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Henrik Strohmayer

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Department of Mathematics
Stockholm University

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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

THE OPERAD OF TWO COMPATIBLE PRE-LIE PRODUCTS AND POINTED WEIGHTED PARTITIONS

HENRIK STROHMAYER

ABSTRACT. We introduce weighted and pointed weighted partitions and use them to show the Koszulness of $\mathcal{L}ie_2$ and $\mathcal{P}re\mathcal{L}ie_2$, the operads governing two compatible Lie brackets and two compatible pre-Lie products, respectively.

1. Introduction

In [Val07] B. Vallette introduced a new method to show the Koszulness of a class of set theoretic operads and their associated algebraic operads. By associating a certain poset to a set theoretic operad, $\mathcal{P}$, and then studying its Cohen-Macaulay properties, one gets a concrete recipe for checking whether $\mathcal{P}$, and thus also its Koszul dual operad, is Koszul or not. Studying the posets of unordered and ordered pointed and multipointed partitions in [CV06], B. Vallette and F. Chapoton were able to prove the Koszulness of several important operads such as $\mathcal{P}erm$, $\mathcal{P}re\mathcal{L}ie$, $\mathcal{C}om\mathcal{T}rias$, $\mathcal{P}ost\mathcal{L}ie$, $\mathcal{D}ias$, $\mathcal{D}end$, $\mathcal{T}rias$, $\mathcal{T}ri\mathcal{D}end$ over a field of any characteristic and over $\mathbb{Z}$. In [DK07], A. Khoroshkin and V. Dotsenko constructed a new operad, $\mathcal{L}ie_2$, by considering two compatible Lie brackets (compatible in the sense that any linear combination of the two Lie brackets is a Lie bracket). In this note we construct an operad, $\mathcal{P}re\mathcal{L}ie_2$, describing two compatible pre-Lie products. To show the Koszulness of $\mathcal{L}ie_2$ and $\mathcal{P}re\mathcal{L}ie_2$ by the poset method of Vallette we introduce weighted and pointed weighted partition posets. These posets are not totally semimodular, therefore we need to refine the arguments of [CV06] in order to show that they are Cohen-Macaulay.

All vector spaces and tensor products are considered over $\mathbb{K}$, where $\mathbb{K}$ is a field of characteristic 0 or $\mathbb{F}_p$. For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \dots, n\}$ and given a finite set S we denote the cardinality of S by $|S|$.

2. $\mathcal{P}re\mathcal{L}ie_2$, $\mathcal{L}ie_2$ and their Koszul dual operads

In this section we introduce a new operad, $\mathcal{P}re\mathcal{L}ie_2$, governing two compatible pre-Lie products and explicitly describe its Koszul dual operad. We also recall the definition of the operad $\mathcal{L}ie_2$ from [DK07] as well as some definitions from [Val07] related to set theoretic operads.

Definition 2.1. A *pre-Lie algebra* is a vector space V over $\mathbb{K}$ equipped with a mapping $\circ\colon V \otimes V \to V$ called a pre-Lie product such that

$$(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b) \ ,$$

for any $a, b, c \in V$.

In [Ger63] M. Gerstenhaber, in his study of the Hochschild cochain complex of an associative algebra, found a structure on the cochains satisfying the above condition and gave it the name pre-Lie because the operation

$$[a, b]_\circ := a \circ b - b \circ a$$

defines a Lie algebra. The Lie bracket obtained in this way is a part of the Gerstenhaber structure on the Hochschild complex. The same structure also appeared in a paper [Vin63] of È. Vinberg in his study of convex homogeneous cones, thus it has also been referred to as Vinberg algebra.

Given two pre-Lie products $\circ$ and $\bullet$ on V we say that they are *compatible* if any linear combination of the two products, $(\alpha\circ+\beta\bullet)(a,b) := \alpha a\circ b+\beta a\bullet b$, again is a pre-Lie product for any $\alpha,\beta\in\mathbb{K}$. This property is equivalent to the condition that

$$(a\circ b)\bullet c-a\circ(b\bullet c)+(a\bullet b)\circ c-a\bullet(b\circ c) = (a\circ c)\bullet b-a\circ(c\bullet b)+(a\bullet c)\circ b-a\bullet(c\circ b),$$

for any $a,b,c\in V$.

We now want to describe the operad encoding this structure. To fix the notation we first give two definitions concerning operads. For an introduction to operads see e.g. [MSS02].

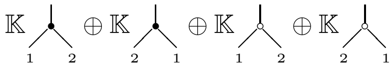
Definition 2.2. A quadratic operad $F(E)/\langle R\rangle$ is the free operad on a Σ-module E modulo relations $R\subset F_{(2)}(E)$, where $F_{(2)}(E)$ is the weight two part of $F(E)$, i.e. trees decorated with exactly two elements of E.

Definition 2.3. Let $\mathcal{P}=F(E)/\langle R\rangle$ be a quadratic operad. Then the *Koszul dual operad* $\mathcal{P}^!$ of $\mathcal{P}$ is defined as $\mathcal{P}^! = F(E^\vee)/\langle R^\perp\rangle$. Here the Czech dual Σ-module $E^\vee$ is given by $E^\vee(n) = E(n)^*\otimes\mathrm{sgn}_n$, sgn_n is the sign representation of Σ_n and $R^\perp$ are the relations orthogonal to R w.r.t. the natural pairing $\langle\_,\_\rangle\colon F_{(2)}(E^\vee)\otimes F_{(2)}(E)\to\mathbb{K}$.

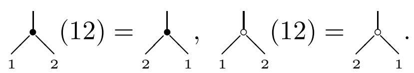
Definition 2.4. Translating the properties of two compatible pre-Lie products into the language of operads we have that $\mathcal{P}re\mathcal{L}ie_2$ is the quadratic operad $F(E)/\langle R\rangle$, where the Σ-module E is given by

$$E(n) := \begin{cases} \mathbb{K}[\Sigma_2]\oplus\mathbb{K}[\Sigma_2] & \text{if } n=2 \\ 0 & \text{if } n\neq 2\end{cases}$$

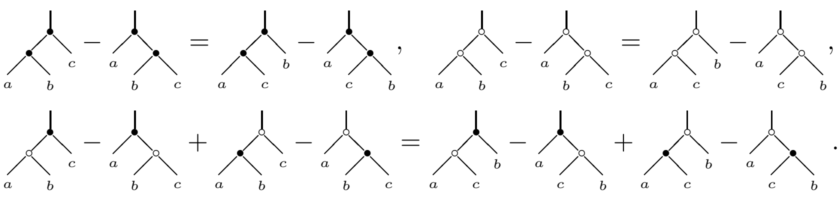
It is useful to represent the natural basis of $E(2)$ as four binary corollas

$\mathbb{K}[\Sigma_2]\oplus\mathbb{K}[\Sigma_2] = $ 

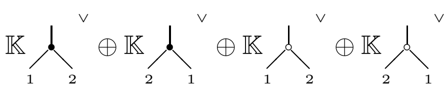
with Σ_2 action defined by



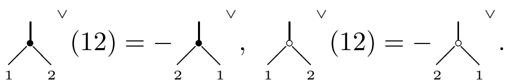
Then the relations R can be represented as follows (when described by planar trees)



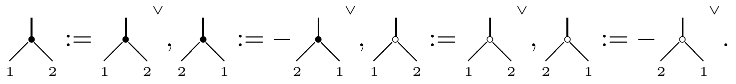
The Koszul dual operad $\mathcal{P}re\mathcal{L}ie_2{}^!$ is then generated by



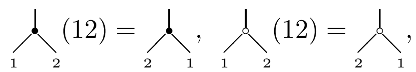
with Σ_2 action given by



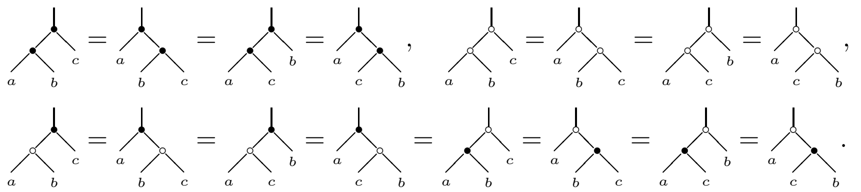
It is in fact more natural to work with a different basis in $\mathcal{P}re\mathcal{L}ie_2{}^!$ defined by



The Σ_2 action is then given on the new basis by



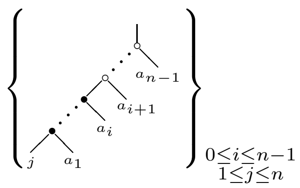
and the relations $R^\perp$ are



Since the Koszul dual operad of $\mathcal{P}re\mathcal{L}ie$ was named $\mathcal{P}erm$ (from *perm*utation) by F. Chapoton [Cha01], we give the name $\mathcal{P}erm_2$ to the Koszul dual operad of $\mathcal{P}re\mathcal{L}ie_2$. It will be clear from the context whether decorated trees belong to $\mathcal{P}re\mathcal{L}ie_2$ or $\mathcal{P}erm_2$.

In [Cha01] $\mathcal{P}erm(n)$ was described as $\mathcal{P}erm(n) = \mathbb{K}^n$ with Σ_n acting on the standard basis $\{e_1, \dots, e_n\}$ by $e_i\sigma = e_{\sigma^{-1}(i)}$ for $\sigma \in \Sigma_n$.

Proposition 2.5. *$\mathcal{P}erm_2(n) = \mathcal{P}erm(n) \oplus \cdots \oplus \mathcal{P}erm(n)$, where the sum consists of n terms. In terms of trees decorated with $E^\vee$ a basis for $\mathcal{P}erm_2(n)$ is given by*

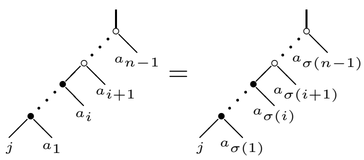


where $(a_1, \dots, a_{n-1}) = (1, \dots, j-1, j+1, \dots, n)$.

Denote by $C^n_{i,j}$ the basis element in $\mathcal{P}erm_2(n)$ corresponding to a given pair (i, j). The composition product in $\mathcal{P}erm_2$ is then given by

$$\mu(C^n_{i,j}; C^{m_1}_{i_1,j_1}, \dots, C^{m_n}_{i_n,j_n}) = C^{m_1+\cdots+m_n}_{i+i_1+\cdots+i_n, m_1+\cdots+m_{l-1}+j_l}$$

Proof. Writing an element of $F(E^\vee)$ (i.e. a tree whose vertices are decorated with elements of $E^\vee$) in the plane, we see that the relations $R^\perp$ yield that any decorated tree is equivalent to one of the above form. The relations also imply that on any such tree we may permute all but the leftmost index, i.e.



for any $\sigma \in \Sigma_{n-1}$.

Since the relations are homogenous in the number of white and black dots, this number is also an invariant under the relations. As there are no other relations, the class of any decorated tree in $\mathcal{P}erm_2(n)$ is completely determined by its leftmost index j, which ranges over $[n]$, and the number of black dots i, of which there can be 0 to $n-1$. Note that $C^n_{i,j}$ corresponds to $(0, \dots, 0, e_j, 0, \dots, 0)$ with e_j in the $i+1$th component of the direct sum.

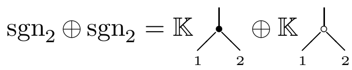
The definition of the composition in the free operad as grafting of trees with the obvious numbering of the indices gives the second claim. ☐

For completeness we recall here the definition of the operad of two compatible Lie brackets of [DK07].

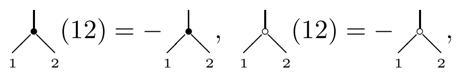
Definition 2.6. $\mathcal{L}ie_2$ is the quadratic operad $F(E)/\langle R\rangle$ where the Σ-module E is given by

$$E(n) := \begin{cases} \mathrm{sgn}_2 \oplus \mathrm{sgn}_2 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

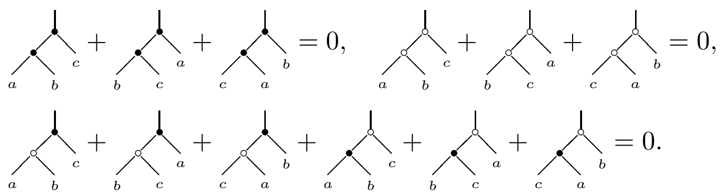
We represent a natural basis of $E(2)$ as two binary corollas



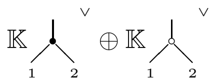
with Σ_2 action defined by



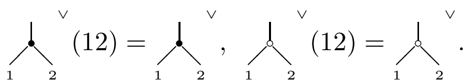
Then the relations R are as follows



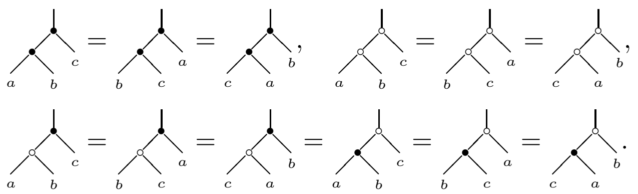
${\mathcal{L}ie_2}^!$ is generated by



with Σ_2 action given by

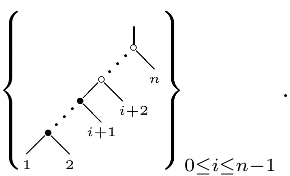


From now on we will skip the $^\vee$. The relations $R^\perp$ are then given by



${\mathcal{L}ie_2}^!$ was given the name $\mathcal{C}om_2$ in [DK07]. Though we use the same notation for $\mathcal{C}om_2$ as we did for $\mathcal{P}erm_2$ no confusion should arise.

Proposition 2.7. *$\mathcal{C}om_2(n) = \mathbb{1}_n \oplus \cdots \oplus \mathbb{1}_n$, where the sum consists of n terms and $\mathbb{1}_n$ denotes the trivial representation of Σ_n. In terms of trees decorated with $E^\vee$ a basis for $\mathcal{C}om_2(n)$ is given by*



Denote by D_i^n the basis element in $\mathcal{C}om_2(n)$ corresponding to i black dots. The composition product in $\mathcal{C}om_2$ is then given by

$$\mu(D_i^n; D_{i_1}^{m_1}, \ldots, D_{i_n}^{m_n}) = D_{i+i_1+\cdots+i_n}^{m_1+\cdots+m_n}$$

Proof. Obvious. □

A Σ-set is a collection of sets, $S = (S_n)_{n\in\mathbb{N}}$, equipped with a right action of the symmetric group Σ_n on S_n. Define a monoidal product in the category of Σ-sets by:

$$S \circ T_n = \bigsqcup_{1\le k\le n} \left(\bigsqcup_{i_1+\cdots+i_k=n} S_k \times (T_{i_1} \times \cdots \times T_{i_k}) \times_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}} \Sigma_n \right)_{\Sigma_k},$$

where we consider the coinvariants with respect to the action of Σ_k given by $(s,(t_{i_1},\ldots,t_{i_k}),\sigma)\tau = (s\tau,(t_{i_{\tau(1)}},\ldots,t_{i_{\tau(k)}}),\bar{\tau}^{-1}\sigma)$ and $\bar{\tau}$ is the induced block permutation. A unit I with respect to this product is given by the Σ-set defined by

$$I_n := \begin{cases} [1] & \text{if } n = 1 \\ \emptyset & \text{if } n \neq 1. \end{cases}$$

Definition 2.8. A *set operad* is a monoid $(\mathcal{P}, \mu\colon \mathcal{P}\circ\mathcal{P} \to \mathcal{P}, \varepsilon\colon I \to \mathcal{P})$ in the monoidal category $(\Sigma\text{-sets}, \circ, I)$.

To any set operad $\mathcal{P}$ one can associate an algebraic operad $\widetilde{\mathcal{P}}$ by considering formal linear combinations of the elements, i.e. $\widetilde{\mathcal{P}}(n) = \mathbb{K}[\mathcal{P}_n]$. To an element $(\nu_1,\ldots,\nu_r) \in \mathcal{P}_{i_1}\times\cdots\times\mathcal{P}_{i_k}$ one can associate a map $\mu_{\nu_1,\ldots,\nu_k}\colon \mathcal{P}_k \to \mathcal{P}_{i_1+\cdots+i_k}$ defined as

$$\mu_{\nu_1,\ldots,\nu_k}(\nu) = \mu(\nu;\nu_1,\ldots,\nu_k).$$

The following definition was introduced in [Val07] since it is a crucial property for set theoretic operads in order to use the poset method.

Definition 2.9. A set operad $\mathcal{P}$ is called a *basic-set operad* if the map $\mu_{\nu_1,\ldots,\nu_r}$ is injective for all $(\nu_1,\ldots,\nu_r) \in \mathcal{P}(i_1)\times\cdots\times\mathcal{P}(i_r)$.

Lemma 2.10. *$\mathcal{P}erm_2$ and $\mathcal{C}om_2$ come from basic-set operads.*

Proof. First we note that $\mathcal{P}erm_2$ comes from a set theoretic operad, $\mathcal{P}erm_2 = \widetilde{\mathcal{P}}$, where $\mathcal{P}_n = \{C^n_{i,j}\}$ and the $C^n_{i,j}$ are the basis elements given in Proposition 2.5. The map $\mu_{C^{m_1}_{i_1,j_1},\ldots,C^{m_n}_{i_n,j_n}}$ sends $C^n_{k,l}$ to $C^{m_1+\cdots+m_n}_{k+i_1+\cdots+i_n, m_1+\cdots+m_{l-1}+j_l}$. Since $m_s \ge 1$ and $0 \le j_s \le m_s - 1$, clearly this map is injective.

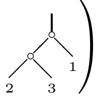
Also $\mathcal{C}om_2$ comes from a set operad $\mathcal{Q}$, where $\mathcal{Q}_n = \{D^n_i\}$ and the D^n_i are as in Proposition 2.7. The proof is immediate from the definiton of the composition product. □

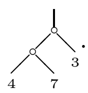
3. Operadic partition posets

To a set operad $\mathcal{P}$ one can associate a certain poset encoding important properties of $\mathcal{P}$, as was done in [Val07]. We present it slightly differently and then recall the definition of the poset of pointed partitions. See [BW83, Val07] for definitions of the various notions related to posets.

Definition 3.1. Let $\mathcal{P}$ be a set operad. A *$\mathcal{P}$-partition* of $[n]$ is the following data $\{(B_1,p_1),\ldots,(B_s,p_s)\}$, where $\{B_1,\ldots,B_s\}$ is a partition of $[n]$ and $p_i \in \mathcal{P}(|B_i|)$. We let $\Pi_{\mathcal{P}}(n)$ denote the set of all $\mathcal{P}$-partitions of $[n]$ and let $\Pi_{\mathcal{P}}$ denote the collection $\{\Pi_{\mathcal{P}}(n)\}_{n\in\mathbb{N}}$. For an algebraic operad $\mathcal{O}$ coming from a set operad $\mathcal{P}$, i.e. $\mathcal{O} = \widetilde{\mathcal{P}}$, we will write $\Pi_{\mathcal{O}}$ for $\Pi_{\mathcal{P}}$.

Remark 3.2. One can think of this as enriching a partition with elements of an operad or, shifting the perspective, as labeling the input of the operation that an element $p_i \in \mathcal{P}(|B_i|)$ describes with the elements of the block B_i instead of with $[|B_i|]$. E.g. one can identify

$\Big(\{3,4,7\},$ 

$\Big) \sim$ 

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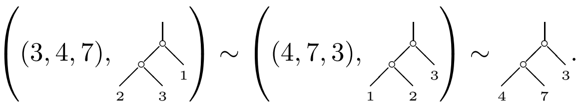
The definition in [Val07] uses ordered sequences of elements of the blocks instead of unordered blocks and then considers equivalence classes of pairs (S_B, p), where S_B is an ordered sequence of the elements of a block B where each element appears exactly once and $p \in \mathcal{P}(|S_B|)$. E.g.



Our definition corresponds to choosing the representative of a class with the elements of the sequence in ascending order. In the following we will assume that, given a partition $\alpha = \{(A_1, p_1), \ldots, (A_r, p_r)\}$, the elements of a block $A_i = \{a^i_1, \ldots, a^i_{m_i}\}$ are indexed in ascending order, i.e. $a^i_j < a^i_{j+1}$.

Next we define a partial order on $\Pi_{\mathcal{P}}(n)$.

Definition 3.3. Let $\alpha = \{(A_1, p_1), \ldots, (A_r, p_r)\}$ and $\beta = \{(B_1, q_1), \ldots, (B_s, q_s)\}$ be two $\mathcal{P}$-partitions of $[n]$. We let $\alpha \leq \beta$ if

(i) $\{A_1, \ldots, A_r\}$ is a refinement of $\{B_1, \ldots, B_s\}$, i.e. each B_j is the union of one or more A_i.

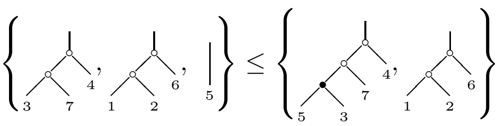
(ii) when $B_j = A_{i_1} \cup \cdots \cup A_{i_t}$ then there exists a $p \in \mathcal{P}_t$ such that $q_j = \mu(p; p_{i_1}, \ldots, p_{i_t})\sigma^{-1}$, where $\sigma \in \Sigma_{|B_j|}$ is the obvious permutation associated to

$$\begin{pmatrix} b^j_1 \ldots b^j_{|B_j|} \\ a^{i_1}_1 \ldots a^{i_t}_{m_{i_t}} \end{pmatrix}.$$

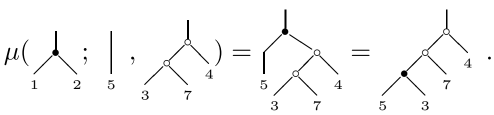
We call $\Pi_{\mathcal{P}}$ together with this partial order the *operadic partition poset* of $\mathcal{P}$.

Remark 3.4. We define the order in the opposite way to the one in [Val07] to make it correspond to the way it is defined in [CV06]. Note that with this in mind our definition leads to the same ordering of the corresponding equivalence classes.

Example 3.5. Using the identification in Remark 3.2 we see that in $\Pi_{\mathcal{P}erm_2}(7)$



since



Example 3.6. $\Pi_{\mathcal{P}erm}(3)$ can be depicted as in Figure 1, with greater elements above.

In [Val07] pointed partitions were introduced to describe Π_{Perm}.

Definition 3.7. A *pointed partition* of $[n]$ is a partition $\beta = \{B_1, \ldots, B_s\}$ of $[n]$ together with a distinguished element b_i in each block B_i. This element is emphasized by $\overline{\mathbf{b_i}}$ and we define $\mathrm{p}(B_i) := b_i$. The set $\{\mathrm{p}(B_i) | B_i \in \beta\}$ of pointed elements of β is denoted by $\mathrm{p}(\beta)$. We denote the set of all pointed partions of $[n]$ by Π^{p}_n and denote the collection $\{\Pi^{\mathrm{p}}_n\}_{n \in \mathbb{N}}$ by Π^{p}.

We define a partial order relation on Π^{p}_n by $\alpha \leq \beta$ if α is a refinement of β as a partition and $\mathrm{p}(\beta) \subset \mathrm{p}(\alpha)$. Π^{p} together with this partial order is called the *poset of pointed partitions*.

Remark 3.8. What we call a pointed partition here is precisely what is called a pointed partition of type A in [CV06].

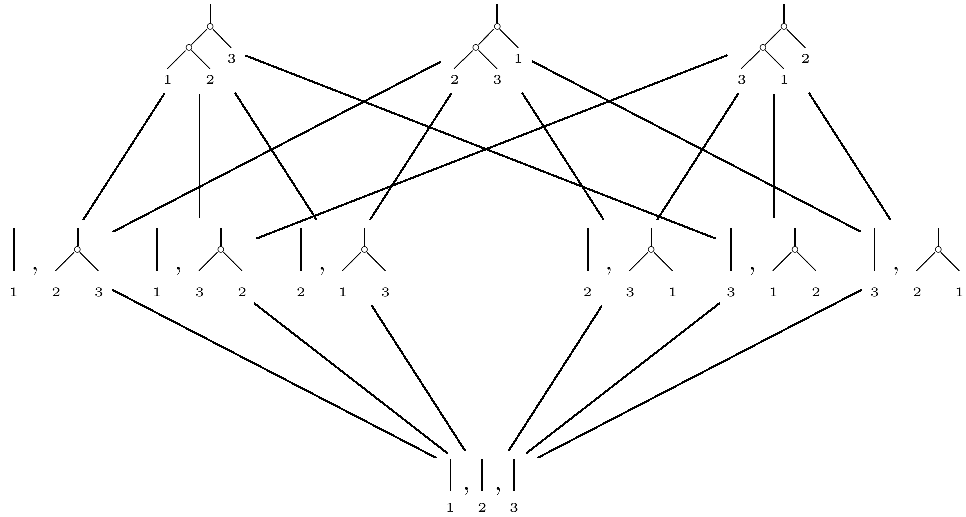


FIGURE 1. The poset $\Pi_{\mathcal{P}erm}(3)$

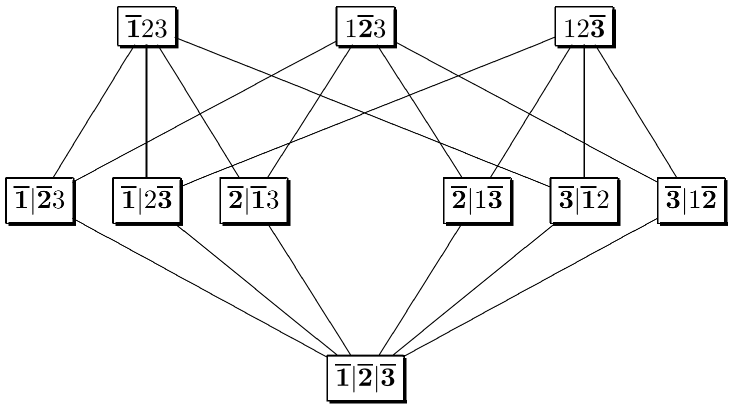


FIGURE 2. The poset Π_3^{p}

In [Val07], Vallette studied homological properties of the order complex associated to the partition poset of an operad. The following is the main result.

Theorem 3.9 (Theorem 9 of [Val07]). *Let $\mathcal{P}$ be a basic-set quadratic operad, then the associated algebraic operad $\widetilde{\mathcal{P}}$ is Koszul iff each subposet $[\hat{0}, \gamma]$ of each $\Pi_{\mathcal{P}}(n)$ is Cohen-Macaulay, where γ is a maximal element of $\Pi_{\mathcal{P}}(n)$.*

This theorem was used in [CV06] to show the Koszulness of $\mathcal{P}erm$ (over a field of any characteristic and over $\mathbb{Z}$, it was shown for a field of characteristic 0 in [CL01]). There it was shown that for each Π_n^{p} (isomorphic to $\Pi_{\mathcal{P}erm}(n)$ by [Val07]) all subposets $[\hat{0}, \gamma]$ of the form in the above theorem were totally semimodular. Hence by Corollary 5.2 of [BW83] they are CL-shellable and by Proposition 2.3 of the same paper shellable from whence it follows that they are Cohen-Macaulay by Theorem 4.2 of [Gar80]. The chain of implications is

(3.10)

$$\text{totally semimodular} \implies \text{CL-shellable} \implies \text{shellable} \implies \text{Cohen-Macaulay}.$$

4. Weighted partitions, $\mathcal{C}om_2$ and $\mathcal{L}ie_2$

Contrary to the claims in [DK07], the maximal chains of $\Pi_{\mathcal{C}om_2}$ are not totally semimodular as we will see. To handle posets of this type we introduce a new kind of partitions which we call *weighted partitions*. We then use this poset to show the Koszulness of $\mathcal{C}om_2$ and $\mathcal{L}ie_2$ via Vallette's poset method.

Definition 4.1. Given a partition $\beta = \{B_1, \dots, B_s\}$ of $[n]$, we assign a weight w_i to each block $B_i = \{b^i_1, \dots, b^i_{k_i}\}$, with $0 \le w_i \le k_i - 1$. The weight of the block is denoted by $\mathrm{w}(B_i) := w_i$. The weight of a partition β is $\mathrm{w}(\beta) := \mathrm{w}(B_1) + \cdots + \mathrm{w}(B_s)$. A partition with this extra structure we call a *weighted partition* and we denote the set of weighted partions of $[n]$ by Π^{w}_n. The collection $\{\Pi^{\mathrm{w}}_n\}_{n\in\mathbb{N}}$ is denoted by Π^{w}.

Let $\mathrm{n}(\beta)$ be the number of blocks of β. Then we can define a partial order on Π^{w}_n by letting $\alpha \le \beta$ if

(i) the partition of α is a refinement of the partition of β and
(ii) $\mathrm{w}(\beta) - \mathrm{w}(\alpha) \le \mathrm{n}(\alpha) - \mathrm{n}(\beta)$.

We call Π^{w} together with this partial order the *poset of weighted partitions*.

Remark 4.2. We see that the covering relation $\prec$ of the above partial order is given by $\alpha \prec \beta$ if

(i) the partition of α is a refinement of that of β obtained by splitting exactly one block of β into two and
(ii) $0 \le \mathrm{w}(\beta) - \mathrm{w}(\alpha) \le 1$.

We denote the maximal elements of Π^{w}_n by μ_t, $0 \le t \le n-1$, where t denotes the weight. Any element α of Π^{w}_n can be described by $\alpha = \{(A_1, w_1), \dots, (A_m, w_m)\}$ where $\{A_1, \dots A_r\}$ is a partition of $\{1, \dots, n\}$ and $w_i = \mathrm{w}(A_i)$.

We observe that Π^{w}_n is a pure poset, i.e. all maximal chains have the same length.

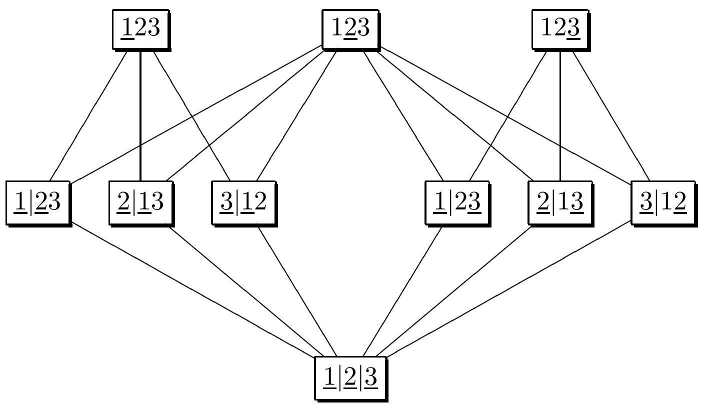


Figure 3. The poset Π^{w}_3

Remark 4.3. In Figure 3. the weight w of a block $B = \{b_1, \dots, b_k\}$ is indicated by $\underline{b}_{w+1}$, recall that we order the elements b_i of a block in increasing order. E.g. the block $\{\underline{1}, 2\}$ has weight 0 whereas the block $\{1, \underline{3}\}$ has weight 1.

Lemma 4.4. *The poset $\Pi_{\mathcal{C}om_2}(n)$ is isomorphic to Π^{w}_n.*

Proof. There is an obvious bijection between the elements of $\Pi_{\mathcal{C}om_2}(n)$ and Π^{w}_n where a block B enriched with an element $D^{|B|}_i$ with i black product(s) corresponds to the same block B with weight i in Π^{w}_n.

Now let $\alpha = \{(A_1, p_1), \ldots, (A_m, p_m)\}$ be a $\mathcal{C}om_2$-partition, then β covers α iff

$$\beta = \{(A_j \cup A_k, \mu(\curlyvee; p_j, p_k)), (A_1, p_1), \ldots, \widehat{(A_j, p_j)}, \ldots, \widehat{(A_k, p_k)}, \ldots, (A_m, p_m)\}$$

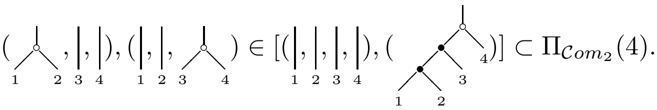
or

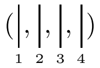
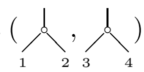
$$\beta = \{(A_j \cup A_k, \mu(\curlyvee; p_j, p_k)), (A_1, p_1), \ldots, \widehat{(A_j, p_j)}, \ldots, \widehat{(A_k, p_k)}, \ldots, (A_m, p_m)\}.$$

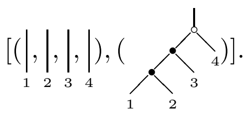
The first case corresponds to increasing the weight by one when uniting two blocks of a weighted partition and the second case to keeping it constant, which precisely is the covering relation of Π_n^{w}. ☐

Definition 4.5. A finite poset P is called *semimodular* if it is bounded and for any distinct $\kappa, \lambda \in P$ covering a $\nu \in P$ there exists a $\omega \in P$ covering both κ and λ. P is said to be *totally semimodular* if it is bounded and all intervals $[\zeta, \xi]$ are semimodular.

Remark 4.6. $\Pi_{\mathcal{C}om_2}$ has maximal intervals which are not totally semimodular. E.g. consider the elements



They both cover  but the only element covering both of them is 

which does not belong to the interval .

Remembering the chain of implications (3.10) at the end of the previous section we see that it is in fact sufficient to show that the maximal intervals of $\Pi_{\mathcal{C}om_2}$ are CL-shellable. By Theorem 3.2 of [BW83], showing CL-shellability of a poset is equivalent to showing that it allows a recursive atom ordering. Recall that the atoms of a poset are the elements covering $\hat{0}$.

Definition 4.7. A graded poset P admits a recursive atom ordering if the length of the poset is 1 or if the length is greater than 1 and there is an ordering $\alpha_1, \ldots, \alpha_m$ of the atoms of P satisfying

(i) For all $j \in [m]$, $[\alpha_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[\alpha_j, \hat{1}]$ that come first in the ordering are those that cover some α_i, where $i < j$.

(ii) For all $i < j$, if $\alpha_i, \alpha_j < \lambda$ then there is a $k < j$, not necessarily distinct from i, and an element $\kappa \leq \lambda$ such that κ covers both α_j and α_k

Lemma 4.8. *Π_n^w allows a recursive atom ordering for any n.*

Proof. Since Π_n^{w} is pure, $[\hat{0}, \mu_t]$ is graded. Now suppose the length of $[\hat{0}, \mu_t]$ is greater than 1, otherwise we are done. We may also assume that $0 < t < n-1$, since if $t = 0$ or $t = n-1$ we have that $[\hat{0}, \mu_t]$ is isomorphic to Π_n, the poset of ordinary partitions of n. This poset is easily seen to be totally semimodular, analogously to the proof in [CV06] that Π_n^{p} is totally semimodular. Thus by Theorem 5.1 of [BW83] any ordering of the atoms is a recursive atom ordering.

When denoting pointed weighted partitions we will supress the blocks only containing one element e.g.

$$\{(\{i,j\}, w), (\{k,l\}, w')\} = \{(\{i,j\}, w), (\{k,l\}, w'), (\{1\}, 0), \ldots, \widehat{(\{i\}, 0)}, \ldots, \widehat{(\{j\}, 0)}, \ldots, \widehat{(\{k\}, 0)}, \ldots, \widehat{(\{l\}, 0)}, \ldots, (\{n\}, 0)\}.$$

Denote the atom $\{(\{i,j\},w)\}$ by $\alpha^w_{i,j}$, where the upper index indicates the weight.

We claim that any atom ordering of the form

$$\alpha^0_{i_1,j_1} \dashv \alpha^1_{i_1,j_1} \dashv \alpha^0_{i_2,j_2} \dashv \alpha^1_{i_2,j_2} \dashv \cdots \dashv \alpha^0_{i_r,j_r} \dashv \alpha^1_{i_r,j_r} \tag{4.9}$$

fullfills the second criterion of being a recursive atom ordering, with $\alpha \dashv \beta$ meaning that α is less than β in the atom ordering.

Let $\alpha^{w_1}_{i,j}$ and $\alpha^{w_2}_{k,l}$ be distinct atoms with $\alpha^{w_1}_{i,j} \dashv \alpha^{w_2}_{k,l}$ and suppose $\alpha^{w_1}_{i,j},\alpha^{w_2}_{k,l} \leq \gamma$. We want to show that there is a $\delta \leq \gamma$ and a $\alpha^{w'}_{i',j'} \dashv \alpha^{w_2}_{k,l}$ such that $\alpha^{w'}_{i',j'}, \alpha^{w_2}_{k,l} \prec \delta$. Lemma 1 of [DK07] shows that this is true, with $\alpha^{w'}_{i',j'} = \alpha^{w_1}_{i,j}$, for all cases except when i,j,k,l are distinct and $w_1 = w_2$.

Now consider this case, i.e. $w_1 = w_2 =: w$, and let $\tilde{w}$ be the element of $\{0,1\} \setminus \{w\}$. By the ordering (4.9) of the atoms, $\alpha^w_{i,j} \dashv \alpha^w_{k,l}$ implies $\alpha^{\tilde{w}}_{i,j} \dashv \alpha^w_{k,l}$ and either $\{(\{i,j\},w),(\{k,l\},w)\} \leq \gamma$ and covers $\alpha^w_{i,j}$ and $\alpha^w_{k,l}$, in which case we take $\alpha^{w'}_{i',j'} = \alpha^w_{i,j}$ and $\delta = \{(\{i,j\},w),\ (\{k,l\},w)\}$, or $\{(\{i,j\},\tilde{w}),(\{k,l\},w)\} \leq \gamma$ and covers $\alpha^{\tilde{w}}_{i,j}$ and $\alpha^w_{k,l}$, in which case we take $\alpha^{w'}_{i',j'} = \alpha^{\tilde{w}}_{i,j}$ and $\delta = \{(\{i,j\},\tilde{w}),(\{k,l\},w)\}$.

We also have to show that any interval $[\alpha^w_{i,j}, \mu_t]$ allows a recursive atom ordering in which the atoms that come first are those that cover some $\alpha^w_{k,l} \dashv \alpha^w_{i,j}$.

We also need to show that, given the ordering (4.9), $[\alpha^w_{i,j}, \mu_t]$ satisfies the first criterion of being a recursive atom ordering. We may identify

$$\{\{i,j\},1,\ldots,\hat{i},\ldots,\hat{j},\ldots,n\} \sim [n-1]$$

and it is easily seen that $[\alpha^w_{i,j}, \mu_t]$ is isomorphic to a maximal interval $[\hat{0}, \mu_{t-w}]$ in Π^{pw}_{n-1}. Thus checking the above step is readily done if we may order the atoms in the same way as above. We only need to show that some way of ordering the atoms of $[\alpha^w_{i,j}, \mu_t]$ in pairs as above satisfies that the first atoms are the ones covering some atom $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$. After that we can proceed by induction.

We may assume that the length of $[\alpha^w_{i,j}, \mu_t]$ is greater than 1, since otherwise we are done. We may also assume that $0 < t - w < n - 2$, since if $t = w$ or $t = n - 2 + w$ the interval $[\alpha^w_{i,j}, \mu_t]$ is isomorphic to the interval $[\{i,j\},[n]]$ in the poset of ordinary partitions. In the same way as above we see that any such interval is is totally semimodular whereby any ordering of the atoms is a recursive atom ordering. We may therefore freely order the atoms of $[\alpha^w_{i,j}, \mu_t]$ so that the atoms that come first are those that cover some atom less than $\alpha^w_{i,j}$ in the ordering (4.9).

Now the atoms are either of the form $\{(\{i,j\},w),(\{k,l\},v)\}$ which we denote by $\beta^v_{k,l}$ or of the form $\{(\{i,j,k\},w+v)\}$ which we denote by β^v_k, where $v \in \{0,1\}$.

We have that $\beta^v_{k,l}$ covers some $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$, namely $\alpha^{w'}_{i',j'} = \alpha^v_{k,l}$, iff $\alpha^v_{k,l} \dashv \alpha^w_{i,j}$. Since by the atom ordering of $[\hat{0}, \mu_t]$ we have that $\alpha^v_{k,l} \dashv \alpha^w_{i,j}$ iff $\alpha^{\tilde{v}}_{k,l} \dashv \alpha^w_{i,j}$, we have that $\beta^v_{k,l}$ covers some $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$ iff $\beta^{\tilde{v}}_{k,l}$ covers some $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$.

Similarily we have that β^v_k may cover some $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$, where $\{i',j'\} \subset \{i,j,k\}$. Again $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$ iff $\alpha^{\tilde{w}'}_{i',j'} \dashv \alpha^w_{i,j}$. Hence β^v_k covers some $\alpha^{w'}_{i',j'} \dashv \alpha^w_{i,j}$ iff $\beta^{\tilde{v}}_k$ does.

Thus we may order the atoms of $[\alpha^w_{i,j}, \mu_t]$ by first putting all pairs of atoms, differing only in weight, covering some atom less than $\alpha^w_{i,j}$ followed by all pairs of atoms not covering any atom less than $\alpha^w_{i,j}$. Using the aforementioned identification $[\alpha^w_{i,j}, \mu_t] \cong [\hat{0}, \mu_{t-w}]$, we just proceed by induction.

□

Theorem 4.10. *$\mathcal{C}om_2$ and $\mathcal{L}ie_2$ are Koszul.*

Proof. By Lemma 4.8 Π^{w}_n allows a recursive atom ordering and therefore is CL-shellable. The chain of implications (3.10) thus gives us that Π^{w}_n is Cohen-Macaulay. Lemma 4.4 yields that this also is true for $\Pi_{\mathcal{C}om_2}$. By Lemma 2.10 we have that the set operad associated to $\mathcal{C}om_2$ is a basic-set operad. Thus we may apply Theorem 3.9 and conclude that $\mathcal{C}om_2$ is Koszul and so also its Koszul dual operad $\mathcal{L}ie_2$. □

5. Pointed weighted partitions, $\mathcal{P}erm_2$ and $\mathcal{P}re\mathcal{L}ie_2$

In this section we will prove that $\mathcal{P}re\mathcal{L}ie_2$ is Koszul by considering its Koszul dual operad $\mathcal{P}erm_2$. To prove the Koszulness of $\mathcal{P}erm_2$ via Vallette's poset method we have to introduce a new kind of partitions, *pointed weighted partitions*. The poset of such partitions combine properties of the poset of pointed partitions of [CV06] and the poset of weighted partitions from the previous section. By this we obtain a poset structure which keeps track of both the distinguished input and the number of occurrences of $\curlywedge$ and $\blacktriangle$ in $\mathcal{P}erm_2$. This process is completely analogous to what we did in the previous section. We include it in detail for the sake of completeness.

Definition 5.1. Given a partition $\beta = \{B_1, \dots, B_s\}$ of $[n]$, then in a block $B_i = \{b_1^i, \dots, b_{k_i}^i\}$ one of the b_l^i is pointed out. We denote this by $\mathrm{p}(B_i) := b_l^i$. We denote the set of pointed elements of a partition β by $\mathrm{p}(\beta)$. We also assign a weight w_i to each block B_i, with $0 \leq w_i \leq k_i - 1$. The weight of the block is denoted by $\mathrm{w}(B_i) := w_i$. The weight of a partition β is $\mathrm{w}(\beta) := \mathrm{w}(B_1) + \cdots + \mathrm{w}(B_s)$. A partition with this extra structure we call a *pointed weighted partition* and we denote the set of pointed weighted partions of $[n]$ by Π_n^{pw}. The collection $\{\Pi_n^{\mathrm{pw}}\}_{n\in\mathbb{N}}$ is denoted by Π^{pw}

Let $\mathrm{n}(\beta)$ be the number of blocks of β. We define a partial order on Π_n^{pw} by letting $\alpha \leq \beta$ if

(i) the partition of α is a refinement of the partition of β,
(ii) $\mathrm{p}(\beta) \subset \mathrm{p}(\alpha)$ and
(iii) $\mathrm{w}(\beta) - \mathrm{w}(\alpha) \leq \mathrm{n}(\alpha) - \mathrm{n}(\beta)$.

We call Π^{pw} together with this partial order the *poset of pointed weighted partitions.*

Remark 5.2. We see that the covering relation $\prec$ is given by $\alpha \prec \beta$ if

(i) the partition of α is a refinement of that of β obtained by splitting exactly one block of β into two,
(ii) $\mathrm{p}(\beta) \subset \mathrm{p}(\alpha)$ and
(iii) $0 \leq \mathrm{w}(\beta) - \mathrm{w}(\alpha) \leq 1$.

We denote the maximal elements of Π_n^{pw} by $\mu_{s,t}$, $0 \leq s, t \leq n-1$, where the first index is the pointed element and the second is the weight. Any element α of Π_n^{pw} can be described by $\alpha = \{(A_1, a_1, w_1), \dots, (A_m, a_m, w_m)\}$ where $\{A_1, \dots A_m\}$ is a partition of $\{1, \dots, n\}$, $a_r = \mathrm{p}(A_r)$ and $w_r = \mathrm{w}(A_r)$.

We observe that Π_n^{pw} is a pure poset, i.e. all maximal chains have the same length.

Remark 5.3. In Figure 4. the weight w of a block $B = \{b_1, \dots, b_k\}$ is indicated by $\underline{b}_{w+1}$, recall that we order the elements b_i of a block in increasing order. The pointed element b_j is indicated by $\overline{\mathbf{b_j}}$. E.g. the block $\{1, \underline{\overline{\mathbf{3}}}\}$ has weight 1 and the element 3 is pointed out whereas the block $\{\underline{1}, \overline{\mathbf{2}}\}$ has weight 0 and the element 2 is pointed out.

Lemma 5.4. *The poset $\Pi_{\mathcal{P}erm_2}$ is isomorphic to Π^{pw}.*

Proof. There is an obvious bijection between the elements of $\Pi_{\mathcal{P}erm_2}(n)$ and Π_n^{pw} where a block $B = \{b_1, \dots, b_u\}$ enriched with an element $C_{i,j}^u$ with i black product(s) corresponds to the element in Π_n^{pw} given by the same block B with weight i and b_j pointed out.

Now let $\alpha = \{(A_1, p_1), \dots, (A_m, p_m)\}$ be a $\mathcal{P}erm_2$-partition, then β covers α iff

$$\beta = \{(A_j \cup A_k, \mu(\curlywedge; p_j, p_k)), (A_1, p_1), \dots, \widehat{(A_j, p_j)}, \dots, \widehat{(A_k, p_k)}, \dots, (A_m, p_m)\}$$

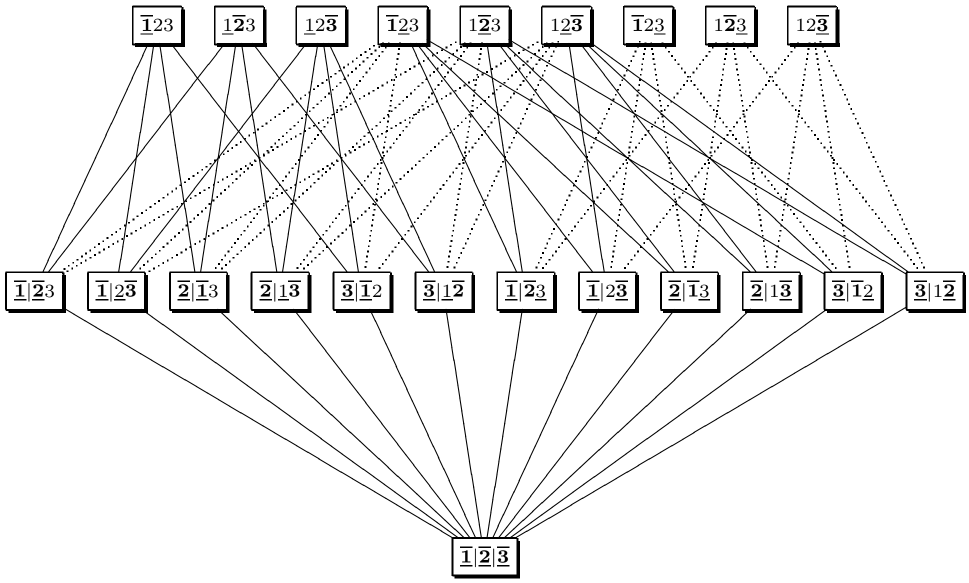


FIGURE 4. The poset Π_3^{pw}.

or

$$\beta = \{(A_j \cup A_k, \mu(\curlywedge; p_j, p_k)), (A_1, p_1), \ldots, \widehat{(A_j, p_j)}, \ldots, \widehat{(A_k, p_k)}, \ldots, (A_m, p_m)\}$$

or β is given as above but with p_j and p_k switching places in the operadic composition. The first case corresponds to increasing the weight by one when uniting two blocks of a weighted partition and the second case to keeping it constant. Which of p_j and p_k that is grafted to the left leg of the corolla corresponds to which of the pointed elements of the united blocks that stay pointed. This is precisely the covering relation of Π_n^{pw}. □

As in the previous section we want to show CL-shellability of the maximal intervals of the poset we study and again we do this by showing that they allow a recursive atom ordering. The proof combines the arguments of Lemma 4.8 and Lemma 1.10. of [CV06].

Lemma 5.5. *Any maximal interval* $[\hat{0}, \mu_{s,t}]$ *of* Π_n^{pw} *allows a recursive atom ordering.*

Proof. Since Π_n^{pw} is pure, $[\hat{0}, \mu_{s,t}]$ is graded. Now suppose the length of $[\hat{0}, \mu_{s,t}]$ is greater than 1, otherwise we are done. We may also assume that $0 < t < n-1$, since if not, we have that $[\hat{0}, \mu_{s,t}]$ is isomorphic to the interval $[\hat{0}, \mu_s]$ in the poset of pointed partitions Π_n^{p}. By [CV06] any such interval is totally semimodular. Thus by Theorem 5.1 of [BW83] any ordering of the atoms is a recursive atom ordering.

When denoting pointed weighted partitions we will supress the blocks only containing one element e.g.

$$\{(\{i,j\}, p, w), (\{k,l\}, p', w')\} = \{(\{i,j\}, p, w), (\{k,l\}, p', w'), (\{1\}, 1, 0), \ldots, \widehat{(\{i\}, i, 0)}, \ldots, \widehat{(\{j\}, j, 0)}, \ldots, \widehat{(\{k\}, k, 0)}, \ldots, \widehat{(\{l\}, l, 0)}, \ldots, (\{n\}, n, 0)\}.$$

Denote the atom $\{(\{i,j\}, p, w)\}$ by $\alpha_{i,j}^{p,w}$, where the first upper index indicates the pointed element and the second the weight.

We claim that any ordering of the form

$$\alpha_{i_1,j_1}^{p_1,0} \dashv \alpha_{i_1,j_1}^{p_1,1} \dashv \alpha_{i_2,j_2}^{p_2,0} \dashv \alpha_{i_2,j_2}^{p_2,1} \dashv \cdots \dashv \alpha_{i_r,j_r}^{p_r,0} \dashv \alpha_{i_r,j_r}^{p_r,1} \tag{5.6}$$

fullfills the second criterion of being a recursive atom ordering, with $\alpha \dashv \beta$ meaning that β is greater than α in the atom ordering.

Let $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ be distinct atoms with $\alpha_{i,j}^{p_1,w_1} \dashv \alpha_{k,l}^{p_2,w_2}$ and suppose $\alpha_{i,j}^{p_1,w_1}$,$\alpha_{k,l}^{p_2,w_2} \leq \gamma$. We want to show that there is a $\delta \leq \gamma$ and a $\alpha_{i',j'}^{p',w'} \dashv \alpha_{k,l}^{p_2,w_2}$ such that $\delta \succ \alpha_{i',j'}^{p',w'}, \alpha_{k,l}^{p_2,w_2}$. We have three main cases to consider.

(i) $\{i,j\}$=$\{k,l\}$. Since the length of $[\hat{0}, \mu_{s,t}]$ is greater than 1 there must be at least one $m \in [n] \setminus \{i,j\}$ such that $\{i,j,m\}$ is a subset of a block B in γ. Then $\delta = \{(\{i,j,m\}, p, w)\}$ is an element covering both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ and which is less than γ, where p and w depend on in which of three subcases we are:
  (a) $p_1 = p_2$ and $w_1 \neq w_2$: $p = \mathrm{p}(B)$ if $\mathrm{p}(B) = p_1$ or $\mathrm{p}(B) = m$ otherwise any of them, and $w = \max(q, v)$.
  (b) $p_1 \neq p_2$ and $w_1 = w_2$: $p = m$ and $w = w_1$ if $\mathrm{w}(B) = w_1$ and $w = w_1 + 1$ otherwise.
  (c) $p_1 \neq p_2$ and $w_1 \neq w_2$: $p = m$ and $w = \max(w_1, w_2)$.

(ii) $\{i,j\} \cap \{k,l\} = \{m\}$, for some $m \in \{i,j\}$. Let n be the element of $\{k,l\} \setminus \{m\}$. Since both atoms are less then γ we must have that $\{i,j,n\}$ is a subset of a block B in γ. Then $\delta = \{(\{i,j,n\}, p, w)\}$ is an element covering both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ and which is less than γ, where p and w depend on in which of four subcases we are:
  (a) $p_1 = p_2$ and $w_1 = w_2$: $p = p_1 = m$, and $w = w_1$ if $\mathrm{w}(B) = w_1$ and $w = w_1 + 1$ otherwise.
  (b) $p_1 = p_2$ and $w_1 \neq w_2$: $p = p_1 = m$ and $w = \max(w_1, w_2)$.
  (c) $p_1 \neq p_2$ and $w_1 = w_2$: p is any of p_1 and p_2, and $w = w_1$ if $\mathrm{w}(B) = w_1$ and $w = w_1 + 1$ otherwise.
  (d) $p_1 \neq p_2$ and $w_1 \neq w_2$: p is any of p_1 and p_2, and $w = \max(w_1, w_2)$.

(iii) $\{i,j\} \cap \{k,l\} = \emptyset$. Here we have two subcases:
  (a) $w_1 \neq w_2$: $\delta = \{(\{i,j\}, p_1, w_1), (\{k,l\}, p_2, w_2)\}$ covers both $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ and will always be less than or equal to any γ greater than both atoms.
  (b) $w_1 = w_2$: By the ordering of the atoms $\alpha_{i,j}^{p_1,w_1} \dashv \alpha_{k,l}^{p_2,w_2}$ implies $\alpha_{i,j}^{p_1,\tilde{w_1}} \dashv \alpha_{k,l}^{p_2,w_2}$, where $\tilde{w_1}$ is the element in $\{0,1\} \setminus \{w_1\}$. Now since $\alpha_{i,j}^{p_1,w_1}$,$\alpha_{k,l}^{p_2,w_2} \leq \gamma$ either $\delta = \{(\{i,j\}, p_1, w_1), (\{k,l\}, p_2, w_2)\} \leq \gamma$ or $\tilde{\delta} = \{(\{i,j\}, p_1, \tilde{w_1}), (\{k,l\}, p_2, w_2)\} \leq \gamma$, where δ covers $\alpha_{i,j}^{p_1,w_1}$ and $\alpha_{k,l}^{p_2,w_2}$ whereas $\tilde{\delta}$ covers $\alpha_{i,j}^{p_1,\tilde{w_1}}$ and $\alpha_{k,l}^{p_2,w_2}$.

We also need to show that, given an ordering of the form (5.6), any interval $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ satisfies the first criterion of being a recursive atom ordering. We may identify

$$\{\{i,j\}, 1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\} \sim [n-1]$$

and we see that $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ is isomorphic to a maximal interval $[\hat{0}, \mu_{s',t-w}]$ in Π_{n-1}^{pw}, where s' is the appropriate pointed element after the above identification. Thus checking the above step is readily done if we may order the atoms in the same way as above. We only need to show that some way of ordering the atoms of $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ in pairs as above satisfies that the first atoms are the ones covering some atom $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$. After that we can proceed by induction.

We may assume that the length of $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ is greater than 1, since otherwise we are done. We may also assume that $0 < t - w < n - 2$, since if $t = w$ or $t = n - 2 + w$ then by the same argument as above the interval is totally semimodular whence it follows that any ordering of the atoms is a recursive atom ordering. Thus we may order the atoms of $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ in accordance with the first criterion of being a recursive atom ordering.

Now the atoms are either of the form $\{(\{i,j\},p,w),(\{k,l\},q,v)\}$ which we denote by $\beta_{k,l}^{q,v}$ or of the form $\{(\{i,j,k\},q,w+v)\}$ which we denote by $\beta_k^{q,v}$, where $v \in \{0,1\}$. Let $\tilde{v}$ be the element of $\{0,1\} \setminus \{v\}$.

We have that $\beta_{k,l}^{q,v}$ covers some $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$, namely $\alpha_{i',j'}^{p',w'} = \alpha_{k,l}^{q,v}$, iff $\alpha_{k,l}^{q,v} \dashv \alpha_{i,j}^{p,w}$. Since by the atom ordering of $[\hat{0}, \mu_{s,t}]$ we have that $\alpha_{k,l}^{q,v} \dashv \alpha_{i,j}^{p,w}$ iff $\alpha_{k,l}^{q,\tilde{v}} \dashv \alpha_{i,j}^{p,w}$, we have that $\beta_{k,l}^{q,v}$ covers some $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$ iff $\beta_{k,l}^{q,\tilde{v}}$ covers some $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$.

Similarily we have that $\beta_k^{q,v}$ may cover some $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$, where $\{i',j'\} \subset \{i,j,k\}$. Again $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$ iff $\alpha_{i',j'}^{p',\tilde{w}'} \dashv \alpha_{i,j}^{p,w}$. Hence $\beta_k^{q,v}$ covers some $\alpha_{i',j'}^{p',w'} \dashv \alpha_{i,j}^{p,w}$ iff $\beta_k^{q,\tilde{v}}$ does.

Thus we may order the atoms of $[\alpha_{i,j}^{p,w}, \mu_{s,t}]$ by first putting all pairs of atoms, differing only in weight, covering some atom less than $\alpha_{i,j}^{p,w}$ followed by all pairs of atoms not covering any atom less than $\alpha_{i,j}^{p,w}$. Using the aforementioned identification $[\alpha_{i,j}^{p,w}, \mu_{s,t}] \cong [\hat{0}, \mu_{s',t-w}]$, we just proceed by induction. □

Theorem 5.7. *$\mathcal{P}erm_2$ and $\mathcal{P}re\mathcal{L}ie_2$ are Koszul.*

Proof. Using Lemma 5.5, Lemma 2.10 and Lemma 5.4 the proof is completely analogous to the proof of Theorem 4.10. □

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Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden
E-mail address: henriks@math.su.se