{Mod},$$

where $\Gamma^n(R^{n\times n})$ carries the product multiplication.

More precisely, the functor F corresponds to the abelian group $F(R^n)$, with module structure given by the equation

$$s^{[n]}(x) = F(s)(x)$$

Proof. The proof is virtually identical to the one for numerical functors and therefore omitted. ☐

CHAPTER 5

Mazes

Labyrinth of Fun

The quartet Baues, Dreckmann, Franjou and Pirashvili, [1], discovered in 2001 how to combinatorially encode $\mathbf{Z}$-module functors, and in particular polynomial ones. Their design was to establish a two-way correspondence (category equivalence) between module functors ${}_{\mathbf{Z}}\mathfrak{X}\mathfrak{Mod} \to {}_{\mathbf{Z}}\mathfrak{Mod}$ and Mackey functors $\Omega \to {}_{\mathbf{Z}}\mathfrak{Mod}$, where Ω is the category of finite sets and surjections.

Unfortunately, the argument does not generalize to an arbitrary base ring, as it is not apparent what category should play the rôle of Ω. To remedy this situation, we explore here the theory of mazes. We will later (Chapter 7) recapture the Ω-construction of [1].

The construction we describe is quite general. It does not require the base ring to be numerical, and not even commutative. So until we start discussing polynomiality, R is just assumed to be a unital ring.

1. Mazes

Consider two finite sets X and Y. A **passage** from $x \in X$ to $y \in Y$ is a (formal) arrow p from x to y, tagged with an element of R, denoted by $\overline{p}$. This we write as

$$p\colon x \to y,$$

or

$$x \xrightarrow{\overline{p}} y\ .$$

Definition 1. A **maze** from X to Y is a multiset of passages from X to Y. It is required that there be at least one passage leading from every element of X, and at least one passage leading to every element of Y (we, so to speak, wish to prevent *dead ends* from forming). ◇

Because a maze is a multiset, there can be (and, in general, will be) multiple passages between any two given elements.

Definition 2. We say $P\colon X \to Y$ is a **submaze** of $Q\colon X \to Y$, if $P \subseteq Q$ as multisets. ◇

Definition 3. If $P\colon X \to Y$ is a maze, the **restriction** of P to $X' \to Y'$, for subsets $X' \subseteq X$ and $Y' \subseteq Y$, is the maze (if indeed it is one) from X' to Y' containing only those passages of P that begin in X' and end in Y'. It will be denoted by

$$P\big|_{X' \to Y'}.$$

◇

We shall sometimes abuse notation, and use the symbol $P\big|_{X' \to Y'}$ even when this is not neccessarily a maze. We will take this liberty when summing over submazes, with the tacit understanding, that if $P\big|_{X' \to Y'}$ is not itself a maze, of course it has no submazes either, so the sum will be empty.

Note that $P\big|_{X' \to Y'}$ is not a submaze of P (unless $X' = X$ and $Y' = Y$).

Passages $p\colon y \to z$ and $q\colon x \to y$ are said to be **composable**, because one ends where the other begins.

Definition 4. If $P\colon Y \to Z$ and $Q\colon X \to Y$ are mazes, we define the **cartesian product** $P \quad Q$ to be the multiset of all pairs of composable passages:

$$P \quad Q = \left\{ \left(\left[\, z \xleftarrow{p} y \,\right], \left[\, y \xleftarrow{q} x \,\right] \right) \;\middle|\; \left[\, z \xleftarrow{p} y \,\right] \in P \;\wedge\; \left[\, y \xleftarrow{q} x \,\right] \in Q \right\}.$$

◇

For a subset $U \subseteq P \quad Q$, we shall write

$$U \sqsubseteq P \quad Q$$

to indicate that the projections on P and Q are both onto. Note that such a set U naturally gives rise to a new maze, namely

$$\left\{ \left[\, z \xleftarrow{pq} x \,\right] \;\middle|\; \left(\left[\, z \xleftarrow{p} y \,\right], \left[\, y \xleftarrow{q} x \,\right] \right) \in U \right\}.$$

The surjectivity condition on the projections will prevent dead ends from forming.

When we write $P \quad Q$, we will sometimes refer to the cartesian product, and sometimes its associated maze, and hope the circumstances will make clear which is meant.

Definition 5. The **product** or **composition** of the mazes P and Q is defined as the formal sum

$$PQ = \sum_{U \sqsubseteq P \quad Q} U.$$

◇

That multiplication is associative follows easily from the observation that $(PQ)R$ and $P(QR)$ both equal

$$\sum_{W \sqsubseteq P \quad Q \quad R} W.$$

There exist **identity mazes**

$$I_X = \left\{ \left[x \xrightarrow{1} x \right] \,\middle|\, x \in X \right\}.$$

Note, by the by, that it is perfectly legal to consider the **empty maze**

$$I_\emptyset = \emptyset \colon \emptyset \to \emptyset$$

with no passages. It is the only maze into or out of $\emptyset$.

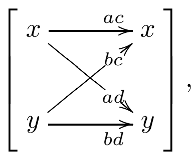
Example 1. Consider the two mazes

$$Q = \begin{bmatrix} x & \searrow^{a} & \\ & & z \\ y & \nearrow_{b} & \end{bmatrix}, \qquad P = \begin{bmatrix} & \nearrow^{c} & x \\ z & & \\ & \searrow_{d} & y \end{bmatrix}.$$

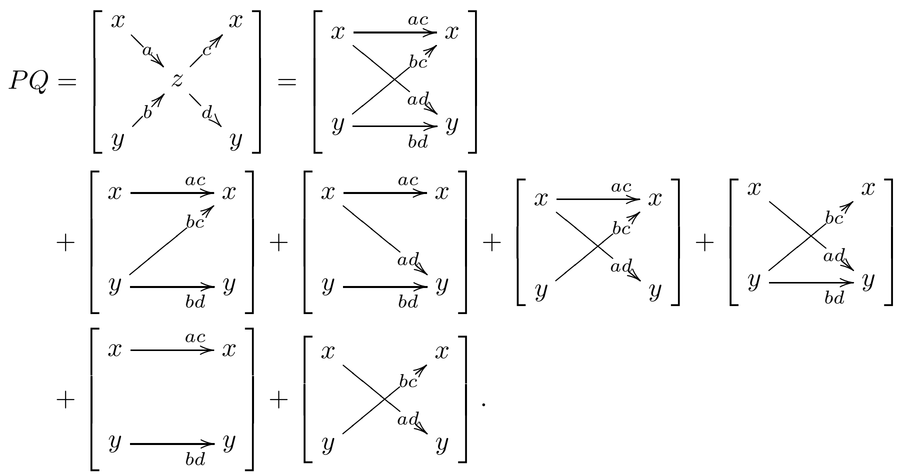
Their cartesian product is

$$P \quad Q = \Big\{ \Big(\big[x \xleftarrow{c} z \big], \big[z \xleftarrow{a} x \big] \Big), \Big(\big[y \xleftarrow{d} z \big], \big[z \xleftarrow{a} x \big] \Big),$$
$$\Big(\big[x \xleftarrow{c} z \big], \big[z \xleftarrow{b} y \big] \Big), \Big(\big[y \xleftarrow{d} z \big], \big[z \xleftarrow{b} y \big] \Big) \Big\},$$

which we identify with the maze



and their product is



△

2. The Labyrinth Category

Definition 6. The **labyrinth category** $\mathfrak{Laby}$ is the R-category[1] obtained in the following way: Its objects are the finite sets. Given two sets, their arrow set is the free module of mazes between them, with the following relations imposed (i. e. divided away), for multiset P of passages:

I.

$$\left[P \cup \{ * \xrightarrow{\;0\;} * \} \right] = 0.$$

II.

$$\left[P \cup \{ * \xrightarrow{a+b} * \} \right] =$$
$$\left[P \cup \{ * \xrightarrow{\;a\;} * \} \right] + \left[P \cup \{ * \xrightarrow{\;b\;} * \} \right] + \left[P \cup \{ * \overset{a}{\underset{b}{\rightrightarrows}} * \} \right].$$

(The unions are to be interpreted in a multiset-theoretic way.) ◇

We first state two elementary formulæ for this category, proved by induction.

Theorem 1. *In the labyrinth category, the following equations hold:*

$$\left[P \cup \left\{ * \xrightarrow{\sum_{i=1}^{n} a_i} * \right\} \right] = \sum_{\emptyset \subset I \subseteq [n]} \left[P \cup \left\{ * \xrightarrow{a_i} * \;\middle|\; i \in I \right\} \right]$$

$$\left[P \cup \left\{ * \xrightarrow{a_i} * \;\middle|\; 1 \le i \le n \right\} \right] = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left[P \cup \left\{ * \xrightarrow{\sum_{i \in I} a_i} * \right\} \right]$$

3. Operations on Mazes

There are some more (as yet nameless) operations on mazes which will occasionally be useful to us.

If $P \colon Y \to Z$ and $Q \colon X \to Y$ are mazes, we define

$$P \quad Q = \left\{ \left[\, z \xleftarrow{\sum pq} x \, \right] \;\middle|\; z \in Z, x \in X \right\},$$

where the sum is taken over all pairs $[z \overset{p}{\leftarrow} y] \in P$ and $[y \overset{q}{\leftarrow} x] \in Q$ of composable passages. This new maze will have at most one passage running between any given $x \in X$ and $z \in Z$.

We immediately have the following formula relating the operations and .

[1] By an *R-category* we understand a category enriched over $\mathfrak{Mod}$, so that its arrow sets are in fact R-modules. A $\mathbf{Z}$-category is just a preadditive category.

Theorem 2.
$$\sum_{V \subseteq P \quad Q} V = \sum_{W \subseteq P \quad Q} W.$$

Proof. For $W \subseteq P \quad Q$, define

$$E(W) = \{V \subseteq P \quad Q \mid \exists [x \to z] \in V \ \leftrightarrow \ \exists [x \to z] \in W\}.$$

Then apply the first of the formulæ of Theorem 1 to each passage of $E(W)$ to show
$$\sum_{V \in E(W)} V = W,$$
which proves the theorem. □

Passages $p\colon x \to y$ and $q\colon x \to y$ are said to be **parallel**, because they share starting and ending points.

Definition 7. We say that mazes $P, Q\colon X \to Y$ are **similar** if they contain no parallel passages and

$$\forall x \in X, y \in Y\colon [x \to y] \in P \ \leftrightarrow \ [x \to y] \in Q.$$

◇

Essentially P and Q have the same passages, except that their labels may differ.

When P and Q are similar mazes, we define

$$P \quad Q = \left\{ \left[x \xrightarrow{p+q} y \right] \,\middle|\, \left[x \xrightarrow{p} y \right] \in P, \left[x \xrightarrow{q} y \right] \in Q \right\},$$

and obtain without effort the following theorem.

Theorem 3. *Let $P_1, \ldots, P_n$ be similar mazes, and let the passages of P_i be $p_{i1}, \ldots, p_{im}$. Then*

$$P_1 \quad \cdots \quad P_n = \sum_K \{p_{ij} \mid (i,j) \in K\},$$

where the sum is taken over all $K \subseteq [n] \times [m]$ such that the projection on the second variable is onto.

4. Module Functors

We shall now establish a remarkable equivalence between two kinds of functors. On the one hand, we consider module functors $\mathfrak{XMod} \to \mathfrak{Mod}$, which may be of an arbitrary nature (additive, polynomial, numerical, and what not). On the other hand, we shall have functors $\mathfrak{Laby} \to \mathfrak{Mod}$. *These shall always be assumed R-linear.*

Given a direct sum R^X and $x \in X$, let 1_x denote the unity of the xth component R. We let

$$\sigma_{yx}\colon R^X \to R^X$$

be the homomorphism that takes 1_x to 1_y and every other 1_z to 0. We shall make extensive use of these maps, as they turn out to be the skeletal components of the module category.

Definition 8. Given a linear map

$$s = \sum_{a \in A, b \in B} s_{ba}\sigma_{ba}\colon R^A \to R^B,$$

(a $B \times A$ matrix) we let its **associated maze** $S\colon A \to B$ be

$$S = \left\{ \left[\ a \xrightarrow{s_{ba}} b \ \right] \ \middle| \ a \in A, b \in B \right\}.$$

◇

Note that if but a single component s_{ba} vanishes, the associated maze $S = 0$. Note also that the associated maze of a composition $s \circ t$ is none other than $S \ T$, and that of a sum $s+t$ is $S \ T$, which motivates our interest in these operations, as well as our choice of notation.

In the continuation, we will make no formal difference between a linear map and its associated maze, and denote them both by the same symbol, as long as it is clear which one is meant.

We wish now to define a functor (which will eventually turn out to be an equivalence)

$$\Phi\colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Fun}(\mathfrak{Laby}, \mathfrak{Mod}).$$

Given a module functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$, the corresponding labyrinth functor should take finite sets to the corresponding cross-effects:

$$X \mapsto F^{\dagger}(R|_X).$$

Also, mazes should be interpreted as deviations, in the following sense:

$$[P\colon X \to Y] \mapsto \left[F\left(\Diamond_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx} \right) \Bigg|_{F^{\dagger}(R|_X) \to F^{\dagger}(R|_Y)} \right].$$

But it is in fact unnecessary to restrict the action to the appropriate cross-effects, as the following lemma shows.

Lemma 1. *The map*

$$F\left(\Diamond_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx} \right) \colon F(R^X) \to F(R^Y)$$

is in fact a map $F^{\dagger}(R|_X) \to F^{\dagger}(R|_Y)$, in the sense that all other components are 0.

Proof. We use Theorem 2 of Chapter 3. Precomposition with $F(\tau_x)$, where τ_x is any insertion with $x \in X$, and postcomposition with $F(\varrho_y)$, where ϱ_y is any retraction with $y \in Y$, both yield 0, because $\sigma_{yx}\tau_x = \varrho_y\sigma_{yx} = 0$. □

The homomorphism

$$F\left(\Diamond_{[p\colon x\to y]\in P} \overline{p}\sigma_{yx}\right)$$

may thus be interpreted both as a map $R^X \to R^Y$, and as a map $F^\dagger(R|_X) \to F^\dagger(R|_Y)$, depending on the circumstances. We hence define $\Phi(F)\colon \mathfrak{Laby} \to \mathfrak{Mod}$ by the following formulæ:

$$X \mapsto F^\dagger(R|_X)$$
$$[P\colon X \to Y] \mapsto \left[F\left(\Diamond_{[p\colon x\to y]\in P} \overline{p}\sigma_{yx}\right) : F^\dagger(R|_X) \to F^\dagger(R|_Y)\right].$$

Lemma 2. *$\Phi(F)$ is a functor from the labyrinth category.*

Proof. That $\Phi(F)$ respects the relations in $\mathfrak{Laby}$ follows from

$$\Phi(F)\left(P \cup \{\, x \xrightarrow{0} y \,\}\right) = F(\cdots \diamond 0) = 0$$

and

$$\begin{aligned}\Phi(F)&\left(P \cup \{\, x \xrightarrow{a+b} y \,\}\right) = F(\cdots \diamond (a+b)\sigma_{yx})\\ &= F(\cdots \diamond a\sigma_{yx}) + F(\cdots \diamond b\sigma_{yx}) + F(\cdots \diamond a\sigma_{yx} \diamond b\sigma_{yx})\\ &= \Phi(F)\left(P \cup \{\, x \xrightarrow{a} y \,\}\right) + \Phi(F)\left(P \cup \{\, x \xrightarrow{b} y \,\}\right)\\ &\quad + \Phi(F)\left(P \cup \{\, x \overset{a}{\underset{b}{\rightrightarrows}} y \,\}\right).\end{aligned}$$

Functoriality follows from the Deviation Formula. □

We now define $\Phi(\zeta)$, for a natural transformation $\zeta\colon F \to G$, by restriction to the appropriate cross-effects:

$$\Phi(\zeta)_X = \zeta_{R^X} : F^\dagger(R|_X) \to G^\dagger(R|_X).$$

Lemma 3. *Φ is a functor.*

Proof. Because natural transformation are linear, they preserve deviations, and hence cross-effects. Hence, for X and Y of different cardinality, the component

$$\zeta\colon F^\dagger(R|_X) \to G^\dagger(R|_Y)$$

is 0. From this, multiplicativity of the functor Φ follows. □

Lemma 4. Φ *is fully faithful.*

Proof. Given $\eta\colon \Phi(F) \to \Phi(G)$, the only possible candidate for a $\zeta\colon F \to G$, such that $\Phi(\zeta) = \eta$, is

$$\zeta_{R^X} = \bigoplus_{Y \subseteq X} \eta_Y.$$

□

Now comes the hard part: showing Φ is essentially surjective. Let an $H\colon \mathfrak{Laby} \to \mathfrak{Mod}$ be given. Define its inverse image $\Phi^{-1}(H)\colon \mathfrak{XMod} \to \mathfrak{Mod}$ by letting

$$\Phi^{-1}(H)(R^A) = \bigoplus_{Y \subseteq A} H(Y)$$

and, given

$$s = \sum_{a \in A, b \in B} s_{ba}\sigma_{ba}\colon R^A \to R^B,$$

letting the $H(Y) \to H(Z)$ part of $\Phi^{-1}(H)(s)$ be given by

$$\sum_{P \subseteq S \big|_{Y \to Z}} H(P),$$

where S is the associated maze of s. Note that

$$\Phi^{-1}(H)(s)\big|_{H(Y) \to H(Z)} = 0$$

if $Y = \emptyset \neq Z$, or conversely, but

$$\Phi^{-1}(H)(s)\big|_{H(\emptyset) \to H(\emptyset)} = H(I_\emptyset).$$

We will show that $\Phi(\Phi^{-1}(H)) = H$. Note, however, that in general $\Phi^{-1}\Phi(F) \neq F$, despite the notation. Φ^{-1} will only be a pseudo-inverse to Φ (inverse up to natural isomorphism).

Lemma 5. $\Phi^{-1}(H)$ *is a functor.*

Proof. Given

$$S = \sum_{b \in B, c \in C} s_{cb}\sigma_{cb}\colon R^B \to R^C, \qquad T = \sum_{a \in A, b \in B} t_{ba}\sigma_{ba}\colon R^A \to R^B,$$

we calculate, for $X \subseteq A$ and $Z \subseteq C$, the $H(X) \to H(Z)$ component of

$$\Phi^{-1}(H)(S) \circ \Phi^{-1}(H)(T)$$

as:

$$\sum_{Y \subseteq B} \left(\sum_{P \subseteq S \big|_{Y \to Z}} H(P) \right) \left(\sum_{Q \subseteq T \big|_{X \to Y}} H(Q) \right) = \sum_{Y \subseteq B} \sum_{\substack{P \subseteq S \big|_{Y \to Z} \\ Q \subseteq T \big|_{X \to Y}}} H(PQ)$$

$$= \sum_{Y \subseteq B} \sum_{\substack{P \subseteq S \big|_{Y \to Z} \\ Q \subseteq T \big|_{X \to Y}}} H\left(\sum_{V \sqsubseteq P \ \ Q} V \right) \quad = \sum_{V \subseteq (S \ \ T) \big|_{X \to Z}} H(V).$$

The last step follows from noting that every submaze of $(S \ \ T)\big|_{X \to Z}$ is obtained as $V \sqsubseteq P \quad Q$, for some P and Q. Since the $H(X) \to H(Z)$ part of

$$\Phi^{-1}(H)(ST) = \Phi^{-1}(H)\left(\sum_{a \in A, c \in C} \left(\sum_{b \in B} s_{cb} t_{ba} \right) \sigma_{ca} \right)$$

is

$$\sum_{W \subseteq (S \ \ T) \big|_{X \to Z}} H(W) = \sum_{W \subseteq (S \ \ T) \big|_{X \to Z}} H(W),$$

the functoriality of $\Phi^{-1}(H)$ follows. □

Lemma 6.

$$\Phi(\Phi^{-1}(H)) = H.$$

Proof. We first write down the well-known (and easily established) formula

$$\sum_{Q \subseteq S \subseteq P} (-1)^{|S|} = \begin{cases} (-1)^{|P|} & \text{if } P = Q, \\ 0 & \text{else,} \end{cases}$$

where P and Q are finite sets.

Given a maze $P \colon X \to Y$, we want to calculate the deviation of the module functor $\Phi^{-1}(H)$ corresponding to the maze P:

$$\Phi^{-1}(H)\left(\mathop{\Diamond}_{[p\colon x \to y] \in P} \overline{p} \sigma_{yx} \right) = \sum_{S \subseteq P} (-1)^{|P|-|S|} \Phi^{-1}(H)\left(\sum_{p \in S} \overline{p} \sigma_{yx} \right). \tag{5.1}$$

The $H(Z_1) \to H(Z_2)$ component of $\Phi^{-1}(H)\left(\sum_{p \in S} \overline{p} \sigma_{yx} \right)$ is

$$\sum_{Q \subseteq S \big|_{Z_1 \to Z_2}} H(Q).$$

The component $H(Z_1) \to H(Z_2)$ of (5.1) is then

$$\sum_{S \subseteq P} (-1)^{|P|-|S|} \sum_{Q \subseteq S \big|_{Z_1 \to Z_2}} H(Q) = \sum_{Q \subseteq P \big|_{Z_1 \to Z_2}} (-1)^{|P|} H(Q) \sum_{Q \subseteq S \subseteq P} (-1)^{|S|}.$$

The inner sum vanishes if $P \neq Q$. It equals $(-1)^{|P|}$ if $P = Q$, but at the same time $Q \subseteq P\big|_{Z_1 \to Z_2}$, so in fact $Z_1 = X$, $Z_2 = Y$ and $Q = P = P\big|_{X \to Y}$. Thus

$$\Phi^{-1}(H)\left(\mathop{\Diamond}_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx}\right)\Bigg|_{H(Z_1) \to H(Z_2)} = \begin{cases} H(P) & \text{if } Z_1 = X \text{ and } Z_2 = Y, \\ 0 & \text{else.} \end{cases}$$

From this it follows instantly, both that

$$\begin{aligned} \Phi(\Phi^{-1}(H))(X) &= \Phi^{-1}(H)^{\dagger}(R|_X) = \operatorname{Im} \Phi^{-1}(H)\left(\mathop{\Diamond}_{x \in X} \pi_x\right) \\ &= \operatorname{Im} H(I_X) = \operatorname{Im} 1_{H(X)} = H(X), \end{aligned}$$

and that

$$\begin{aligned} \Phi(\Phi^{-1}(H))(P) &= \Phi^{-1}(H)\left(\mathop{\Diamond}_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx}\right)\Bigg|_{\Phi^{-1}(H)^{\dagger}(R|_X) \to \Phi^{-1}(H)^{\dagger}(R|_Y)} \\ &= \Phi^{-1}(H)\left(\mathop{\Diamond}_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx}\right)\Bigg|_{H(X) \to H(Y)} = H(P). \end{aligned}$$

□

It is now only a matter of putting these lemmata together, to obtain the following truly marvellous theorem:

Theorem 4. *The functor*

$$\Phi_{\mathfrak{Laby}}\colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Fun}(\mathfrak{Laby}, \mathfrak{Mod}),$$

where $\Phi_{\mathfrak{Laby}}(F)\colon \mathfrak{Laby} \to \mathfrak{Mod}$ *takes*

$$\begin{aligned} X &\mapsto F^{\dagger}(R|_X) \\ [P\colon X \to Y] &\mapsto \left[F\left(\mathop{\Diamond}_{[p\colon x \to y] \in P} \overline{p}\sigma_{yx}\right) : F^{\dagger}(R|_X) \to F^{\dagger}(R|_Y)\right], \end{aligned}$$

is a category equivalence.

5. Polynomial Functors

The preceding section dealt with module functors in general. Since the passages of a maze correspond to deviations, the following simple characterization of polynomiality should come as no surprise.

Theorem 5. *The module functor F is polynomial of degree n iff $\Phi_{\mathfrak{Laby}}(F)$ vanishes on sets with more than n elements.*

Proof. Clearly enough, if F is polynomial functor of degree n, then $\Phi_{\mathfrak{Laby}}(F)$ will vanish on mazes with more than n passages, since applying $\Phi_{\mathfrak{Laby}}(F)$ to such a maze will involve an nth deviation.

Suppose now conversely that $\Phi_{\mathfrak{Laby}}(F)$ vanishes on mazes with more than n passages, and let there be given $n+1$ homomorphisms

$$\alpha_1, \ldots, \alpha_{n+1} \colon R^A \to R^B$$

with associated mazes

$$P_1, \ldots, P_{n+1} \colon A \to B.$$

These mazes can be made similar by adding in extra passages labelled 0, if need be, and we may label the passages of P_i by

$$p_{i1}, \ldots, p_{im}.$$

Let sets $X \subseteq A$ and $Y \subseteq B$ be fixed.

Note that if

$$\{p_{ij} \mid j \in J\}$$

is a legitimate submaze of P_i for one particular i, it is so for all choices of i. When this is the case, we say that the set $J \subseteq [m]$ is *admissible*. Then also

$$\left\{ \sum_{i \in I} p_{ij} \;\middle|\; j \in J \right\}$$

is a legitimate submaze of

$$\Big(\mathop{}_{i \in I} P_i \Big) \Big|_{X \to Y} = \mathop{}_{i \in I} P_i \,\big|_{X \to Y}$$

(the associated maze of the sum $\sum_{i \in I} \alpha_i$) for any $I \subseteq [n+1]$.

We are now ready to calculate the deviation of F:

$$\begin{aligned}
&F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1}) \big|_{F^\dagger(R|_X) \to F^\dagger(R|_Y)} \\
&\quad = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} F\Big(\sum_{i \in I} \alpha_i \Big) \Big|_{F^\dagger(R|_X) \to F^\dagger(R|_Y)} \\
&\quad = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \sum_{Q \subseteq (\mathop{}_{i \in I} P_i) |_{X \to Y}} \Phi_{\mathfrak{Laby}}(F)(Q) \\
&\quad = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \sum_{J \subseteq [m]} \Phi_{\mathfrak{Laby}}(F) \left(\left\{ \sum_{i \in I} p_{ij} \;\middle|\; j \in J \right\} \right),
\end{aligned}$$

where the inner sum is taken over admissible J only. Letting K_l denote the projection of the set $K \subseteq I \times J$ on the lth component, we may use Theorem 3 to transform the latter sum to

$$F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1}) \big|_{F^\dagger(R|_X) \to F^\dagger(R|_Y)}$$

$$= \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \sum_{J \subseteq [m]} \sum_{\substack{K \subseteq I \times J \\ K_2 = J}} \Phi_{\mathfrak{Laby}}(F)(\{p_{ij} \mid (i,j) \in K\})$$

$$= \sum_{K \subseteq [n+1] \times [m]} \left(\sum_{K_1 \subseteq I \subseteq [n+1]} (-1)^{n+1-|I|} \right) \left(\sum_{J = K_2} \Phi_{\mathfrak{Laby}}(F)(\{p_{ij} \mid (i,j) \in K\}) \right)$$

$$= \sum_{\substack{K \subseteq [n+1] \times [m] \\ K_1 = [n+1]}} \Phi_{\mathfrak{Laby}}(F)(\{p_{ij} \mid (i,j) \in K\}).$$

The condition $K_1 = [n+1]$ implies $|K| \geq n+1$, and so all the mazes

$$\{p_{ij} \mid (i,j) \in K\}$$

will contain more than n passages. The sum will therefore equal 0, by the hypothesis on $\Phi_{\mathfrak{Laby}}(F)$. □

6. Numerical Functors

We now investigate how to interpret numericality in the labyrinthine setting.

First some notation. For P a maze and a a scalar, let $a \quad P$ be the maze obtained from P by multiplying the labels of all passages by a:

$$a \quad P = \left\{ \left[\, x \xrightarrow{ap} y \, \right] \,\middle|\, \left[\, x \xrightarrow{p} y \, \right] \in P \right\}.$$

Given a multiset A supported by the maze P, we let E_A denote the maze

$$E_A = \bigcup_{a \in A} \left\{ \, a \xrightarrow{1} a \, \right\},$$

with the passages multiplied according to the degree function of A, and uniformly given the label 1. (This is an example of a *simple* maze; we will see later that the simple mazes form bases for the arrow sets of the labyrinth category.)

Lemma 7. *Let r lie in a numerical ring, n be a natural number, and w_j be positive integers satsfying $w_1 + \cdots + w_q \leq n$. Then*

$$\prod_{j=1}^{q} \binom{r}{w_j} = \sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}.$$

Proof. We prove the formula when r is an integer, and then refer to the Numerical Universality Principle.

$$\sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}$$

$$= \sum_{m=0}^{n} \binom{r}{m} \sum_{K \subseteq [m]} (-1)^{m-|K|} \prod_{j=1}^{q} \binom{|K|}{w_j}$$

$$= \sum_{\substack{M\subseteq[r]\\|M|\leq n}} \sum_{K\subseteq M} (-1)^{|M|-|K|} \prod_{j=1}^{q} \binom{|K|}{w_j}$$

$$= \sum_{K\subseteq[r]} (-1)^{|K|} \prod_{j=1}^{q} \binom{|K|}{w_j} \sum_{\substack{K\subseteq M\subseteq[r]\\|M|\leq n}} (-1)^{|M|}.$$

When $0 \leq r \leq n$, the requirement $|M| \leq n$ is superfluous, and K must equal $[r]$, lest the inner sum vanish. We then have

$$\sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}$$

$$= \sum_{K=[r]} (-1)^{|K|} \prod_{j=1}^{q} \binom{|K|}{w_j} \sum_{K\subseteq M\subseteq[r]} (-1)^{|M|}$$

$$= (-1)^{r} \prod_{j=1}^{q} \binom{r}{w_j} (-1)^{r} = \prod_{j=1}^{q} \binom{r}{w_j}.$$

The formula is thus true when $0 \leq r \leq n$. But then it must hold everywhere, since both sides are polynomials of degree n. □

Theorem 6. *The functor F is polynomial of degree n iff the equation*

$$\Phi_{\mathfrak{Laby}}(F)(P) = \sum_{\substack{\#A=P\\|A|\leq n}} \prod_{p\in P} \binom{\overline{p}}{\deg_A p} \Phi_{\mathfrak{Laby}}(F)(E_A)$$

holds for all mazes P.

Proof. By Theorem 7 of Chapter 3, a polynomial functor will certainly satisfy this. The converse is trickier.

First note that if $\Phi_{\mathfrak{Laby}}(F)$ satisfies the equation, then it will vanish on mazes with more than n elements, whence F is polynomial of degree n. We wish to use Theorem 6 of Chapter 3, and thus evaluate

$$F(r \cdot 1_{R^n}) = \sum_{P\subseteq r\ I_{[n]}} \Phi_{\mathfrak{Laby}}(F)(P).$$

The component

$$\Phi_{\mathfrak{Laby}}(F)(X) \to \Phi_{\mathfrak{Laby}}(F)(Y)$$

of this is 0 if $X \neq Y$. If $X = Y$, we may without loss of generality assume $X = Y = [q]$. Then the component

$$\Phi_{\mathfrak{Laby}}(F)([q]) \to \Phi_{\mathfrak{Laby}}(F)([q])$$

is

$$\Phi_{\mathfrak{Laby}}(F)(r \quad I_{[q]}) = \sum_{\substack{\#A=[q] \\ |A|\leq n}} \prod_{j=1}^{q} \binom{r}{\deg_A j} \Phi_{\mathfrak{Laby}}(F)(E_A)$$

$$= \sum_{w_1+\cdots+w_q\leq n} \prod_{j=1}^{q} \binom{r}{w_j} \Phi_{\mathfrak{Laby}}(F)(E_w),$$

where we let $w_j = \deg_A j \geq 1$. Similarly, the component

$$\Phi_{\mathfrak{Laby}}(F)([q]) \to \Phi_{\mathfrak{Laby}}(F)([q])$$

of

$$\sum_{m=0}^{n} \binom{r}{m} F\left(\underset{m}{\Diamond} 1_{R^n}\right) = \sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} F(k \cdot 1_{R^n})$$

is

$$\sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{w_1+\cdots+w_q\leq n} \prod_{j=1}^{q} \binom{k}{w_j} \Phi_{\mathfrak{Laby}}(F)(E_w).$$

It is now only a matter of using the lemma, to establish the equality

$$F(r \cdot 1_{R^n}) = \sum_{m=0}^{n} \binom{r}{m} F\left(\underset{m}{\Diamond} 1_{R^n}\right).$$

Consequently, F is numerical. □

Definition 9. The **nth quotient labyrinth category** $\mathfrak{Laby}_n$ is defined as the quotient category obtained from $\mathfrak{Laby}$ when the following relations are divided away:

III.

$$P = 0,$$

whenever P contains more than n passages.

IV.

$$P = \sum_{\substack{\#A=P \\ |A|\leq n}} \prod_{p\in P} \binom{\overline{p}}{\deg_A p} E_A,$$

for all mazes P.

◇

The theorem may then be rephrased as: F is numerical of degree n iff $\Phi_{\mathfrak{Laby}}(F)$ factors through $\mathfrak{Laby}_n$. Or, equivalently:

Theorem 7. *The functor $\Phi_{\mathfrak{Laby}}$ induces a category equivalence*

$$\mathfrak{Num}_n \to \mathrm{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod}).$$

A few examples of labyrinth representations are in order. We take $[n]$ as the canonical representative of sets of cardinality n.

Example 2. Let $C(R^n) = K$ be a constant functor. $\Phi_{\mathfrak{Laby}}(C)$ will take $\emptyset \mapsto K$, and all non-empty sets to 0. △

Example 3. Let $F(R^n) = K \oplus L^n$ be a linear functor. $\Phi_{\mathfrak{Laby}}(F)$ will take

$$[0] \mapsto K, \qquad [1] \mapsto L, \qquad [2], [3], \ldots \mapsto 0,$$

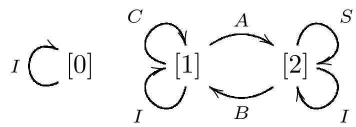
and map the maze

$$\left[\, 1 \xrightarrow{c} 1 \,\right] \mapsto [c \colon L \to L]\,.$$

△

7. Quadratic Functors

We here determine the structure of $\mathfrak{Num}_2$ by classifying the quadratic numerical functors. To find the labyrinthine descriptions of quadratic functors, we first draw the (skeletal) structure of the category $\mathfrak{Laby}_2$:



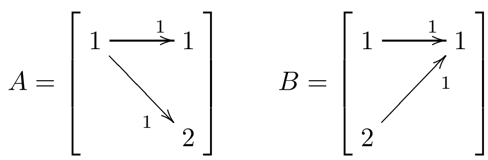
Since we in $\mathfrak{Laby}_2$ have the relations

$$\left[\, * \xrightarrow{a} * \,\right] = \binom{a}{1} \left[\, * \xrightarrow{1} * \,\right] + \binom{a}{2} \left[\, * \overset{1}{\underset{1}{\rightrightarrows}} * \,\right]$$

and

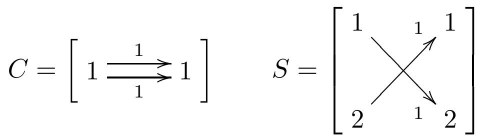
$$\begin{bmatrix} * \xrightarrow{a} * \\ \\ * \xrightarrow{b} * \end{bmatrix} = \binom{a}{1}\binom{b}{1} \begin{bmatrix} * \xrightarrow{a} * \\ \\ * \xrightarrow{b} * \end{bmatrix}$$

(the simple mazes generate the category), every maze in $\mathfrak{Laby}_2$ can be reduced to (linear combinations of) identity mazes and the following:



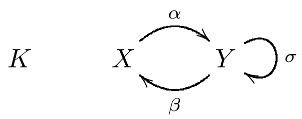
$\circ$	A	B	C	S
A	–	$I+S$	$2A$	–
B	C	–	–	B
C	–	$2B$	$2C$	–
S	A	–	–	I

Table 5.1: Multiplication table for $\mathfrak{Lady}_2$.



Even these are not independent. Their multiplication table is given in Table 5.1. Clearly we can do with only A, B and S, and we obtain the following explicit description of $\mathfrak{Num}_2$.

Theorem 8. *A quadratic numerical functor is equivalent to the following data: modules K, X and Y, together with homomorphisms α, β, σ as indicated:*



These homomorphisms are subject to the following four relations:

$$\alpha\beta = 1 + \sigma, \qquad \beta\sigma = \beta, \qquad \sigma\alpha = \alpha, \qquad \sigma^2 = 1.$$

The reader will note, that we can in fact also dispense with $\sigma = \alpha\beta - 1$, and instead let α and β be subject to a meagre two relations:

$$\beta\alpha\beta = 2\beta, \qquad \alpha\beta\alpha = 2\alpha.$$

We now describe the four classical quadratic functors. Because they are of the second degree, and because they are all pointed[2], the module $K = 0$. We will denote $R_1 = \langle e_1 \rangle$, and $R_2 = \langle e_2 \rangle$.

Example 4. The functor $\Phi_{\mathfrak{Lady}}(T^2)$ will take

$$X = (T^2)^\dagger(R_1) = \langle e_1 \otimes e_1 \rangle, \qquad Y = (T^2)^\dagger(R_1|R_2) = \langle e_1 \otimes e_2, e_2 \otimes e_1 \rangle$$

and map

$$\begin{aligned} \alpha &: \quad e_1 \otimes e_1 \mapsto e_1 \otimes e_2 + e_2 \otimes e_1 \\ \beta &: \quad e_1 \otimes e_2, e_2 \otimes e_1 \mapsto e_1 \otimes e_1 \\ \sigma &: \quad e_1 \otimes e_2 \mapsto e_2 \otimes e_1, \quad e_2 \otimes e_1 \mapsto e_1 \otimes e_2. \end{aligned}$$

$\triangle$

[2]A *pointed* functor maps 0 to 0.

Example 5. The functor $\Phi_{\mathfrak{Laby}}(S^2)$ will take

$$X = \left\langle e_1^2 \right\rangle, \qquad Y = \langle e_1 e_2 \rangle$$

and map

$$\begin{aligned} \alpha\colon &\quad e_1^2 \mapsto 2e_1e_2) \\ \beta\colon &\quad e_1e_2 \mapsto e_1^2 \\ \sigma\colon &\quad e_1e_2 \mapsto e_1e_2. \end{aligned}$$

△

Example 6. The functor $\Phi_{\mathfrak{Laby}}(\Lambda^2)$ will take

$$X = \langle e_1 \wedge e_1 \rangle = 0, \qquad Y = \langle e_1 \wedge e_2 \rangle$$

and map

$$\begin{aligned} \alpha\colon &\quad 0 \\ \beta\colon &\quad 0 \\ \sigma\colon &\quad e_1 \wedge e_2 \mapsto -e_1 \wedge e_2. \end{aligned}$$

△

Example 7. The functor $\Phi_{\mathfrak{Laby}}(\Gamma^2)$ will take

$$X = \left\langle e_1^{[2]} \right\rangle, \qquad Y = \langle e_1 e_2 \rangle$$

and map

$$\begin{aligned} \alpha\colon &\quad e_1^{[2]} \mapsto e_1e_2 \\ \beta\colon &\quad e_1e_2 \mapsto 2e_1^{[2]} \\ \sigma\colon &\quad e_1e_2 \mapsto e_1e_2. \end{aligned}$$

△

CHAPTER 6

Multisets

Är du en enhet eller delar?
Jag bäfvar, mod och sansning felar,
Min fråga gör mig stel och stum.

— Hedvig Charlotta Nordenflycht, *Öfver Andra Mosebok XXXIII:18,20; XXXIV:5,6*

1. Multisets

A multiset is a set with possibly repeated elements. More formally:

Definition 1. A **multiset** is a pair

$$M = (\#M, \deg_M),$$

where $\#M$ is a set and

$$\deg_M : \#M \to \mathbf{Z}^+$$

is a function, called the **degree** (or **multiplicity**) function. $\diamond$

The underlying set $\#M$ is called the **support** of M. We call $\deg_M a$ the **degree** or **multiplicity** of an object $a \in \#M$; it counts the "number of times a occurs in M". The degree of the whole multiset M we define to be

$$\deg M = \prod_{x \in \#M} (\deg x)!.$$

We tacitly assume all multisets under discussion to be *finite*, as these are the only ones we will ever need. The **cardinality** of M is its number of elements, counted with multiplicity:

$$|M| = \sum_{x \in \#M} \deg x.$$

Example 1. The multiset $\{a, a, b\}$ has cardinality 3 and support $\{a, b\}$. We have $\deg a = 2$, $\deg b = 1$ and $\deg c = 0$. $\triangle$

The **union** $A \cup B$ of two multisets A and B is precisely what it should be, namely, the elements of A together with those of B. More formally,

$$A \cup B = (\#A \cup \#B,\ \deg_{A\cup B}\colon x \mapsto \deg_A x + \deg_B x).$$

The **direct product** of two multisets A and B is also precisely what it should be, namely the multiset of all possible pairs of elements of A and B:

$$A \times B = (\#A \times \#B,\ \deg_{A\times B}\colon x \mapsto \deg_A x \cdot \deg_B x).$$

There is also a natural notion of **submultisets**[1]: Say $A \subseteq B$ if $\deg_A x \leq \deg_B x$ for all x, so that all elements of A are in B.

We adopt the following convention: *Whenever we quantify over a multiset each element should be counted as many times as its multiplicity indicates.* (If we do wish to count each element only once, we will quantify over the support.) Thus, for example,

$$\prod\{a, a, b\} = \prod_{x\in\{a,a,b\}} x = a^2 b.$$

Finally, recall that the Principle of Inclusion and Exclusion states, in one form, the following: If f and g are functions such that

$$\sum_{X\subseteq Y} f(X) = g(Y),$$

then

$$f(Y) = \sum_{X\subseteq Y} (-1)^{|Y|-|X|} g(X).$$

Here X and Y range over sets, but a generalization to multisets is immediate.

Theorem 1: The Multiset Principle of Inclusion and Exclusion. *If f and g are functions such that*

$$\sum_{\substack{\#A\subseteq Y\\ |A|=n}} f(A) = g(Y),$$

then

$$\sum_{\substack{\#A=Y\\ |A|=n}} f(A) = \sum_{X\subseteq Y} (-1)^{|Y|-|X|} g(X),$$

where A ranges over multisets, and X and Y over sets.

[1] Some people call these *multisubsets*.

2. Multations

Let A and B be multisets of equal cardinality. A **multation** $\varphi\colon A \to B$ is a pairing of their elements. We shall write multations as two-row matrices, with the elements of A on top of those of B, the way ordinary permutations are usually written:

$$\varphi = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$$

The order of the columns is of course irrelevant.

Observe that φ under no circumstances can be regarded as an ordinary "function", since identical copies of some element of A may very well be paired off with distinct elements of B.

φ will, however, be a *submultiset* of $A \times B$, such that every element of A occurs exactly once as the first component of a pair in φ, and each element of B exactly once as a second component. (This may serve as a formal definition.) The degree $\deg_\varphi(a, b)$ counts the number of times $a \in A$ is paired off with $b \in B$.

As a notational convenience, we adopt the following (purely formal) convention: If

$$\varphi = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$$

is a multation, define

$$\begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} = (\deg \varphi) \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}.$$

Also, given a multation

$$\begin{bmatrix} a_1 & a_1 & \dots & a_2 & a_2 & \dots \\ b_1 & b_1 & \dots & b_2 & b_2 & \dots \end{bmatrix},$$

with m_j appearances of the column $\begin{bmatrix} a_j \\ b_j \end{bmatrix}$, we may sometimes adopt the perspective of viewing it as a formal product

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^{[m_1]} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}^{[m_2]} \cdots$$

of divided powers. Thus, the expression

$$\begin{pmatrix} a_1 & a_1 & \dots & a_2 & a_2 & \dots \\ b_1 & b_1 & \dots & b_2 & b_2 & \dots \end{pmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^{m_1} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}^{m_2} \cdots,$$

will denote the corresponding product of *ordinary* powers. The rationale behind this formalism is that, when composing multations, the round-bracket notation provides a natural way of handling the diverse degrees of the multations involved, which would otherwise be quite cumbersome.

Example 2. There exist two multations from the multiset $\{a, a, b\}$ to itself, namely:

$$\begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix} \qquad \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix}$$

The degree of (a, b) is 0 with respect to the first of these, and 1 with respect to the second.

In this case, we have

$$\begin{pmatrix} a & a & b \\ a & a & b \end{pmatrix} = 2 \begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix}$$

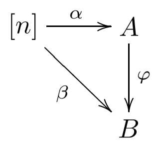
and

$$\begin{pmatrix} a & a & b \\ a & b & a \end{pmatrix} = \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix}.$$

△

3. The Multiset Category

Consider a diagram of multations and multisets of equal cardinality:



We say that the pair (α, β) **induces** the multation φ, if the diagram "commutes", in the sense that

$$\beta(j) = \varphi\alpha(j)$$

for all j. The idea is that the multations α and β provide two compatible "enumerations" of A and B.

We now proceed to define the composition of two multations. We choose to define the composition of two *round-bracket* multations, and then extend by linearity. So consider two such multations

$$\begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \beta(1) & \cdots & \beta(n) \end{pmatrix}, \qquad \begin{pmatrix} \gamma(1) & \cdots & \gamma(n) \\ \delta(1) & \cdots & \delta(n) \end{pmatrix},$$

with the first one going $A \to B$ and being induced by the pair (α, β), and the second going $B \to C$ and induced by (γ, δ). We define their **composition** by "summing over all possibilities of composing them":

$$\begin{pmatrix} \gamma(1) & \cdots & \gamma(n) \\ \delta(1) & \cdots & \delta(n) \end{pmatrix} \circ \begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \beta(1) & \cdots & \beta(n) \end{pmatrix} = \sum_{\sigma} \begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \delta\sigma(1) & \cdots & \delta\sigma(n) \end{pmatrix},$$

where the sum is to be taken over all permutations $\sigma \colon [n] \to [n]$ such that $\beta(j) = \gamma\sigma(j)$ for all j.

Example 3. For example, we have:

$$\begin{pmatrix} p & q & q \\ x & x & y \end{pmatrix} \circ \begin{pmatrix} a & a & b \\ p & q & q \end{pmatrix} = \begin{pmatrix} a & a & b \\ x & x & y \end{pmatrix} + \begin{pmatrix} a & a & b \\ x & y & x \end{pmatrix}.$$

The possible permutations $\sigma\colon [3] \to [3]$ are $()$ and $(2,3)$.

It follows that

$$\begin{aligned} \begin{bmatrix} p & q & q \\ x & x & y \end{bmatrix} \circ \begin{bmatrix} a & a & b \\ p & q & q \end{bmatrix} &= \begin{pmatrix} p & q & q \\ x & x & y \end{pmatrix} \circ \begin{pmatrix} a & a & b \\ p & q & q \end{pmatrix} \\ &= \begin{pmatrix} a & a & b \\ x & x & y \end{pmatrix} + \begin{pmatrix} a & a & b \\ x & y & x \end{pmatrix} \\ &= 2 \begin{bmatrix} a & a & b \\ x & x & y \end{bmatrix} + \begin{bmatrix} a & a & b \\ x & y & x \end{bmatrix}. \end{aligned}$$

△

It is not immediately obvious that composition of two multations will result in an *integer* sum of multations. That this is indeed the case, is a consequence of the following lemma.

Lemma 1: The Multation Lemma. *Let P and Q be multisets, and suppose that the multation $\chi\colon P \to Q$ is induced by the multations $\zeta\colon [n] \to P$ and $\eta\colon [n] \to Q$. The number of permutations $\sigma\colon [n] \to [n]$ such that ζ and $\eta\sigma$ induce the same multation χ is exactly*

$$\frac{\deg P \deg Q}{\deg \chi}.$$

Proof. The multation χ is represented by the array

$$\begin{bmatrix} \zeta(1) & \zeta(2) & \dots \\ \eta(1) & \eta(2) & \dots \end{bmatrix}.$$

The number of permutations $\sigma_1\colon [n] \to [n]$ that leave the first row invariant ($\zeta(j) = \zeta\sigma_1(j)$ for all j) is precisely $\deg P$. Similarly, the number of permutations σ_2 that leave the second row invariant ($\eta(j) = \eta\sigma_2(j)$ for all j) is precisely $\deg Q$. Then every possible permutation $\sigma\colon [n] \to [n]$ will arise as a composition $\sigma_2\sigma_1^{-1}$, and will be counted exactly $\deg \chi$ times. □

The **identity multation** ("identitation") ι_A of a multiset A is the multation in which every element is paired off with itself. It is clear that composition is associative and that the identity multations act as identities. Recalling our long-running convention of a fixed base ring R of scalars, we may thus define:

Definition 2. The **nth multiset category** is defined in the following way. The objects are the multisets of cardinality exactly n. Given two multisets A and B, the arrow set $\mathfrak{MSet}_n(A,B)$ will be the free module generated by the multations $A \to B$. ◇

4. The Divided Power Functors

Multisets have a canonical representation as functors. For A a multiset, we let

$$\Gamma^A = \bigotimes_{a \in A} \Gamma^a.$$

Definition 3. The n**th divided power category** $\mathfrak{DP}_n$ is the full subcategory of $\mathfrak{HPol}_n$ consisting only of the functors Γ^A, where $A \in \mathfrak{MSet}_n$. $\diamond$

Theorem 2. *The functor*

$$\Xi \colon \mathfrak{MSet}_n \to \mathfrak{DP}_n,$$

taking the multiset A to the functor Γ^A, and a multation $\varphi\colon A \to B$ with $\deg_\varphi(a,b) = g_{ab}$ *to the natural transformation* $\overline{\varphi}\colon \Gamma^B \to \Gamma^A$ *given by the formula*

$$\bigotimes_{b \in \#B} y_b^{[\sum_{a\in\#A} \deg_\varphi(a,b)]} \mapsto \bigotimes_{a \in \#A} \prod_{b \in \#B} y_b^{[\deg_\varphi(a,b)]},$$

is a category anti-isomorphism.

Proof. Let the multation $\varphi\colon A \to B$ satisfy $\deg_\varphi(a,b) = g_{ab}$ for $a \in \#A$ and $b \in \#B$, so that it will correspond to the natural transformation $\overline{\varphi}\colon \Gamma^B \to \Gamma^A$ given by

$$\bigotimes_{b \in \#B} y_b^{[\sum_{a\in\#A} g_{ab}]} \mapsto \bigotimes_{a \in \#A} \prod_{b \in \#B} y_b^{[g_{ab}]}.$$

Suppose also that a $\psi\colon B \to C$ is given, with $\deg_\psi(b,c) = h_{bc}$ for $b \in \#B$ and $c \in \#C$, so that it corresponds to the following natural transformation $\overline{\psi}\colon \Gamma^C \to \Gamma^B$:

$$\bigotimes_{c \in \#C} x_c^{[\sum_{b\in\#B} h_{bc}]} \mapsto \bigotimes_{b \in \#B} \prod_{c \in \#C} x_c^{[h_{bc}]}.$$

We first calculate $\psi\varphi$. Let

$$\alpha\colon [n] \to A, \qquad \beta\colon [n] \to B, \qquad \gamma\colon [n] \to B, \qquad \delta\colon [n] \to C$$

be multations, such that (α,β) induces φ, and (γ,δ) induces ψ.

$$\begin{aligned}
\psi\varphi &= \begin{bmatrix} \gamma(1) & \cdots & \gamma(n) \\ \delta(1) & \cdots & \delta(n) \end{bmatrix} \circ \begin{bmatrix} \alpha(1) & \cdots & \alpha(n) \\ \beta(1) & \cdots & \beta(n) \end{bmatrix} \\
&= \frac{1}{\deg\varphi \deg\psi} \begin{pmatrix} \gamma(1) & \cdots & \gamma(n) \\ \delta(1) & \cdots & \delta(n) \end{pmatrix} \circ \begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \beta(1) & \cdots & \beta(n) \end{pmatrix} \\
&= \frac{1}{\deg\varphi \deg\psi} \sum_\sigma \begin{pmatrix} \alpha(1) & \cdots & \alpha(n) \\ \delta\sigma(1) & \cdots & \delta\sigma(n) \end{pmatrix}
\end{aligned}$$

where the sum is taken over all bijections $\sigma\colon [n] \to [n]$ such that $\beta\sigma(j) = \beta(j)$ for all j. Now fix natural numbers k_{abc}, and consider only those σ having exactly k_{abc} indices j for which

$$\alpha(j) = a, \qquad \beta(j) = \beta\sigma(j) = b, \qquad \gamma\sigma(j) = c.$$

By the Multation Lemma, there are exactly

$$\frac{\deg\varphi \deg\psi}{\prod_{a,b,c} k_{abc}!}$$

such bijections, so from these, we get a contribution

$$\frac{1}{\prod_{a,b,c} k_{abc}!} \prod_{a,c} \begin{bmatrix} a \\ c \end{bmatrix}^{\sum_b k_{abc}} = \prod_{a,c} \binom{\sum_b k_{abc}}{\{k_{abc}\}_b} \begin{bmatrix} a \\ c \end{bmatrix}^{\sum_b k_{abc}} \tag{6.1}$$

to $\psi\varphi$.

We now want to find the action of $\overline{\varphi} \circ \overline{\psi}$. Letting $y_b = \sum_c s_{bc} x_c$, we get the following action of $\overline{\varphi}$:

$$\bigotimes_b \left(\sum_c s_{bc} x_c \right)^{[\sum_a g_{ab}]} \mapsto \bigotimes_a \prod_b \left(\sum_c s_{bc} x_c \right)^{[g_{ab}]}. \tag{6.2}$$

To find what $\overline{\varphi} \circ \overline{\psi}$ does to an element

$$\bigotimes_c x_c^{[\sum_b h_{bc}]},$$

we seek first the coefficient of

$$\bigotimes_b \prod_c x_c^{[h_{bc}]}$$

in the left-hand side of (6.2), which is

$$\prod_{b,c} s_{bc}^{h_{bc}}.$$

The answer is then the coefficient of this in the right-hand side of (6.2). This coefficient may be collected in different ways. Choosing $s_{bc}^{k_{abc}}$ from the factor

$$\left(\sum_c s_{bc} x_c \right)^{[g_{ab}]}$$

leads to a term

$$\bigotimes_a \prod_b \prod_c x_c^{[k_{abc}]} = \bigotimes_a \prod_c \binom{\sum_b k_{abc}}{\{k_{abc}\}_b} x_c^{[\sum_b k_{abc}]},$$

in $\overline{\varphi} \circ \overline{\psi}$, which is exactly what (6.1) predicts.

This proves the functoriality of Ξ. It should be more or less clear that every natural transformation $\Gamma^B \to \Gamma^A$ is of the form designated, and uniquely so, which proves Ξ is full and faithful. $\square$

The proof is complicated, and is best understood by means of studying examples. An alternative, conceptually simpler, proof appears in [15].

Example 4. The multation

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

corresponds to the natural transformation $\Gamma^1 \otimes \Gamma^1 \to \Gamma^2$ given by

$$x^{[1]} \otimes y^{[1]} \mapsto x^{[1]}y^{[1]},$$

while the multation

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

corresponds to the transformation $\Gamma^2 \to \Gamma^1 \otimes \Gamma^1$ mapping

$$x^{[2]} \mapsto x^{[1]} \otimes x^{[1]}.$$

For another example, consider the two multisets $\{1, 1, 2\}$ and $\{1, 2, 2\}$. They correspond to the divided power functors $\Gamma^2 \otimes \Gamma^1$ and $\Gamma^1 \otimes \Gamma^2$, respectively. The two multations

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}$$

correspond to the two natural transformations $\Gamma^1 \otimes \Gamma^2 \to \Gamma^2 \otimes \Gamma^1$ given by

$$x^{[1]} \otimes y^{[2]} \mapsto y^{[2]} \otimes x^{[1]} \qquad x^{[1]} \otimes y^{[2]} \mapsto x^{[1]}y^{[1]} \otimes y^{[1]},$$

respectively. △

5. Homogeneous Polynomial Functors

We now turn to combinatorially interpreting homogeneous polynomial functors, and cite [15] as our reference. But first we state and prove yet another Yoneda Lemma.

Theorem 3: The Multihomogeneous Yoneda Lemma. *Let A be a multiset with $|A| = n$ and $\#A = [n]$, and F be a homogeneous functor of degree n. The map*

$$\Upsilon\colon \operatorname{Nat}(\Gamma^A, F) \to F_A^\dagger(R^n)$$
$$\eta \mapsto \eta_{R^n}(1^{\otimes[A]})$$

is an isomorphism of modules.

Proof. We have, by the Homogeneous Yoneda Lemma,

$$\bigoplus_{\substack{\#A \subseteq [n] \\ |A| = n}} F_A^\dagger(R^n) = F(R^n)$$

$$
\begin{aligned}
&\cong \mathrm{Nat}(\Gamma^n \operatorname{Hom}(R^n, -), F) \\
&= \mathrm{Nat}\left(\bigoplus_{\substack{\#A \subseteq [n] \\ |A| = n}} \Gamma^A, F \right) \\
&= \bigoplus_{\substack{\#A \subseteq [n] \\ |A| = n}} \mathrm{Nat}(\Gamma^A, F),
\end{aligned}
$$

and it is easy to see that the map Υ is the A-component of the original Yoneda map. □

Theorem 4. *The functor*

$$\Phi_{\mathfrak{MSet}_n} : \mathfrak{HPol}_n \to \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod}),$$

where $\Phi_{\mathfrak{MSet}_n}(F) \colon \mathfrak{MSet}_n \to \mathfrak{Mod}$ *takes*

$$
\begin{aligned}
A &\mapsto \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^A, F) \cong F(R^{\#A})_A \\
[\varphi \colon A \to B] &\mapsto \left[\varphi^* \colon \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^A, F) \to \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^B, F)\right],
\end{aligned}
$$

is a category equivalence (note that, by virtue of the anti-isomorphism $\mathfrak{DP}_n \cong \mathfrak{MSet}_n^\circ$*, the multation* φ *may also be viewed as a natural transformation* $\Gamma^B \to \Gamma^A$*).*

Proof. Let $\Phi = \Phi_{\mathfrak{MSet}_n}$ map the natural transformation $\theta \colon F \to G$ to $\Phi(\theta) \colon \Psi(F) \to \Psi(G)$, given by

$$\Phi(\theta)_A = \theta_* \colon \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^A, F) \to \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^A, G).$$

Functoriality of Φ is obvious.

Note that the functor

$$\mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^-, F) \colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

corresponds, under the category anti-isomorphism $\mathfrak{DP}_n \cong \mathfrak{MSet}_n^\circ$, to the functor

$$\mathrm{Nat}_{\mathfrak{HPol}_n}(-, F) \colon \mathfrak{DP}_n \to \mathfrak{Mod},$$

and it follows that

$$
\begin{aligned}
\mathrm{Nat}(\Phi(F), \Phi(G)) &= \mathrm{Nat}(\mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^-, F), \mathrm{Nat}_{\mathfrak{HPol}_n}(\Gamma^-, G)) \\
&\cong \mathrm{Nat}(\mathrm{Nat}_{\mathfrak{HPol}_n}(-, G), \mathrm{Nat}_{\mathfrak{HPol}_n}(-, F)) \cong \mathrm{Nat}_{\mathfrak{HPol}_n}(F, G),
\end{aligned}
$$

when applying the (ordinary) Yoneda Lemma. This proves that Φ is fully faithful.

To show Ψ is essentially surjective, let $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$ be given, and define $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ by

$$F(R^X) \mapsto \bigoplus_{\substack{\#A \subseteq X \\ |A|=n}} J(A)$$

(where, of course, X is a set, but A ranges over multisets). Also, given

$$S = \sum s_{yx}\sigma_{yx} \colon R^X \to R^Y,$$

let the $J(A) \to J(B)$ component of $F(S)$ be given by

$$\sum_{\varphi\colon A \to B} \left(\prod s_{\varphi(a)a}\right) J(\varphi),$$

(the sum is taken over all multations $\varphi\colon A \to B$). Here, as before, we let $\sigma_{yx}\colon R^X \to R^Y$ denote the homomorphism that takes 1_x to 1_y and every other 1_z to 0.

Showing this is a functor is left for the reader, and we instead concentrate on showing $\Phi(F) = J$. For a multation $\varphi\colon A \to B$, define a (formal) divided power by

$$\sigma^{[\varphi]} = \prod \sigma_{yx}^{[\deg_\varphi(x,y)]}.$$

A little thought shows that φ^* takes

$$\mathrm{Nat}(\Gamma^A, F) \cong F(R^{\#A})_A \ni y \mapsto F_{\sigma^{[\varphi]}}(y) \in F(R^{\#B})_B \cong \mathrm{Nat}(\Gamma^B, F),$$

and also that $J(\varphi) = F_{\sigma^{[\varphi]}}$. Hence

$$\Phi(F)(A) = \mathrm{Im}\, F_{\pi^{[A]}} = \mathrm{Im}\, F_{\sigma^{[1_A]}} = \mathrm{Im}\, J(1_A) = \mathrm{Im}\, 1_{J(A)} = J(A)$$

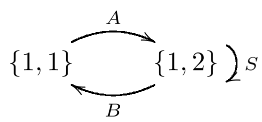
and

$$\Phi(F)(\varphi) = \varphi^* = F_{\sigma^{[\varphi]}} = J(\varphi).$$

□

6. Homogeneous Quadratic Functors

We here determine the structure of $\mathfrak{HPol}_2$ by classifying the quadratic functors. To find the multiset descriptions of quadratic functors, we first draw the (skeletal) structure of the category $\mathfrak{MSet}_2$:



$\circ$	A	B	S
A	$-$	$\iota + S$	$-$
B	2ι	$-$	B
S	A	$-$	ι

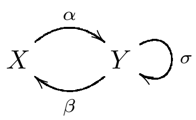
Table 6.1: Multiplication table for $\mathfrak{MSet}_2$.

Every multation reduces to a linear combination of identity multations and the following:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The multiplication table is given in Table 6.1. Compare this with Table 5.1 — the only difference lies in the value of the product BA.

Theorem 5. *A quadratic homogeneous functor is equivalent to the following data: modules X and Y, together with homomorphisms α, β, σ as indicated:*



These homomorphisms are subject to the following four relations:

$$\alpha\beta = 1 + \sigma \qquad \beta\sigma = \beta \qquad \sigma\alpha = \alpha \qquad \sigma^2 = 1$$

Evidently $\sigma = \alpha\beta - \iota$ is dispensable. It is enough to have α and β, subject to the single relation

$$\beta\alpha = 2.$$

CHAPTER 7

Numerical versus Strict Polynomial Functors

> [...] le plus beau projet de notre académie,
> Une entreprise noble et dont je suis ravie,
> Un dessein plein de gloire, et qui sera vanté
> Chez tous les beaux esprits de la postérité [...]
>
> — Molière, *Les Femmes savantes*

1. The Ariadne Functor

To state and prove the main result of this section, we need some heavy notation. For the duration of this section, let n be a fixed natural number.

Let P be a maze. A **multiplicity assignment** (of degree n) is a function

$$\mu\colon P \to \mathbf{Z}^+,$$

such that

$$\sum_{p\in P} \mu(p) = n.$$

Note that P is a multiset; when we say "function", we must therefore imagine the passages of P to be labelled and distinguished, for example by some multation $[n] \to P$. Exactly how this is done will not matter, since we will always sum over all possible multiplicity assignments.

If P had been a set, a multiplicity assignment would amount to no more than specifying a multiset structure. But P is not a set, and we certainly wish to avoid speaking of multisets supported by multisets, hence the new terminology.

The **degree** of the multiplicity assignment μ is defined to be

$$\deg\mu = \prod_{p\in P} \mu(p)!$$

(as for multisets).

To a given P with multiplicity assignment $\mu\colon P \to \mathbf{Z}^+$, we associate a multation

$$\prod_{[p\colon x\to y]\in P} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} = \deg\mu \cdot \prod_{[p\colon x\to y]\in P} \begin{bmatrix} x \\ y \end{bmatrix}^{[\mu(p)]}.$$

Because $\sum_{p\in P} \mu(p) = n$, this will always be a multation on a set with n elements (but not always on the same set).

We now define our main object of study. Given a maze P, we let $A_n(P)$ be the following sum of multations:

$$A_n(P) = \sum_{\mu\colon P\to \mathbf{Z}^+} \left(\prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} \right). \tag{7.1}$$

This will provide a functor from $\mathfrak{Laby}$ to $\mathfrak{MSet}_n$, which we now set out to prove. We first prove that A_n respects the relations in $\mathfrak{Laby}$. It is clear that $A_n(P) = 0$ if a single passage of P is labelled 0. Now to show that

$$A_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) =$$
$$A_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) + A_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) + A_n\left(P \cup \left\{ u \underset{b}{\overset{a}{\rightrightarrows}} v \right\}\right).$$

This is an immediate consequence of the equation

$$\begin{aligned}
(a+b)^{[m]} \begin{bmatrix} u \\ v \end{bmatrix}^m & \prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} \\
&= a^{[m]} \begin{bmatrix} u \\ v \end{bmatrix}^m \prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} \\
&+ b^{[m]} \begin{bmatrix} u \\ v \end{bmatrix}^m \prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} \\
&+ \sum_{\substack{i+j=m \\ i,j\geq 1}} a^{[i]} b^{[j]} \begin{bmatrix} u \\ v \end{bmatrix}^m \prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)},
\end{aligned}$$

where, for a fix multiplicity assignment

$$\mu\colon P \cup \left\{ u \xrightarrow{a+b} v \right\} \to \mathbf{Z}^+,$$

we have let

$$m = \mu(u \xrightarrow{a+b} v).$$

Finally, let $P\colon Y \to Z$ and $Q\colon X \to Y$ be two mazes. To show that A_n is functorial, we calculate

$$A_n(PQ) = A_n\left(\sum_{S \sqsubseteq P\ Q} S \right)$$

$$= \sum_{S \sqsubseteq P\ Q} \sum_{\xi\colon S \to \mathbf{Z}^+} \left(\prod_{[s\colon x \to z] \in S} \overline{s}^{[\xi(s)]} \begin{bmatrix} x \\ z \end{bmatrix}^{\xi(s)} \right)$$

$$= \sum_{S \sqsubseteq P\ Q} \sum_{\xi} \frac{1}{\deg \xi} \prod_{[s\colon x \to z] \in S} \left(\overline{s} \begin{bmatrix} x \\ z \end{bmatrix} \right)^{\xi(s)}.$$

Similarly,

$$A_n(P) \circ A_n(Q)$$

$$= \sum_{\mu,\nu} \left(\prod_{[p\colon y \to z] \in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} y \\ z \end{bmatrix}^{\mu(p)} \circ \prod_{[q\colon x \to y] \in Q} \overline{q}^{[\nu(q)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\nu(q)} \right)$$

$$= \sum_{\mu,\nu} \frac{1}{\deg \mu \deg \nu} \prod_{[p\colon y \to z] \in P} \left(\overline{p} \begin{bmatrix} y \\ z \end{bmatrix} \right)^{\mu(p)} \circ \prod_{[q\colon x \to y] \in Q} \left(\overline{q} \begin{bmatrix} x \\ y \end{bmatrix} \right)^{\nu(q)}.$$

Using the Multation Lemma, these two expressions are easily seen to be equal. We thus obtain:

Theorem 1. *The formulæ*

$$A_n(X) = \bigoplus_{\substack{\# A \subseteq X \\ |A| = n}} A$$

$$A_n(P) = \sum_{\mu\colon P \to \mathbf{Z}^+} \left(\prod_{[p\colon x \to y] \in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix}^{\mu(p)} \right),$$

for X a set and P a maze, provide a linear functor

$$A_n\colon \mathfrak{Laby} \to \mathfrak{MSet}_n.$$

Definition 1. This functor is called the nth **Ariadne functor** ◇

Theorem 2. *Over a numerical base ring, the Ariadne functor factors through the quotient category $\mathfrak{Laby}_n$:*

$$A_n\colon \mathfrak{Laby}_n \to \mathfrak{MSet}_n.$$

Proof. We must show that A_n respects the relations defining the quotient category $\mathfrak{Laby}_n$. It is clear that $A_n(P) = 0$ when $|P| > n$, for then no multiplicity assignments on P exist.

To prove that A_n respects the relation

$$P = \sum_{\substack{\# A = P \\ |A| \leq n}} \prod_{p \in P} \binom{\overline{p}}{\deg_A p} E_A,$$

we note first of all, that it is enough to prove it respects the special case

$$P \cup \left\{ u \xrightarrow{a} v \right\} = \sum_{k=1}^{\infty} \binom{a}{k} \left[P \cup \bigcup_k \{ u \xrightarrow{1} v \} \right].$$

To do that, we apply the Ariadne functor. By considering only those multiplicity assignments μ satisfying $\mu(a) = m$, for some fixed m, and acting in a certain fixed manner on P, we see (after some computation) that our task reduces to establishing the equality

$$a^{[m]} = \sum_{k=1}^{\infty} \binom{a}{k} \sum_{\delta\colon [k] \to \mathbf{Z}^+} \prod_{j \in [k]} 1^{[\delta(j)]},$$

where the sum is taken over those δ fulfilling $\sum \delta(j) = m$. But

$$m! \sum_{k=1}^{\infty} \binom{a}{k} \sum_{\delta\colon [k] \to \mathbf{Z}^+} \prod_{j \in [k]} 1^{[\delta(j)]} = \sum_{k=1}^{\infty} \binom{a}{k} \sum_{\delta\colon [k] \to \mathbf{Z}^+} \binom{m}{\{\delta(j)\}_j} = a^m.$$

This is because the inner sum counts the number of ways m distinct objects may be placed in k distinct boxes, with no box left empty. The total sum then counts the number of ways to distribute the m objects into a total of a boxes. The proof is finished. □

2. Out of the Labyrinth

The Ariadne functor leads the way out of the labyrinth category. More precisely, it leads to the following theorem on how to pass from a multiset functor $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$ to a labyrinth functor $H\colon \mathfrak{Laby} \to \mathfrak{Mod}$. The functor $(A_n)^*$ is in effect the forgetful functor

$$\mathfrak{HPol}_n \to \mathfrak{Num}_n,$$

and reflects combinatorially what happens when we take a homogeneous functor, and view it simply as a numerical one.

Theorem 3.

$$\Phi_{\mathfrak{Laby}} \circ \Phi^{-1}_{\mathfrak{MSet}_n} = (A_n)^*.$$

Proof. We must show that, for a functor $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$,

$$\Phi_{\mathfrak{Laby}} \Phi^{-1}_{\mathfrak{MSet}_n}(J) = J \circ A_n.$$

Denoting $H = \Phi_{\mathfrak{Laby}} \Phi^{-1}_{\mathfrak{MSet}_n}(J)$, we have, for a finite set X,

$$H(X) = \Phi^{-1}_{\mathfrak{MSet}_n}(J)^{\dagger}(R|_X) = \operatorname{Im} \Phi^{-1}_{\mathfrak{MSet}_n}(J) \left(\Diamond_{x \in X} \pi_x \right)$$

$$= \operatorname{Im} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} \Phi^{-1}_{\mathfrak{MSet}_n}(J) \left(\sum_{y \in Y} \pi_y \right).$$

The $J(A) \to J(B)$ component of

$$\sum_{Y \subseteq X} (-1)^{|X|-|Y|} \Phi^{-1}_{\mathfrak{MSet}_n}(J) \left(\sum_{y \in Y} \pi_y \right)$$

is

$$\sum_{Y \subseteq X} (-1)^{|X|-|Y|} \sum_{\varphi \colon A \to B} \left(\prod \delta^Y_{\varphi(a)a} \right) J(\varphi),$$

where we have defined

$$\delta^Y_{ba} = \begin{cases} 1 & \text{if } a = b \in Y \\ 0 & \text{else.} \end{cases}$$

The only surviving components will therefore be those where $A = B$, $\varphi = \iota_A$, and $\#A \subseteq Y$. Hence

$$\begin{aligned} H(X) &= \operatorname{Im} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} \sum_{\substack{\#A \subseteq Y \\ |A|=n}} J(\iota_A) \\ &= \operatorname{Im} \sum_{Y \subseteq X} (-1)^{|X|-|Y|} \sum_{\substack{\#A \subseteq Y \\ |A|=n}} 1_{J(A)} \\ &= \operatorname{Im} \sum_{\substack{\#A = X \\ |A|=n}} 1_{J(A)} = \bigoplus_{\substack{\#A = X \\ |A|=n}} J(A) = JA_n(X). \end{aligned}$$

The fourth step was due to the Multiset Principle of Inclusion and Exclusion.

Turning to $H(P)$, where $P \colon X \to Y$ is a maze, we first suppose that P has no parallel passages. We may label the passages as $p_i \colon x_i \to y_i$, for $1 \le i \le k$.

$$\begin{aligned} H(P) &= \Phi^{-1}_{\mathfrak{MSet}_n}(J) \left(\Diamond\, \overline{p_i} \sigma_{y_i x_i} \right) \\ &= \sum_{I \subseteq [k]} (-1)^{k-|I|} \Phi^{-1}_{\mathfrak{MSet}_n}(J) \left(\sum_{i \in I} \overline{p_i} \sigma_{y_i x_i} \right), \end{aligned}$$

of which the $J(A) \to J(B)$ component is

$$\begin{aligned} &\sum_{I \subseteq [k]} (-1)^{k-|I|} \sum_{\varphi \colon A \to B} \left(\prod \overline{p}^I_{\varphi(a)a} \right) J(\varphi) \\ &\qquad = \sum_{\varphi \colon A \to B} \left(\sum_{I \subseteq [k]} (-1)^{k-|I|} \prod \overline{p}^I_{\varphi(a)a} \right) J(\varphi), \quad (7.2) \end{aligned}$$

where we have defined

$$\overline{p}^I_{ba} = \begin{cases} \overline{p}_i & \text{if } a = x_i \text{ and } b = y_i \text{ for } i \in I \\ 0 & \text{else.} \end{cases}$$

We see that, for the coefficient of $J(\varphi)$ to be non-zero, all elements of the multation φ must "correspond" to passages in P. The converse also holds, namely that all passages of P must be represented in φ. This is because, if a passage p_j be "missing" from φ, sets I with and without j in (7.2) will give rise to terms of alternating signs, which will cancel each other out. Hence the coefficient of $J(\varphi)$ will survive only if φ is of the form

$$\varphi = \prod_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}^{[m_i]},$$

for positive integers $m_1 + \cdots + m_k = n$. Furthermore, we observe that only $I = [k]$ will yield a non-zero contribution in (7.2), so consequently,

$$\begin{aligned} H(P) &= \sum_{m_1+\cdots+m_k=n} \left(\prod \overline{p}_i^{m_i}\right) J\left(\prod_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}^{[m_i]}\right) \\ &= \sum_{m_1+\cdots+m_k=n} \left(\prod \overline{p}_i^{[m_i]}\right) J\left(\prod_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}^{m_i}\right) \\ &= JA_n(P). \end{aligned}$$

Consider now a maze with a pair of parallel passages

$$\begin{aligned} Q &= P \cup \left\{ u \overset{a}{\underset{b}{\rightrightarrows}} v \right\} \\ &= P \cup \left\{ u \xrightarrow{a+b} v \right\} - P \cup \left\{ u \xrightarrow{a} v \right\} - P \cup \left\{ u \xrightarrow{b} v \right\}, \end{aligned}$$

where we inductively assume the equations

$$\begin{aligned} H\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) &= JA_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) \\ H\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) &= JA_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) \\ H\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) &= JA_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) \end{aligned}$$

hold. Then

$$\begin{aligned} H(Q) &= JA_n(P \cup \{a+b\}) - JA_n(P \cup \{a\}) - JA_n(P \cup \{b\}) \\ &= \sum_{\mu\colon P \cup \{a+b\} \to \mathbf{Z}^+} J\left(\prod_{[p\colon x \to y] \in P} \overline{p}^{[\mu(p)]} \begin{bmatrix} x \\ y \end{bmatrix} \cdot (a+b)^{[\mu(a+b)]} \begin{bmatrix} u \\ v \end{bmatrix}^{\mu(a+b)}\right) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\mu\colon P\cup\{a\}\to\mathbf{Z}^+} J\left(\prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]}\begin{bmatrix}x\\y\end{bmatrix}\cdot a^{[\mu(a)]}\begin{bmatrix}u\\v\end{bmatrix}^{\mu(a)}\right)\\
&- \sum_{\mu\colon P\cup\{b\}\to\mathbf{Z}^+} J\left(\prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]}\begin{bmatrix}x\\y\end{bmatrix}\cdot b^{[\mu(b)]}\begin{bmatrix}u\\v\end{bmatrix}^{\mu(b)}\right)\\
&= \sum_{\mu\colon P\cup\{a,b\}\to\mathbf{Z}^+} J\left(\prod_{[p\colon x\to y]\in P} \overline{p}^{[\mu(p)]}\begin{bmatrix}x\\y\end{bmatrix}\cdot a^{[\mu(a)]}b^{[\mu(b)]}\begin{bmatrix}u\\v\end{bmatrix}^{\mu(a)+\mu(b)}\right)\\
&= JA_n(Q),
\end{aligned}
$$

as desired. □

3. Simple Mazes

In the preceding section we saw how the Ariadne functor provides the bridge between homogeneous and numerical functors. We shall here see how it may be used as a numerical invariant, which can shed light on the internal structure of the labyrinth categories.

Definition 2. A maze of which all passages carry the label 1, is called a **simple maze**. ◇

Theorem 4. *Given finite sets X and Y, the simple mazes are linearly independent in the module $\mathfrak{Laby}(X, Y)$.*

Proof. Suppose we have a relation

$$\sum_j a_{n,j} P_{n,j} + \sum_j a_{n+1,j} P_{n+1,j} + \cdots = 0$$

in $\mathfrak{Laby}(X, Y)$, where $a_{i,j} \in R$ and each $P_{i,j}$ denotes a simple maze of cardinality i. All $P_{i,j}$ are of course assumed to be distinct. The nth Ariadne functor will kill all mazes with cardinality greater than n, and the end result after application will be

$$\sum_j a_{n,j} A_n(P_{n,j}) = 0.$$

But since the $P_{n,j}$ are distinct simple mazes, the $A_n(P_{n,j})$ will all denote distinct multations. Hence all $a_{n,j} = 0$. The claim now follows by induction. □

Theorem 5. *Let the base ring be numerical. Given finite sets X and Y, the simple mazes constitute a basis for the module $\mathfrak{Laby}_n(\mathbf{Z})(X, Y)$, which is thus free.*

Proof. The above proof for linear independence goes through exactly as before, because the Ariadne functor factors through the quotient category $\mathfrak{Laby}_n$. Using the defining equation for $\mathfrak{Laby}_n$, we see that any maze will reduce to simple ones. □

And as an immediate corollary:

Theorem 6.

$$\mathfrak{Labh}_n(R) \cong R \otimes_{\mathbf{Z}} \mathfrak{Labh}_n(\mathbf{Z}).$$

4. The Wedge Category

For reference, we devote this section to investigating the connection between our mazes and the category of surjections explored by Pirashvili et al. in [1].

Let C be a category possessing *weak pullbacks*; that is, a finite number of universal ways to complete an incomplete pullback square. For two objects $X, Y \in C$, a **wedge**[1] from X to Y is a diagram (read from left to right):

$$X \leftarrow U \rightarrow Y$$

We identify the top and bottom wedges in the following commutative diagram, with the middle column an isomorphism:

$$\begin{array}{ccccc} X & \longleftarrow & U & \longrightarrow & Y \\ \| & & \updownarrow & & \| \\ X & \longleftarrow & V & \longrightarrow & Y \end{array}$$

Define the **wedge category** $\hat{C}$, based on C, in the following way: Its objects will be those of C. Its arrows will be formal sums of wedges of C (identified under the just described equivalence relation), in the free monoid they generate. Composition of wedges amounts to summing weak pullbacks:

$$[\, X \leftarrow U \rightarrow Y \leftarrow V \rightarrow Z \,] = \sum [\, X \leftarrow W \rightarrow Z \,]$$

where the sum is taken over all weak pullbacks:

$$\begin{array}{ccccccc} & & & W & & & \\ & & \swarrow & & \searrow & & \\ & U & & & & V & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

(If C does indeed possess pullbacks, there is no need to revert to these formal sums, and composition can be defined simply as the pullback.) It will now be observed, confer [1], that $\hat{C}$ is a preadditive category.

The category Ω of finite sets and surjections possesses weak pullbacks. Namely, the square:

$$\begin{array}{ccc} W & \dashrightarrow & B \\ \downarrow & & \downarrow \beta \\ A & \xrightarrow[\alpha]{} & P \end{array}$$

[1] [1] uses *flèche*, a word which is usually used to denote a *single* arrow.

is a weak pullback iff

$$W \sqsubseteq A \times_P B = \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\},$$

so that the projections on A and B are both onto. We call $A \times_P B$ (the pullback in $\mathfrak{Set}$) the **principal pullback**.

The existence of weak pullbacks ensures that the wedge category $\hat{\Omega}$ may be created. We form a quotient category $\hat{\Omega}_n$ by forcing all wedges:

$$X \leftarrow U \rightarrow Y$$

of which $|U| > n$, to equal 0. It turns out that this category is already known to us as $\mathfrak{Laby}_n(\mathbf{Z})$.

Theorem 7.

$$\hat{\Omega}_n \cong \mathfrak{Laby}_n(\mathbf{Z}).$$

Proof. The objects of both categories are finite sets, and each set will of course correspond to itself. Wedges will correspond to simple mazes; more precisely, the wedge

$$\varphi = \left[\, X \xleftarrow{\varphi^*} U \xrightarrow{\varphi_*} Y \,\right]$$

in $\hat{\Omega}_n$ will correspond to the simple maze $X \to Y$, of which the passages $x \to y$ number exactly

$$\left|(\varphi^*, \varphi_*)^{-1}(x,y)\right|$$

(the cardinality of the fibre above $(x,y) \in X \times Y$). Since the simple mazes from X to Y form a basis, this correspondence is full and faithful.

It remains to show functoriality. Suppose

$$\varphi = \left[\, X \xleftarrow{\varphi^*} U \xrightarrow{\varphi_*} Y \,\right], \qquad \psi = \left[\, Y \xleftarrow{\psi^*} V \xrightarrow{\psi_*} Z \,\right]$$

are two wedges, corresponding to the mazes $P\colon X \to Y$ and $Q\colon Y \to Z$, where the number of passages $x \to y$ in P equals

$$|(\varphi^*, \varphi_*)^{-1}(x,y)|,$$

and the number of passages $y \to z$ in Q equals

$$|(\psi^*, \psi_*)^{-1}(y,z)|.$$

The theorem then follows from the observation that $U \times_Y V$ may be naturally identified with $Q \quad P$, and subsets $W \sqsubseteq U \times_Y V$ with submazes $R \subseteq Q \quad P$. □

The main result of [1] is, in our language, the following[2]:

[2]They restrict their attention to *pointed* functors, that is, functors that take 0 to itself. We have circumvented this restriction by considering $\emptyset$ to be a finite set.

Theorem 8.

$$\mathfrak{Num}_n(\mathbf{Z}) \sim \mathrm{Fun}(\hat{\Omega}_n, {}_{\mathbf{Z}}\mathfrak{Mod}).$$

Proof. Follows immediately from the preceding theorem and the equivalence $\mathfrak{Num}_n \sim \mathrm{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod})$. □

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