ISSN: 1401-5617

Ground state solutions for some indefinite problems

Andrzej Szulkin
Tobias Weth

Research Reports in Mathematics
Number 7, 2007

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2007/7

Date of publication: October 24, 2007

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Ground state solutions for some indefinite problems

Andrzej SZULKIN*
Department of Mathematics, Stockholm University
106 91 Stockholm, Sweden
andrzejs@math.su.se

Tobias WETH
Mathematisches Institut, Universität Giessen
Arndtstr. 2, 35392 Giessen, Germany
Tobias.Weth@math.uni-giessen.de

Abstract

We consider the nonlinear stationary Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$. Here f is a superlinear, subcritical nonlinearity, and we mainly study the case where both V and f are periodic in x and 0 belongs to a spectral gap of $-\Delta + V$. Inspired by previous work of Li et al. [11] and Pankov [13], we develop an approach to find ground state solutions, i.e., nontrivial solutions with least possible energy. The approach is based on a direct and surprisingly simple reduction of the indefinite variational problem to a definite one and gives rise to a new minimax characterization of the corresponding critical value. Our method works for merely continuous nonlinearities f which are allowed to have weaker asymptotic growth than usually assumed. For odd f, we obtain infinitely many geometrically distinct solutions. The approach also yields new existence and multiplicity results for the Dirichlet problem for the same type of equations in a bounded domain.

1 Introduction

In this paper we will be concerned with the semilinear Schrödinger equation

$$(1.1) \qquad \begin{cases} -\Delta u + V(x)u = f(x,u) \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Here $H^1(\mathbb{R}^N)$ is the usual Sobolev space. Our assumptions on V and f stated below imply that the Schrödinger operator $-\Delta + V$ is selfadjoint and semi-bounded in $L^2(\mathbb{R}^N)$ and solutions of (1.1) are critical points of the functional

$$(1.2) \qquad \Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R}), \qquad \Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

*Supported in part by the Swedish Research Council.

In the last part of the paper, we will also consider a related variational problem associated with a semilinear elliptic boundary value problem in a bounded domain. We will be mainly interested in the case where these problems are indefinite in the sense that 0 is not a local minimum for the corresponding functionals, but some of our results are new also in the definite case. In the case of the full space problem (1.1) we focus on periodic data – another setting will be discussed briefly in Section 4 below. For the Schrödinger operator $-\Delta + V$ we assume:

(S_1) V is continuous, 1-periodic in $x_1, \dots, x_N$ and $0 \notin \sigma(-\Delta + V)$, the spectrum of $-\Delta + V$.

Starting with the seminal work of Angenent [2], Coti Zelati and Rabinowitz [6], and Alama-Li [1], this case has attracted immense attention in the last 15 years. Setting $F(x,u) := \int_0^u f(x,s)\,ds$, we suppose that f satisfies the following assumptions:

(S_2) f is continuous, 1-periodic in $x_1, \dots, x_N$ and $|f(x,u)| \le a(1 + |u|^{p-1})$ for some $a > 0$ and $p \in (2, 2^*)$, where $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := +\infty$ if $N = 1$ or 2.

(S_3) $f(x,u) = o(u)$ uniformly in x as $|u| \to 0$.

(S_4) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$.

(S_5) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

While (S_1)$-$(S_3) are standard assumptions in this context, the following Ambrosetti-Rabinowitz type superlinearity condition is commonly used in place of (S_4) and (S_5):

(AR) $\eta F(x,u) \ge f(x,u)u > 0$ for some $\eta > 2$ and all $u \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}^N$.

We recall that (AR) implies $F(x,u) \ge c|u|^\eta > 0$ for $|u| \ge 1$ and all $x \in \mathbb{R}^N$, so it is a stronger condition than (S_4). To state our results, we fix some notation. Let $E := H^1(\mathbb{R}^N)$. By (S_1) there is an equivalent inner product $\langle .,. \rangle$ in E such that

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

where $E = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum, and $u = u^+ + u^- \in E^+ \oplus E^-$. If $\sigma(-\Delta + V) \subset (0, +\infty)$, then $\dim E^- = 0$, otherwise E^- is infinite-dimensional. The following set has been introduced by Pankov [13]:

$$\mathcal{M} := \left\{ u \in E \setminus E^- \; : \; \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^- \right\}. \tag{1.3}$$

By definition, $\mathcal{M}$ contains all nontrivial critical points of Φ. The following is our first main result.

Theorem 1.1 *Suppose (S_1)-(S_5) are satisfied and let $c := \inf_{u \in \mathcal{M}} \Phi(u)$. Then c is attained, $c > 0$ and if $u_0 \in \mathcal{M}$ satisfies $\Phi(u_0) = c$, then u_0 is a solution of (1.1).*

Since c is the lowest level for Φ at which there are nontrivial solutions of (1.1), u_0 will be called a ground state. Theorem 1.1 is due to Pankov [13, Section 5] under the following additional assumptions on the nonlinearity: $f \in C^1$, $|f'_u(x,u)| \le \tilde{a}(1+|u|^{p-1})$ and

$$0 < \frac{f(x,u)}{u} < \theta f'_u(x,u) \quad \text{for some } \theta \in (0,1) \text{ and all } u \neq 0. \tag{1.4}$$

The existence of a nontrivial solution has been obtained in [1, 9, 17] under assumption (AR) and different additional conditions, but it is new under assumptions (S_1)-(S_5). We point out that (1.4) is stronger than both (AR) and (S_5). In the definite case where $\sigma(-\Delta+V) \subset (0,\infty)$, Theorem 1.1 is a slight extension of a recent result by Li et al. [11] which, together with Pankov's work [13], inspired us to consider the indefinite problem.

Let us briefly sketch Pankov's approach. He first shows that $\mathcal{M}$ is a C^1-manifold, and it is a natural constraint in the sense that u is a critical point of Φ if and only if $u \in \mathcal{M}$ and it is a critical point of $\Phi|_{\mathcal{M}}$. Since $c := \inf \Phi|_{\mathcal{M}} > -\infty$, Ekeland's variational principle yields a Palais-Smale sequence for $\Phi|_{\mathcal{M}}$ at the level c. Pankov then uses (1.4) to show that this Palais-Smale sequence is bounded, and he finds a minimizer by concentration-compactness arguments. Since we are not assuming f is differentiable and satisfies (1.4), $\mathcal{M}$ need not be of class C^1 in our case, and therefore Pankov's method does not apply. Nevertheless, $\mathcal{M}$ is still a topological manifold, naturally homeomorphic to the unit sphere in E^+. To explain this in detail, we define for $u \in E \setminus E^-$ the subspace

$$E(u) := E^- \oplus \mathbb{R}u = E^- \oplus \mathbb{R}u^+ \tag{1.5}$$

and the convex subset

$$\hat{E}(u) := E^- \oplus \mathbb{R}^+ u = E^- \oplus \mathbb{R}^+ u^+, \tag{1.6}$$

of E, where as usual, $\mathbb{R}^+ = [0,\infty)$. Our approach is based on the following key observations.

I.) For each $u \in E \setminus E^-$, the set $\mathcal{M}$ intersects $\hat{E}(u)$ in exactly one point $\hat{m}(u)$ which is the unique global maximum point of $\Phi|_{\hat{E}(u)}$. Moreover, the map $u \mapsto \hat{m}(u)$ is continuous, and the restriction of $\hat{m}$ to the unit sphere S^+ in E^+ defines a homeomorphism between S^+ and $\mathcal{M}$.

II.) The composed functional $\Phi \circ \hat{m} : S^+ \to \mathbb{R}$ is of class C^1 (even though $\hat{m}$ might not be differentiable) and coercive on S^+. Moreover, critical points of $\Phi \circ \hat{m}$ are in $1-1$ correspondence with nontrivial critical points of Φ.

We point out that, as a consequence of I.), the least energy value c has a minimax characterization given by

$$c = \inf_{w \in E^+ \setminus \{0\}} \max_{u \in \hat{E}(w)} \Phi(u). \tag{1.7}$$

Note that this minimax principle is much simpler than the usual characterizations related to the concept of linking, see e.g. [18]. In the case where f is odd, the characterization reduces to a mere minimax over *linear* subspaces, i.e.,

$$c = \inf_{w \in E^+ \setminus \{0\}} \max_{u \in E(w)} \Phi(u). \tag{1.8}$$

This equality resembles characterizations of the lowest eigenvalue of a linear selfadjoint operator in a spectral gap, see e.g. [7, 8]. We also note that (1.7) and (1.8) could be used numerically to compute the least energy c (and possibly also minimizers). For a related computational minimax algorithm, see Li and Zhou [10].

Next we consider the multiplicity of solutions of (1.1). We note that if u_0 is a solution of (1.1), then so are all elements of the orbit of u_0 under the action of $\mathbb{Z}^N$, $\mathcal{O}(u_0) := \{u_0(\cdot - k) : k \in \mathbb{Z}^N\}$. Two solutions u_1 and u_2 are said to be *geometrically distinct* if $\mathcal{O}(u_1)$ and $\mathcal{O}(u_2)$ are disjoint.

Theorem 1.2 *Suppose* (S_1)-(S_5) *are satisfied and* f *is odd in* u. *Then (1.1) admits infinitely many pairs* $\pm u$ *of geometrically distinct solutions.*

Again this result is new under assumptions (S_1)-(S_5) – even in the definite case. For f satisfying (AR) and a Lipschitz condition, infinitely many geometrically distinct solutions have been obtained in [9]. See also [1] where a stronger result (existence of multibumps) has been proved for a pure power nonlinearity.

We remark that our method simplifies considerably in the definite case $E^- = \{0\}$. On the other hand, when $E^- \neq \{0\}$, then our approach also yields existence and multiplicity results in the case where the nonlinearity f in (1.1) is replaced by $-f$, see Theorem 4.1 below.

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in Section 2. In Section 3 we consider a semilinear elliptic problem with zero Dirichlet boundary data in a bounded domain. The results are the same as above but now some parts of the proofs become simpler because the Nemytskii operator corresponding to the nonlinearity f is compact. Finally, in Section 4 we add some remarks on variants of problem (1.1).

2 Proof of the main results

We assume that $(S_1) - (S_5)$ are satisfied from now on. We start with some elementary observations. First, (S_2) and (S_3) imply that

$$\text{for each } \varepsilon > 0 \text{ there is } C_\varepsilon > 0 \text{ such that } |f(x,u)| \leq \varepsilon|u| + C_\varepsilon |u|^{p-1} \text{ for all } u \in \mathbb{R}. \tag{2.1}$$

Lemma 2.1 $F(x,u) > 0$ *if* $u \neq 0$ *and* $\frac{1}{2}f(x,u)u \geq F(x,u)$ *for all* $u \in \mathbb{R}$.

This follows immediately from (S_3) and (S_5).

Lemma 2.2 *Let $u, s, v \in \mathbb{R}$ be numbers with $s \geq -1$ and $w := su + v \neq 0$, and let $x \in \mathbb{R}^N$. Then*

$$f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,u+w) < 0.$$

The proof of this estimate is elementary but not straightforward. We postpone it to the appendix. In the following we assume E^- is nontrivial and for $u \notin E^-$ we consider the subspace $E(u)$ and the convex subset $\hat{E}(u)$ as defined in (1.5) and (1.6).

Proposition 2.3 *If $u \in \mathcal{M}$, then*

$$\Phi(u+w) < \Phi(u) \qquad \text{for every } w \in \mathcal{Z} := \{su + v \;:\; s \geq -1,\ v \in E^-\}, \quad w \neq 0.$$

Hence u is the unique global maximum of $\Phi|_{\hat{E}(u)}$.

Proof We let $B : E \times E \to \mathbb{R}$ denote the symmetric bilinear form given by

$$B(v_1, v_2) = \int_{\mathbb{R}^N} (\nabla v_1 \nabla v_2 + V(x) v_1 v_2)\, dx \qquad \text{for } v_1, v_2 \in E.$$

Let $w = su + v \in \mathcal{Z}$; i.e., $v \in E^-$ and $s \geq -1$. Then $u + w = (1+s)u + v \in \hat{E}(u)$. We calculate

$$\begin{aligned}
\Phi(u+w) - \Phi(u) &= \frac{1}{2}[B(u+w, u+w) - B(u,u)] + \int_{\mathbb{R}^N} (F(x,u) - F(x,u+w))\, dx \\
&= \frac{1}{2}[B((1+s)u+v, (1+s)u+v) - B(u,u)] + \int_{\mathbb{R}^N} (F(x,u) - F(x,u+w))\, dx \\
&= \frac{1}{2}\Big([(1+s)^2 - 1]B(u,u) + 2(1+s)B(u,v) + B(v,v)\Big) + \int_{\mathbb{R}^N} (F(x,u) - F(x,u+w))\, dx \\
&= -\frac{\|v\|^2}{2} + B(u, s(\frac{s}{2}+1)u + (1+s)v) + \int_{\mathbb{R}^N} (F(x,u) - F(x,u+w))\, dx \\
&= -\frac{\|v\|^2}{2} + \int_{\mathbb{R}^N} \Big(f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,u+w)\Big)\, dx,
\end{aligned}$$

where in the last step we used the fact that, since $u \in \mathcal{M}$,

$$0 = \Phi'(u)z = B(u,z) - \int_{\mathbb{R}^N} f(x,u)z(x)\, dx \qquad \text{for all } z \in E(u).$$

Since w is nonzero on a set of of positive measure, we conclude by Lemma 2.2 that $\Phi(u+w) < \Phi(u)$, as claimed. $\square$

Lemma 2.4
(a) There exists $\alpha > 0$ such that $c = \inf_{\mathcal{M}} \Phi \geq \inf_{S_\alpha} \Phi(u) > 0$, where $S_\alpha := \{u \in E^+ \;:\; \|u\| = \alpha\}$.
(b) $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2c}\}$ for every $u \in \mathcal{M}$.

Proof (a) For $u \in E^+$ we have $\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx$ and $\int_{\mathbb{R}^N} F(x,u)\,dx = o(\|u\|^2)$ as $u \to 0$ by (2.1), hence the second inequality follows if $\alpha > 0$ is sufficiently small. The first inequality is a consequence of Proposition 2.3, since for every $u \in \mathcal{M}$ there is $s > 0$ such that $su^+ \in \hat{E}(u) \cap S_\alpha$.
(b) For $u \in \mathcal{M}$ we have

$$c \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} F(x,u)\,dx \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2),$$

hence $\|u^+\| \geq \max\{\sqrt{2c}, \|u^-\|\}$. □

Lemma 2.5 *If $\mathcal{V} \subset E^+ \setminus \{0\}$ is a compact subset, then there exists $R > 0$ such that $\Phi \leq 0$ on $E(u) \setminus B_R(0)$ for every $u \in \mathcal{V}$.*

Proof Without loss of generality, we may assume that $\|u\| = 1$ for every $u \in \mathcal{V}$. Suppose by contradiction that there exist $u_n \in \mathcal{V}$ and $w_n \in E(u_n)$, $n \in \mathbb{N}$ such that $\Phi(w_n) \geq 0$ for all n and $\|w_n\| \to \infty$ as $n \to \infty$. Passing to a subsequence, we may assume that $u_n \to u \in E^+$, $\|u\| = 1$. Set $v_n = w_n/\|w_n\| = s_n u_n + v_n^-$, then

$$0 \leq \frac{\Phi(w_n)}{\|w_n\|^2} = \frac{1}{2}(s_n^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x,w_n)}{w_n^2} v_n^2\,dx. \tag{2.2}$$

Hence $\|v_n^-\|^2 \leq s_n^2 = 1 - \|v_n^-\|^2$ and therefore $\frac{1}{\sqrt{2}} \leq s_n \leq 1$. So, for a subsequence, $s_n \to s > 0$, $v_n \rightharpoonup v$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Hence $v = su + v^- \neq 0$ and, since $|w_n(x)| \to \infty$ if $v(x) \neq 0$, it follows from (S_4) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{F(x,w_n)}{w_n^2} v_n^2\,dx \to \infty, \tag{2.3}$$

contrary to (2.2). □

Lemma 2.6 *For each $u \notin E^-$ the set $\mathcal{M} \cap \hat{E}(u)$ consists of precisely one point $\hat{m}(u)$ which is the unique global maximum of $\Phi|_{\hat{E}(u)}$.*

Proof By Proposition 2.3, it suffices to show that $\mathcal{M} \cap \hat{E}(u) \neq \varnothing$. Since $\hat{E}(u) = \hat{E}(u^+)$, we may assume that $u \in E^+$, $\|u\| = 1$. By Lemma 2.5, there exists $R > 0$ such that $\Phi \leq 0$ on $E(u) \setminus B_R(0)$. By Lemma 2.4(a), $\Phi(tu) > 0$ for small $t > 0$, and since $\Phi \leq 0$ on $\hat{E}(u) \setminus B_R(0)$, $0 < \sup_{\hat{E}(u)} \Phi < \infty$. It is easy to see that Φ is weakly upper semicontinuous on $\hat{E}(u)$, therefore $\Phi(u_0) = \sup_{\hat{E}(u)} \Phi$ for some $u_0 \in \hat{E}(u) \setminus \{0\}$. This u_0 is a critical point of $\Phi|_{E(u)}$, so $\langle \Phi'(u_0), u_0 \rangle = \langle \Phi'(u_0), v \rangle = 0$ for all $v \in E(u)$. Consequently, $u_0 \in \mathcal{M} \cap \hat{E}(u)$, as required. □

Proposition 2.7 *Φ is coercive on $\mathcal{M}$, i.e., $\Phi(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{M}$.*

Proof Arguing by contradiction, suppose there exists a sequence $(u_n)_n \subset \mathcal{M}$ such that $\|u_n\| \to \infty$ and $\Phi(u_n) \le d$ for some $d \in [c, \infty)$. Let $v_n := u_n/\|u_n\|$. Then $v_n \rightharpoonup v$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. By Lemma 2.4(b), $\|v_n^+\|^2 \ge 1/2$. Let $y_n \in \mathbb{R}^N$ satisfy

$$\int_{B_1(y_n)} (v_n^+)^2\,dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} (v_n^+)^2\,dx. \tag{2.4}$$

Since Φ and $\mathcal{M}$ are invariant under translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. Suppose

$$\int_{B_1(y_n)} (v_n^+)^2\,dx \to 0 \qquad \text{as } n \to \infty. \tag{2.5}$$

Then $v_n^+ \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ according to P.L. Lions' lemma [18, Lemma 1.21], and therefore (2.1) implies that $\int_{\mathbb{R}^N} F(x, sv_n^+)\,dx \to 0$ for every $s \in \mathbb{R}$. Since $sv_n^+ \in \hat{E}(u_n)$ for $s \ge 0$, Proposition 2.3 implies that

$$d \ge \Phi(u_n) \ge \Phi(sv_n^+) = \frac{s^2}{2}\|v_n^+\|^2 - \int_{\mathbb{R}^N} F(x, sv_n^+)\,dx \ge \frac{s^2}{4} - \int_{\mathbb{R}^N} F(x, sv_n^+)\,dx \;\to\; \frac{s^2}{4}.$$

This yields a contradiction if $s > \sqrt{4d}$. Hence (2.5) is false and since $v_n^+ \to v^+$ in $L^2_{loc}(\mathbb{R}^N)$, $v^+ \ne 0$. Since $|u_n(x)| \to \infty$ if $v(x) \ne 0$, it follows again from (S_4) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} v_n^2\,dx \to \infty$$

and therefore

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} v_n^2\,dx \to -\infty$$

as $n \to \infty$, a contradiction. The proof is finished. □

Lemma 2.8 *The map $E^+ \setminus \{0\} \to \mathcal{M}$, $u \mapsto \hat{m}(u)$ (see Lemma 2.6) is continuous.*

Proof Let $u \in E^+ \setminus \{0\}$. By a standard argument, the continuity of $\hat{m}$ in u is reduced to the following assertion:

(2.6) if $u_n \to u$ for a sequence $(u_n)_n \subset E^+ \setminus \{0\}$, then $\hat{m}(u_n) \to \hat{m}(u)$ for a subsequence.

To prove (2.6), we let $(u_n)_n \subset E^+ \setminus \{0\}$ be a sequence with $u_n \to u$. Without loss of generality, we may assume that $\|u_n\| = \|u\| = 1$ for all n, so that $\hat{m}(u_n) = \|\hat{m}(u_n)^+\|\, u_n + \hat{m}(u_n)^-$. By Lemma 2.5 there exists $R > 0$ such that

$$\Phi(\hat{m}(u_n)) = \sup_{E(u_n)} \Phi \le \sup_{B_R(0)} \Phi \le \sup_{u\in B_R(0)} \|u^+\|^2 = R^2 \qquad \text{for every } n.$$

Hence $\hat{m}(u_n)$ is bounded by Proposition 2.7. Passing to a subsequence, we may assume that

$$t_n := \|\hat{m}(u_n)^+\| \to t \qquad \text{and} \qquad \hat{m}(u_n)^- \rightharpoonup u_*^- \quad \text{in } E \text{ as } n \to \infty,$$

where $t \geq \sqrt{2c} > 0$ by Lemma 2.4(b). Moreover, by Lemma 2.6,

$$\Phi(\hat{m}(u_n)) \geq \Phi(t_n u_n + \hat{m}(u)^-) \to \Phi(tu + \hat{m}(u)^-) = \Phi(\hat{m}(u)).$$

Therefore by Fatou's lemma and the weak lower semicontinuity of the norm,

$$\begin{aligned}\Phi(\hat{m}(u)) &\leq \lim_{n\to\infty} \Phi(\hat{m}(u_n)) = \lim_{n\to\infty}\left(\frac{1}{2}t_n^2 - \frac{1}{2}\|\hat{m}(u_n)^-\|^2 - \int_{\mathbb{R}^N} F(x, \hat{m}(u_n))\,dx\right)\\ &\leq \frac{1}{2}t^2 - \frac{1}{2}\|u_*^-\|^2 - \int_{\mathbb{R}^N} F(x, tu + u_*^-)\,dx = \Phi(tu + u_*^-) \leq \Phi(\hat{m}(u)).\end{aligned}$$

Hence all inequalities above must be equalities and it follows that $\hat{m}(u_n)^- \to u_*^-$, and by Lemma 2.6, $u_*^- = \hat{m}(u)^-$. So $\hat{m}(u_n) \to \hat{m}(u)$. □

We now consider the functional

$$\hat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}, \qquad \hat{\Psi}(u) := \Phi(\hat{m}(u)),$$

which is continuous by Lemma 2.8. The following somewhat surprising observation is crucial for our approach.

Proposition 2.9 *$\hat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, and*

$$\hat{\Psi}'(w)z = \frac{\|\hat{m}(w)^+\|}{\|w\|}\,\Phi'(\hat{m}(w))z \qquad \text{for } w, z \in E^+,\ w \neq 0.$$

Proof We put $u = \hat{m}(w) \in \mathcal{M}$, so we have $u = u^- + \frac{\|u^+\|}{\|w\|}w$. Let $z \in E^+$. Choose $\delta > 0$ such that $w_t := w + tz \in E^+ \setminus \{0\}$ for $|t| < \delta$ and put $u_t = \hat{m}(w_t) \in \mathcal{M}$. We may write $u_t = u_t^- + s_t w_t$ with $s_t > 0$. Then $s_0 = \frac{\|u^+\|}{\|w\|}$, and the function $(-\delta, \delta) \to \mathbb{R}$, $t \mapsto s_t$, is continuous by Lemma 2.8. Lemma 2.6 and the mean value theorem now imply that

$$\begin{aligned}\hat{\Psi}(w_t) - \hat{\Psi}(w) &= \Phi(u_t) - \Phi(u) = \Phi(u_t^- + s_t w_t) - \Phi(u^- + s_0 w)\\ &\leq \Phi(u_t^- + s_t w_t) - \Phi(u_t^- + s_t w) = \Phi'(u_t^- + s_t[w + \tau_t(w_t - w)])s_t(w_t - w)\\ &= s_0\Phi'(u)tz + o(t) \qquad \text{as } t \to 0,\end{aligned}$$

with some $\tau_t \in (0,1)$. By a similar reasoning, we also have that

$$\begin{aligned}\hat{\Psi}(w_t) - \hat{\Psi}(w) &= \Phi(u_t^- + s_t w_t) - \Phi(u^- + s_0 w) \geq \Phi(u^- + s_0 w_t) - \Phi(u^- + s_0 w)\\ &= \Phi'(u^- + s_0[w + \eta_t(w_t - w)])s_0(w_t - w)\\ &= s_0\Phi'(u)tz + o(t) \qquad \text{as } t \to 0,\end{aligned}$$

with some $\eta_t \in (0,1)$. Combining these inequalities, we conclude that the directional derivative $\partial_z \hat{\Psi}(w)$ exists and is given by

$$\partial_z \hat{\Psi}(w) = \lim_{t \to 0} \frac{\hat{\Psi}(w_t) - \hat{\Psi}(w)}{t} = s_0 \Phi'(u) z = \frac{\|\hat{m}(w)^+\|}{\|w\|} \Phi'(\hat{m}(w)) z.$$

Hence $\partial_z \hat{\Psi}(w)$ is linear (and continuous) in z and depends continuously on w. So the assertion follows by [18, Proposition 1.3]. $\square$

Next we consider the unit sphere $S^+ := \{w \in E^+ \ : \ \|w\| = 1\}$ in E^+. We note that the restriction of the map $\hat{m}$ to S^+ is a homeomorphism with inverse given by

$$\check{m} : \mathcal{M} \to S^+, \qquad \check{m}(u) = \frac{u^+}{\|u^+\|}. \tag{2.7}$$

We also consider the restriction $\Psi : S^+ \to \mathbb{R}$ of $\hat{\Psi}$ to S^+.

Corollary 2.10

(a) $\Psi \in C^1(S^+)$, *and*

$$\Psi'(w) z = \|\hat{m}(w)^+\| \, \Phi'(\hat{m}(w)) z \qquad \textit{for} \qquad z \in T_w S^+ = \{v \in E^+ \ : \ \langle w, v\rangle = 0\}.$$

(b) $(w_n)_n$ *is a Palais-Smale sequence for* Ψ *if and only if* $(\hat{m}(w_n))_n$ *is a Palais-Smale sequence for* Φ.

(c) We have

$$\inf_{S^+} \Psi = \inf_{\mathcal{M}} \Phi = c.$$

Moreover, $u \in S^+$ *is a critical point of* Ψ *if and only if* $\hat{m}(u) \in \mathcal{M}$ *is a critical point of* Φ*, and the corresponding critical values coincide.*

Proof (a) is a direct consequence of Proposition 2.9.

To prove (b), let $(w_n)_n$ be a sequence such that $C := \sup_n \Psi(w_n) = \sup_n \Phi(\hat{m}(w_n)) < \infty$, and let $u_n := \hat{m}(w_n) \in \mathcal{M}$. Since for every n we have an orthogonal splitting

$$E = E(w_n) \oplus T_{w_n} S^+ = E(u_n) \oplus T_{w_n} S^+ \qquad \text{(with respect to } \langle \cdot\,, \cdot \rangle)$$

and $\Phi'(u_n) v = 0$ for all $v \in E(u_n)$, we find that $\nabla \Phi(u_n) \in T_{w_n} S^+$ and using (a),

$$\|\Psi'(w_n)\| = \sup_{\substack{z \in T_{w_n} S^+ \\ \|z\| = 1}} \Psi'(w_n) z = \sup_{\substack{z \in T_{w_n} S^+ \\ \|z\| = 1}} \|u_n^+\| \Phi'(w_n) z = \|u_n^+\| \, \|\Phi'(u_n)\|. \tag{2.8}$$

According to Lemma 2.4(b) and Proposition 2.7, $\sqrt{2c} \le \|u_n^+\| \le \sup_n \|u_n^+\| < \infty$. Hence $(w_n)_n$ is a Palais-Smale sequence for Ψ if and only if $(u_n)_n$ is a Palais-Smale sequence for Φ.

(c) The proof is similar as in (b) but easier. $\square$

Proof of Theorem 1.1 (completed). From Lemma 2.4(a) we know that $c > 0$. Moreover, if $u_0 \in \mathcal{M}$ satisfies $\Phi(u_0) = c$, then $\check{m}(u_0) \in S^+$ is a minimizer of Ψ and therefore a critical point of Ψ, so that u_0 is a critical point of Φ by Corollary 2.10. It remains to show that there exists a minimizer $u \in \mathcal{M}$ of $\Phi|_{\mathcal{M}}$. By Ekeland's variational principle [18], there exists a sequence $(w_n)_n \subset S^+$ with $\Psi(w_n) \to c$ and $\Psi'(w_n) \to 0$ as $n \to \infty$. Put $u_n = \hat{m}(w_n) \in \mathcal{M}$ for $n \in \mathbb{N}$. Then $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$ by Corollary 2.10(b). By Proposition 2.7, (u_n) is bounded and hence $u_n \rightharpoonup u$ after passing to a subsequence. Let $y_n \in \mathbb{R}^N$ satisfy

$$\int_{B_1(y_n)} u_n^2\,dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} u_n^2\,dx.$$

Using once more that Φ and $\mathcal{M}$ are invariant under translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} u_n^2\,dx \to 0 \qquad \text{as } n \to \infty, \tag{2.9}$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2^*$, again by [18, Lemma 1.21]. From (2.1) and the Sobolev embeddings $E \to L^2(\mathbb{R}^N)$, $E \to L^p(\mathbb{R}^N)$, we infer that $\int_{\mathbb{R}^N} f(x,u_n)u_n^+\,dx = o(\|u_n^+\|)$ as $n \to \infty$, hence

$$o(\|u_n^+\|) = \Phi'(u_n)u_n^+ = \|u_n^+\|^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n^+\,dx = \|u_n^+\|^2 - o(\|u_n^+\|)$$

and therefore $\|u_n^+\| \to 0$, contrary to Lemma 2.4(b). It follows that (2.9) cannot hold, so $u_n \rightharpoonup u \neq 0$ and $\Phi'(u) = 0$.

It remains to show that $\Phi(u) = c$. By Lemma 2.1, Fatou's lemma and since $(u_n)_n$ is bounded,

$$\begin{aligned} c + o(1) = \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n &= \int_{\mathbb{R}^N} \left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\geq \int_{\mathbb{R}^N} \left(\frac{1}{2}f(x,u)u - F(x,u)\right) dx + o(1) = \Phi(u) - \frac{1}{2}\Phi'(u)u + o(1) = \Phi(u) + o(1). \end{aligned}$$

Hence $\Phi(u) \leq c$. The reverse inequality follows from the definition of c since $u \in \mathcal{M}$. □

The remainder of this section is devoted to the proof of Theorem 1.2. So from now on we assume that – in addition to $(S_1) - (S_5)$ – the nonlinearity $f = f(x,u)$ is odd in u. We need the following simple fact.

Lemma 2.11 *The map $\check{m}$ defined in* (2.7) *is Lipschitz continuous.*

Proof For $u, v \in \mathcal{M}$ we have, by Lemma 2.4(b),

$$\begin{aligned} \|\check{m}(u) - \check{m}(v)\| &= \left\| \frac{u^+}{\|u^+\|} - \frac{v^+}{\|v^+\|} \right\| = \left\| \frac{u^+ - v^+}{\|u^+\|} + \frac{(\|v^+\| - \|u^+\|)v^+}{\|u^+\|\,\|v^+\|} \right\| \\ &\leq \frac{2}{\|u^+\|}\|(u-v)^+\| \leq \sqrt{\frac{2}{c}}\,\|u - v\|. \end{aligned}$$

□

Remark 2.12 It is easy to see that both maps $\hat{m}, \check{m}$ are equivariant with respect to the $\mathbb{Z}^N$-action given by $u \mapsto u(\cdot - k)$ for $k \in \mathbb{Z}^N$. So, by Corollary 2.10(c), the orbits $\mathcal{O}(u) \subset \mathcal{M}$ consisting of critical points of Φ are in $1-1$ correspondence with the orbits $\mathcal{O}(w) \subset S^+$ consisting of critical points of Ψ.

We need some more notation. For $d \geq e \geq c$ we put

$$\begin{aligned}
&\Phi^d := \{u \in \mathcal{M} \ : \ \Phi(u) \leq d\}, && \Phi_e := \{u \in \mathcal{M} \ : \ \Phi(u) \geq e\}, && \Phi_e^d = \Phi_e \cap \Phi^d, \\
&\Psi^d := \{w \in S^+ \ : \ \Psi(w) \leq d\}, && \Psi_e := \{w \in S^+ \ : \ \Psi(w) \geq e\}, && \Psi_e^d = \Psi_e \cap \Psi^d, \\
&K := \{w \in S^+ \ : \ \Psi'(w) = 0\}, && K_d := \{w \in K \ : \ \Psi(w) = d\} && \text{and} \\
&\nu(d) := \sup\{\|u\| \ : \ u \in \Phi^d\}.
\end{aligned}$$

We point out that $\nu(d) < \infty$ for every d by Proposition 2.7. We may choose a subset $\mathcal{F}$ of K such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset K$ has a unique representative in $\mathcal{F}$. By Remark 2.12, it suffices to show that the set $\mathcal{F}$ is infinite. So from now on we assume by contradiction that

$$\mathcal{F} \text{ is a finite set.} \tag{2.10}$$

Lemma 2.13 $\kappa := \inf\{\|v - w\| : v, w \in K, \ v \neq w\} > 0.$

Proof Choose $v_n, w_n \in \mathcal{F}$ and $k_n, l_n \in \mathbb{Z}^N$ such that $v_n(\cdot - k_n) \neq w_n(\cdot - l_n)$ for all n and

$$\|v_n(\cdot - k_n) - w_n(\cdot - l_n)\| \to \kappa \qquad \text{as } n \to \infty.$$

Put $m_n = k_n - l_n$. Passing to a subsequence, $v_n = v \in \mathcal{F}$, $w_n = w \in \mathcal{F}$ and either $m_n = m \in \mathbb{Z}^N$ for almost all n or $|m_n| \to \infty$. In the first case,

$$0 < \|v_n(\cdot - k_n) - w_n(\cdot - l_n)\| = \|v - w(\cdot - m)\| = \kappa \qquad \text{for all } n.$$

In the second case $w(\cdot - m_n) \rightharpoonup 0$ and therefore $\kappa = \lim_{n\to\infty} \|v - w(\cdot - m_n)\| \geq \|v\| = 1.$ □

We need the following key lemma.

Lemma 2.14 (Discreteness of PS-sequences) *Let $d \geq c$. If $(v_n^1)_n, (v_n^2)_n \subset \Psi^d$ are two Palais-Smale sequences for Ψ, then either $\|v_n^1 - v_n^2\| \to 0$ as $n \to \infty$ or $\limsup_{n\to\infty} \|v_n^1 - v_n^2\| \geq \rho(d) > 0$, where $\rho(d)$ depends on d but not on the particular choice of Palais-Smale sequences.*

This property is related to the notion of a *discrete Palais-Smale attractor* as considered by Bartsch and Ding [3] for the functional Φ (under somewhat different assumptions). However, it is not clear that a discrete Palais-Smale attractor for Φ gives rise to a corresponding one for Ψ since $\hat{m}$ might not be Lipschitz continuous. Moreover, the discreteness property stated above is somewhat simpler and directly yields nice properties of the corresponding pseudo-gradient flow, see Lemma 2.15 below.

Proof We put $u_n^1 := \hat{m}(v_n^1)$ and $u_n^2 := \hat{m}(v_n^2)$ for $n \in \mathbb{N}$. Then both sequences $(u_n^1)_n, (u_n)_n^2 \subset \Phi^d$ are bounded Palais-Smale sequences for Φ. We distinguish two cases.

Case 1: $\|u_n^1 - u_n^2\|_p \to 0$ as $n \to \infty$. By a result of Troestler [16], see also [5, Proposition 2.3], the orthogonal projection of E on E^+ is continuous in the L^p-norm, so $\|(u_n^1 - u_n^2)^+\|_p \to 0$. Using (S_2), (S_3),

$$\begin{aligned}\|(u_n^1 - u_n^2)^+\|^2 &= \Phi'(u_n^1)(u_n^1 - u_n^2)^+ - \Phi'(u_n^2)(u_n^1 - u_n^2)^+ \\ &\quad + \int_{\mathbb{R}^N} [f(x, u_n^1) - f(x, u_n^2)](u_n^1 - u_n^2)^+ \, dx \\ &\le \varepsilon\|(u_n^1 - u_n^2)^+\| + \int_{\mathbb{R}^N} \left(\varepsilon(|u_n^1| + |u_n^2|) + C_\varepsilon(|u_n^1|^{p-1} + |u_n^2|^{p-1})\right) |(u_n^1 - u_n^2)^+| \, dx \\ &\le (1 + C_0)\varepsilon\|(u_n^1 - u_n^2)^+\| + D_\varepsilon\|(u_n^1 - u_n^2)^+\|_p\end{aligned}$$

for all $n \ge n_\varepsilon$, where $\varepsilon > 0$ is arbitrary, C_ε, D_ε, n_ε do and C_0 does not depend on the choice of ε. Hence $\limsup_{n\to\infty} \|(u_n^1 - u_n^2)^+\|^2 \le \limsup_{n\to\infty}(1 + C_0)\varepsilon\|(u_n^1 - u_n^2)^+\|$ for each $\varepsilon > 0$ and therefore $\|(u_n^1 - u_n^2)^+\| \to 0$. Similarly, $\|(u_n^1 - u_n^2)^-\| \to 0$, so $\|u_n^1 - u_n^2\| \to 0$ as $n \to \infty$ and Lemma 2.11 yields $\|v_n^1 - v_n^2\| = \|\check{m}(u_n^1) - \check{m}(u_n^2)\| \to 0$ as $n \to \infty$.

Case 2: $\|u_n^1 - u_n^2\|_p \not\to 0$ as $n \to \infty$. Then – again by [18, Lemma 1.21] – there exists $\varepsilon > 0$ and $y_n \in \mathbb{R}^N$ such that, after passing to a subsequence,

$$\int_{B_1(y_n)} (u_n^1 - u_n^2)^2 \, dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} (u_n^1 - u_n^2)^2 \, dx \ge \varepsilon \qquad \text{for all } n. \tag{2.11}$$

Using that $\hat{m}$, $\check{m}$ and $\nabla\Phi$, $\nabla\Psi$ are all equivariant with respect to translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. We may pass to a subsequence such that

$$u_n^1 \rightharpoonup u^1 \in E, \qquad u_n^2 \rightharpoonup u^2 \in E, \qquad \text{where } u^1 \neq u^2 \text{ by (2.11) and } \Phi'(u^1) = \Phi'(u^2) = 0,$$

and

$$\|(u_n^1)^+\| \to \alpha^1, \qquad \|(u_n^2)^+\| \to \alpha^2,$$

where $\sqrt{2c} \le \alpha_i \le \nu(d)$ for $i = 1, 2$ by Lemma 2.4(b). We first consider the case where $u^1 \neq 0$ and $u^2 \neq 0$, so that $u_1, u_2 \in \mathcal{M}$ and

$$v^1 := \check{m}(u^1) \in K, \qquad v^2 := \check{m}(u^2) \in K, \qquad v^1 \neq v^2.$$

We then have

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| = \liminf_{n\to\infty} \left\| \frac{(u_n^1)^+}{\|(u_n^1)^+\|} - \frac{(u_n^2)^+}{\|(u_n^2)^+\|} \right\| \ge \left\| \frac{(u^1)^+}{\alpha^1} - \frac{(u^2)^+}{\alpha^2} \right\| = \|\beta_1 v_1 - \beta_2 v_2\|,$$

where $\beta_1 := \frac{\|(u^1)^+\|}{\alpha_1} \geq \frac{\sqrt{2c}}{\nu(d)}$ and $\beta_2 := \frac{\|(u^2)^+\|}{\alpha_2} \geq \frac{\sqrt{2c}}{\nu(d)}$. Since $\|v^1\| = \|v^2\| = 1$, an elementary geometric argument and the inequalities above imply that

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| \geq \|\beta_1 v^1 - \beta_2 v^2\| \geq \min\{\beta_1, \beta_2\}\|v_1 - v_2\| \geq \frac{\sqrt{2c}\kappa}{\nu(d)}.$$

It remains to consider the case where either $u^1 = 0$ or $u^2 = 0$. If $u^2 = 0$, then $u^1 \neq 0$, and

$$\liminf_{n\to\infty} \|v_n^1 - v_n^2\| = \liminf_{n\to\infty} \left\| \frac{(u_n^1)^+}{\|(u_n^1)^+\|} - \frac{(u_n^2)^+}{\|(u_n^2)^+\|} \right\| \geq \frac{\|(u^1)^+\|}{\alpha^1} \geq \frac{\sqrt{2c}}{\nu(d)}.$$

The case $u^1 = 0$ is treated similarly. The proof is finished. $\square$

It is known (see e.g. [15, Lemma II.3.9]) that Ψ admits a pseudo-gradient vector field, i.e., there exists a Lipschitz continuous map $H : S^+\backslash K \to TS^+$ with $H(w) \in T_w S^+$ for all $w \in S^+\backslash K$ and

$$\left.\begin{aligned} &\|H(w)\| < 2\|\nabla\Psi(w)\| \\ &\langle H(w), \nabla\Psi(w)\rangle > \frac{1}{2}\|\nabla\Psi(w)\|^2 \end{aligned}\right\} \quad \text{for all } w \in S^+ \setminus K. \tag{2.12}$$

Let $\eta : \mathcal{G} \to S^+ \setminus K$ be the corresponding (Ψ-decreasing) flow defined by

$$\begin{cases} \frac{d}{dt}\,\eta(t,w) &= -H(\eta(t,w)), \\ \quad \eta(0,w) &= w, \end{cases} \tag{2.13}$$

where

$$\mathcal{G} := \big\{(t,w)\ :\ w \in S^+ \setminus K,\ \ T^-(w) < t < T^+(w)\big\} \quad \subset \quad \mathbb{R} \times (S^+ \setminus K)$$

and $T^-(w) < 0$, $T^+(w) > 0$ are the maximal existence times of the trajectory $t \mapsto \eta(t,w)$ in negative and positive direction. Note that Ψ is strictly decreasing along trajectories of η.

For deformation type arguments, the following lemma is crucial.

Lemma 2.15 *For every $w \in S^+$ the limit $\lim\limits_{t\to T^+(w)} \eta(t,w)$ exists and is a critical point of Ψ.*

Proof Fix $w \in S^+$ and put $d := \Psi(w)$.
Case 1: $T^+(w) < \infty$. For $0 \leq s < t < T^+(w)$ we have by (2.12) and (2.13)

$$\begin{aligned} \|\eta(t,w) - \eta(s,w)\| &\leq \int_s^t \|H(\eta(\tau,w))\|\,d\tau \leq 2\sqrt{2}\int_s^t \sqrt{\langle H(\eta(\tau,w)), \nabla\Psi(\eta(\tau,w))\rangle}\,d\tau \\ &\leq 2\sqrt{2(t-s)}\Big(\int_s^t \langle H(\eta(\tau,w)), \nabla\Psi(\eta(\tau,w))\rangle\,d\tau\Big)^{\frac{1}{2}} \\ &= 2\sqrt{2(t-s)}[\Psi(\eta(s,w)) - \Psi(\eta(t,w))]^{\frac{1}{2}} \leq 2\sqrt{2(t-s)}[\Psi(w) - c]^{\frac{1}{2}}. \end{aligned}$$

Since $T^+(w) < \infty$, this implies that $\lim_{t\to T^+(w)} \eta(t,w)$ exists and then it must be a critical point of Ψ (otherwise the trajectory $t \mapsto \eta(t,w)$ could be continued beyond $T^+(w)$).
Case 2: $T^+(w) = \infty$. To prove that $\lim_{t\to\infty} \eta(t,w)$ exists, it clearly suffices to establish the following property:

(2.14) for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ with $\|\eta(t_\varepsilon, w) - \eta(t,w)\| < \varepsilon$ for $t \ge t_\varepsilon$.

We suppose by contradiction that (2.14) is false. Then there exists $0 < \varepsilon < \frac{1}{2}\rho(d)$ – where $\rho(d)$ is given in Lemma 2.14 – and a sequence $(t_n)_n \subset [0,\infty)$ with $t_n \to \infty$ and $\|\eta(t_n,w) - \eta(t_{n+1},w)\| = \varepsilon$ for every n. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that $\|\eta(t_n,w) - \eta(t_n^1,w)\| = \frac{\varepsilon}{3}$ and let $\kappa_n := \min_{s\in[t_n,t_n^1]} \|\nabla\Psi(\eta(s,w))\|$. Then

$$
\begin{aligned}
\frac{\varepsilon}{3} &= \|\eta(t_n^1,w) - \eta(t_n,w)\| \le \int_{t_n}^{t_n^1} \|H(\eta(s,w))\|\,ds \le 2\int_{t_n}^{t_n^1} \|\nabla\Psi(\eta(s,w))\|\,ds \\
&\le \frac{2}{\kappa_n}\int_{t_n}^{t_n^1} \|\nabla\Psi(\eta(s,w))\|^2\,ds \le \frac{4}{\kappa_n}\int_{t_n}^{t_n^1} \langle H(\eta(s,w)), \nabla\Psi(\eta(s,w))\rangle\,ds \\
&= \frac{4}{\kappa_n}\left(\Psi(\eta(t_n,w)) - \Psi(\eta(t_n^1,w))\right).
\end{aligned}
$$

Since $\Psi(\eta(t_n,w)) - \Psi(\eta(t_n^1,w)) \to 0$ as $n \to \infty$, $\kappa_n \to 0$ and there exist $s_n^1 \in [t_n, t_n^1]$ such that $\nabla\Psi(w_n^1) \to 0$, where $w_n^1 := \eta(s_n^1,w)$. Similarly we find a largest $t_n^2 \in (t_n^1, t_{n+1})$ for which $\|\eta(t_{n+1},w) - \eta(t_n^2,w)\| = \frac{\varepsilon}{3}$ and then $w_n^2 := \eta(s_n^2,w)$ satisfying $\nabla\Psi(w_n^2) \to 0$. As $\|w_n^1 - \eta(t_n,w)\| \le \frac{\varepsilon}{3}$ and $\|w_n^2 - \eta(t_{n+1},w)\| \le \frac{\varepsilon}{3}$, $(w_n^1)_n$, $(w_n^2)_n$ are two Palais-Smale sequences such that

$$\frac{\varepsilon}{3} \le \|w_n^1 - w_n^2\| \le 2\varepsilon < \rho(d).$$

This however contradicts Lemma 2.14, hence (2.14) is true. So $\lim_{t\to\infty}\eta(t,w)$ exists, and obviously it must be a critical point of Ψ. □

In the following, for a subset $P \subset S^+$ and $\delta > 0$, we put

$$U_\delta(P) := \{w \in S^+ \;:\; \mathrm{dist}(w,P) < \delta\}.$$

Lemma 2.16 *Let $d \ge c$. Then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*
(a) $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$ *and*
(b) $\lim\limits_{t\to T^+(w)} \Psi(\eta(t,w)) < d - \varepsilon$ *for* $w \in \Psi^{d+\varepsilon} \setminus U_\delta(K_d)$.

Proof In view of (2.10), (a) is obviously satisfied for $\varepsilon > 0$ small enough. Without loss of generality, we may assume $U_\delta(K_d) \subset \Phi^{d+1}$ and $\delta < \rho(d+1)$. In order to find ε such that (b) holds, we put

$$\tau := \inf\{\|\nabla\Psi(w)\| \;:\; w \in U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)\}$$

and claim that $\tau > 0$. Indeed, suppose by contradiction that there exists a sequence $(v_n^1)_n \subset U_\delta(K_d) \backslash U_{\frac{\delta}{2}}(K_d)$ such that $\nabla\Psi(v_n^1) \to 0$. Passing to a subsequence, using the finiteness condition (2.10) and the $\mathbb{Z}^N$-invariance of Ψ, we may assume $v_n^1 \in U_\delta(w_0) \setminus U_{\frac{\delta}{2}}(w_0)$ for some $w_0 \in K_d$. Let $v_n^2 \to w_0$. Then $\nabla\Psi(v_n^2) \to 0$ and

$$\frac{\delta}{2} \le \limsup_{n\to\infty} \|v_n^1 - v_n^2\| \le \delta < \rho(d+1),$$

contrary to Lemma 2.14. Hence τ is positive. Let $A := \sup\{\|\nabla\Psi(w)\| \,:\, w \in U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)\}$ and choose $\varepsilon < \frac{\delta\tau^2}{8A}$ such that (a) holds. By Lemma 2.15 and (a), the only way (b) can fail is that $\eta(t,w) \to \widetilde{w} \in K_d$ as $t \to T^+(w)$ for some $w \in \Psi^{d+\varepsilon} \setminus U_\delta(K_d)$. In this case we let

$$t_1 := \sup\{t \in [0, T^+(w)) \,:\, \eta(t,w) \notin U_\delta(\widetilde{w})\} \quad \text{and} \quad t_2 := \inf\{t \in (t_1, T^+(w)) \,:\, \eta(t,w) \in U_{\frac{\delta}{2}}(\widetilde{w})\}.$$

Then

$$\frac{\delta}{2} = \|\eta(t_1,w) - \eta(t_2,w)\| \le \int_{t_1}^{t_2} \|H(\eta(s,w))\|\,ds \le 2\int_{t_1}^{t_2} \|\nabla\Psi(\eta(s,w))\|\,ds \le 2A(t_2 - t_1),$$

and

$$\begin{aligned}\Psi(\eta(t_2,w)) - \Psi(\eta(t_1,w)) &= -\int_{t_1}^{t_2} \langle \nabla\Psi(\eta(s,w)), H(\eta(s,w))\rangle\,ds \\ &\le -\frac{1}{2}\int_{t_1}^{t_2} \|\nabla\Psi(\eta(s,w))\|^2\,ds \le -\frac{1}{2}\tau^2(t_2-t_1) \le -\frac{\delta\tau^2}{8A}.\end{aligned}$$

Hence $\Psi(\eta(t_2,w)) \le d + \varepsilon - \frac{\delta\tau^2}{8A} < d$ and therefore $\eta(t,w) \not\to \widetilde{w}$, contrary to our assumption. □

Proof of Theorem 1.2 (completed). For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $A \subset S^+$ (i.e., $A = -A = \overline{A}$) with $\gamma(A) \ge j$, where γ denotes the usual Krasnoselski genus (see, e.g., [14, 15]). Moreover, we consider the nondecreasing sequence of Lusternik-Schnirelman values for Ψ defined by

$$c_k := \inf\{d \in \mathbb{R} \,:\, \gamma(\Psi^d) \ge k\} \qquad (k \in \mathbb{N}).$$

We claim:

$$K_{c_k} \ne \varnothing \text{ and } c_k < c_{k+1} \text{ for all } k \in \mathbb{N}. \tag{2.15}$$

To prove this, let $k \in \mathbb{N}$ and put $d = c_k$. By Lemma 2.13, $\gamma(K_d) = 0$ or 1 (depending on whether K_d is empty or not). By the continuity property of the genus, there exists $\delta > 0$ such that $\gamma(\overline{U}) = \gamma(K_d)$, where $U := U_\delta(K_d)$ and $\delta < \frac{\kappa}{2}$. Choose $\varepsilon = \varepsilon(\delta) > 0$ such that the properties of Lemma 2.16 hold. Then for every $w \in \Psi^{d+\varepsilon} \setminus U$ there exists $t \in [0, T^+(w))$ with $\Phi(\eta(t,w)) < d - \varepsilon$. Hence we may define the *entrance time map*

$$e : \Psi^{d+\varepsilon} \setminus U \to [0,\infty), \qquad e(w) := \inf\{t \in [0, T^+(w)) \,:\, \Psi(\eta(t,w)) \le d - \varepsilon\}, \tag{2.16}$$

which satisfies $e(w) < T^+(w)$ for every $w \in \Psi^{d+\varepsilon} \setminus U$. Since $d-\varepsilon$ is not a critical value of Ψ by Lemma 2.16, it is easy to see that e is a continuous (and even) map. Consequently, the map

$$h : \Psi^{d+\varepsilon} \setminus U \to \Psi^{d-\varepsilon}, \qquad h(w) = \eta(e(w), w)$$

is odd and continuous. Hence $\gamma(\Psi^{d+\varepsilon} \setminus U) \le \gamma(\Psi^{d-\varepsilon}) \le k-1$ and therefore

$$\gamma(\Psi^{d+\varepsilon}) \le \gamma(\overline{U}) + k - 1 = \gamma(K_d) + k - 1.$$

The definition of $d = c_k$ and of c_{k+1} implies that $\gamma(K_d) \ge 1$ if $c_{k+1} > c_k$ and $\gamma(K_d) > 1$ if $c_{k+1} = c_k$. Since $\gamma(\mathcal{F}) = \gamma(K_d) \le 1$, (2.15) follows.

It follows now from (2.15) that there is an infinite sequence $(\pm w_k)$ of pairs of geometrically distinct critical points of Ψ with $\Psi(w_k) = c_k$, contrary to (2.10). The proof is finished. $\square$

3 A semilinear problem on a bounded domain

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the boundary value problem

$$\begin{cases} -\Delta u - \lambda u = f(x,u) \\ u \in H_0^1(\Omega), \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\begin{cases} |f(x,u)| \le c(1+|u|^{p-1}) \text{ for some } c > 0 \text{ and } p \in (2, 2^*); \\ f(x,u) = o(u) \text{ uniformly in } x \text{ as } |u| \to 0; \\ F(x,u)/u^2 \to \infty \text{ uniformly in } x \text{ as } |u| \to \infty; \\ u \mapsto f(x,u)/|u| \text{ is strictly increasing on } (-\infty, 0) \text{ and on } (0,\infty). \end{cases} \tag{3.2}$$

The corresponding functional is

$$\Phi(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \int_\Omega F(x,u)\, dx,$$

it is of class C^1 in $E := H_0^1(\Omega)$ and critical points of Φ correspond to solutions of (3.1). Setting $E = E^+ \oplus E^0 \oplus E^-$ and $u = u^+ + u^0 + u^-$, where E^+, E^0, E^- correspond to the positive, zero and negative part of the spectrum of $-\Delta - \lambda$ in E, we can define an equivalent inner product $\langle . , . \rangle$ in such a way that

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega F(x,u)\, dx.$$

Let

$$\mathcal{M} := \left\{ u \in E \setminus (E^0 \oplus E^-) \; : \; \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^0 \oplus E^- \right\}.$$

Theorem 3.1 *Suppose assumptions* (3.2) *are satisfied and let* $c := \inf_{u \in \mathcal{M}} \Phi(u)$. *Then c is attained, $c > 0$ and if $\Phi(u_0) = c$, then u_0 is a solution of (3.1).*

Proof If $E^0 = \{0\}$ (i.e., $0 \notin \sigma(-\Delta - \lambda)$), the argument is similar to that in Theorem 1.1 but simpler because E^- is finite-dimensional and the embedding $E \to L^q(\Omega)$ is compact for $1 \le q < 2^*$. Note in particular that in Proposition 2.7 this compactness replaces translation invariance by elements of $\mathbb{Z}^N$ and in the final step $(u_n) \subset \mathcal{M}$ is a Palais-Smale sequence which is bounded by Proposition 2.7, hence $u_n \to u$ after passing to a subsequence, by compactness again.

If $\dim E^0 > 0$, the same is true except that the proofs of Lemma 2.5 and Proposition 2.7 require small modifications. We still have (2.2) but now this implies that $\|v_n^-\|^2 \le s_n^2 \le 1 - \|v_n^0\|^2 - \|v_n^-\|^2$ (where v_n^0 denotes the orthogonal projection of v_n on E^0). If $s_n \to s > 0$ after passing to a subsequence, then (2.3) follows as before. Otherwise $s_n \to 0$, so up to a subsequence, $v_n^- \to 0$ and $v_n^0 \to v^0 \neq 0$. Hence (2.3) follows again. In a similar way, the proof of Proposition 2.7 is adjusted. We leave the details to the reader. □

Theorem 3.2 *Suppose f is odd in u and assumptions* (3.2) *are satisfied. Then (3.1) has infinitely many pairs of solutions $\pm u_k$ such that $|u_k|_\infty \to \infty$.*

Proof The functional Ψ is of class C^1 on S^+ according to Corollary 2.10, it is obviously even and $\Psi'(w) = 0$ implies $\hat{m}(w)$ is a critical point of Φ. We shall show Ψ satisfies the Palais-Smale condition. Suppose $\Psi(w_n)$ is bounded and $\Psi'(w_n) \to 0$. Then $\Phi(\hat{m}(w_n))$ is bounded, hence so is $\hat{m}(w_n)$ by Proposition 2.7. We may assume taking a subsequence that $\hat{m}(w_n)$ is weakly convergent in E and strongly convergent in $L^q(\Omega)$, $1 \le q < 2^*$. Employing Corollary 2.10 again,

$$\Phi'(\hat{m}(w_n)) = \langle \hat{m}(w_n)^+ - \hat{m}(w_n)^-, \cdot\rangle - \int_\Omega f(x, \hat{m}(w_n)) \cdot dx \to 0 \tag{3.3}$$

and we see from (3.3) that $\hat{m}(w_n) \to u$ for some $u \in \mathcal{M}$. Hence $w_n \to \check{m}(u) = u^+/\|u^+\|$ (see (2.7) for the definition of $\check{m}$).

Let

$$c_k := \inf_{\gamma(A) \ge k} \sup_{w \in A} \Psi(w),$$

where the infimum is taken over all closed subsets $A \subset S^+$ with $A = -A$. Since $\inf_{\mathcal{M}} \Psi > 0$, c_k are well defined and positive for all $k \ge 1$. Now standard arguments using the deformation lemma, see e.g. [14, 15, 18] imply that all c_k are critical values and $c_k \to \infty$ (that $c_k \to \infty$ is seen as in [14, Proposition 9.33]). Hence, setting $u_k := \hat{m}(w_k)$, we have

$$c_k = \Psi(w_k) = \Phi(u_k) = \Phi(u_k) - \frac{1}{2}\Phi'(u_k)u_k = \int_\Omega \left(\frac{1}{2} f(x, u_k) - F(x, u_k)\right) dx.$$

By Lemma 2.1, the integrand above is nonnegative, so $c_k \to \infty$ implies $|u_k|_\infty \to \infty$. □

For continuous f, Theorem 3.2 is new even if $E^- = E^0 = \{0\}$ (i.e., $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in E). In this case it extends [12, Theorem 2.3], where f needs to be differentiable and f'_u satisfies (1.4) with $\theta = 1$.

4 Remarks on variants of (1.1)

Here we briefly discuss different assumptions on f and V in the nonlinear Schrödinger equation (1.1).

I. Suppose that f is replaced by $-f$ in (1.1) and f still satisfies (S_2)-(S_5). Then we can consider $-\Phi$ instead of Φ and replace $\mathcal{M}$ by

$$\widetilde{\mathcal{M}} := \left\{ u \in E \setminus E^+ \ : \ \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^+ \right\}.$$

If $\sigma(-\Delta + V) \subset (0, \infty)$, then $E = E^+$, $\widetilde{\mathcal{M}} = \varnothing$ and (1.1) has only the trivial solution $u = 0$. Indeed, since F is strictly convex, so is Φ and $\Phi'(u) = 0$ if and only if $u = 0$. However, it is easy to see applying our arguments to $-\Phi$ that the following holds:

Theorem 4.1 *Suppose the assumptions (S_1)-(S_5) are satisfied, $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \varnothing$ and let $c := \inf_{u \in \widetilde{\mathcal{M}}}(-\Phi(u))$. Then the conclusions of Theorems 1.1 and 1.2 hold for (1.1) with f replaced by $-f$.*

Note that the functional Ψ will now be defined on the unit sphere $S^- \subset E^-$.

II. Suppose (S_2)-(S_5) are satisfied, except that f need not be periodic, and (S_1) is replaced by

(S_1') V is continuous and $V(x) \to \infty$ as $|x| \to \infty$.

Let

$$H^1_V(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\, dx < \infty\}$$

and consider the problem

$$\begin{cases} -\Delta u + V(x)u = f(x, u) \\ u \in H^1_V(\mathbb{R}^N). \end{cases} \tag{4.1}$$

It is well known that the embedding $H^1_V(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is compact for $2 \leq p < 2^*$; therefore $\sigma(-\Delta + V)$ in $L^2(\mathbb{R}^N)$ consists of eigenvalues $\lambda_n \to \infty$. It follows that $H^1_V(\mathbb{R}^N) = E^+ \oplus E^0 \oplus E^-$, where E^0, E^- are finite-dimensional, and it is easy to see that the conclusions of Theorems 3.1 and 3.2 are valid for (4.1), with the same proofs. Different conditions (including (S_1') as a special case) under which the above embedding is compact have been discussed in [4]. Also under these conditions there exists a ground state solution and, for odd f, infinitely many solutions to (4.1). The details are left to the reader.

Remark 4.2 Consider (3.1) and (4.1) with f replaced by $-f$ and let

$$\mathcal{M} := \left\{ u \in E \setminus (E^+ \oplus E^0) \ : \ \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^+ \oplus E^0 \right\}.$$

Then the functional Ψ corresponding to $-\Phi$ is defined on the unit sphere $S^- \subset E^-$ and $\dim S^- = k-1$, where k is the number of negative eigenvalues of $-\Delta - \lambda$ in $H_0^1(\Omega)$ or $-\Delta + V$ in $H_V^1(\mathbb{R}^N)$ (counted with their multiplicities). If $k \geq 1$, the conclusion of Theorem 3.1 remains valid for (3.1) and (4.1). For odd f, the number of pairs of solutions will be at least k because $\gamma(S^-) = k$. However, in this case our method only provides a somewhat unusual finite-dimensional reduction because existence and multiplicity results under weaker assumptions on f can be obtained by other methods. For (3.1) no growth restriction is necessary and it suffices to assume that $f(x,u) = o(u)$ uniformly in x as $|u| \to 0$ and there is $u > 0$ such that $f(x,u) \geq \lambda u$ for all x. Then a truncation argument together with a minimax principle can be used as in [14, Theorem 9.6]. For (4.1), if one sets

$$\varphi(u) := \min_{w \in E^+ \oplus E^0} \Phi(u^- + w),$$

then the minimizer above is unique and $\varphi \in C^1(E^-, \mathbb{R})$ under appropriate convexity assumptions on F. Again, a suitable minimax principle can be used. We omit the details.

A Appendix

Here we give the proof of Lemma 2.2. We fix $x \in \mathbb{R}^N$, $u, s, v \in \mathbb{R}$ with $s \geq -1$, $w := su + v \neq 0$ and put

$$g := f(x,u)[s(\frac{s}{2} + 1)u + (1+s)v] + F(x,u) - F(x,u+w).$$

We need to show $g < 0$. We first consider the case $u = 0$. Then $w = v \neq 0$ by assumption, and hence $g = -F(x,w) < 0$ by Lemma 2.1. We may therefore assume $u \neq 0$ from now on. We define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(t) = \frac{f(x,t)}{t}$ for $t \neq 0$ and $\tilde{f}(0) = 0$. We have

$$\begin{aligned}
g &= f(x,u)[(\frac{s}{2}+1)w + \frac{s}{2}v] - \int_0^1 f(x, u+tw)w \, dt \\
&= \tilde{f}(u)[uw + \frac{su}{2}(v+w)] - \int_0^1 \tilde{f}(u+tw)(uw + tw^2) \, dt \\
&= \tilde{f}(u)[uw + \frac{1}{2}(w^2 - v^2)] - \int_0^1 \tilde{f}(u+tw)(uw+tw^2) \, dt \\
&= -\frac{1}{2}\tilde{f}(u)v^2 + \int_0^1 h(t) \, dt, \qquad \text{(A.1)}
\end{aligned}$$

where $h(t) := [\tilde{f}(u) - \tilde{f}(u+tw)](uw+tw^2)$ for $0 \leq t \leq 1$. We now distinguish different cases.

Case 1: $uw > 0$. Then $uw + tw^2 > 0$ and $\tilde{f}(u) - \tilde{f}(u+tw) < 0$ for $t > 0$ by (S_5), so that $h(t) < 0$ for $t > 0$ and therefore $g < -\frac{1}{2}\tilde{f}(u)v^2 \leq 0$.

Case 2: $uw < 0$ and $|w| \leq |u|$. Then, for $t \in (0,1)$, $uw + tw^2 < 0$ and by (S_5), $\tilde{f}(u) - \tilde{f}(u+tw) > 0$. So $h(t) < 0$ as $t \in (0,1)$ and again, $g < -\frac{1}{2}\tilde{f}(u)v^2 \leq 0$.

Case 3: $uw < 0$ and $|w| > |u|$. For $0 \leq t \leq -\frac{u}{w}$ we have $\tilde{f}(u) - \tilde{f}(u+tw) \geq 0$, $uw + tw^2 \leq 0$

and therefore $h(t) \leq 0$, hence

$$\int_0^1 h(t)\,dt \leq \int_{-\frac{u}{w}}^1 h(t)\,dt \tag{A.2}$$

For $-\frac{u}{w} < t \leq 1$ we have $uw + tw^2 > 0$ and $\tilde{f}(u + tw) > 0$, so that

$$\begin{aligned}\int_{-\frac{u}{w}}^1 h(t)\,dt &= \int_{-\frac{u}{w}}^1 [\tilde{f}(u) - \tilde{f}(u+tw)](uw+tw^2)\,dt < \int_{-\frac{u}{w}}^1 \tilde{f}(u)(uw+tw^2)\,dt \\ &= \frac{1}{2}\tilde{f}(u)((1+s)u+v)^2. \end{aligned}\tag{A.3}$$

Next we claim that

$$((1+s)u+v)^2 \leq v^2. \tag{A.4}$$

To see this, we distinguish the cases $w > -u > 0$ and $w < -u < 0$. If $su + v = w > -u > 0$, we have $v \geq (1+s)u + v > 0$ and thus (A.4) holds. If $su + v = w < -u < 0$, we have $v \leq (1+s)u + v < 0$ and therefore (A.4) holds again. Combining (A.1), (A.2), (A.3) and (A.4), we obtain

$$g = -\frac{1}{2}\tilde{f}(u)v^2 + \int_0^1 h(t)\,dt < 0,$$

as claimed. This completes the proof of Lemma 2.2.

References

[1] S. Alama and Y. Y. Li, *On "multibump" bound states for certain semilinear elliptic equations*, Indiana Univ. Math. J. 41 (1992), 983–1026.

[2] S. Angenent, *The shadowing lemma for elliptic PDE,* Dynamics of infinite-dimensional systems (Lisbon, 1986), volume 37 of NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 7–22. Springer, Berlin, 1987.

[3] T. Bartsch and Y.H. Ding, *On a nonlinear Schrödinger equation with periodic potential*, Math. Ann. 313 (1999), 15–37.

[4] T. Bartsch, A. Pankov and Z.Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Comm. Contemp. Math. 3 (2001), 549–569.

[5] J. Chabrowski and A. Szulkin, *On a semilinear Schrödinger equation with critical Sobolev exponent*, Proc. AMS 130 (2001), 85–93.

[6] V. Coti Zelati and P.H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on R^n*, Comm. Pure Appl. Math. 45 (1992), 1217–1269.

[7] J. Dolbeault, M.J. Esteban, E. Séré, *On the eigenvalues of operators with gaps, application to Dirac operators*, J. Funct. Anal. 174 (2000), 208–226.

[8] M. Griesemer and H. Siedentop, *A minimax principle for the eigenvalues in spectral gaps*, J. London Math. Soc. 60 (1999), 490–500.

[9] W. Kryszewski and A. Szulkin, *Generalized linking teorem with an application to a semilinear Schrödinger equation*, Adv. Diff. Eq. 3 (1998), 441–472.

[10] Y. Li and J. Zhou, *A minimax method for finding multiple critical points and its applications to semilinear elliptic PDEs*, SIAM Sci. Comp. 23 (2001), 840-865.

[11] Y.Q. Li, Z.Q. Wang and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. IHP - Analyse Non Linéaire 23 (2006), 829–837.

[12] Z.L. Liu, Z.Q. Wang, *On the Ambrosetti-Rabinowitz superlinear condition*, Adv. Nonl. Stud. 4 (2004), 563–574.

[13] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals*, Milan J. Math. 73 (2005), 259–287.

[14] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS 65, Amer. Math. Soc., Providence, R.I., 1986.

[15] M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, second edition, 1996.

[16] C. Troestler, *Bifurcation into spectral gaps for a noncompat semilinear Schrödinger equation with nonconvex potential*, Preprint.

[17] C. Troestler, M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*, Comm. Partial Differential Equations 21 (1996) 1431-1449.

[18] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.