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TRIPLET EXTENSIONS I: SEMIBOUNDED OPERATORS WITH DEFECT ONE

PAVEL KURASOV

Abstract. The extension theory for semibounded symmetric operators is generalized by including operators acting in a triplet of Hilbert spaces. We concentrate our attention on the case where the minimal operator is essentially self-adjoint in the basic Hilbert space and construct a family of its self-adjoint extensions inside the triplet. All such extensions can be described by certain boundary conditions and a natural counterpart of Krein's resolvent formula is obtained.

Contents

1. Introduction

The extension theory for symmetric operators [1, 30] developed originally by J. von Neumann [27] gives an affirmative answer to the question under which conditions does a symmetric densely defined operator possess self-adjoint (canonical) extensions, and describes all such extensions as restrictions of the adjoint operator. The family of self-adjoint operators may be then parameterized by Krein's resolvent formula [20], where spectral properties of the operators are encoded in certain Nevanlinna functions, usually known as Krein's Q-function.

In the current article we are going to study so-called triplet extensions of symmetric operators. Consider a triplet of Hilbert spaces [8] (see rigorous definition below)

$$G \subset H \subset G^\dagger, \tag{1}$$

and an operator B, which is symmetric both as an operator in G and in H. Surely the deficiency indices for these two operators could be different and the case which attracted our attention is when the operator B is semibounded and essentially self-adjoint in H, but has nontrivial deficiency indices as operator in G. In this case there is a unique self-adjoint extension of B in H, but inside G there is a non-trivial family of extensions. This family can be characterized by classical extension theory and therefore is not particular interesting. Such extensions do not fully use the structure of the Gelfand triplet (1), more precisely the space $G^\dagger$ plays no role in this construction. On the other hand every densely defined operator B in G determines the triplet adjoint operator $B^\dagger$ acting in $G^\dagger$. It is therefore interesting to study (generalized) extensions of B which are at the same time restrictions of $B^\dagger$. We call such operators triplet extensions. This construction generalizes naturally von Neumann approach and clearly coincides with it in the degenerate case $G = H = G^\dagger$. Defining triplet extensions we naturally would like to exclude canonical extensions of B in G, which can as well be obtained as restrictions of $B^\dagger$ since $B^* \subset B^\dagger$, where B^* denotes the operator which is adjoint to B in G (see Definition 1 for details).

Starting with the most general definition of triplet extensions we continue with the case where G is one of the spaces from the scale of Hilbert spaces associated with the unique self-adjoint extension of B in H. This assumption is satisfied in several examples demonstrated below. In addition we restrict our considerations to the case where the deficiency indices of B in G are $(1,1)$ in order to make our presentation more transparent, but most of the formulas can easily be generalized to the case of any equal deficiency indices. For the same reason we treat just the case when B is essentially self-adjoint in H, since in this case triplet extensions are of particular interest. As expected formulas generalizing Krein's resolvent formula play the central role in the characterization of the corresponding operator families. In particular we obtain an extension of Krein's formula where the role of the Q-function is played by certain generalized Nevanlinna function. It is especially surprising since the corresponding operator is self-adjoint in a certain Hilbert space, not in a Pontryagin, where appearance of generalized Nevanlinna functions is standard.

In the rest of the introduction we would like to discuss few examples showing that triplet extensions are important in certain applications. The extension theory for symmetric operators plays an important role in modern mathematical physics, especially in quantum mechanics. This role is two-fold:

- On one side extension theory is sometimes needed to describe the family of self-adjoint operators corresponding to a formal differential expression obtained from a classical Hamiltonian via the correspondence principle.
- On the other hand extension theory can be used to introduce interactions which are specific for quantum mechanics and do not have classical analogs, so-called contact interactions.

Probably the most important example connected with these two approaches concerns the Sturm-Liouville operator on the half-line, where in order to determine self-adjoint operators one usually needs in addition to a formally symmetric differential expression also certain boundary conditions at the origin. Then the spectral properties of the corresponding operators are described by the Titchmarsh-Weyl coefficient, which is identical to Krein's Q-function in this case - a certain Nevanlinna type function. It appears natural to try to generalize this approach in order to include the singular case (limit point case in accordance with H. Weyl classification [30]) when the minimal operator has trivial deficiency indices. Allmost all constructions considered so far lead to operators in Pontryagin (with indefinite metrics) and generalized Nevanlinna functions [12, 14, 15, 24].

Another class of problems where we meet similar difficulties is the theory of singular interactions [31, 17, 19, 3, 4, 5, 22, 23, 25, 26] where one of the first questions is the rigorous definition of the operator formally given by

$$L + \alpha \langle \varphi, \cdot \rangle \varphi, \quad \alpha \in \mathbb{R}, \tag{2}$$

where L is a self-adjoint operator in Hilbert space H (with the scalar product denoted by $\langle \cdot, \cdot \rangle$) and φ is a certain vector from the scale of Hilbert spaces $\mathcal{H}_n(L)$ associated with the self-adjoint operator L (see definition below). The interesting case is when $\varphi \notin \mathcal{H}$, otherwise the perturbation term is just a bounded operator. One can define such perturbation in the case $\varphi \in \mathcal{H}_{-n}, n = 1, 2$ by associating (2) with one of the self-adjoint extensions of the operator $L_{\min}$ - the restriction of L to the set of functions ψ satisfying additional condition $\langle \varphi, \psi \rangle = 0$.

The corresponding family of operators is again described by Krein's resolvent formula [20, 5] with a Nevanlinna function encoding their spectral properties. But this approach does not work in the case $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-2},\ n \geq 3$ due to the fact that the corresponding restricted operator is essentially self-adjoint (in H) so that the original operator L is its unique self-adjoint extension. Attempts to define a nontrivial family of operators in this case are again connected with generalized Nevanlinna functions and operators in Pontryagin spaces [14, 22].

It has been noted [25, 26, 22] that the restricted operator has nontrivial deficiency indices considered in the Hilbert space $\mathcal{H}_{n-2}$ instead of $\mathcal{H}$. It follows that we may try to define the operator given formally by (2) as a triplet extension with respect to the following triplet of Hilbert spaces from the scale associated with L in H

$$\mathcal{H}_{n-2} \subset H \subset \mathcal{H}_{-n+2}. \tag{3}$$

The operator corresponding to (2) has been defined on a certain vector space of singular elements forming a sort of cascade belonging to different spaces from the scale [22, 12]. It was shown that the vector space can be turned into a Hilbert space to obtain a family of self-adjoint operators corresponding to formal singular interactions. Unfortunately this model was not optimal in the sense that the parameters (certain normalization points $\mu_j < 0$ and a Gramm matrix Γ) have to be chosen satisfying certain restrictions which origin was hard to understand. In particular in order to obtain a Hilbert space model it is necessary to choose all μ_j pairwise different and Γ could not be chosen diagonal. In the model presented here the gram matrix Γ is diagonal and an explicit explanation for the choice of normalization points μ_j is given. Since the basis elements in the new model have the same order of singularity, one may call it *peak model* (in order to distinguish it from the *cascade model* given in [12]).

A similar approach has already been developed for singular Sturm-Liouville operators of hydrogen atom type in [24] interpreting the generalized Titchmarsh-Weyl coefficient as Krein's Q-function even in the case if it is of generalized Nevanlinna type.

The current article is organized as follows. As already mentioned our approach is a direct extension of classical J. von Neumann theory, or more precisely its version developed by M. Krein, M. Birman and M. Vishik known as Birman-Krein-Vishik theory [9, 21, 32, 6] being very useful in the physical case of semibounded operators. This theory is briefly discussed in Section 2. The following two sections are devoted to the definition of triplet extensions respectively in the general case and in the case where the triplet is formed just by three spaces from the scale. The minimal extended space is introduced. The corresponding minimal and maximal operators acting in the extended space are described in Section 5. The self-adjoint family of triplet extensions is finally obtained in Section 6 as restrictions of the maximal operator. Their resolvents are calculated explicitly. In Section7 we obtaine a new extended resolvent formula and describe the class of functions appearing in its denominator. It is shown that these functions can also be obtained by certain renormalization procedure demonstrated in Section 8. The last two sections are devoted to an application of the developed approach to the theory of singular perturbations. In particular a new family of point interactions for the Laplacian in $\mathbb{R}^3$ is presented.

In this article we are going to use the scale of Hilbert spaces associated with a certain positive self-adjoint operator L.

$$\ldots \subset \mathcal{H}_3 \subset \underset{\mathrm{Dom}\,(L)}{\overset{\|}{\mathcal{H}_2}} \subset \mathcal{H}_1 \subset \underset{H}{\overset{\|}{\mathcal{H}_0}} \subset \mathcal{H}_{-1} \subset \underset{(\mathrm{Dom}\,(L))^*}{\overset{\|}{\mathcal{H}_{-2}}} \subset \mathcal{H}_{-3} \subset \ldots \tag{4}$$

The spaces $\mathcal{H}_{-n}$, $n = 1, 2, ...$ can be considered as completions of $H = \mathcal{H}_0$ with respect to the following norms

$$\| U \|^2_{\mathcal{H}_n} = \langle U, (L+1)^n U \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the original Hilbert space H. Then the spaces with positive indices are just dual spaces

$$\mathcal{H}_s = \mathcal{H}^*_{-s},$$

so that the spaces $\mathcal{H}_m \subset H \subset \mathcal{H}_{-m}$, $m = 1, 2, ...$ form a Gelfand triplet (of Hilbert spaces). The operator $L + 1$ acts as isometric shift in the scale of Hilbert spaces mapping $\mathcal{H}_{n+2}$ onto $\mathcal{H}_n$. Let us denote by L_n the restriction for $n > 0$ and extension for $n < 0$ of the operator L to the domain $\mathrm{Dom}\,(L_n) = \mathcal{H}_{n+2}$. The operator L_n so defined is self-adjoint in $\mathcal{H}_n$. In particular the operator $L_0 = L$ and its domain is the space $\mathcal{H}_2$.

Note that an equivalent norm in $\mathcal{H}_n$ can be introduced using the scalar product

$$\langle U, V \rangle_{\mathcal{H}_n} = \langle U, b(L) V \rangle_H \,, \tag{5}$$

where b is any polynomial of order n positive on $\mathbb{R}_+ \cup \{0\}$. In what follows we choose b equal to

$$b(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2)...(\lambda - \mu_m), \tag{6}$$

where $\mu_1, ..., \mu_m < 0$ are arbitrary pairwise different negative numbers.

2. Perturbations and extensions of semibounded operators: classical theory

The aim of the current article is to generalize the extension theory for symmetric operators by including the case where the role of the adjoint operator is played by the triplet adjoint operator. It is clear that the classical extension theory, originally developed by J. von Neumann [27] (see also [10]), should appear as a special case where the triplet of Hilbert spaces degenerated into just one Hilbert space, so that we have $G = H = G^\dagger$. Since all operators appearing in this article are semibounded it is natural to use a special version of the extension procedure given by M.Sh. Birman, M.G. Krein and M.I. Vishik [9, 21, 32] and usually called Birman-Krein-Vishik theory [6]. In this section we recall the main ideas of this approach.

Let B be a nonnegative symmetric operator acting in a Hilbert space H. Assume that it has deficiency indices $(1,1)$. Then the Hilbert space possesses the decomposition

$$H = \mathrm{Range}\,(B-\mu) \oplus \mathrm{Ker}\,(B^*-\mu), \quad \mu \in \mathbb{R}_-, \tag{7}$$

where B^* denotes the operator adjoint to B in the space H and the negative parameter μ can be chosen arbitrarily. Then any self-adjoint extension A is at the same time a restriction of the adjoint operator, so that the following formula holds

$$B \subset A \subset B^*. \tag{8}$$

We denote by A any nonnegative self-adjoint extension of B. Such extension can be obtained for example by closing the domain of B with respect to the graph norm $\parallel U \parallel_B = \parallel (B+I)U \parallel_H$, where I is the identity operator. This extension is usually called Friedrichs extension, but the role of A can be played by any other nonnegative extension.

Let us denote by $G(\lambda)$ the family of deficiency elements satisfying the following indentity

$$G(\lambda) = \frac{A-\mu}{A-\lambda}G(\mu) = G(\mu) + (\lambda-\mu)\frac{1}{A-\lambda}G(\mu), \quad \lambda,\mu \in \mathbb{C}\setminus\mathbb{R}_+. \tag{9}$$

Then the domain of the adjoint operator is given by

$$\mathrm{Dom}\,(B^*) = \mathrm{Dom}\,(A) \dot{+} \mathcal{L}\{G(\mu)\}, \tag{10}$$

where $\mu < 0$ is a certain fixed point on the negative half-axis and $\mathcal{L}$ denotes the linear span. The sum here is direct, since $G(\mu) \notin \mathrm{Dom}\,(A)$ and therefore every $U \in \mathrm{Dom}\,(B^*)$ can be written as

$$U = U_r + uG(\mu),\ U_r \in \mathrm{Dom}\,(A), u \in \mathbb{C} \tag{11}$$

and such representation is unique. The action of the operator B^* is given by

$$B^*(U_r + uG(\mu)) = AU_r + \mu u G(\mu). \tag{12}$$

The domain of the maximal operator can also be described as

$$\mathrm{Dom}\,(B^*) = \mathrm{Dom}\,(A) + \mathcal{L}\{G(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}_+\}, \tag{13}$$

where the parameter λ runs over all complex numbers excluding the positive half-axis. Note that in this representation the sum is not direct anymore, since

$$G(\mu) - G(\lambda) \in \mathrm{Dom}\,(A). \tag{14}$$

This formula shows that (10) and (13) describe exactly the same linear sets.

Now every self-adjoint restriction of the maximal operator can be described by imposing the following boundary condition

$$\langle G(\mu), (L-\mu)U_r\rangle = \gamma u,\ \gamma \in \mathbb{R} \cup \{\infty\} \tag{15}$$

on the functions possessing representation (11). Let us denote the corresponding operator by A_γ. In this parameterization we have $A = A_\infty$. The boundary condition in particular guarantees that the resolvent equation

$$(A_\gamma - \lambda)U = F, \quad F \in H \tag{16}$$

is solvable for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. And this solution leads to Krein's resolvent formula

$$\frac{1}{A_\gamma - \lambda} = \frac{1}{A-\lambda} - \frac{1}{\langle G(\mu), (\lambda-\mu)G(\lambda)\rangle - \gamma}\left\langle G(\overline{\lambda}), \cdot\right\rangle G(\lambda). \tag{17}$$

In this formula the function

$$Q(\lambda) = \langle G(\mu), (\lambda-\mu)G(\lambda)\rangle \tag{18}$$

is a Nevanlinna function, *i.e.* is analytic outside the real axis, is symmetric with respect to the real axis and has nonnegative imaginary in the upper half-plane. This function is usually called **Krein's Q-function** and the parameter γ describes all possible self-adjoint extensions of B. Note that in general the parameter γ depends on the chosen regularization point μ.

3. Self-adjoint extensions in a triplet of Hilbert spaces

Consider a triplet of Hilbert spaces

$$G \subset H \subset G^\dagger, \tag{19}$$

satisfying the following properties:

- $G, H, G^\dagger$ are Hilbert spaces;
- the space G is a dense subspace of H;
- the space H is a dense subspace of $G^\dagger$;
- the space $G^\dagger$ is dual to G with respect to the norm in H.

It is natural to extend the notation $\langle \cdot, \cdot \rangle_H$, denoting originally the scalar product in H, to the pairing between elements from G and $G^\dagger$, so that

$$\langle U, V \rangle_H$$

is well-defined whenever $U \in G, V \in G^\dagger$ or $U \in G^\dagger, V \in G$. Let B be a densely defined operator in G. Then the **triplet adjoint** operator $B^\dagger$ is defined in the Hilbert space $G^\dagger$ on the domain

$$\begin{aligned} \mathrm{Dom}\,(B^\dagger) \;\; = \;\; & \big\{ V \in G^\dagger : \exists C_V > 0 : U \in \mathrm{Dom}\,(B) \\ & \Rightarrow |\langle V, BU \rangle_H| \leq C_V \parallel U \parallel_G \big\}, \end{aligned}$$

by the equality

$$\langle V, BU \rangle_H = \langle B^\dagger V, U \rangle_H, \;\; \forall U \in \mathrm{Dom}\,(B).$$

In the current article we are going to investigate so-called triplet extensions of symmetric operators B in G having nontrivial deficiency indices. We consider the case where B is semibounded and has defect one. Our main interest lies in the situation when the operator B is essentially self-adjoint in H, *i.e.* its closure in H is a self-adjoint operator. In this case considering B in H we cannot get any interesting extension theory and triplet extensions start to play an interesting role.

Under a triplet extension of B we understand a generalized extension of B to a certain Hilbert space $\mathbf{H}$ inside $G^\dagger$ which is simultaneously a restriction of the triplet adjoint operator $B^\dagger$. In order to exclude canonical extensions (inside G) we assume that $\mathbf{H}$ contains $\mathrm{Ker}\,(B^\dagger - \mu), \forall \mu \in \mathbb{C} \setminus R_+$. More precisely the introduce the following definition

Definition 1. *Let $G \subset H \subset G^\dagger$ be a triplet of Hilbert spaces and let B be a densely defined symmetric operator in G. An operator $\mathbf{A}$ acting in a Hilbert space $\mathbf{H}$ is a self-adjoint* **triplet extension** *of the operator B if and only if*

- *the space $\mathbf{H}$*
  1. *is a subset of $G^\dagger$;*
  2. *contains G as a Hilbert subspace;*

(3) *contains* $\operatorname{Ker}(B^\dagger - \mu)$, *for all* $\mu \in \mathbb{C} \setminus \mathbb{R}_+$;
i.e.

$$G \subset \mathbf{H} \subset G^\dagger, \tag{20}$$

and

- *the operator* $\mathbf{A}$ *is self-adjoint in* $\mathbf{H}$ *and satisfies*

$$B \subset \mathbf{A} \subset B^\dagger, \tag{21}$$

i.e. it is an extension of B *and a restriction of* $B^\dagger$.

Formula (20) follows directly from assumptions 1 and 2 and implies together with (21) that every triplet extension of B is an operator acting inside the triplet and this operator acts as the triplet adjoint. This is a direct generalization of the classical formula (8) valid for self-adjoint extensions inside the space (when the triplet (19) reduces to just one Hilbert space).

If the operator B in G has nontrivial deficiency indices, then the kernel $\operatorname{Ker}(B^\dagger - \lambda)$ is always nontrivial. In particular Definition 1 implies that

$$\mathbf{H} \supset \operatorname{Range}(B - \mu) \dot{+} \operatorname{Ker}(B^\dagger - \mu), \quad \mu \in \mathbb{R}_- \tag{22}$$

where the sum is orthogonal with respect to the scalar product in $G^\dagger$ (but not necessarily in $\mathbf{H}$). This formula is a natural generalization of (7) valid for extensions inside the space.

If B is essentially self-adjoint in G, then the kernel $\operatorname{Ker}(B^\dagger - \mu),\ \mu \in \mathbb{R}_-$ is trivial, since $B^\dagger$ is a positive self-adjoint operator in $G^\dagger$. Condition 3 makes no further restriction on $\mathbf{H}$ in this case and the closure of B in H satisfies all conditions in the definition. Therefore in what follows only operators with nontrivial deficiency indices will be considered.

We are not aiming to describe the whole family of triplet extensions in the current article. The family we are going to construct is the minimal one in the sence that the space $\mathbf{H}$ is the minimal vector space satisfying assumptions 1-3. In addition we shall assume that the spaces G and $G^\dagger$ are from the scale of Hilbert spaces associated with a certain nonnegative operator and B is a restriction of this operator to a subspace of G. The corresponding triplet extensions can be refered to as extensions in the scale of Hilbert spaces and are considered further in the following section.

4. Triplet extensions in the scale of Hilbert spaces

In this section we consider the case where the Hilbert space G from the triplet (19) coincides with one of the Hilbert spaces associated with the closure of B in H (remember that B is essentially self-adjoint in

H). It will be more convenient to change slightly our point of view. Let L be a nonnegative self-adjoint operator in the Hilbert space H. Consider the triplet

$$\mathcal{H}_m \subset H \subset \mathcal{H}_{-m} \tag{23}$$

and the minimal operator $L_{\min}$ satisfying the following conditions

(1) $L_{\min}$ is a restriction of L;
(2) $L_{\min}$ has deficiency indices $(1,1)$ in $\mathcal{H}_m$;
(3) $L_{\min}$ is essentially self-adjoint in H.

It follows in particular that $L_{\min}$ is a restriction of L_m, which is self-adjoint in $\mathcal{H}_m$. The triplet adjoint to the operator L_m coincides with L_{-m}. It follows that the maximal operator $L_{\max} = L^{\dagger}_{\min}$ - the triplet adjoint to $L_{\min}$ is, - an extension of L_{-m} in $\mathcal{H}_{-m}$.

Hence we are in the situation described at the beginning of the preceding section with

$$G = \mathcal{H}_m, \; G^{\dagger} = \mathcal{H}_{-m}, \; B = L_{\min} \text{ and } B^{\dagger} = L_{\max}.$$

In what follows we are going to investigate the possibility to construct a self-adjoint triplet extension $\mathbb{A}$ of $L_{\min}$ in a certain Hilbert space $\mathbb{H}$. In our approach such extensions will be constructed by first specifying the linear space $\mathbb{H}$ and then defining the operator $\mathbb{A}_{\max}$ in it as a restriction of $L_{\max}$. Such operator possesses self-adjoint restrictions if and only if the minimal operator $\mathbb{A}_{\min} = \mathbb{A}^*_{\max}$ - the adjoint of $\mathbb{A}_{\max}$ in $\mathbb{H}$,- is also a restriction of $\mathbb{A}_{\max}$. It will be shown that $\mathbb{A}_{\min}$ is also an extension of $L_{\min}$ so that the following inclusions hold

$$L_{\min} \subset \mathbb{A}_{\min} \subset \mathbb{A}_{\max} \subset L_{\max}. \tag{24}$$

This condition implies certain restrictions on the scalar product, that now may be introduced on $\mathbb{H}$ to turn it into a Hilbert space. If formula (24) holds, then a triplet extension $\mathbb{A}$ of $L_{\min}$ can be obtained using standard extension theory inside the new Hilbert space $\mathbb{H}$ so that

$$L_{\min} \subset \mathbb{A}_{\min} \subset \mathbb{A} \subset \mathbb{A}_{\max} \subset L_{\max} \tag{25}$$

holds.

Let us denote by $G(\lambda)$ the family of deficiency elements for the operator $L_{\min}$ in $\mathcal{H}_m$, which are solutions to the equation $(L^*_{\min} - \lambda) G(\lambda) = 0$.[1] Then the self-adjoint canonical extensions of $L_{\min}$ inside the space $\mathcal{H}_m$ can be constructed following the original scheme described in Section 2 and in particular formula (10) implies

$$\mathrm{Dom}\,(L^*_{\min}) = \mathrm{Dom}\,(L_m) \dot{+} \mathcal{L}\{G(\mu)\}, \quad \mu < 0. \tag{26}$$

[1] $L^*_{\min}$ denotes here the operator adjoint to $L_{\min}$ in the Hilbert space $\mathcal{H}_m$.

This construction does not bring any new ideas and the corresponding operator is not a triplet extension of $L_{\min}$, since condition 3 in Definition 1 is not fulfilled.

Let us determine the domain of the maximal operator $L_{\max} = L_{\min}^{\dagger}$. The domain of $L_{\min}^{\dagger}$ consists of all $V \in \mathcal{H}_{-m}$ such that the form $\langle V, LU\rangle_H$, $U \in \mathrm{Dom}\,(L_{\min})$ determines a bounded linear functional with respect to $U \in \mathcal{H}_m$, *i.e.* the following estimate holds

$$|\langle V, LU\rangle_H| \leq C_V \parallel U \parallel_{\mathcal{H}_m}, \text{ for some } C_V > 0.$$

Let us recall that the domain of $L_{\min}^*$ described by (26) is precisely the set of all $W \in \mathcal{H}_m$ such that

$$|\langle W, LU\rangle_{\mathcal{H}_m}| = |\langle W, b(L)LU\rangle_H| \leq \tilde{C}_W \parallel U \parallel_{\mathcal{H}_m}, \text{ for some } \tilde{C}_W > 0,$$

holds. It follows that $\mathrm{Dom}\,(L_{\max}) = b(L)\mathrm{Dom}\,(L_{\min}^*)$ and hence every $V \in \mathrm{Dom}\,(L_{\max})$ possesses the representation

$$V = \tilde{V} + vg(\mu), \tilde{V} \in \mathcal{H}_{-m+2}, v \in \mathbb{C}, \tag{27}$$

where

$$g(\mu) = b(L)G(\mu) \in \mathcal{H}_{-m}, \tag{28}$$

and μ is a fixed negative number. Since the operators L and $b(L)$ commute, the action of $L_{\max}$ is given by

$$L_{\max}\left(\tilde{V} + vg(\mu)\right) = L\tilde{V} + \mu v g(\mu). \tag{29}$$

In particular the function $g(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ solves the equation

$$(L_{\max} - \lambda)g(\lambda) = 0$$

and every other solution is a multiple of $g(\lambda)$. The functions $g(\lambda)$ resemble the deficiency elements appearing in the classical extension theory, but do not belong to the original Hilbert space $\mathcal{H}$, since the operator $L_{\min}$ is essentially self-adjoint there.[2] In our approach $g(\lambda)$ will be a deficiency element for the minimal operator defined in a certain extension of the original Hilbert space.

Lemma 1. *In the above situation the minimal vector space $\mathbb{H}$ satisfying assumptions 1, 2 and 3 of Definition 1 is an m-dimensional extension of $\mathcal{H}_m$ which can be described as*

$$\mathbb{H} = \mathcal{H}_m \dot{+} \mathcal{L}\{g(\mu_1), g(\mu_2), ..., g(\mu_m)\}, \tag{30}$$

where μ_j are different negative numbers

$$\mu_j < 0, \; \mu_j \neq \mu_i, \; i, j = 1, 2, ..., m. \tag{31}$$

[2]In fact $g(\lambda)$ is a deficiency element for the operator L_{-m} restricted to the set of functions $U \in \mathcal{H}_{-m+2}$ satisfying one further condition $\langle U, (L - \lambda)G(\lambda)\rangle = 0$.

Proof. The set $\mathbb{H}$ contains at least the following set

$$\mathbb{H} \supset \mathcal{H}_m + \mathcal{L}\{g(\lambda), \lambda \in \mathbb{C} \setminus R_+\}.$$

Let us prove that this extension is finite dimensional. It is sufficient to show, that every $g(\lambda)$ can be written as a linear combination of $g(\mu_j),\ j = 1, ..., m$ and a function from $\mathcal{H}_m$. Indeed this representation is given by

$$g(\lambda) = \sum_{j=1}^{m} \frac{b_j(\lambda)}{b_j(\mu_j)} g(\mu_j) + b(\lambda) G(\lambda), \tag{32}$$

where the polynomial b is determined by (6) and

$$b_j(\lambda) = \prod_{i \neq j} (\lambda - \mu_i). \tag{33}$$

The sum in (30) is direct since

$$\mathcal{H}_m \cap \mathcal{L}\{g(\mu_1), ..., g(\mu_m)\} = \{0\}.$$

Hence every vector space satisfying the assumptions of Definition 1 contains the vector space given by (30). □

The last Lemma describes $\mathbb{H}$ as a vector space. It can be turned into a Hilbert space by introducing a scalar product using a certain positive definite Gramm matrix Γ

$$\langle \mathbb{U}, \mathbb{V} \rangle_{\mathbb{H}} = \langle U, b(L) V \rangle_H + \langle \vec{u}, \Gamma \vec{v} \rangle_{\mathbb{C}^m}, \tag{34}$$

where we use the following notation

$$\vec{u} = (u_1, u_2, ..., u_m) \in \mathbb{C}^m. \tag{35}$$

With this scalar product the Hilbert space $\mathbb{H}$ can be identified with the orthogonal sum

$$\mathbb{H} \cong \mathcal{H}_m \oplus \mathbb{C}^m, \tag{36}$$

with the natural identification

$$\mathbb{U} = (U, \vec{u}) = U + \sum_{j=1}^{m} u_j g(\mu_j). \tag{37}$$

Therefore in what follows elements from $\mathbb{H}$ will be considered both as functions from $\mathcal{H}_{-m}$ and as a pair $\mathbb{U} = (U, \vec{u}),\ U \in \mathcal{H}_m, \vec{u} \in \mathbb{C}^m$.

It will be shown later that the matrix Γ has to be chosen diagonal in order to satisfy inclusion (24), but in order to explore all possibilies we assume for the moment, that Γ is just a Hermitian matrix with positive eigenvalues.

5. Maximal and minimal operators

In this section we described the maximal and minimal operators acting in the extension space $\mathbb{H}$. The operator $\mathbb{A}_{\max}$ acting in $\mathbb{H}$ is defined as the restriction of the linear operator $L_{\max}$ to the space $\mathbb{H}$.

Lemma 2. *Let $\mathbb{A}_{\max}$ be the restriction of $L_{\max}$ to the space $\mathbb{H}$. Then it acts on the domain*

$$\begin{gathered} \mathrm{Dom}\,(\mathbb{A}_{\max}) = \mathcal{H}_{m+2} \dot{+} \mathcal{L}\{g(\mu), g(\mu_1), g(\mu_2), ..., g(\mu_m)\} \ni \mathbb{U} \\ \mathbb{U} = U_r + ug(\mu) + \sum_{j=1}^{m} u_j g(\mu_j),\ U_r \in \mathcal{H}_{m+2}, u, u_j \in \mathbb{C}, \end{gathered} \tag{38}$$

as

$$\mathbb{A}_{\max}\left(U_r + ug(\mu) + \sum_{j=1}^{m} u_j g(\mu_j)\right) = LU_r + \mu u g(\mu) + \sum_{j=1}^{m} \mu_j u_j g(\mu_j), \tag{39}$$

where $\mu < 0$ is an arbitrary negative number and μ_j satisfy (31).

Proof. By definition the domain of $\mathbb{A}_{\max}$ is given by

$$\mathrm{Dom}\,(\mathbb{A}_{\max}) = \{\mathbb{U} \in \mathbb{H} \cap \mathrm{Dom}\,(L_{\max}) : L_{\max}\mathbb{U} \in \mathbb{H}\},$$

where the first condition actually gives no restriction since every $\mathbb{U} \in \mathbb{H}$ possesses the representation (27)

$$\mathbb{U} = U + \sum_{j=1}^{m} u_j g(\mu_j),\ U \in \mathcal{H}_m, u_j \in \mathbb{C}$$

and therefore it belongs to $\mathrm{Dom}\,(L_{\max})$. Hence such a function belongs to the domain of $\mathbb{A}_{\max}$ if and only if it is mapped by the operator $L_{\max} - \mu,\ \mu < 0$ to a certain $\mathbb{V} \in \mathbb{H}$. Since $\mathbb{V}$ again possesses the representation (37) we get the equality

$$(L-\mu)U + \sum_{j=1}^{m} u_j(\mu_j - \mu) g(\mu_j) = V + \sum_{j=1}^{m} v_j g(\mu_j),$$

which implies

$$U = (L-\mu)^{-1} V + \sum_{j=1}^{m} (v_j - u_j(\mu_j - \mu))(L-\mu)^{-1} g(\mu_j).$$

Every such function U can be written as a linear combination of the functions

$$g(\mu), g(\mu_1), g(\mu_2), ..., g(\mu_m)$$

and a function from $\mathcal{H}_{m+2}$. It follows that $\mathbb{U}$ possesses the representation (38). It is clear that the sum is direct, since no nontrivial linear combination of $g(\mu), g(\mu_1), .., g(\mu_m)$ belongs to $\mathcal{H}_{m+2}$.

Taking into account that $L_{\max} g(\lambda) = \lambda g(\lambda)$ and $L_{\max} U_r = LU_r$ we get formula (39). $\square$

We have just proven that the maximal operator $\mathbb{A}_{\max}$ is given by formulas (38) and (39). But in what follows it will be usefull to obtain a description of the domain and the action of this operator compatible with the orthogonal decomposition (36).

Let us introduce the diagonal $m \times m$ matrix

$$\mathcal{M} = \mathrm{diag}\,\{\mu_1, \mu_2, ..., \mu_m\} \tag{40}$$

and the m-dimensional vector $\vec{b} \in \mathbb{C}^m$ with the coordinates

$$b^j = \frac{1}{b_j(\mu_j)}, \quad j = 1, 2, ..., m, \tag{41}$$

where the polynomials b_j are given by (33).

Lemma 3. *For the maximal operator $\mathbb{A}_{\max}$ given by (38) and (39) it holds*

$$\begin{array}{rcl} \mathrm{Dom}\,(\mathbb{A}_{\max}) & = & \{\mathbb{U} = (U, \vec{u}) : U = U_r + uG(\mu)\;, \\ & & U_r \in \mathcal{H}_{m+2}, u \in \mathbb{C}, \vec{u} \in \mathbb{C}^m\} \end{array} \tag{42}$$

and

$$\mathbb{A}_{\max} \left(\begin{array}{c} U_r + ug \\ \vec{u} \end{array} \right) = \left(\begin{array}{c} LU_r + \mu ug \\ \mathcal{M}\vec{u} + u\vec{b} \end{array} \right). \tag{43}$$

Proof. It follows from

$$G(\mu) = \sum_{j=1}^{m} \frac{1}{b_j(\mu_j)} \frac{1}{\mu_j - \mu} \left(g(\mu_j) - g(\mu) \right) \in \mathcal{H}_m, \tag{44}$$

that any function $\mathbb{U}$ from the domain of $\mathrm{Dom}\,(\mathbb{A}_{\max})$ given by (38) can equivalently be written in the form

$$U_r + uG(\mu) + \sum_{j=1}^{m} u_j g(\mu_j),\; U_r \in \mathcal{H}_{m+2}$$

with some coefficients $u, u_j \in \mathbb{C}$. Hence formula (42) holds.

Let us calculate the action of the operator. Using (44) and (39) we get

$$\begin{aligned}
& \mathbb{A}_{\max}\left(U_r + uG(\mu) + \sum_{j=1}^{m} u_j g(\mu_j)\right) \\
= \; & \mathbb{A}_{\max}\left(U_r + u\sum_{j=1}^{m}\frac{1}{b_j(\mu_j)}\frac{1}{\mu-\mu_j}(g(\mu)-g(\mu_j)) + \sum_{j=1}^{m} u_j g(\mu_j)\right) \\
= \; & LU_r + u\sum_{j=1}^{m}\frac{1}{b_j(\mu_j)}\frac{1}{\mu-\mu_j}(\mu g(\mu)-\mu_j g(\mu_j)) + \sum_{j=1}^{m} u_j \mu_j g(\mu_j) \\
= \; & LU_r + \mu u G(\mu) + \sum_{j=1}^{m}\left(\mu_j u_j + u\frac{1}{b_j(\mu_j)}\right) g(\mu_j),
\end{aligned}$$

which accomplishes the proof. □

The maximal operator $\mathbb{A}_{\max}$ is going to play the role of the adjoint operator for the extension problem in $\mathbb{H}$. In what follows we shall need its boundary form which shows in particular that this operator is not symmetric

(45)

$$\begin{aligned}
& \langle \mathbb{A}_{\max}\mathbb{U}, \mathbb{V}\rangle_{\mathbb{H}} - \langle \mathbb{U}, \mathbb{A}_{\max}\mathbb{V}\rangle_{\mathbb{H}} \\
& = \left\langle \begin{pmatrix} LU_r + \mu u G(\mu) \\ \mathcal{M}\vec{u} + u\vec{b} \end{pmatrix}, \begin{pmatrix} V_r + vG(\mu) \\ \vec{v} \end{pmatrix} \right\rangle_{\mathbb{H}} \\
& \qquad - \left\langle \begin{pmatrix} U_r + uG(\mu) \\ \vec{u} \end{pmatrix}, \begin{pmatrix} LV_r + \mu v G(\mu) \\ \mathcal{M}\vec{v} + v\vec{b} \end{pmatrix} \right\rangle_{\mathbb{H}} \\
& = \langle LU_r + \mu u G(\mu), b(L)\,(V_r + vG(\mu))\rangle_H \\
& \quad - \langle U_r + uG(\mu), b(L)\,(LV_r + \mu v G(\mu))\rangle_H \\
& \quad + \left\langle \mathcal{M}\vec{u} + u\vec{b}, \Gamma\vec{v}\right\rangle_{\mathbb{C}^m} - \left\langle \vec{u}, \Gamma\left(\mathcal{M}\vec{v} + v\vec{b}\right)\right\rangle_{\mathbb{C}^m} \\
& = \overline{u}\left(\langle \Gamma\vec{b}, \vec{v}\rangle_{\mathbb{C}^m} - \langle g(\mu), (L-\mu)V_r\rangle_H\right) - \overline{\left(\langle \Gamma\vec{b}, \vec{u}\rangle_{\mathbb{C}^m} - \langle g(\mu), (L-\mu)U_r\rangle_H\right)} v \\
& \quad + \langle \vec{u}, (\mathcal{M}\Gamma - \Gamma\mathcal{M})\,\vec{v}\rangle_{\mathbb{C}^m}\,.
\end{aligned}$$

Note that if Γ is diagonal, then the last term in the formula vanishes.

Any triplet extension of the operator $L_{\min}$ can now be characterized as a self-adjoint restriction of $\mathbb{A}_{\max}$. Consider the minimal operator $\mathbb{A}_{\min}$ acting in $\mathbb{H}$ - the operator adjoint to $\mathbb{A}_{\max}$ in $\mathbb{H}$. The operator $\mathbb{A}_{\max}$ possesses symmetric restrictions if and only if the minimal operator $\mathbb{A}_{\min}$ is symmetric, or in other words, is a restriction of the maximal operator. This necessary property of the new minimal operator puts certain restrictions on the Gramm matrix Γ which defines the scalar product in $\mathbb{H}$. Knowing that the new minimal operator $\mathbb{A}_{\rm min}$ is symmetric then all triplet extensions of $L_{\rm min}$ can be determined using standard extension theory as described in Section 2.

Let us calculate now the minimal operator $\mathbb{A}_{\rm min}$ for arbitrary choice of the Gramm matrix Γ.

Lemma 4. *The operator* $\mathbb{A}_{\rm min}$ *is defined on the functions from* $\mathrm{Dom}\,(\mathbb{A}_{\rm max})$ *(given by (42)) satisfying the two following additional conditions*

$$\left\{ \begin{array}{l} u = 0, \\ \langle \Gamma \vec{b}, \vec{u} \rangle_{\mathbb{C}^m} = \langle g(\mu), (L-\mu) U_r \rangle_H, \end{array} \right. \tag{46}$$

and acts as

$$\mathbb{A}_{\rm min} \left(\begin{array}{c} U_r \\ \vec{u} \end{array} \right) = \left(\begin{array}{c} LU_r \\ \Gamma^{-1} \mathcal{M} \Gamma \vec{u} \end{array} \right). \tag{47}$$

Proof. We calculate the operator adjoint to $\mathbb{A}_{\rm max} - \mu$, where μ is an arbitrary negative number. Let us consider two arbitrary vectors: $\mathbb{U} \in \mathrm{Dom}\,(\mathbb{A}_{\rm max})$ and $\mathbb{V} \in \mathbb{H}$. The sesquilinear form of the operator $\mathbb{A}_{\rm max} - \mu$ is

$$\begin{array}{l} \langle \mathbb{V}, (\mathbb{A}_{\rm max} - \mu) \mathbb{U} \rangle_{\mathbb{H}} \\ = \langle V, b(L)(L-\mu) U_r \rangle_H + \langle \vec{v}, \Gamma(\mathcal{M} - \mu) \vec{u} \rangle_{\mathbb{C}^m} + \overline{\langle \vec{b}, \Gamma \vec{v} \rangle_{\mathbb{C}^m}} u. \end{array}$$

Consider first vectors $\mathbb{U}$ of the type $\mathbb{U} = (U_r, \vec{0})$, then the sesquilinear form reduces to

$$\langle \mathbb{V}, (\mathbb{A}_{\rm max} - \mu) \mathbb{U} \rangle_{\mathbb{H}} = \langle V, b(L)(L-\mu) U_r \rangle_H$$

and it determines a bounded linear functional with respect to $\mathbb{U}$ in the norm of $\mathbb{H}$ only if $V = V_r \in \mathcal{H}_n$. Using this fact the boundary form now for arbitrary $\mathbb{U} \in \mathrm{Dom}\,(\mathbb{A}_{\rm max})$ can be written as

$$\begin{array}{rcl} \langle \mathbb{V}, (\mathbb{A}_{\rm max} - \mu) \mathbb{U} \rangle_{\mathbb{H}} & = & \langle V, b(L)(L-\mu) U_r + ug \rangle_H + \langle (\mathcal{M} - \mu) \Gamma \vec{v}, \vec{u} \rangle_{\mathbb{C}^m} \\ & & + \overline{\{ \left\langle \vec{b}, \Gamma \vec{v} \right\rangle_{\mathbb{C}^m} - \langle G(\mu), (L-\mu) b(L) V_r \rangle_H \}} u. \end{array}$$

The first two summands determine bounded linear functionals with respect to $\mathbb{U} \in \mathbb{H}$, but the functional $(U_r + ug, \vec{u}) \mapsto u$ is not bounded in the norm of $\mathbb{H}$. Thus the expression in the curly brackets has to be equal to zero. In other words every function $\mathbb{V} \in \mathrm{Dom}\,(\mathbb{A}_{\rm min})$ should satisfy the following condition

$$\left\langle \Gamma \vec{b}, \vec{v} \right\rangle_{\mathbb{C}^m} = \langle g(\mu), (L-\mu) V_r \rangle_H. \tag{48}$$

Summing up we have proven that the domain of the adjoint operator is determined by (46) and the sesquilinear form is

$$\langle U_r + uG(\mu), (L-\mu)b(L)V\rangle_H + \langle \vec{u}, (\mathcal{M}-\mu)\Gamma\vec{v}\rangle_{\mathbb{C}^m} .$$

It follows that the action of the operator $\mathbb{A}_{\max}$ is given by formula (47). $\square$

Note that the operator $\mathbb{A}_{\min}$ is an extension of the operator $L_{\min}$.

It is easy to see that the operator $\mathbb{A}_{\min}$ is symmetric in $\mathbb{H}$ or, in other words, is a restriction of $\mathbb{A}_{\max}$, if and only if the matrices $\mathcal{M}$ and Γ commute

$$\Gamma\mathcal{M} = \mathcal{M}\Gamma. \tag{49}$$

Here the matrix $\mathcal{M}$ is diagonal with all diagonal elements pairwise different, the matrix Γ is Hermitian and positive definite. Hence in order to satisfy (49) the matrix Γ has to be diagonal as well and all diagonal elements should be positive numbers. Therefore in what follows we are going to assume that the matrix Γ is diagonal and positive definite. Under this assumption formula (47) takes the form

$$\mathbb{A}_{\min} \begin{pmatrix} U_r \\ \vec{u} \end{pmatrix} = \begin{pmatrix} LU_r \\ \mathcal{M}\vec{u} \end{pmatrix}. \tag{50}$$

Now we are in the situation described by (24) and all self-adjoint restrictions of $\mathbb{A}_{\max}$ can be obtained using classical Birman-Krein-Vishik extension theoy for symmetric operators in a Hilbert space.

6. The self-adjoint family of extensions

In this section we are calculating explicitly the one-parameter family of self-adjoint operators in $\mathbb{H}$ satisfying (25) with $\mathbb{A}_{\min}, \mathbb{A}_{\max}$ from the previous section. A particular such extension, denoted by $\mathbb{A}_0$, is not hard to guess

$$\mathbb{A}_0 = L \oplus \mathcal{M}, \ \ \mathrm{Dom}\,(\mathbb{A}_0) = \mathcal{H}_{m+2} \oplus \mathbb{C}^{n-2}.$$

Let us calculate the defect and deficiency element for the symmetric operator $\mathbb{A}_{\min}$. The operator $\mathbb{A}_{\max}$ is closed and therefore it is the adjoint operator to $\mathbb{A}_{\min}$. Any deficiency element $\mathbb{F}(\lambda),\ \Im\lambda \neq 0$ is a nontrivial solution to the equation

$$(\mathbb{A}_{\max} - \lambda)\mathbb{F}(\lambda) = 0,$$

which can be written in the form

$$\begin{cases} (L-\lambda)F_r + (\mu-\lambda)G(\mu)f = 0, \\ (\mathcal{M}-\lambda)\vec{f} + f\vec{b} = 0. \end{cases}$$

Any solution to this system is a multiple of

$$\mathbb{F}(\lambda) = \begin{pmatrix} \frac{L-\mu}{L-\lambda} G(\mu) \\ -\frac{1}{\mathcal{M}-\lambda}\vec{b} \end{pmatrix} = \begin{pmatrix} G(\lambda) \\ -\frac{1}{\mathcal{M}-\lambda}\vec{b} \end{pmatrix}. \tag{51}$$

We conclude that the operator $\mathbb{A}_{\min}$ has deficiency indices $(1,1)$. With this parameterization the family of deficiency elements has the important property

$$\mathbb{F}(z) = \frac{\mathbb{A}_0 - \lambda}{\mathbb{A}_0 - z}\mathbb{F}(\lambda). \tag{52}$$

Every element $\mathbb{F}(\lambda)$ can also be viewed as a function from $\mathcal{H}_m$

$$\mathbb{F}(\lambda) = \frac{1}{b(\lambda)} g(\lambda). \tag{53}$$

To prove this one can use the formula

$$\frac{1}{b(L)} = \sum_{j=1}^{n-2} \frac{1}{b_j(\mu_j)} \frac{1}{L-\mu_j}, \tag{54}$$

with b_j given by (33) and its natural modification:

$$\begin{aligned}
\mathbb{F}(\lambda) &= G(\lambda) - \sum_{j=1}^{m} \frac{1}{\mu_j - \lambda} \frac{1}{b_j(\mu_j)} g(\mu_j) \\
&= \frac{1}{b(L)} g(\lambda) + \sum_{j=1}^{m} \frac{1}{b_j(\mu_j)} \frac{1}{\lambda - \mu_j} g(\mu_j) \\
&= \sum_{j=1}^{m} \left(\frac{1}{b_j(\mu_j)} \frac{1}{L-\mu_j} g(\lambda) + \frac{1}{b_j(\mu_j)} \frac{1}{\lambda - \mu_j} g(\mu_j) \right) \\
&= \sum_{j=1}^{m} \frac{1}{b_j(\mu_j)} \frac{1}{\lambda-\mu_j} g(\lambda) \\
&= \frac{1}{b(\lambda)} g(\lambda).
\end{aligned}$$

It follows that the functions $g(\lambda)$ (multiplied by a scalar factor) play the role of deficiency elements in our construction.

Now it is standard to determine the family of self-adjoint extensions of the minimal operator $\mathbb{A}_{\min}$, which are at the same moment restrictions of the maximal operator $\mathbb{A}_{\max}$. All such restrictions and their resolvents are described by Theorem 1.

Such operators can always be determined by certain boundary conditions connecting the "boundary values"

$$u, \vec{u} \text{ and } \langle g(\mu), (L-\mu)U_r\rangle_H.$$

Under the condition that Γ is diagonal, the boundary form of the maximal operator $\mathbb{A}_{\max}$ is (see (45))

(55)

$$\begin{aligned}&\langle \mathbb{A}_{\max}\mathbb{U}, \mathbb{V}\rangle_{\mathbb{H}} - \langle \mathbb{U}, \mathbb{A}_{\max}\mathbb{V}\rangle_{\mathbb{H}} \\ &= \overline{u}\left(\langle \Gamma\vec{b}, \vec{v}\rangle_{\mathbb{C}^{n-2}} - \langle g(\mu), (L-\mu)V_r\rangle_H\right) - \overline{\left(\langle \Gamma\vec{b}, \vec{u}\rangle_{\mathbb{C}^{n-2}} - \langle g(\mu), (L-\mu)U_r\rangle_H\right)}v.\end{aligned}$$

We define then the following restrictions of the maximal operator and show that these are exactly the self-adjoint extensions.

Definition 2. *The domain of the operator $\mathbb{A}_\theta$, $\theta \in [0,\pi)$ consists of functions $\mathbb{U} \in \mathbb{H}$ possessing the representation*

$$\mathbb{U} = \begin{pmatrix} U \\ \vec{u} \end{pmatrix} = \begin{pmatrix} U_r + uG(\mu) \\ \vec{u} \end{pmatrix}, \quad U_r \in \mathcal{H}_n, u \in \mathbb{C}, \vec{u} \in \mathbb{C}^{n-2}, \tag{56}$$

and satisfying the boundary condition

$$\sin\theta\,\langle g(\mu), (L-\mu)U_r\rangle_H + \cos\theta\, u - \sin\theta\langle \Gamma\vec{b}, \vec{u}\rangle_{\mathbb{C}^{n-2}} = 0. \tag{57}$$

The action of $\mathbb{A}_\theta$ is given by the formula

$$\mathbb{A}_\theta\mathbb{U} = \mathbb{A}_\theta \begin{pmatrix} U_r + uG(\mu) \\ \vec{u} \end{pmatrix} = \begin{pmatrix} LU_r + \mu uG(\mu) \\ \mathcal{M}\vec{u} + u\vec{b} \end{pmatrix}, \tag{58}$$

where the matrix $\mathcal{M}$ and the vector $\vec{b}$ are determined by (40) and (41).

Note that the operator $\mathbb{A}_0$ introduced earlier coincides with $\mathbb{A}_\theta$ for $\theta = 0$. It is straightforward to calculate the resolvent of $\mathbb{A}_\theta$.

Theorem 1. *Let Γ be a positive diagonal matrix. Then the family of self-adjoint restrictions of the maximal operator $\mathbb{A}_{\max}$ coincides with the family $\mathbb{A}_\theta$, $\theta \in [0,\pi)$. The resolvent of the operator $\mathbb{A}_\theta$ for $\Im\lambda \neq 0$ is given by*

(59)

$$\begin{aligned}&\frac{1}{\mathbb{A}_\theta - \lambda} - \frac{1}{\mathbb{A}_0 - \lambda} \\ &= -\frac{1}{Q(\lambda) + \cot\theta}\begin{pmatrix} \langle g(\bar{\lambda}), \cdot\rangle_H\, G(\lambda) & \left\langle \frac{-1}{\mathcal{M}-\lambda}\vec{b}, \Gamma\cdot\right\rangle_{\mathbb{C}^m} G(\lambda) \\ \langle g(\bar{\lambda}), \cdot\rangle_H \frac{-1}{\mathcal{M}-\lambda}\vec{b} & \left\langle \frac{-1}{\mathcal{M}-\lambda}\vec{b}, \Gamma\cdot\right\rangle_{\mathbb{C}^m} \frac{-1}{\mathcal{M}-\lambda}\vec{b} \end{pmatrix},\end{aligned}$$

where

$$Q(\lambda) = \langle g(\mu), (\lambda-\mu)G(\lambda)\rangle_H + \langle \vec{b}, \Gamma\frac{1}{\mathcal{M}-\lambda}\vec{b}\rangle_{\mathbb{C}^m}. \tag{60}$$

Proof. Since the matrix Γ is diagonal the boundary form of the maximal operator is given by (55) and it is then clear that the restriction of $\mathbb{A}_{\max}$ to the set of functions satisfying (57) is a symmetric operator (the boundary form vanishes).

Let us calculate directly the resolvent of $\mathbb{A}_\theta$. Consider the resolvent equation for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{1}{\mathbb{A}_\theta - \lambda}\mathbb{V} = \mathbb{U} \Rightarrow \mathbb{V} = (\mathbb{A}_\theta - \lambda)\mathbb{U},$$

where $\mathbb{V} \in \mathbb{H}$ and $\mathbb{U} \in \mathrm{Dom}\,(\mathbb{A}_\theta)$. The last equation implies

$$\begin{cases} V = (L-\lambda)U_r + (\mu - \lambda)G(\mu)u, \\ \vec{v} = (\mathcal{M} - \lambda)\vec{u} + \vec{b}u; \end{cases}$$

$$\Rightarrow \begin{cases} U_r = \dfrac{1}{L-\lambda}V - \dfrac{\mu-\lambda}{L-\lambda}G(\mu)u \\ \vec{u} = \dfrac{1}{\mathcal{M}-\lambda}\vec{v} - \dfrac{1}{\mathcal{M}-\lambda}\vec{b}u. \end{cases}$$

Substitution into the boundary condition (57) allows to calculate u

$$u = \frac{-\left\langle g(\overline{\lambda}), V\right\rangle_H + \langle \Gamma\vec{b}, \frac{1}{\mathcal{M}-\lambda}\vec{v}\rangle_{\mathbb{C}^m}}{\langle g(\mu), (\lambda-\mu)G(\lambda)\rangle_H + \left\langle \vec{b}, \Gamma\frac{1}{\mathcal{M}-\lambda}\vec{b}\right\rangle_{\mathbb{C}^m} + \cot\theta}.$$

It is natural to denote the Nevanlinna function appearing in the denominator by $Q(\lambda)$ (see (60). Then all components of the function $\mathbb{U}$ can be calculated

$$\begin{cases} U = \frac{1}{L-\lambda}V - \frac{1}{Q(\lambda)+\cot\theta}\left\{\left\langle g(\bar{\lambda}), V\right\rangle_H G(\mu) + \left\langle \frac{-1}{\mathcal{M}-\lambda}\vec{b}, \Gamma\vec{v}\right\rangle_{\mathbb{C}^{n-2}} G(\mu)\right\}, \\ \vec{u} = \frac{1}{\mathcal{M}-\lambda}\vec{v} - \frac{1}{Q(\lambda)+\cot\theta}\left\{\langle g(\lambda), V\rangle_H \frac{-1}{\mathcal{M}-\lambda}\vec{b} + \left\langle \frac{-1}{\mathcal{M}-\lambda}\vec{b}, \Gamma\vec{v}\right\rangle_{\mathbb{C}^{n-2}} \frac{-1}{\mathcal{M}-\lambda}\vec{b}\right\}, \end{cases}$$

which implies formula (59). Hence every operator $\mathbb{A}_\theta$ is symmetric and the range of $\mathbb{A}_\theta - \lambda$, $\Im\lambda \neq 0$ coincides with the whole $\mathbb{H}$. Hence every such operator is self-adjoint.

On the other hand formula (59) can be written in Krein's form

$$\frac{1}{\mathbb{A}_\theta - \lambda} - \frac{1}{\mathbb{A}_0 - \lambda} = -\frac{1}{Q(\lambda) + \cot\theta}\left\langle \mathbb{F}(\overline{\lambda}), \cdot \right\rangle_{\mathbb{H}} \mathbb{F}(\lambda), \tag{61}$$

where $\mathbb{F}(\lambda)$ is given by (51). This proves that the family $\mathbb{A}_\theta$ indeed is the family of all possible self-adjoint restrictions of $\mathbb{A}_{\max}$. □

The resolvent formulas just proven (59,61) are classical Krein's formulas and the Q-function appearing in the denominator is a Nevanlinna function, since the operators $\mathbb{A}_0, \mathbb{A}_\theta$ are self-adjoint in the Hilbert space

$\mathbb{H}$. In the following subsection we present another resolvent formula associated with the particular structure of the triplet extensions.

7. Extended resolvent formula of Krein type

The Hilbert space $\mathbb{H}$ is naturally decomposed into the orthogonal sum of the infinite dimensional space $\mathcal{H}_m$ and a finite dimensional space $\mathbb{C}^m$ in accordance to (36). It is therefore clear that the compression of the resolvent to the infinite dimensional component is given by Krein's formula for generalized resolvents with the denominator equal to a sum of two Nevanlinna functions:

$$\begin{aligned} & P_{\mathcal{H}_m} \frac{1}{\mathbb{A}_\theta - \lambda} |_{\mathcal{H}_m} \\ & = \frac{1}{L-\lambda} - \frac{1}{Q(\lambda)+\cot\theta} \left\langle g(\bar\lambda), \cdot \right\rangle_H G(\lambda) \\ & = \frac{1}{L-\lambda} - \frac{1}{q(\lambda)+q_\Gamma(\lambda)+\cot\theta} \left\langle G(\bar\lambda), \cdot \right\rangle_{\mathcal{H}_m} G(\lambda), \end{aligned} \tag{62}$$

where

- $q(\lambda) = \langle g(\mu), (\lambda-\mu) G(\lambda)\rangle_H$ is the Q-function associated with the operators L and $L_{\min}$ in $\mathcal{H}_m$;
- $q_\Gamma(\lambda) = \left\langle \vec{b}, \Gamma \frac{1}{\mathcal{M}-\lambda} \vec{b} \right\rangle_{\mathbb{C}^m}$ is the Q-function associated with the operator $\mathcal{M}$ and vector $\vec{b}$ in $\mathbb{C}^m$.

This resolvent formula shows once more that the operator $\mathbb{A}_\theta$ is indeed a generalized extension of the operator $L_{\min}$.

Let us consider now another type of resolvent formula - just the restriction of the resolvent of $\mathbb{A}_\theta$ to $\mathcal{H}_m$, but written in the functional representation

$$\begin{aligned} \frac{1}{\mathbb{A}_\theta - \lambda} |_{\mathcal{H}_m} & = \frac{1}{L-\lambda} - \frac{1}{Q(\lambda)+\cot\theta} \left\langle g(\bar\lambda), \cdot \right\rangle_H \mathbb{F}(\lambda) \\ & = \frac{1}{L-\lambda} - \frac{1}{b(\lambda)(Q(\lambda)+\cot\theta)} \left\langle g(\bar\lambda), \cdot \right\rangle_H g(\lambda). \end{aligned} \tag{63}$$

The function appearing in the denominator

$$\begin{aligned} Q_m(\lambda) & = b(\lambda)(Q(\lambda)+\cot\theta) \\ & = b(\lambda) \left(\left\langle \varphi, \frac{1}{L-\lambda} \frac{\lambda-\mu}{L-\mu} \frac{1}{b(L)} \varphi \right\rangle_H + \left\langle \vec{b}, \Gamma \frac{1}{\mathcal{M}-\lambda} \vec{b} \right\rangle_{\mathbb{C}^{n-2}} + \cot\theta \right). \end{aligned} \tag{64}$$

is a generalized Nevanlinna function (see [13]). In the following section we show, that precisely this function can be obtained by just regularizing the classical formula (18) valid for canonical extensions.

8. Renormalization of the Q-function

For canonical extensions (inside the original Hilbert space) the functions $g(\lambda)$ and $G(\lambda)$ coincide, since this case corresponds to $m = 0$ and $b \equiv 1$. The function (18) appearing in the denominator of Krein's formula in the case of canonical extensions can be considered as a renormalization of the Nevanlinna function

$$Q(\lambda) = \langle g(\mu), \frac{(L-\mu)^2}{L-\lambda} g(\mu)\rangle_H = \langle g(\mu), (L-\mu) g(\lambda)\rangle_H,$$

which is correctly defined only if $g(\mu) \in \mathcal{H}_1$. In fact precisely this function appears in the resolvent formula when the perturbed operator is a bounded rank one perturbation (see [5] and Section 9).

If $g(\mu) \in \mathcal{H} \setminus \mathcal{H}_1$ then the function Q can be obtained using the following renormalization procedure

(65)
$$\begin{array}{rcl} Q(\lambda) & \overset{\text{formally}}{=} & \langle g(\mu), \dfrac{(L-\mu)^2}{L-\lambda} g(\mu)\rangle_H - \langle g(\mu), (L-\mu) g(\mu)\rangle_H + p \\ & = & \langle g(\mu), (\lambda-\mu) g(\lambda)\rangle_H + p, \end{array}$$

with the renormalization point $\mu < 0$ and renormalization parameter $p \overset{\text{formally}}{=} \langle g(\mu), (L-\mu) g(\mu)\rangle_H \in \mathbb{R}$. If $g(\mu) \in \mathcal{H}_1$, then the renormalization parameter is uniquely determined by the last formula; if $g(\mu) \in \mathcal{H} \setminus \mathcal{H}_1$, then this parameter can be chosen arbitrarily.[3]

This renormalization procedure can be continued in order to include more and more singular elements $g(\mu)$. For example if $g(\mu) \in \mathcal{H}_{-1} \setminus \mathcal{H}$, then the scalar product $\langle g(\mu), g(\lambda)\rangle_H$ is not defined and one needs one further renormalization with a certain $\mu_1 < 0$ and $p_1 \in \mathbb{R}$

(66)
$$\begin{array}{rcl} Q_1(\lambda) & \overset{\text{formally}}{=} & (\lambda-\mu)\left\{\langle g(\mu), g(\lambda)\rangle_H - \langle g(\mu), g(\mu_1)\rangle_H + p_1\right\} + p \\ & = & \left\langle g(\mu), (\lambda-\mu)\dfrac{\lambda-\mu_1}{L-\mu_1} g(\lambda)\right\rangle_H + (\lambda-\mu)p_1 + p, \end{array}$$

The renormalization parameter p_1 is formally equal to $\langle g(\mu), g(\mu_1)\rangle_H$ and if $g(\mu) \in \mathcal{H}$, then this parameter is uniquely defined and the Q-function coincides with the function given by (65). The function Q_1 contains two renormalization parameters p and p_1 and may not be anymore a Nevanlinna function, but a generalized Nevanlinna function so far with one negative square.

[3]This is connected with the fact that $\mathcal{H}_{-1}$-perturbations are uniquely determined, but $\mathcal{H}_{-2}$-perturbations not (see Section 9).

Continuing this renormalization procedure we arrive to the following formula for the Q-function in the case $g(\mu) \in \mathcal{H}_{-m} \setminus H_{-m+1}$, $m \geq 1$

$$\begin{aligned} Q_m(\lambda) &= \left\langle g(\mu), (\lambda-\mu)\frac{\lambda-\mu_1}{L-\mu_1}...\frac{\lambda-\mu_m}{L-\mu_m}g(\lambda)\right\rangle_H + p(\lambda) \\ &= \left\langle g(\mu), (\lambda-\mu)\frac{b(\lambda)}{b(L)}g(\lambda)\right\rangle_H + p(\lambda) \end{aligned}, \tag{67}$$

where $p(\lambda)$ is the following polynomial of degree m

$$p(\lambda) = (\lambda-\mu)\sum_{j=1}^{m}(\lambda-\mu_1)...(\lambda-\mu_{j-1})p_j + p. \tag{68}$$

The renormalization points μ_j are all chosen negative $\mu_j < 0$ and the real renormalization parameters are formally equal to the following scalar products

$$p_j \overset{\text{formally}}{=} \langle g(\mu), \frac{1}{(L-\mu_j)(L-\mu_{j-1})...(L-\mu_2)}g(\mu_1)\rangle_H. \tag{69}$$

As before this sequence of Q-functions is constructed in such a way that if the deficiency element is less singular so that some of the scalar products in (69) are well-defined, then the renormalization parameters p_j can be properly chosen to get the Q-function corresponding to the less singular deficiency elements.

The function Q_m is a generalized Nevanlinna function with at most $\left[\frac{m+1}{2}\right]$ negative squares and contains $m+1$ arbitrary real parameters p, p_j. If the deficiency elements are less singular, for example if $g(\mu) \in \mathcal{H}_{-l}, l < m-1$ then the parameters can be chosen in such a way that the Q-function has less negative squares. In particular if $g(\mu) \in \mathcal{H}$, then the function Q_m can be made a Nevanlinna function by choosing the renormalization parameters properly. Moreover if $g(\mu) \in \mathcal{H}_{-m} \setminus \mathcal{H}$, then Q_m is not a usual Nevanlinna function independently of how the parameters p, p_j are chosen.

Formulas (67) and (64) give the same function if and only if the polynomial $p(\lambda)$ is chosen equal to

$$p(\lambda) = -\sum_{j=1}^{m}\frac{\gamma_{jj}^2}{(b_j(\mu_j))^2}b_j(\lambda) + b(\lambda)\cot\theta, \tag{70}$$

where b_j are given by (33) and γ_{jj} are the entries of Γ.

9. Triplet extensions and supersingular perturbations

Let L be a positive self-adjoint operator acting in the Hilbert space H. Let φ be an element from the scale of Hilbert spaces $\mathcal{H}_n$ associated with the operator L

$$\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}. \tag{71}$$

Then a rank one perturbation of the operator L is given by the following formal expression

$$L + \alpha \langle \varphi, \cdot \rangle_H \varphi, \quad \alpha \in \mathbb{R}. \tag{72}$$

In this section we are going to construct self-adjoint operators corresponding to formal expressions (72) in the case $n > 2$. Such perturbations are usually called **supersingular** in order to distinguish them from **singular** perturbations given by $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$.

Let us recall first the main ideas of the theory of regular ($n = 0$) and singular ($n = 1, 2$) perturbations. If $\varphi \in \mathcal{H}_0 = H$, then the perturbation in (72) is bounded and the perturbed operator, denoted by L_α, is self-adjoint on $\mathrm{Dom}\,(L)$. If $\varphi \in \mathcal{H}_{-1}$ then (72) determines a unique operator L_α, since the perturbation is relatively form bounded in this case and the method of quadratic forms can be applied. It is also possible to use the extension theory for symmetic operators and identify the operator given by (72) with one particular self-adjoint extension of the symmetric operator $L_{\min} = L|_{\{U \in \mathrm{Dom}\,(L): \langle \varphi, U \rangle_H = 0\}}$. If $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, then the quadratic form approach cannot be applied but the extension theory approach can be used, since $L_{\min}$ is correctly defined as a symmetric (not essentially self-adjoint operator) in H. Then the operator corresponding to (72) is usually defined as one particular operator from the one-parameter family of self-adjoint extensions of $L_{\min}$. In general it is impossible to decide which particular extension corresponds to formula (72), which is understood formally and in order to underline this the corresponding operator will be denoted by $A_\gamma, \gamma \in \mathbb{R} \cup \{\infty\}$ instead of L_α. Using the fact that the deficiency elements for the symmetric operator $L_{\min}$ are

$$g(\lambda) = \frac{1}{L - \lambda} \varphi, \tag{73}$$

the resolvent of any operator A_γ is described by Krein's formula

$$\frac{1}{A_\gamma - \lambda} = \frac{1}{L - \lambda} - \frac{1}{Q(\lambda) + \gamma} \left\langle \frac{1}{L - \bar{\lambda}} \varphi, \cdot \right\rangle_H \frac{1}{L - \lambda} \varphi, \quad \Im \lambda \neq 0; \tag{74}$$

where $Q(\lambda)$ is given by

$$Q(\lambda) = \left\langle \varphi, \frac{\lambda - \mu}{(L-\lambda)(L-\mu)} \varphi \right\rangle_H + p, \tag{75}$$

where $\mu < 0$ and $p \in \mathbb{R}$ are arbitrary parameters. The same formula (74) gives the resolvent of the operator L_α in the case of $\mathcal{H}_{-1}$-perturbations if the parameters are properly chosen as

$$p = \langle \varphi, \frac{1}{L-\mu} \varphi \rangle_H \text{ and } \gamma = 1/\alpha.$$

In this case the Q-function is just equal to

$$Q(\lambda) = \left\langle \varphi, \frac{1}{L-\lambda} \varphi \right\rangle_H . \tag{76}$$

Precisely this renormalization procedure has been generalized in Section 8.

Summing up to define singular perturbations given by (72) classical extension theory of symmetric operators may be used. The function $Q(\lambda)$ is a Nevanlinna function and contains information about spectral properties of the operator A_γ.

Let us discuss now supersingular perturbations given by vectors $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1},\ n > 2$. The formal expression (72) naturally leads to the minimal operator

$$L_{\min} = L|_{\{U \in \mathcal{H}_n : \langle \varphi, U \rangle_H = 0\}}. \tag{77}$$

The operator $L_{\min}$ is a symmetric operator in the Hilbert space $\mathcal{H}_{n-2}$ and has deficiency indices $(1,1)$ if considered in this Hilbert space. It is clear that the functions $g(\lambda)$ and $G(\lambda)$ are given by the following formulas in this case

$$g(\lambda) = \frac{1}{L-\lambda} \varphi \in \mathcal{H}_{-n+2}, \quad G(\lambda) = \frac{1}{b(L)} \frac{1}{L-\lambda} \varphi \in \mathcal{H}_{n-2}. \tag{78}$$

Since $\varphi \notin \mathcal{H}_{-2}$ the operator $L_{\min}$ is essentially self-adjoint in the original Hilbert space H and therefore satisfies the assumptions formulated in Section 3 with $m = n-2$. Any self-adjoint operator associated with (72) should be an extension of $L_{\min}$ and it is natural to associate with it the family of triplet extensions constructed in Section 6. The kernel of the operator $L_{\max} - \lambda = L_{\min}^\dagger - \lambda$ is spanned by the functions $g(\lambda)$. The model space $\mathbb{H}$ is then given by

$$\mathbb{H} = \mathcal{H}_{n-2} \dot{+} \mathcal{L}\{(L-\mu_1)^{-1}\varphi, (L-\mu_2)^{-1}\varphi, ..., (L-\mu_{n-2})\varphi\} \ni \mathbb{U} = (U, \vec{u}), \tag{79}$$

and endowed with the scalar product given by (34). The elements from the space can also be viewed as elements from $\mathcal{H}_{-n+2}$

$$\mathbb{U} = U + \sum_{j=1}^{n-2} u_j (L-\mu_j)^{-1}\varphi,$$

since $(L-\mu_j)^{-1}\varphi \in \mathcal{H}_{-n+2}$. Constructing the space we have chosen $n-2$ arbitrary negative renormalization points $\mu_j < 0,\ j = 1,2,..,n-2$ and $n-2$ arbitrary positive parameters $\gamma_{jj} > 0,\ j = 1,2,...,n-2$, since the Gramm matrix Γ in (34) has to be chosen diagonal and positive to guarantee the existence of self-adjoint triplet extensions.

The one-parameter family $\mathbb{A}_\theta,\ \theta \in [0,\pi)$ of triplet extensions of $L_{\min}$ in $\mathbb{H}$ is then defined on the domain of functions possessing the representation

$$\mathbb{U} = (U_r + ub^{-1}(L)(L-\mu)^{-1}\varphi, \vec{u}),\ U_r \in \mathcal{H}_n, u \in \mathbb{C}, \vec{u} \in \mathbb{C}^{n-2}, \tag{80}$$

and the boundary condition

$$\sin\theta\langle\varphi, U_r\rangle_H + \cos\theta u - \sin\theta\langle\Gamma\vec{b}, \vec{u}\rangle_{\mathbb{C}^{n-2}} = 0. \tag{81}$$

Note that $b^{-1}(L)(L-\mu)^{-1}\varphi \in \mathcal{H}_{n-2}$. The action of the operator $\mathbb{A}_\theta$ is given by

$$\mathbb{A}_\theta\mathbb{U} = \mathbb{A}_\theta \begin{pmatrix} U_r + ub^{-1}(L)(L-\mu)^{-1}\varphi \\ \vec{u} \end{pmatrix} = \begin{pmatrix} LU_r + \mu ub^{-1}(L)(L-\mu)^{-1}\varphi \\ \mathcal{M}\vec{u} + u\vec{b} \end{pmatrix}, \tag{82}$$

where the matrix $\mathcal{M}$ and the vector $\vec{b}$ are determined by (40) and (41). Taking into account (78) the resolvent of the self-adjoint operator $\mathbb{A}_\theta$ can be calculated

$$\begin{aligned} &\frac{1}{\mathbb{A}_\theta - \lambda} - \frac{1}{\mathbb{A}_0 - \lambda} \\ &= -\frac{1}{Q(\lambda)+\cot\theta} \begin{pmatrix} \left\langle \frac{1}{L-\bar\lambda}\varphi, \cdot \right\rangle_H \frac{1}{b(L)(L-\lambda)}\varphi & \left\langle \frac{-1}{\mathcal{M}-\bar\lambda}\vec{b}, \Gamma\cdot \right\rangle_{\mathbb{C}^{n-2}} \frac{1}{b(L)(L-\lambda)}\varphi \\ \left\langle \frac{1}{L-\bar\lambda}\varphi, \cdot \right\rangle_H \frac{-1}{\mathcal{M}-\lambda}\vec{b} & \left\langle \frac{-1}{\mathcal{M}-\bar\lambda}\vec{b}, \Gamma\cdot \right\rangle_{\mathbb{C}^{n-2}} \frac{-1}{\mathcal{M}-\lambda}\vec{b} \end{pmatrix}, \end{aligned} \tag{83}$$

with

$$Q(\lambda) = \left\langle \varphi, \frac{1}{b(L)} \frac{\lambda-\mu}{(L-\lambda)(L-\mu)}\varphi \right\rangle_H + \left\langle \vec{b}, \Gamma\frac{1}{\mathcal{M}-\lambda}\vec{b} \right\rangle_{\mathbb{C}^{n-2}}. \tag{84}$$

The extended resolvent formula (63) takes the form

$$\frac{1}{\mathbb{A}_\theta - \lambda}|_{\mathcal{H}_{n-2}} = \frac{1}{L-\lambda} - \frac{1}{b(\lambda)(Q(\lambda)+\cot\theta)} \left\langle \frac{1}{L-\bar\lambda}\varphi, \cdot \right\rangle_H \frac{1}{L-\lambda}\varphi, \tag{85}$$

which is a natural generalization of (74). The denominator is given by the generalized Nevanlinna function

$$Q_{n-2}(\lambda) = \left\langle \varphi, \frac{1}{L-\lambda} \frac{\lambda-\mu}{L-\mu} \frac{b(\lambda)}{b(L)} \varphi \right\rangle_H + p(\lambda) \tag{86}$$

with the polynomial $p(\lambda)$ given by (70).

Summing up supersingular perturbations of at least of semibounded operators are given by triplet extensions. Their spectral properties are described by generalized Nevanlinna functions.

The peak model for supersingular perturbations presented here is a generalization of the cascade model described in [22, 12]. The advantage of the new model is that the Gramm matrix Γ is now diagonal, which lieads to simplification of all formulas and makes the studies of the corresponding operator more transparent. In the new model it is clear that all μ_j have to be pairwise different, since otherwise the functions $(L-\mu_j)\varphi$ are not linearly independent.

10. Point interactions in $\mathbb{R}^3$: new family

In this section we are going to apply methods developed in the previous section to construct a new family of point interactions in $\mathbb{R}^3$. Classical point interaction goes back to E. Fermi [16] and in the three-dimensional space can formally be described by the following expression

$$-\Delta + \alpha\delta = -\Delta + \alpha\delta\langle\delta, \cdot\rangle, \tag{87}$$

where δ is Dirac's delta function in $\mathbb{R}^3$ and $\alpha \in \mathbb{R} \cup \{\infty\}$ is a coupling constant. F.A. Berezin and L.D. Faddeev [7] suggested to interpret this operator using restriction-extension procedure. This procedure can be summarized as follows. Consider the restriction of the Laplace operator to the set of functions from $W_2^2(\mathbb{R}^3)$ equal to zero at the origin.[4] Note that the perturbation term is vanishing on such functions and therefore the Laplace operator and any self-adjoint operator corresponding to (87) are two (different) extensions of the restricted operator. This construction provides rigorous mathematical foundation for the celebrated Fermi pseudopotential [16] widely used in physics and chemistry [11]. The most hard limitation of this method is connected with the fact that the deficiency element $g(\mu) = e^{i\sqrt{\mu}|\mathbf{x}|}/4\pi|\mathbf{x}|$ is spherically symmetric and therefore the original Laplace operator and its perturbation given by (87) differ only on the subspace of spherically symmetric functions, *i.e.* this method allows one to introduce interaction in the

[4]This restriction is possible due to Sobolev embedding theorem.

s-channel only. The perturbed operator is defined on the set of functions $U = \tilde{U} + ug(\mu)$, $\tilde{U} \in W_2^2(\mathbb{R}^3), u \in \mathbb{C}$ possessing the following asymptotics at the origin

$$U(\mathbf{x}) = \frac{u_-}{4\pi|\mathbf{x}|} + u_0 + o(1), \ \mathbf{x} \to 0, \tag{88}$$

where $u_-, u_0 \in \mathbb{C}$ can be considered as certain boundary values of the function U. More precisely, every self-adjoint operator corresponding to the formal expression (87) coincides with the (differential) Laplace operator $-\Delta$ defined on the domain of functions from $W_2^2(\mathbb{R}^3 \setminus \{0\})$ satisfying the boundary condition

$$u_0 = \gamma u_-, \ \ \gamma \in \mathbb{R} \cup \{\infty\}. \tag{89}$$

Without any additional assumption it is impossible to establish the connection between the real parameters α in (87) and γ in (89) (except the fact that $\alpha = 0$ should correspond to $\gamma = \infty$), since this perturbation is from the class $\mathcal{H}_{-2} : \delta \in \mathcal{H}_{-2}(-\Delta)$.[5] This approach has been generalized to study numerous problems of mathematical physics and its applications [2, 11, 28, 29].

Described limitations of the method are connected first of all with the fact that the singular element determining the perturbation in (87) - Dirac's δ-function, - is spherically symmetric. Assume that we are interested in getting similar models where point interactions are not spherically symmetric.[6] Intuitively it is clear that one has to consider singular elements which are not spherically symmetric. Restricting our consideration to singular elements given by distributions we take into account that any generalized function with the support at the origin can be written as a linear combination of the derivatives of the delta function. In the current article we shall consider just the first order derivatives, which gives us the following formal expression for the perturbed Laplacian

$$L_{\vec{\alpha}} = -\Delta + \sum_{i=1}^{3} \alpha^i \langle \partial_{x_i} \delta, \cdot \rangle \partial_{x_i} \delta, \tag{90}$$

where α^i, $i = 1, 2, 3$ are real coupling constants and $\partial_{x_i} = \partial / \partial x_i$ denote the derivative with respect to the variable x_i. The perturbation term is from the class $\mathcal{H}_{-3}$, since $\varphi_j = \partial_{x_i} \delta \in \mathcal{H}_{-3}(\Delta)$ and it follows, that to determine the operator $L_{\vec{\alpha}}$ the theory of supersingular perturbations has to be applied (see previous Section). The rank of the perturbation

[5] See Section 1.5 in [5] and [3] where such relation is established using the homogeneity properties of the Laplace operator and the delta-function.

[6] Such models are needed to describe small objects having complicated geometry.

is equal to three, but the elements $\varphi_i = \partial_{x_i}\delta,\ i = 1,2,3$ and the operator $-\Delta$ generate three mutually orthogonal subspaces and thus the developed approach needs just a slight modification. Let us restrict our consideration to interactions commuting with the permutation of the coordinates, $\imath.e$ $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$. It will be shown later that the corresponding interaction is spherically symmetric but influences the p-channel (instead of the s-channel like in the classical Fermi pseudopotential).

For reader's convenience we present here our model using function representation. Since the perturbation is from the class $\mathcal{H}_{-3}$ one needs to consider just one renormalization point $\mu_1 = -\beta_1^2,\ \beta_1 > 0$. The functions $g(\lambda)$ and $G(\lambda)$ can easily be calculated using (78) and using that $\frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}, k = \sqrt{\lambda}$ is the Green's function for the Laplacian:

(91)

$$\begin{aligned} g_j(\lambda) &= \frac{\partial}{\partial x_j}\frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|} = \frac{ik|\mathbf{x}|-1}{4\pi|\mathbf{x}|^3}e^{ik|\mathbf{x}|}x_j, \\ G_i(\lambda) &= \frac{1}{\beta_1^2+k^2}\left(g_i(\lambda) - g_i(-\beta_1^2)\right) \qquad\qquad j = 1,2,3. \\ &= \frac{1}{\beta_1^2+k^2}\frac{(ik|\mathbf{x}|-1)e^{ik|\mathbf{x}|}+(\beta_1|\mathbf{x}|+1)e^{-\beta_1|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}x_i \end{aligned}$$

The functions $g_i(\lambda)$ are pairwise orthogonal in $W_1^{-1}(\mathbb{R}^3) = \mathcal{H}_{-1}(-\Delta)$ (just as functions having different symmetries) as well as the functions $G_i(\lambda)$.

Let us introduce the notation

$$h(k,r) = \frac{ikr-1}{4\pi r^3}e^{ikr}, \tag{92}$$

which will allow us to simplify the formulas

$$g_j(\lambda) = h(k,|\mathbf{x}|)x_j \ \text{ and } \ G_j(\lambda) = \frac{h(k,|\mathbf{x}|) - h(i\beta_1,|\mathbf{x}|)}{\beta_1^2+k^2}x_j.$$

Let us remember that the function h is just a combination of elementary functions.

The model Hilbert space can now be chosen equal to

(93)

$$\mathbb{H} = W_2^1(\mathbb{R}^3)\dot{+}\mathcal{L}\{g_1(-\beta_1^2), g_2(-\beta_1^2), g_3(-\beta_1^2)\} \ni \mathbb{U} = U + \sum_{i=1}^3 u_1^i h(i\beta_1,|\mathbf{x}|),$$

and therefore every function from $\mathbb{H}$ possesses the representation

$$(94)\quad \mathbb{U} = U + \sum_{i=1}^{3} u_1^i \frac{-\beta_1|\mathbf{x}| - 1}{4\pi|\mathbf{x}|^3} e^{-\beta_1|\mathbf{x}|} x_i \equiv U - \frac{\beta_1|\mathbf{x}| + 1}{4\pi|\mathbf{x}|^3} e^{-\beta_1|\mathbf{x}|} \mathbf{x}^t \mathbf{u}_1,$$

where we introduced the vectors $\mathbf{u}_1 = (u_1^1, u_1^2, u_1^3)^t \in \mathbb{C}^3$ and $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, so that we have

$$\mathbf{x}^t \mathbf{u}_1 = x_1 u_1^1 + x_2 u_1^2 + x_3 u_1^3.$$

In (94) $U \in W_2^1(\mathbb{R}^3)$ and therefore this representation is unique, hence every element from $\mathbb{H}$ can be viewed not only as a function on $\mathbb{R}^3$, but as a pair $\mathbb{U} = (U, \mathbf{u}_1),\ U \in W_2^1(\mathbb{R}^3), \mathbf{u}_1 \in \mathbb{C}^3$. In the latter case it will be called vector representation.

The norm in $\mathbb{H}$ can be chosen equal to

$$(95)\qquad \| \mathbb{U} \|_{\mathbb{H}}^2 = \| U + \sum_{i=1}^{3} u_1^i g_i(-\beta_1^2) \|_{\mathbb{H}}^2 = \| U \|_{W_2^1(\mathbb{R}^3)}^2 + \gamma \| \mathbf{u}_1 \|^2,$$

where γ is an arbitrary positive parameter.

To define the self-adjoint operator in $\mathbb{H}$ corresponding to formal expression (90) consider another one negative parameter $\mu = -\beta^2,\ \beta > 0$ and extension parameter $\theta \in [0, \pi)$. Then the operator $\mathbb{A}_\theta$ is defined on the set of functions possessing the representation

(96)

$$\begin{aligned}
\mathbb{U} &= U_r + \sum_{i=1}^{3} u^i G_i(-\beta^2) + \sum_{i=1}^{3} u_1^i g_i(-\beta_1^2), \\
&= U_r + \tfrac{-(1+\beta|\mathbf{x}|)e^{-\beta|\mathbf{x}|} + (1+\beta_1|\mathbf{x}|)e^{-\beta_1|\mathbf{x}|}}{(\beta_1^2 - \beta^2)4\pi|\mathbf{x}|^3} \mathbf{x}^t \mathbf{u} - \tfrac{1+\beta_1|\mathbf{x}|}{4\pi|\mathbf{x}|^3} e^{-\beta_1|\mathbf{x}|} \mathbf{x}^t \mathbf{u}_1, \\
&= U_r + \tfrac{h(i\beta, |\mathbf{x}|) - h(i\beta_1, |\mathbf{x}|)}{\beta_1^2 - \beta^2} \mathbf{x}^t \mathbf{u} + h(i\beta_1, |\mathbf{x}|) e^{-\beta_1|\mathbf{x}|} \mathbf{x}^t \mathbf{u}_1, \\
& \quad U_r \in W_2^3(\mathbb{R}^3), \mathbf{u}, \mathbf{u}_1 \in \mathbb{C}^3,
\end{aligned}$$

and the boundary conditions[7]

$$(97)\qquad \sin\theta\, \langle \partial_i \delta, U_r \rangle + \cos\theta\ u^i - \sin\theta\ \gamma\ u_1^i = 0, \quad i = 1, 2, 3.$$

The last condition can also be written using vector notations as

$$(98)\qquad \sin\theta\, (\boldsymbol{\nabla} U_r(0) + \gamma \mathbf{u}_1) = \cos\theta\ \mathbf{u}.$$

[7]In principle it is possible to choose three different real extension parameters $\theta_i, i = 1, 2, 3$ independently, but our aim is to construct a model operator corresponding to the formal expression (90) with all α_i all equal, *i.e.* with the interaction commuting with permutations of the coordinates.

The action of the operator $\mathbb{A}_\theta$ is determined by the formula

(99)
$$\mathbb{A}_\theta \left(U_r + \frac{-(1+\beta|\mathbf{x}|)e^{-\beta|\mathbf{x}|} + (1+\beta_1|\mathbf{x}|)e^{-\beta_1|\mathbf{x}|}}{(\beta_1^2-\beta^2)4\pi|\mathbf{x}|^3}\mathbf{x}^t\mathbf{u} - \frac{1+\beta_1|\mathbf{x}|}{4\pi|\mathbf{x}|^3}e^{-\beta_1|\mathbf{x}|}\mathbf{x}^t\mathbf{u}_1 \right)$$
$$= -\Delta U_r - \beta^2 \frac{-(1+\beta|\mathbf{x}|)e^{-\beta|\mathbf{x}|} + (1+\beta_1|\mathbf{x}|)e^{-\beta_1|\mathbf{x}|}}{(\beta_1^2-\beta^2)4\pi|\mathbf{x}|^3}\mathbf{x}^t\mathbf{u}$$
$$-\frac{1+\beta_1|\mathbf{x}|}{4\pi|\mathbf{x}|^3}e^{-\beta_1|\mathbf{x}|}\left(\mathbf{x}^t\mathbf{u} - \beta_1^2\mathbf{x}^t\mathbf{u}_1\right),$$

which implies that outside the origin it acts pointwise just as the usual Laplacian

$$(\mathbb{A}_\theta \mathbb{U})(\mathbf{x}) = -(\mathbb{U}_{x_1x_1} + \mathbb{U}_{x_2x_2} + \mathbb{U}_{x_3x_3})(\mathbf{x}), \quad \mathbf{x} \neq 0. \tag{100}$$

Using notation (92) the action of the operator can be presented by

(101)
$$\mathbb{A}_\theta \left(U_r + \frac{h(i\beta,|\mathbf{x}|) - h(i\beta_1,|\mathbf{x}|)}{\beta_1^2-\beta^2}\mathbf{x}^t\mathbf{u} + h(i\beta_1,|\mathbf{x}|)\mathbf{x}^t\mathbf{u}_1 \right)$$
$$= -\Delta U_r - \beta^2 \frac{h(i\beta,|\mathbf{x}|) - h(i\beta_1,|\mathbf{x}|)}{\beta_1^2-\beta^2}\mathbf{x}^t\mathbf{u} + h(i\beta_1,|\mathbf{x}|)\left(\mathbf{x}^t\mathbf{u} - \beta_1^2\mathbf{x}^t\mathbf{u}_1\right),$$

As a result we are getting a spherically symmetric interaction, *i.e.* the corresponding operator commutes with the rotations around the origin and reflections in planes passing the origin. This implies in particular that the operator commutes with permutations of the coordinates.

Lemma 5. *The operator $\mathbb{A}_\theta$, $\theta \in [0,\pi)$ commutes with the rotations around the origin and reflections in planes passing through the origin.*

Proof. To see this we just need to prove, that the domain $\mathrm{Dom}\,(\mathbb{A}_\theta)$ is invariant under discussing transformations of $\mathbb{R}^3$, since we already know that the action of the operator is given by the Laplacian (100), which is invariant under rotations and reflections. The boundary condition for $\mathbb{A}_\theta$ can be written in the vector form (98). To prove that the domain of $\mathbb{A}_\theta$ is invariant under rotations we have to prove that every function $\mathbb{U}$ possessing representation (96) possesses the same representation after rotation and that every function satisfying the boundary condition (98) satisfy this condition after the rotation.

Let $\mathcal{R}$ be any 3×3 rotation matrix, then define $\mathcal{R}F$ for any function $F = F(\mathbf{x})$, $\mathbf{x} = (x_1,x_2,x_3)^t \in \mathbb{R}^3$ by

$$(\mathcal{R}F)(\mathbf{x}) = F(\mathcal{R}^{-1}\mathbf{x}).$$

Let us prove first that the linear space $\mathbb{H}$ is invariant under rotations. It is clear that the subspace $W_2^1(\mathbb{R}^3)$ is invariant. Using the fact that

the function $h(k, |\mathbf{x}|)$ is invariant under rotations it is easy to see that
(102)

$$\mathcal{R}\left(h(i\beta_1, |\mathbf{x}|)\mathbf{x}^t \mathbf{u}_1\right) = h(i\beta_1, |\mathbf{x}|)\left(\mathcal{R}^{-1}\mathbf{x}\right)^t \mathbf{u}_1 = h(i\beta, |\mathbf{x}|)\mathbf{x}^t(\mathcal{R}\mathbf{u}_1).$$

Hence $\mathbb{H}$ is not only invariant under rotations, but in addition any rotation around the origin in $\mathbb{R}^3$ corresponds to the rotation of the vector $\mathbf{u}_1$. The rotation matrix $\mathcal{R}$ induces a unitary transformation in $\mathbb{H}$.

Similarly we have that the set of functions possessing representation (96) is invariant under rotations and that rotation of the function $\mathbb{U}$ corresponds to the rotations of the vector $\mathbf{u}_1$ and $\mathbf{u}$. Taking into account that

$$\mathcal{R}\left(\boldsymbol{\nabla} U(0)\right) = \left(\boldsymbol{\nabla}\mathcal{R}U\right)(0),$$

we conclude that the boundary conditions (98) are preserved under rotations as well.

The proof for reflections in planes through the origin follows the same lines. □

We would like to underline that the operator $\mathbb{A}_\theta$ does not coincide with the classical point interaction Hamiltonian and even is not unitary equivalent to it. We have obtained a new family of models which can easily be generalized to include higher derivatives and hence more and more singular interactions.

The three Q-functions describing spectral properties of the model are all given by formula (84) which takes the following form now

$$Q^i(\lambda) = \left\langle \partial_{x_i}\delta, \frac{1}{L+\beta_1^2}\frac{\lambda+\beta^2}{(L-\lambda)(L+\beta^2)}\partial_{x_i}\delta\right\rangle_{L_2(\mathbb{R}^3)} + \frac{\gamma}{-\beta_1^2-\lambda}.$$

Taking into account that these functions are all equal $Q^1(\lambda) = Q^2(\lambda) = Q^3(\lambda) \equiv Q(\lambda)$, the last formula can be re-written and the function can be calculated explicitly
(103)

$$\begin{aligned} Q(\lambda) &= \frac{1}{3}\left\langle \delta, \frac{L}{L+\beta_1^2}\frac{\lambda+\beta^2}{(L-\lambda)(L+\beta^2)}\delta\right\rangle_{L_2(\mathbb{R}^3)} + \frac{\gamma}{-\beta_1^2-k^2} \\ &= \frac{1}{12\pi}\left\{ik + \frac{\beta_1^2}{ik-\beta_1} + \beta + \frac{\beta_1^2}{\beta+\beta_1}\right\} + \frac{\gamma}{-\beta_1^2-k^2}. \end{aligned}$$

Then the resolvent formula (83) has to be modified as follows

$$\frac{1}{\mathbb{A}_\theta - \lambda} = \begin{pmatrix} \frac{1}{L-\lambda} & 0 \\ 0 & \frac{1}{-\beta_1^2-\lambda} \end{pmatrix} - \frac{1}{Q(\lambda)+\cot\theta} \times$$

$$\times \sum_{i=1}^{3} \begin{pmatrix} \left\langle \frac{1}{L-\lambda}\partial_i\delta, \cdot \right\rangle_{L_2(\mathbb{R}^3)} \frac{1}{(L+\beta_1^2)(L-\lambda)}\partial_i\delta & \frac{\gamma}{\beta_1^2+\lambda}\langle \mathbf{e}_i, \cdot\rangle_{\mathbb{C}^3} \frac{1}{(L+\beta_1^2)(L-\lambda)}\partial_i\delta \\ \left\langle \frac{1}{L-\lambda}\partial_i\delta, \cdot \right\rangle_{L_2(\mathbb{R}^3)} \frac{1}{\beta_1^2+\lambda}\mathbf{e}_i & \frac{\gamma}{\beta_1^2+\lambda}\langle \mathbf{e}_i, \cdot\rangle_{\mathbb{C}^3} \frac{1}{\beta_1^2+\lambda}\mathbf{e}_i \end{pmatrix},$$

where $\mathbf{e}_i \in \mathbb{C}^3,\ i = 1,2,3$ are standard basis vectors in $\mathbb{C}^3$ and the first term on the right hand side is the resolvent of $\mathbb{A}_0 = L \oplus (-\beta_1^2)$. Using vector notations the resolvent is presented by the following expression

(104)

$$\frac{1}{\mathbb{A}_\theta - \lambda} = \begin{pmatrix} \frac{1}{L-\lambda} \\ \frac{1}{-\beta_1^2-\lambda} \end{pmatrix} - \frac{1}{Q(\lambda)+\cot\theta} \times$$

$$\times \begin{pmatrix} -\frac{h(k,|\mathbf{x}|)-h(i\beta_1,|\mathbf{x}|)}{\beta_1^2+k^2}\mathbf{x}^t \boldsymbol{\nabla}\left(\frac{1}{-\Delta-\lambda}\cdot\right)(0) & \frac{\gamma}{\beta_1^2+k^2}\frac{h(k,|\mathbf{x}|)-h(i\beta_1,|\mathbf{x}|)}{\beta_1^2+k^2}\mathbf{x}^t\cdot \\ \frac{-1}{\beta_1^2+k^2}\boldsymbol{\nabla}\left(\frac{1}{L-\lambda}\cdot\right)(0) & \frac{\gamma}{(\beta_1^2+k^2)^2}\cdot \end{pmatrix}.$$

The same formula can be written using the function representation as follows

(105)

$$\begin{aligned} &(\mathbb{A}_\theta - \lambda)^{-1}\left(U + h(i\beta_1,|\mathbf{x}|)\mathbf{x}^t\mathbf{u}_1\right) \\ &= \frac{1}{-\Delta-\lambda}U + \frac{1}{-\beta_1^2-\lambda}h(i\beta_1,|\mathbf{x}|)\mathbf{x}^t\mathbf{u}_1 \\ &-\frac{1}{Q(\lambda)+\cot\theta}\frac{h(k,|\mathbf{x}|)}{k^2+\beta_1^2}\mathbf{x}^t\left(-\boldsymbol{\nabla}\left(\frac{1}{-\Delta-\lambda}U\right)(0) + \frac{\gamma}{\beta_1^2+k^2}\mathbf{u}_1\right) \end{aligned}$$

As in the general case it appears natural to consider the restriction of this resolvent to the infinite dimensional component $U \in W_2^1(\mathbb{R}^3) \subset \mathbb{H}$

$$\begin{aligned} &(\mathbb{A}_\theta - \lambda)^{-1}U = \tfrac{1}{-\Delta-\lambda}U \\ &-\tfrac{1}{(k^2+\beta_1^2)(Q(\lambda)+\cot\theta)}\left(\int_{\mathbb{R}^3}\tfrac{(ik|\mathbf{y}|-1)e^{ik|\mathbf{y}|}}{4\pi|\mathbf{y}|^3}\mathbf{y}^t U(\mathbf{y})d^3\mathbf{y}\right)\tfrac{(ik|\mathbf{x}|-1)e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}\mathbf{x}. \end{aligned} \tag{106}$$

Looking at this form of the resolvent formula it is hard not to notice striking similarity to the original Krein's resolvent formula (17). Krein's Q-function should be substituted with the function appearing in the denominator of the last formula

$$\begin{aligned} Q_1(\lambda) &= (\lambda+\beta_1^2)(Q_0(\lambda)+\cot\theta) \\ &= (\lambda+\beta_1^2)\left(\tfrac{1}{12\pi}\left\{i\sqrt{\lambda} + \tfrac{\beta_1^2}{i\sqrt{\lambda}-\beta_1} + \beta + \tfrac{\beta_1^2}{\beta+\beta_1}\right\} + \tfrac{\gamma}{-\beta_1^2-\lambda} + \cot\theta\right). \end{aligned}$$

This function does not belong to Nevanlinna class, since it is growing like $\lambda^{3/2}$ as $\lambda \to \infty$. But it is a generalized Nevanlinna function as a product of the polynomial $\lambda + \beta_1^2$ and the Nevanlinna function $Q(\lambda) + \cot\theta$ [13].

The operator $\mathbb{A}_\theta$ is a finite rank perturbation (in the resolvent sense) of the operator $\mathbb{A}_0 = -\Delta \oplus -\beta_1^2$. Therefore the spectrum of the operator $\mathbb{A}_\theta$ contains the branch $[0,\infty)$ of the absolutely continuous spectrum (inherited from the Laplacian in $W_2^1(\mathbb{R}^3)$). In addition the spectrum may contain several negative eigenvalues. The negative eigenvalues correspond to zeroes of the Q-function on the negative axis. Let $\lambda_0 < 0$ be a solution of the equation

$$Q(\lambda_0) + \cot\theta = 0. \tag{107}$$

The function $Q(\lambda)$ is piecewise increasing with just one singularity at $\lambda = -\beta_1^2$ and $Q(\lambda) \to_{\lambda\to-\infty} -\infty$. Then the function $Q(\lambda)+\cot\theta$ has one or two zeroes on $\mathbb{R}_-$ if $Q(0)+\cot\theta = \frac{1}{12\pi}\frac{\beta^2}{\beta+\beta_1} - \frac{\gamma}{\beta_1^2} + \cot\theta$ is less than or greater than zero respectively. The corresponding eigenfunctions are given just by

$$\mathbb{V}_{\lambda_0} = h(k_0,|\mathbf{x}|)\mathbf{x}^t\mathbf{a} = \frac{ik_0|\mathbf{x}|-1}{4\pi|\mathbf{x}|^3}e^{ik_0|\mathbf{x}|}\mathbf{x}^t\mathbf{a}, \quad \lambda_0 = k_0^2, ik_0 \in \mathbb{R}_-, \tag{108}$$

where $\mathbf{a} = (a_1,a_2,a_3)^t \in \mathbb{C}^3$ is the vector parameterizing the three-dimensional space of eigenfunctions. Let us prove that such function is an eigenfunction for $\mathbb{A}_\theta$ provided $Q(\lambda_0) + \cot\theta = 0$. First of all let us see that this function possesses representation (96)

$$\begin{aligned}
&\frac{ik_0|\mathbf{x}|-1}{4\pi|\mathbf{x}|^3}e^{ik_0|\mathbf{x}|}\mathbf{x}^t\mathbf{a} \\
= \; & -\frac{\beta_1|\mathbf{x}|+1}{4\pi|\mathbf{x}|^3}e^{-\beta_1|\mathbf{x}|}\mathbf{x}^t\mathbf{a} \\
& +(\lambda+\beta_1^2)\frac{1}{\beta_1^2-\beta^2}\frac{-(\beta|\mathbf{x}|+1)e^{-\beta|\mathbf{x}|}+(\beta_1|\mathbf{x}|+1)e^{-\beta_1|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}\mathbf{x}^t\mathbf{a} \\
& +(\lambda+\beta_1^2)\left\{\frac{1}{\beta_1^2+k_0^2}\frac{(ik_0|\mathbf{x}|-1)e^{ik_0|\mathbf{x}|}+(\beta_1|\mathbf{x}|+1)e^{-\beta_1|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}\right. \\
& \left.-\frac{1}{\beta_1^2-\beta^2}\frac{-(\beta|\mathbf{x}|+1)e^{-\beta|\mathbf{x}|}+(\beta_1|\mathbf{x}|+1)e^{-\beta_1|\mathbf{x}|}}{4\pi|\mathbf{x}|^3}\right\}\mathbf{x}^t\mathbf{a}.
\end{aligned} \tag{109}$$

It follows that the boundary values of the function $\mathbb{V}_{\lambda_0}$ are

$$\left\{\begin{array}{l}
\mathbf{v}_1 = \mathbf{a} \\
\mathbf{v} = (\lambda_0+\beta_1^2)\mathbf{a} \\
-\boldsymbol{\nabla}V_r(0) = (\lambda_0+\beta_1^2)\frac{1}{12\pi}\left\{ik_0 + \frac{\beta_1^2}{ik_0-\beta_1} + \beta + \frac{\beta_1^2}{\beta_1+\beta}\right\}\mathbf{a}
\end{array}\right.$$

where V_r denotes the last term in formula (109). It is easy to see that the boundary values satisfy the boundary conditions (98) due to (107). Since the action of the operator coincides with the action of the Laplacian, $\mathbb{V}_{\lambda_0}$ solves the equation $-\Delta \mathbb{V}_{\lambda_0}(\mathbf{x}) - \lambda_0 \mathbb{V}_{\lambda_0}(\mathbf{x}) = 0,\ \mathbf{x} \neq 0$ and clearly belongs to $\mathbb{H}$, we conclude that $\mathbb{V}_{\lambda_0}$ is really an eigenfunction for $\mathbb{A}_\theta$. Summing up the operator $\mathbb{A}_\theta$ has one or two negative eigenvalues having multiplicity 3 with the eigenfunctions given by (108).

In a similar way continuous spectrum eigenfunctions may be calculated. We are going to use the following Ansatz

$$\mathbb{V}(\lambda, \mathbf{k}/k, \mathbf{x}) = e^{i\mathbf{k}^t\mathbf{x}} + \frac{ik|\mathbf{x}| - 1}{4\pi|\mathbf{x}|^3} e^{ik|\mathbf{x}|}\mathbf{x}^t\mathbf{a}(\mathbf{k}),\ \lambda > 0 \tag{110}$$

where $\mathbf{k} \in \mathbb{R}^3,\ |\mathbf{k}| = k = \sqrt{\lambda}$, is the direction the incoming plane wave and the scattering amplitude $\mathbf{a}(\mathbf{k}) \in \mathbb{C}^3$ has to be calculated from the boundary conditions. Substituting the boundary values of $\mathbb{V}(\lambda)$

(111)

$$\begin{cases} \mathbf{v}_1 = \mathbf{a} \\ \mathbf{v} = (\lambda + \beta_1^2)\mathbf{a} \\ -\boldsymbol{\nabla} V_r(0) = -i\mathbf{k} + (\lambda + \beta_1^2)\dfrac{1}{12\pi}\left\{ ik + \dfrac{\beta_1^2}{ik - \beta_1} + \beta + \dfrac{\beta_1^2}{\beta_1 + \beta} \right\}\mathbf{a} \end{cases}$$

into (98) the scattering amplitude $\mathbf{a}$ may be calculated

$$\mathbf{a} = \frac{i}{Q_1(\lambda)}\mathbf{k}, \tag{112}$$

which leads to the following formula for the generalized eigenfunction corresponing to the absolutely continuous spectrum

$$\mathbb{V}(\lambda, \mathbf{k}/k, \mathbf{x}) = e^{i\mathbf{k}^t\mathbf{x}} + \frac{i}{Q_1(\lambda)} \frac{ik|\mathbf{x}| - 1}{4\pi|\mathbf{x}|^3} e^{ik|\mathbf{x}|}\mathbf{x}^t\mathbf{k} \tag{113}$$

The scattering matrix corresponding to this eigenfunction depends only on the energy and the angle between the directions of the incoming and outgoing waves, which shows another one time that the developed model is spherically symmetric. The model determines nontrivial scattering in the p-channel, since the scattering amplitude depends on the angle between the vectors $\mathbf{k}$ and $\mathbf{x}$. Rigorous proof of the eigenfunction expansion and spectral theorem for $\mathbb{A}_\theta$ can be carried out by integrating the jump of the resolvent at the real axis. We would like to mention that the extended resolvent formula leads to a new family of eigenfunction expansions described for example in [24].

This model can be generalized to include point interactions in any other channel or a combination of such interactions in different channels. We are planning to return to this question as well as to the spectral analysis of the operator $\mathbb{A}_\theta$ in one of the future publications. We would like to mention that it might be interesting to study the relations between the model presented here and the model suggested by Yu. Karpeshina [18].

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