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# Poincaré series of some hypergraph algebras

Eric Emtander
Ralf Fröberg
Fatemeh Mohammadi
Somayeh Moradi

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Department of Mathematics
Stockholm University

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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

# Poincaré series of some hypergraph algebras

E. Emtander, Stockholm university
erice@math.su.se

R. Fröberg, Stockholm university
ralff@math.su.se

F. Mohammadi, Amirkabir University, Tehran
f_mohammadi@aut.ac.ir

S. Moradi Amirkabir University, Tehran
s_moradi@aut.ac.ir

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**Abstract**

A hypergraph $H = (V, E)$, where $V = \{x_1, \ldots, x_n\}$ and $E \subseteq 2^V$ defines a hypergraph algebra $R_H = k[x_1, \ldots, x_n]/(x_{i_1} \cdots x_{i_k}; \{i_1, \ldots, i_k\} \in E)$. All our hypergraphs are $d$-uniform, i.e., $|e_i| = d$ for all $e_i \in E$. We determine the Poincaré series $P_{R_H}(t) = \sum_{i=1}^{\infty} \dim_k \operatorname{Tor}_i^{R_H}(k,k)t^i$ for some hypergraphs generalizing lines, cycles, and stars. We finish by calculating the graded Betti numbers and the Poincaré series of the graph algebra of the wheel graph.

## 1 Introduction

A line is a graph $L_n = (V, E)$, where

$$V = \{x_1, \ldots, x_{n+1}\} \text{ and } E = \{(x_1, x_2), \ldots, (x_n, x_{n+1})\},$$

a cycle a graph $C_n = (V, E)$, where

$$V = \{x_1, \ldots, x_n\} \text{ and } E = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\},$$

and a star a graph $S_n = (V, E)$, where

$$V = \{x_1, \ldots, x_{n+1}\} \text{ and } E = \{(x_1, x_2), \ldots, (x_1, x_{n+1})\}.$$

In [J 04, Chapter 7] the Betti numbers of their graph algebras,

$$k[x_1, \ldots, x_{n+1}]/(x_1x_2, x_2x_3, \ldots, x_nx_{n+1}),$$

$$k[x_1,\dots,x_n]/(x_1x_2,x_2x_3,\dots,x_{n-1}x_n,x_nx_1),$$

and

$$k[x_1,\dots,x_{n+1}]/(x_1x_2,x_1x_3,\dots,x_1x_{n+1})$$

are determined. This is generalized to certain "hyperlines", "hypercycles", and "hyperstars" in [E-M-M 08]. Here a hyperline is hypergraph with $nd-(n-1)\alpha$ vertices and $n$ edges $e_1,\dots,e_n$, where all edges $e_1,\dots,e_n$ have size $d$, and $e_i\cap e_j\neq\emptyset$ and has size $\alpha$ if and only if $|i-j|=1$, a hypercycle is hypergraph with $n(d-\alpha)$ vertices and $n$ edges $e_1,\dots,e_n$, where all edges have size $d$, and $e_i\cap e_j\neq\emptyset$ and has size $\alpha$ if and only if $|i-j|=1 \pmod n$, and the hyperstar is hypergraph with $n(d-\alpha)$ vertices and $n$ edges $e_1,\dots,e_n$, where all edges have size $d$, and for all $i,j$ $|e_i\cap e_j|=|\cap_{i=1}^n e_i|=\alpha>0$. We denote the line hypergraph and its algebra with $L_n^{d,\alpha}$, the cycle hypergraph and its algebra with $C_n^{d,\alpha}$, and the star hypergraph and its algebra $S_n^{d,\alpha}$. Their Betti numbers were determined in [E-M-M 08, Chapter 3] (in the first two cases with the restriction $2\alpha\leq d$). In this paper we will determine the Poincaré series for the same algebras. The Poincaré series of a graded $k$-algebra $R=k[x_1,\dots,x_n]/I$ is $P_R(t)=\sum_{i=1}^{\infty}\dim_k \mathrm{Tor}_i^R(k,k)t^i$. [G-L 69] is an excellent source for results on Poincaré series.

## 2 Hypercycles and hyperlines when $d=2\alpha$

We start with the case $d=2\alpha$. If $e_i=\{v_{i1},\dots,v_{i\alpha},v'_{i1},\dots,v'_{i\alpha}\}$, where $\{v'_{ij}\}\in e_{i+1}$, we start by factoring out all $v_{ik}-v_{il}$ and $v'_{ik}-v'_{il}$. This is a linear regular sequence of length $(n+1)(\alpha-1)$ for the hyperline and of length $n(\alpha-1)$ for the hypercycle. The results are

$$L'_{n,a}=k[x_1,\dots,x_{n+1}]/(x_1^\alpha x_2^\alpha,x_2^\alpha x_3^\alpha,\dots,x_n^\alpha x_{n+1}^\alpha)$$

and

$$C'_{n,a}=k[x_1,\dots,x_n]/(x_1^\alpha x_2^\alpha,x_2^\alpha x_3^\alpha,\dots,x_{n-1}^\alpha x_n^\alpha,x_n^\alpha x_1^\alpha).$$

Then

$$P_{L_n^{2a,a}}(t)=(1+t)^{(n+1)(\alpha-1)}P_{L'_{n,a}}(t)$$

and

$$P_{C_n^{2a,a}}(t)=(1+t)^{n(\alpha-1)}P_{C'_{n,a}}(t).$$

[G-L 69, Theorem 3.4.2 (ii)]. Now $L_n^{2\alpha,\alpha}$ and $C_n^{2\alpha,\alpha}$ have the same Poincaré series as the graph algebras

$$L_n=L_n^{2,1}=k[x_1,\dots,x_{n+1}]/(x_1x_2,x_2x_3,\dots,x_nx_{n+1})$$

and

$$C_n=C_n^{2,1}=k[x_1,\dots,x_n]/(x_1x_2,x_2x_3,\dots,x_{n-1}x_n,x_nx_1)$$

respectively.

For a graded $k$-algebra $\oplus_{i=0}^{\infty} R_i$, the Hilbert series of $R$ is defines as $H_R(t) = \sum_{i=0}^{\infty} \dim_k(R_i)t^i$. The exact sequences

$$0 \longrightarrow (x_{n+1}) \longrightarrow L_n \xrightarrow{x_{n+1}\cdot} L_n \longrightarrow L_n/(x_{n+1}) \longrightarrow 0$$

and

$$0 \longrightarrow (x_{n+1}) \longrightarrow L_n \longrightarrow L_n/(x_{n+1}) \longrightarrow 0$$

and $L_n/(x_{n+1}) \simeq L_{n-1}$ and $(x_{n+1}) \simeq L_{n-2} \otimes k[x]$ gives

$$H_{L_n}(t) = H_{L_{n-1}}(t) + \frac{t}{1-t} H_{L_{n-2}}(t).$$

The exact sequences

$$0 \longrightarrow (x_1, x_{n-1}) \longrightarrow C_n \xrightarrow{x_n\cdot} C_n \longrightarrow L_{n-2} \longrightarrow 0$$

and

$$0 \longrightarrow (x_1, x_{n-1}) \longrightarrow C_n \longrightarrow C_n/(x_1, x_{n-1}) \longrightarrow 0$$

and $C_n/(x_1, x_{n-1}) \simeq L_{n-4} \otimes k[x]$ gives

$$H_{C_n}(t) = H_{L_{n-2}}(t) - \frac{t}{(1-t)} H_{L_{n-4}}(t).$$

Now $C_n$ and $L_n$ are (as all graph algebras) Koszul algebras [F 75, Corollary 2], so $P_{C_n}(t) = 1/H_{C_n}(-t)$ and $P_{L_n}(t) = 1/H_{L_n}(-t)$. Since $L_0 = k[x_1]$ and $L_1 = k[x_1, x_2]/(x_1 x_2)$, we have $H_{L_0}(t) = 1/(1-t)$ and $H_{L_1}(t) = (1+t)/(1-t)$. We give the first Hilbert series:
$H_{L_2}(t) = (1+t-t^2)/(1-t)^2$, $H_{L_3}(t) = (1+2t)/(1-t)^2$,
$H_{L_4}(t) = (1+2t-t^2-t^3)/(1-t)^3$, $H_{L_5}(t) = (1+3t+t^2-t^3)/(1-t)^3$,
$H_{C_3}(t) = (1+2t)/(1-t)$, $H_{C_4}(t) = (1+2t-t^2)/(1-t)^2$,
$H_{C_5}(t) = (1+3t+t^2)/(1-t)^3$, $H_{C_6}(t) = (1+3t-2t^3)/(1-t)^3$.

**Remark** We note that it is probably hard to get one formula for $H_{L_n}(t)$ for all $n$. An indication is that we get the Fibonacci numbers from $H_{L_n}(t)$. For $t = 1/2$ we get $H_{L_n}(1/2) = F_{n+2}$, the $(n+2)$th Fibonacci number if $F_0 = F_1 = 1$.

Thus we get
$P_{L_2}(t) = (1+t)^2/(1-t-t^2)$, $P_{L_3}(t) = (1+t)^2/(1-2t)$,
$P_{L_4}(t) = (1+t)^3/(1-2t-t^2+t^3)$, $P_{L_5}(t) = (1+t)^3/(1-3t+t^2+t^3)$,
$P_{C_3}(t) = (1+t)/(1-2t)$, $P_{C_4}(t) = (1+t)^2/(1-2t-t^2)$,
$P_{C_5}(t) = (1+t)^2/(1-3t+t^2)$, $P_{C_6}(t) = (1+t)^3/(1-3t+2t^3)$.

We collect the results in

**Theorem 2.1** *The Poincaré series of $L_n$ and $C_n$ satisfy the recursion formulas*

$$P_{L_n}(t) = \frac{(1+t)P_{L_{n-1}}(t)P_{L_{n-2}}(t)}{(1+t)P_{L_{n-2}}(t) - tP_{L_{n-1}}(t)}$$

*where* $P_{L_0}(t) = 1 + t$ *and* $P_{L_1}(t) = (1+t)/(1-t)$ *and*

$$P_{C_n}(t) = \frac{(1+t)P_{L_{n-2}}(t)P_{L_{n-4}}(t)}{P_{L_{n-2}}(t) + (1+t)P_{L_{n-4}}(t)}.$$

*Furthermore*

$$P_{L_n^{2\alpha,\alpha}}(t) = (1+t)^{(n+1)(\alpha-1)}P_{L_n}(t)$$

*and*

$$P_{C_n^{2\alpha,\alpha}}(t) = (1+t)^{n(\alpha-1)}P_{C_n}(t).$$

# 3 Hypercycles and hyperlines when $2\alpha < d$

Next we turn to the case $2\alpha < d$. Now each edge has a free vertex, i.e. a vertex which does not belong to any other edge. Then the Taylor resolution is minimal. In this case there is a formula for the Poincaré series in terms of the graded homology of the Koszul complex [F 78, Corollary to Proposition 2]. Let $R$ be a monomial ring for which the Taylor resolution is minimal. Then the homology of the Koszul complex $H(K_R)$ is of the form $H(K_R) = k[u_1, \ldots, u_N]/I$, where $I$ is generated by a set of monomials of degree 2. Define a bigrading induced by $\deg(u_i) = (1, |u_i|)$, where $|u_i|$ is the homological degree. Then $P_R(t) = (1+t)^e/H_R(-t,t)$, where $e$ is the embedding dimension and $H_R(x,y)$ is the bigraded Hilbert series of $H(K_R)$, see [F 78].

We begin with the hypercycle. The homology of the Koszul complex is generated by $\{z_I\}$, where $I = \{i, i+1, \ldots, j\}$ corresponds to a path $\{e_i, e_{i+1}, \ldots, e_j\}$ in $C_n^{d,\alpha}$ (indices counted $\pmod n$). Thus there are $n$ generators in all homological degrees $< n$ and one generator in homological degree $n$. We have $z_I z_J = 0$ if $I \cap J \neq \emptyset$. Thus the surviving monomials are of the form $m = z_{I_1} \cdots z_{I_r}$, where $I_i \cap I_j = \emptyset$ if $i \neq j$. The bidegree of $m$ is $(r, \sum_{j=1}^r |I_j|)$. Let $\sum_{j=1}^r |I_j| = i$. Then $m$ lies in $H(K)_{i,di-(i-r)\alpha}$. The graded Betti numbers are determined in [E-M-M 08, Chapter 3]. The nonzero Betti numbers are $\beta_{i,di-(i-r)\alpha} = \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}$ if $1 \leq r \leq i < n$ and $\beta_{n,n(d-\alpha)} = 1$. (As usual $\binom{a}{b} = 0$ if $b > a$.) This gives the Poincaré series.

Next we consider the hyperline. The homology of the Koszul complex is generated by $\{z_I\}$, where $I = \{i, i+1, \ldots, j\}$ corresponds to a path $\{e_i, e_{i+1}, \ldots, e_j\}$ in $L(n,d,\alpha)$. Thus there are $n+1-i$ generators of homological degree $i$. We have $z_I z_J = 0$ if $I \cap J \neq \emptyset$. The graded Betti numbers are determined in [E-M-M 08, Chapter 3]. The nonzero Betti numbers are $\beta_{i,di-(i-r)\alpha} = \binom{i-1}{r-1}\binom{n-i+1}{r}$ if $1 \leq r \leq i \leq n$. The same reasoning as above gives the Poincaré series. We state the results in a theorem.

**Theorem 3.1** *If* $2\alpha < d$*, then*

$$P_{C_n}(t) = \frac{(1+t)^{n(d-\alpha)}}{1 + \sum_{1 \leq r \leq i < n} (-1)^r \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1} t^{i+r} - t^{n+1}},$$

*and*

$$P_{L_n}(t) = \frac{(1+t)^{n(d-\alpha)+\alpha}}{1+\sum_{1\le r\le i\le n}(-1)^r\binom{i-1}{r-1}\binom{n-i+1}{r}t^{i+r}}.$$

# 4 The hyperstar

We conclude with a hypergraph generalizing the star graph. Suppose $|e_i| = d$ for all $i$, $1 \le i \le n$, and that if $i \neq j$, then $|e_i \cap e_j| = |\cap_{i=1}^n e_i| = \alpha < d$. Then the ideal is of the form $m(m_1, \ldots, m_n)$, where $m$ is a monomial of degree $\alpha$. Then the hypergraph ring $S_n^{d,\alpha}$ is Golod [G-L 69, Theorem4.3.2]. This means that

**Theorem 4.1**

$$P_{S_n^{d,\alpha}}(t) = (1+t)^{|V|}/(1-\sum\beta_i t^{i+1}) = (1+t)^{n(d-\alpha)+\alpha}/(1-\sum\binom{n}{i}t^{i+1}).$$

# 5 The wheel graph

Finally we consider the wheel graph $W_n$, which is $C_n$ with an extra vertex (the center) which is connected to all vertices in $C_n$. We let $W_n$ also denote the graph algebra $k[x_0, \ldots, x_n]/(x_1x_2, x_2x_3, \ldots, x_nx_1, x_0x_1, \ldots, x_0x_n)$.

**Theorem 5.1** *Let $W_n$ be a wheel graph on $n+1$ vertices. Then the Betti numbers of $W_n$ are as follows:*

(i) *If $j > i+1$, then $\beta_{i,j}(k[\Delta_{W_n}]) = \beta_{i,j}(C_n) + \beta_{i-1,j-1}(C_n)$.*
(ii) *If $j = i+1$, then $\beta_{i,i+1}(W_n) = \beta_{i,i+1}(C_n) + \beta_{i-1,i}(C_n) + \binom{n}{i}$.*

*Proof.* Assume that $V(W_n) = \{x_0, x_1, \ldots, x_n\}$ and $C_n = W_n \setminus \{x_0\}$. It is easy to see that $\Delta_{W_n} = \Delta_{C_n} \cup \{x_0\}$, where $\Delta_{W_n}$ and $\Delta_{C_n}$ are the independence complexes of $W_n$ and $C_n$. It implies that for any $i \ge 1$, $H_i(\Delta_{W_n}) = H_i(\Delta_{C_n})$. Thus, if $j > i+1$, from Hochster's formula ([B-H 98, Theorem 5.5.1]) and the observation above one has the result. Now assume that $j = i+1$. Then $\beta_{i,i+1}(W_n) = \sum_{S\subseteq V(W_n),|S|=i+1}\dim(\widetilde{H}_0(\Delta_S)) = \sum_{S\subseteq V(C_n),|S|=i+1}\dim(\widetilde{H}_0(\Delta_S))$ $+\sum_{S\subseteq V(W_n),S=S'\cup\{x_0\}}\dim(\widetilde{H}_0(\Delta_S))$. For any $S \subseteq V(W_n)$ and $S_0 \subseteq V(C_n)$, let $r_S$ and $r'_{S_0}$ denotes the number of connected components of $\Delta_S$ in $V(W_n)$ and $\Delta_{S_0}$ in $V(C_n)$ respectively. Then we have $\sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}}\dim(\widetilde{H}_0(\Delta_S)) = \sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}}(r_S-1)$. For any $S \subseteq V(W_n)$ such that $S = S_0 \cup \{x_0\}$, we have $r_S = r'_{S_0} + 1$. Therefore
$\sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}}\dim(\widetilde{H}_0(\Delta_S)) = \sum_{S_0\subseteq V(C_n),|S_0|=i}\dim(\widetilde{H}_0(\Delta_{S_0})) + \binom{n}{i} = \beta_{i-1,i}(C_n) + \binom{n}{i}$.

The term $\binom{n}{i}$ is the number of subsets $S_0$ of $V(C_n)$ of cardinality $i$.

Substituting the $\beta_{i,j}(C_n)$ from of [J 04, Theorem 7.6.28] we have the following corollary.

**Corollary 5.2** *Let $W_n$ be the wheel graph on $n+1$ vertices. Then the Betti numbers of $W_n$ are as follows:*
*(i) If $n=3$, then $\beta_{2,3}(W_3)=8$, $\beta_{3,4}(W_3)=3$. If $n=4$, then $\beta_{3,4}(W_4)=9$, $\beta_{4,5}(W_4)=2$. Otherwise $\beta_{i,i+1}(W_n)=n\binom{2}{i-1}+\binom{n}{i}$.*
*(ii) If $n=3m$, then $\beta_{2m,n}(W_n)=3m+2$, $\beta_{2m+1,n+1}(W_n)=2$. If $n=3m+1$, then $\beta_{2m+1,n}(W_n)=3m+2$, $\beta_{2m+2,n+1}(W_n)=1$. If $n=3m+2$, then $\beta_{2m,n}(W_n)=\beta_{2m+1,n+1}(W_n)=1$. Otherwise, if $j>i+1$, then $\beta_{i,j}(W_n)=\frac{n}{n-2(j-i)}\binom{n-2(j-i)}{j-i}\binom{j-i-1}{2i-j}$.*

We can also determine the Poincaré series for the wheel graph algebra. This is also a Koszul algebra, and $H_{W_n}(t)=H_{C_n}(t)+t/(1-t)$. Since $P_{W_n}(t)=1/H_{W_n}(-t)$ and $P_{C_n}(t)=1/H_{C_n}(-t)$, this gives

**Theorem 5.3**
$$P_{W_n}(t)=\frac{P_{C_n}(t)(1+t)}{1+t-tP_{C_n}(t)}$$

# References


[B-H 98] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge University Press, 1998.

[E-M-M 08] E. Emtander, F. Mohammadi, S. Moradi, *Some algebraic properties of hypergraphs*, arXiv: 0812.2366

[F 75] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37**, 29–39 (1975).

[F 78] R. Fröberg, *Some complex constructions with applications to Poincaré series*, Springer Lect. Notes in Math. **740**, 272–284 (1978).

[G-L 69] T. H. Gulliksen, G. Levin, *Homology of local rings*, Queen's paper in pure and appl. Math. **20** (1969).

[J 04] S. Jacques, *Betti Numbers of Graph Ideals*, Dissertation, Univ. of Sheffield (2004), arXiv:math/0410107.