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Poincaré series of some hypergraph algebras

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Abstract

A hypergraph $H = (V, E)$, where $V = \{x_1, \ldots, x_n\}$ and $E \subseteq 2^V$ defines a hypergraph algebra $R_H = k[x_1, \ldots, x_n]/(x_{i_1} \cdots x_{i_k}; \{i_1, \ldots, i_k\} \in E)$. All our hypergraphs are d-uniform, i.e., $|e_i| = d$ for all $e_i \in E$. We determine the Poincaré series $P_{R_H}(t) = \sum_{i=1}^{\infty} \dim_k \operatorname{Tor}_i^{R_H}(k, k)t^i$ for some hypergraphs generalizing lines, cycles, and stars. We finish by calculating the graded Betti numbers and the Poincaré series of the graph algebra of the wheel graph.

1 Introduction

A line is a graph $L_n = (V, E)$, where

$$V = \{x_1, \ldots, x_{n+1}\} \text{ and } E = \{(x_1, x_2), \ldots, (x_n, x_{n+1})\},$$

a cycle a graph $C_n = (V, E)$, where

$$V = \{x_1, \ldots, x_n\} \text{ and } E = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\},$$

and a star a graph $S_n = (V, E)$, where

$$V = \{x_1, \ldots, x_{n+1}\} \text{ and } E = \{(x_1, x_2), \ldots, (x_1, x_{n+1})\}.$$

In [J 04, Chapter 7] the Betti numbers of their graph algebras,

$$k[x_1, \ldots, x_{n+1}]/(x_1x_2, x_2x_3, \ldots, x_nx_{n+1}),$$

$$k[x_1,\dots,x_n]/(x_1x_2,x_2x_3,\dots,x_{n-1}x_n,x_nx_1),$$

and

$$k[x_1,\dots,x_{n+1}]/(x_1x_2,x_1x_3,\dots,x_1x_{n+1})$$

are determined. This is generalized to certain "hyperlines", "hypercycles", and "hyperstars" in [E-M-M 08]. Here a hyperline is hypergraph with $nd-(n-1)\alpha$ vertices and n edges $e_1,\dots,e_n$, where all edges $e_1,\dots,e_n$ have size d, and $e_i\cap e_j\neq\emptyset$ and has size α if and only if $|i-j|=1$, a hypercycle is hypergraph with $n(d-\alpha)$ vertices and n edges $e_1,\dots,e_n$, where all edges have size d, and $e_i\cap e_j\neq\emptyset$ and has size α if and only if $|i-j|=1 \pmod n$, and the hyperstar is hypergraph with $n(d-\alpha)$ vertices and n edges $e_1,\dots,e_n$, where all edges have size d, and for all i,j $|e_i\cap e_j|=|\cap_{i=1}^n e_i|=\alpha>0$. We denote the line hypergraph and its algebra with $L_n^{d,\alpha}$, the cycle hypergraph and its algebra with $C_n^{d,\alpha}$, and the star hypergraph and its algebra $S_n^{d,\alpha}$. Their Betti numbers were determined in [E-M-M 08, Chapter 3] (in the first two cases with the restriction $2\alpha\leq d$). In this paper we will determine the Poincaré series for the same algebras. The Poincaré series of a graded k-algebra $R=k[x_1,\dots,x_n]/I$ is $P_R(t)=\sum_{i=1}^\infty \dim_k \operatorname{Tor}_i^R(k,k)t^i$. [G-L 69] is an excellent source for results on Poincaré series.

2 Hypercycles and hyperlines when $d=2\alpha$

We start with the case $d=2\alpha$. If $e_i=\{v_{i1},\dots,v_{i\alpha},v'_{i1},\dots,v'_{i\alpha}\}$, where $\{v'_{ij}\}\in e_{i+1}$, we start by factoring out all $v_{ik}-v_{il}$ and $v'_{ik}-v'_{il}$. This is a linear regular sequence of length $(n+1)(\alpha-1)$ for the hyperline and of length $n(\alpha-1)$ for the hypercycle. The results are

$$L'_{n,a}=k[x_1,\dots,x_{n+1}]/(x_1^\alpha x_2^\alpha,x_2^\alpha x_3^\alpha,\dots,x_n^\alpha x_{n+1}^\alpha)$$

and

$$C'_{n,a}=k[x_1,\dots,x_n]/(x_1^\alpha x_2^\alpha,x_2^\alpha x_3^\alpha,\dots,x_{n-1}^\alpha x_n^\alpha,x_n^\alpha x_1^\alpha).$$

Then

$$P_{L_n^{2a,a}}(t)=(1+t)^{(n+1)(\alpha-1)}P_{L'_{n,a}}(t)$$

and

$$P_{C_n^{2a,a}}(t)=(1+t)^{n(\alpha-1)}P_{C'_{n,a}}(t).$$

[G-L 69, Theorem 3.4.2 (ii)]. Now $L_n^{2\alpha,\alpha}$ and $C_n^{2\alpha,\alpha}$ have the same Poincaré series as the graph algebras

$$L_n=L_n^{2,1}=k[x_1,\dots,x_{n+1}]/(x_1x_2,x_2x_3,\dots,x_nx_{n+1})$$

and

$$C_n=C_n^{2,1}=k[x_1,\dots,x_n]/(x_1x_2,x_2x_3,\dots,x_{n-1}x_n,x_nx_1)$$

respectively.

For a graded k-algebra $\oplus_{i=0}^{\infty} R_i$, the Hilbert series of R is defines as $H_R(t) = \sum_{i=0}^{\infty} \dim_k(R_i)t^i$. The exact sequences

$$0 \longrightarrow (x_{n+1}) \longrightarrow L_n \stackrel{x_{n+1}\cdot}{\longrightarrow} L_n \longrightarrow L_n/(x_{n+1}) \longrightarrow 0$$

and

$$0 \longrightarrow (x_{n+1}) \longrightarrow L_n \longrightarrow L_n/(x_{n+1}) \longrightarrow 0$$

and $L_n/(x_{n+1}) \simeq L_{n-1}$ and $(x_{n+1}) \simeq L_{n-2} \otimes k[x]$ gives

$$H_{L_n}(t) = H_{L_{n-1}}(t) + \frac{t}{1-t} H_{L_{n-2}}(t).$$

The exact sequences

$$0 \longrightarrow (x_1, x_{n-1}) \longrightarrow C_n \stackrel{x_n\cdot}{\longrightarrow} C_n \longrightarrow L_{n-2} \longrightarrow 0$$

and

$$0 \longrightarrow (x_1, x_{n-1}) \longrightarrow C_n \longrightarrow C_n/(x_1, x_{n-1}) \longrightarrow 0$$

and $C_n/(x_1, x_{n-1}) \simeq L_{n-4} \otimes k[x]$ gives

$$H_{C_n}(t) = H_{L_{n-2}}(t) - \frac{t}{(1-t)} H_{L_{n-4}}(t).$$

Now C_n and L_n are (as all graph algebras) Koszul algebras [F 75, Corollary 2], so $P_{C_n}(t) = 1/H_{C_n}(-t)$ and $P_{L_n}(t) = 1/H_{L_n}(-t)$. Since $L_0 = k[x_1]$ and $L_1 = k[x_1, x_2]/(x_1x_2)$, we have $H_{L_0}(t) = 1/(1-t)$ and $H_{L_1}(t) = (1+t)/(1-t)$. We give the first Hilbert series:
$H_{L_2}(t) = (1+t-t^2)/(1-t)^2$, $H_{L_3}(t) = (1+2t)/(1-t)^2$,
$H_{L_4}(t) = (1+2t-t^2-t^3)/(1-t)^3$, $H_{L_5}(t) = (1+3t+t^2-t^3)/(1-t)^3$,
$H_{C_3}(t) = (1+2t)/(1-t)$, $H_{C_4}(t) = (1+2t-t^2)/(1-t)^2$,
$H_{C_5}(t) = (1+3t+t^2)/(1-t)^3$, $H_{C_6}(t) = (1+3t-2t^3)/(1-t)^3$.

Remark We note that it is probably hard to get one formula for $H_{L_n}(t)$ for all n. An indication is that we get the Fibonacci numbers from $H_{L_n}(t)$. For $t = 1/2$ we get $H_{L_n}(1/2) = F_{n+2}$, the $(n+2)$th Fibonacci number if $F_0 = F_1 = 1$.

Thus we get
$P_{L_2}(t) = (1+t)^2/(1-t-t^2)$, $P_{L_3}(t) = (1+t)^2/(1-2t)$,
$P_{L_4}(t) = (1+t)^3/(1-2t-t^2+t^3)$, $P_{L_5}(t) = (1+t)^3/(1-3t+t^2+t^3)$,
$P_{C_3}(t) = (1+t)/(1-2t)$, $P_{C_4}(t) = (1+t)^2/(1-2t-t^2)$,
$P_{C_5}(t) = (1+t)^2/(1-3t+t^2)$, $P_{C_6}(t) = (1+t)^3/(1-3t+2t^3)$.

We collect the results in

Theorem 2.1 *The Poincaré series of L_n and C_n satisfy the recursion formulas*

$$P_{L_n}(t) = \frac{(1+t)P_{L_{n-1}}(t)P_{L_{n-2}}(t)}{(1+t)P_{L_{n-2}}(t) - tP_{L_{n-1}}(t)}$$

where $P_{L_0}(t) = 1 + t$ *and* $P_{L_1}(t) = (1 + t)/(1 - t)$ *and*

$$P_{C_n}(t) = \frac{(1 + t)P_{L_{n-2}}(t)P_{L_{n-4}}(t)}{P_{L_{n-2}}(t) + (1 + t)P_{L_{n-4}}(t)}.$$

Furthermore

$$P_{L_n^{2\alpha,\alpha}}(t) = (1 + t)^{(n+1)(\alpha-1)}P_{L_n}(t)$$

and

$$P_{C_n^{2\alpha,\alpha}}(t) = (1 + t)^{n(\alpha-1)}P_{C_n}(t).$$

3 Hypercycles and hyperlines when $2\alpha < d$

Next we turn to the case $2\alpha < d$. Now each edge has a free vertex, i.e. a vertex which does not belong to any other edge. Then the Taylor resolution is minimal. In this case there is a formula for the Poincaré series in terms of the graded homology of the Koszul complex [F 78, Corollary to Proposition 2]. Let R be a monomial ring for which the Taylor resolution is minimal. Then the homology of the Koszul complex $H(K_R)$ is of the form $H(K_R) = k[u_1, \dots, u_N]/I$, where I is generated by a set of monomials of degree 2. Define a bigrading induced by $\deg(u_i) = (1, |u_i|)$, where $|u_i|$ is the homological degree. Then $P_R(t) = (1 + t)^e/H_R(-t, t)$, where e is the embedding dimension and $H_R(x, y)$ is the bigraded Hilbert series of $H(K_R)$, see [F 78].

We begin with the hypercycle. The homology of the Koszul complex is generated by $\{z_I\}$, where $I = \{i, i+1, \dots, j\}$ corresponds to a path $\{e_i, e_{i+1}, \dots, e_j\}$ in $C_n^{d,\alpha}$ (indices counted $\pmod n$). Thus there are n generators in all homological degrees $< n$ and one generator in homological degree n. We have $z_Iz_J = 0$ if $I \cap J \neq \emptyset$. Thus the surviving monomials are of the form $m = z_{I_1} \cdots z_{I_r}$, where $I_i \cap I_j = \emptyset$ if $i \neq j$. The bidegree of m is $(r, \sum_{j=1}^{r} |I_j|)$. Let $\sum_{j=1}^{r} |I_j| = i$. Then m lies in $H(K)_{i,di-(i-r)\alpha}$. The graded Betti numbers are determined in [E-M-M 08, Chapter 3]. The nonzero Betti numbers are $\beta_{i,di-(i-r)\alpha} = \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}$ if $1 \le r \le i < n$ and $\beta_{n,n(d-\alpha)} = 1$. (As usual $\binom{a}{b} = 0$ if $b > a$.) This gives the Poincaré series.

Next we consider the hyperline. The homology of the Koszul complex is generated by $\{z_I\}$, where $I = \{i, i+1, \dots, j\}$ corresponds to a path $\{e_i, e_{i+1}, \dots, e_j\}$ in $L(n, d, \alpha)$. Thus there are $n+1-i$ generators of homological degree i. We have $z_Iz_J = 0$ if $I \cap J \neq \emptyset$. The graded Betti numbers are determined in [E-M-M 08, Chapter 3]. The nonzero Betti numbers are $\beta_{i,di-(i-r)\alpha} = \binom{i-1}{r-1}\binom{n-i+1}{r}$ if $1 \le r \le i \le n$. The same reasoning as above gives the Poincaré series. We state the results in a theorem.

Theorem 3.1 *If* $2\alpha < d$*, then*

$$P_{C_n}(t) = \frac{(1 + t)^{n(d-\alpha)}}{1 + \sum_{1\le r\le i<n}(-1)^r \frac{n}{r}\binom{i-1}{r-1}\binom{n-i-1}{r-1}t^{i+r} - t^{n+1}},$$

and

$$P_{L_n}(t) = \frac{(1+t)^{n(d-\alpha)+\alpha}}{1+\sum_{1\le r\le i\le n}(-1)^r\binom{i-1}{r-1}\binom{n-i+1}{r}t^{i+r}}.$$

4 The hyperstar

We conclude with a hypergraph generalizing the star graph. Suppose $|e_i| = d$ for all i, $1 \le i \le n$, and that if $i \ne j$, then $|e_i \cap e_j| = |\cap_{i=1}^n e_i| = \alpha < d$. Then the ideal is of the form $m(m_1, \ldots, m_n)$, where m is a monomial of degree α. Then the hypergraph ring $S_n^{d,\alpha}$ is Golod [G-L 69, Theorem4.3.2]. This means that

Theorem 4.1

$$P_{S_n^{d,\alpha}}(t) = (1+t)^{|V|}/(1-\sum \beta_i t^{i+1}) = (1+t)^{n(d-\alpha)+\alpha}/(1-\sum \binom{n}{i} t^{i+1}).$$

5 The wheel graph

Finally we consider the wheel graph W_n, which is C_n with an extra vertex (the center) which is connected to all vertices in C_n. We let W_n also denote the graph algebra $k[x_0, \ldots, x_n]/(x_1x_2, x_2x_3, \ldots, x_nx_1, x_0x_1, \ldots, x_0x_n)$.

Theorem 5.1 *Let W_n be a wheel graph on $n+1$ vertices. Then the Betti numbers of W_n are as follows:*

(i) If $j > i+1$, then $\beta_{i,j}(k[\Delta_{W_n}]) = \beta_{i,j}(C_n) + \beta_{i-1,j-1}(C_n)$.
(ii) If $j = i+1$, then $\beta_{i,i+1}(W_n) = \beta_{i,i+1}(C_n) + \beta_{i-1,i}(C_n) + \binom{n}{i}$.

Proof. Assume that $V(W_n) = \{x_0, x_1, \ldots, x_n\}$ and $C_n = W_n \setminus \{x_0\}$. It is easy to see that $\Delta_{W_n} = \Delta_{C_n} \cup \{x_0\}$, where Δ_{W_n} and Δ_{C_n} are the independence complexes of W_n and C_n. It implies that for any $i \ge 1$, $H_i(\Delta_{W_n}) = H_i(\Delta_{C_n})$. Thus, if $j > i+1$, from Hochster's formula ([B-H 98, Theorem 5.5.1]) and the observation above one has the result. Now assume that $j = i+1$. Then $\beta_{i,i+1}(W_n) = \sum_{S\subseteq V(W_n),|S|=i+1} \dim(\widetilde{H}_0(\Delta_S)) = \sum_{S\subseteq V(C_n),|S|=i+1} \dim(\widetilde{H}_0(\Delta_S))$ $+\sum_{S\subseteq V(W_n),S=S'\cup\{x_0\}} \dim(\widetilde{H}_0(\Delta_S))$. For any $S \subseteq V(W_n)$ and $S_0 \subseteq V(C_n)$, let r_S and r'_{S_0} denotes the number of connected components of Δ_S in $V(W_n)$ and Δ_{S_0} in $V(C_n)$ respectively. Then we have $\sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}} \dim(\widetilde{H}_0(\Delta_S)) = \sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}}(r_S - 1)$. For any $S \subseteq V(W_n)$ such that $S = S_0 \cup \{x_0\}$, we have $r_S = r'_{S_0} + 1$. Therefore
$\sum_{S\subseteq V(W_n),S=S_0\cup\{x_0\}} \dim(\widetilde{H}_0(\Delta_S)) = \sum_{S_0\subseteq V(C_n),|S_0|=i} \dim(\widetilde{H}_0(\Delta_{S_0})) + \binom{n}{i} = \beta_{i-1,i}(C_n) + \binom{n}{i}$.

The term $\binom{n}{i}$ is the number of subsets S_0 of $V(C_n)$ of cardinality i.

Substituting the $\beta_{i,j}(C_n)$ from of [J 04, Theorem 7.6.28] we have the following corollary.

Corollary 5.2 *Let W_n be the wheel graph on $n+1$ vertices. Then the Betti numbers of W_n are as follows:*
(i) If $n=3$, then $\beta_{2,3}(W_3)=8$, $\beta_{3,4}(W_3)=3$. If $n=4$, then $\beta_{3,4}(W_4)=9$, $\beta_{4,5}(W_4)=2$. Otherwise $\beta_{i,i+1}(W_n)=n\binom{2}{i-1}+\binom{n}{i}$.
(ii) If $n=3m$, then $\beta_{2m,n}(W_n)=3m+2$, $\beta_{2m+1,n+1}(W_n)=2$. If $n=3m+1$, then $\beta_{2m+1,n}(W_n)=3m+2$, $\beta_{2m+2,n+1}(W_n)=1$. If $n=3m+2$, then $\beta_{2m,n}(W_n)=\beta_{2m+1,n+1}(W_n)=1$. Otherwise, if $j>i+1$, then $\beta_{i,j}(W_n)=\frac{n}{n-2(j-i)}\binom{n-2(j-i)}{j-i}\binom{j-i-1}{2i-j}$.

We can also determine the Poincaré series for the wheel graph algebra. This is also a Koszul algebra, and $H_{W_n}(t)=H_{C_n}(t)+t/(1-t)$. Since $P_{W_n}(t)=1/H_{W_n}(-t)$ and $P_{C_n}(t)=1/H_{C_n}(-t)$, this gives

Theorem 5.3

$$P_{W_n}(t)=\frac{P_{C_n}(t)(1+t)}{1+t-tP_{C_n}(t)}$$

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