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This thesis consists of two papers:

Paper A. *Chord properties of digital straight line segments*

This paper treats digital straight line segments in two different cases, in the 8-connected plane and in the Khalimsky plane. We investigate them using a new classification, dividing them into a union of horizontal and diagonal segments. Then we study necessary and sufficient conditions for straightness in both cases, using vertical distances for certain points. We also establish necessary and sufficient conditions in the 8-connected plane as well as in the Khalimsky plane by transforming their chain codes. Using this technique we can transform Khalimsky lines to the 8-connected case.

Paper B. *The number of Khalimsky-continuous functions on intervals*

This paper deals with Khalimsky-continuous functions. We consider these functions when they have two, three or four points in their codomain. In the case of two points in the codomain, we see a new example of the classical Fibonacci sequence. In the study of functions with three and four points in their codomain, we find some new sequences, the asymptotic behavior of which we investigate. Finally, we consider Khalimsky-continuous functions with one fixed endpoint. In this case, we get a sequence which has the same recursion relation as the Pell numbers but different initial values. We also obtain a new example of the Delannoy numbers.

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Chord properties of digital straight line segments

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Abstract

We exhibit the structure of digital straight line segments in the 8-connected plane and in the Khalimsky plane by considering vertical distances and unions of two segments.

1. Introduction

In the field of digital geometry one of the themes which has been studied extensively is digital straight lines. Maloň and Freeman (1961) and Freeman (1970) introduced the chain code as a technique for representing 8-connected arcs and lines. The most important problem related to straightness is how to recognize the sets of pixels or codes representing a digital straight line. Rosenfeld (1974) characterized straightness by the chord property and found two fundamental properties of run lengths in a digital line. He stated that the digitized line can only contain runs of two different lengths and these run lengths must be consecutive integers. Hung and Kasvand (1984) gave a necessary and sufficient condition for a digital arc to have the chord property. This condition made the chord property easier to check. Kim (1982) characterized it by convexity, and showed that a digital straight line segment is a digital arc which is digitally convex.

In the present paper we deal with grid points in the 8-connected plane as well as the plane equipped with the Khalimsky topology. Digital straight line segments are special cases of digital arcs. We shall investigate Rosenfeld's digitization and his chord property in section 1.1. Melin (2005) introduced a modified version of the chord property of Rosenfeld. He established necessary and sufficient conditions for straightness in the Khalimsky plane. We mention his results in section 1.3 and put them into another framework using instead vertical distances in section 2.1. Bruckstein (1991) presented some transformations on sequences composed of two symbols, 0 and 1. These transformations can be described by matrices which form a well-known group called $GL(2,\mathbb{Z})$. The main results in his paper is that the image of the chain code under one of these transformations represents a digital straight line segment if and only if the original sequence is the chain code of a digital straight line segment. Similar transformations have been used by Jamet and Toutant (2006:231) in the case of three dimensions.

In section 2 we shall investigate sets of 8-connected and Khalimsky-connected points by dividing them into unions of horizontal and diagonal segments. Then we shall present necessary and sufficient conditions in both cases using vertical distances for certain points. We shall also establish necessary and sufficient conditions in the 8-connected plane as well as in the Khalimsky plane by transforming the sequences of their chain codes. Using this technique we transform Khalimsky lines to the 8-connected case.

1.1. Rosenfeld's digitization of straight lines

We present here Rosenfeld's digitization of straight lines in the digital plane $\mathbb{Z}^2$. First we define the set

$$C(0) = \{x; x_1 = 0 \text{ and } -1/2 < x_2 \leqslant 1/2\} \cup \{x; x_2 = 0 \text{ and } -1/2 < x_1 \leqslant 1/2\}.$$

For each $p \in \mathbb{Z}^2$, let $C(p) = C(0) + p$, which we shall call the *cross* with center p. Now the Rosenfeld digitization in $\mathbb{R}^2$ is:

$$D_R\colon \mathscr{P}(\mathbb{R}^2) \to \mathscr{P}(\mathbb{Z}^2),\ D_R(A) = \{p \in \mathbb{Z}^2; C(p) \cap A \neq \emptyset\}. \tag{1.1}$$

This digitization is based on the one-dimensional digitization

$$\mathbb{R} \ni x \mapsto \lceil x - 1/2 \rceil \in \mathbb{Z}.$$

The union of all crosses $C(p)$ for $p \in \mathbb{Z}^2$ is equal to the set of all grid lines $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$, so that every straight line has a nonempty digitization. Note that the family of all crosses is disjoint, which implies that the digitization of a point is either empty or a singleton set. In the real plane, the concept of a straight line is well-known: it is a set of the form $\{(1-t)a + tb; t \in \mathbb{R}\}$, where a and b are two distinct points in the plane. A straight line segment is a connected subset of a straight line (perhaps the whole line).

We shall consider in particular closed segments of finite length and we write them as $\{(1-t)a + tb; 0 \leqslant t \leqslant 1\}$, where a and b are the endpoints. We shall denote this segment by $[a, b]$. Like Rosenfeld, we will consider lines and straight line segments with slope between 0 and 45° in the 8-connected case and in the Khalimsky plane.

We shall say that D is a *digital straight line segment*, and write $D \in DSLS_8$, if and only if there exists a real line segment the Rosenfeld digitization of which is equal to D.

Rosenfeld (1974) introduced the chord property to characterize digital straight line segments in $\mathbb{Z}^2$:

Definition 1.1. A subset $D \subseteq \mathbb{R}^2$ is said to have the *chord property* if for all points $p, q \in D$ the segment $[p, q]$ is contained in $D + B^{\infty}_{<}(0, 1)$, the dilation of D by the open unit ball for the l^{∞} metric.

Rosenfeld's digitization of a subset in the plane $\mathbb{Z}^2$ is 8-connected, but if we consider it in the Khalimsky plane, it is not necessarily connected for that topology. Also, we do not have the chord property with respect to the l^∞ distance for certain Khalimsky-connected sets which are digitizations of straight line segments. Melin (2005) solved these problems by suggesting another digitization and modified Rosenfeld's chord property. To explain this, we shall start with the definition of the Khalimsky plane and then continue with Melin's digitization.

1.2. The Khalimsky topology

There are several different ways to introduce the Khalimsky topology on the integer line. We present the Khalimsky topology by a topological basis. For every even integer m, the set $\{m-1, m, m+1\}$ is open, and for every odd integer n the singleton $\{n\}$ is open. A basis is given by

$$\big\{\{2n+1\}, \{2n-1, 2n, 2n+1\}; n \in \mathbb{Z}\big\}.$$

It follows that even points are closed.

A digital interval $[a,b]_\mathbb{Z} = [a,b] \cap \mathbb{Z}$ with the subspace topology is called a *Khalimsky interval*, and a homeomorphic image of a Khalimsky interval into a topological space is called a *Khalimsky arc*.

On the digital plane $\mathbb{Z}^2$, the Khalimsky topology is given by the product topology. A point with both coordinates odd is open. If both coordinates are even, the point is closed. These types of points are called *pure*. Points with one even and one odd coordinate are neither open nor closed; these are called *mixed*. Note that the mixed points are only connected to their 4-neighbors, whereas the pure points are connected to all eight neighbors. More information on the Khalimsky plane and the Khalimsky topology can be found in Kiselman (2004).

1.3. Continuous Khalimsky digitization

The Rosenfeld digitization in $\mathbb{R}^2$ does not work well when $\mathbb{Z}^2$ is equipped with the Khalimsky topology. This means that the Rosenfeld digitization of a straight line segment is not in general connected for the Khalimsky topology. Melin (2005) introduced a Khalimsky-continuous digitization. This digitization gives us Khalimsky-connected digital straight line segments.

Here we recall his definition and related results. Let

$$D(0) = \{(t,t) \in \mathbb{R}^2; -1/2 < t \leqslant 1/2\} \cup \{(t,-t) \in \mathbb{R}^2; -1/2 < t \leqslant 1/2\}.$$

For each pure point $p \in \mathbb{Z}^2$, define $D(p) = D(0) + p$. Note that $D(p)$ is a cross, rotated 45°, with center at p, and that $D(p)$ is contained in the Voronoi cell $\{x \in \mathbb{R}^2; \|x-p\|_\infty \leqslant 1/2\}$. This means that a digitization with $D(p)$ as a cross with nucleus p is a Voronoi digitization. We define the *pure digitization* $D_P(A)$ of a subset A of $\mathbb{R}^2$ as

$$D_P(A) = \{p \in \mathbb{Z}^2; p \text{ is pure and } D(p) \cap A \neq \emptyset\}.$$

This digitization is the basis for the continuous digitization. The continuous digitization $D(L)$ of L is defined as follows: If L is horizontal or vertical $D(L) = D_R(L)$, the Rosenfeld digitization defined in (1.1). Otherwise define $D_M(L)$ as

$$D_M(L) = \{p \in \mathbb{Z}^2; (p_1 \pm 1, p_2) \in D_P(L)\} \cup \{p \in \mathbb{Z}^2; (p_1, p_2 \pm 1) \in D_P(L)\}$$

and let $D(L) = D_P(L) \cup D_M(L)$. In this digitization we add mixed points (p_1, p_2) if the two points $(p_1 \pm 1, p_2)$ or the two points $(p_1, p_2 \pm 1)$ belong to the pure digitization. Melin (2005) characterized digital straight line segments in the 8-connected and the Khalimsky-connected cases by using a function which he called chord measure.

Definition 1.2. Let $A \in \mathscr{P}_{\text{finite}}(\mathbb{Z}^2)$ be a finite set. Then the *chord measure of* A, denoted by $\xi(A)$, is defined by:

$$\xi(A) = \max_{p,q \in A} H(A, p, q),$$

where $H(A, p, q)$ is the distance from the line segment $[p, q]$ to A, which is defined by

$$H\colon \mathscr{P}_{\text{finite}}(\mathbb{Z}^2) \times \mathbb{Z}^2 \times \mathbb{Z}^2 \to [0, +\infty], \quad H(A, p, q) = \sup_{x \in [p,q]} \min_{m \in A} d(m, x).$$

The distance function H is related to the Hausdorff distance between A and $[p, q]$ as two subsets of the metric space $(\mathbb{Z}^2, d)$.

Definition 1.3. Let $A \in \mathscr{P}_{\text{finite}}(\mathbb{Z}^2)$. We say that A has the *chord property for the metric* d if $\xi(A) < 1$.

As to the Rosenfeld digitization, Melin (2005) showed that a continuous Khalimsky digitization satisfies the chord property for a certain metric and, conversely, a Khalimsky arc satisfying this chord property is the digitization of a straight line segment. He considered a special metric. Let δ^∞ be the metric on $\mathbb{R}^2$ defined by

$$\delta^\infty(x, y) = \max\left(\tfrac{1}{2}|x_1 - y_1|, |x_2 - y_2|\right);$$

it is the l^∞-metric rescaled in the first coordinate. For each positive α, we may define a metric $\delta^\infty_\alpha(x, y) = \max(\alpha|x_1 - y_1|, |x_2 - y_2|)$, but Melin (2005) showed by examples that the choice $\alpha = \frac{1}{2}$ is suitable.

We shall call D a *digital straight line segment* in the Khalimsky plane, and write $D \in DSLS_{\text{Kh}}$ if and only if there exists a real straight line segment whose Khalimsky digitization is equal to D. Melin (2005) proved two theorems that characterize $DSLS_{\text{Kh}}$.

Theorem 1.4. (Melin 2005: Theorem 6.3) *The continuous Khalimsky digitization of a straight line segment is a Khalimsky arc (possibly empty) having the chord property for the δ^∞-metric (when the slope is between $0°$ and $45°$) or the metric $\check{\delta}(x, y) = \delta((x_2, x_1), (y_2, y_1))$ (for lines with slope between $45°$ and $90°$).*

Theorem 1.5. (Melin 2005: Theorem 6.4) *Suppose that a Khalimsky arc*

$$D = \{(x, f(x)); x \in I\} \subseteq \mathbb{Z}^2$$

is the graph of a monotone, continuous function f, and that D has pure endpoints. If D has the chord property for the δ^∞-metric, then D is the Khalimsky-continuous digitization of a straight line segment.

Remark 1.6. Melin (2005) defined another way to distinguish $DSLS_{\mathrm{Kh}}$ in the proof of Theorem 1.5. He defined a strip $S(\alpha, \beta, \rho)$ for given $\alpha, \beta, \rho \in \mathbb{R}$ by

$$S(\alpha, \beta, \rho) = \{x \in \mathbb{R}^2; \alpha x_1 + \beta - \rho(1+\alpha) \leqslant x_2 \leqslant \alpha x_1 + \beta + \rho(1+\alpha)\},$$

He called the number ρ the *diagonal half-width* of the strip. The boundary of the strip consists of two components given by the lines $x_2 = \alpha x_1 + \beta \pm \rho(1+\alpha)$, i.e., the center line, $x_2 = \alpha x_1 + \beta$, translated by the vectors $(-\rho, \rho)$ and $(\rho, -\rho)$. As a consequence of the digitization of pure points, we can see easily that a set of pure points is a subset of a digital straight line segment if and only if they are contained in a strip with a diagonal width strictly less than $\frac{1}{2}$.

2. Boomerangs and digital straight line segment

In this paper we want to characterize the digital straight line segments, so we consider the collection of monotone functions on a bounded interval. We may restrict attention to monotone functions, because a function which is not monotone can never represent a straight line segment.

In the case of the Khalimsky topology, it is clear that the graph of a discontinuous function cannot have the chord property, so we do not need to consider such functions. We consider increasing functions; the case of decreasing functions is similar. For this case we have the chord property which we introduced in Definition 1.3.

If $P = (p^i)_{i=0}^n$ is a sequence of points which is the graph of a function f, thus with $p_2^i = f(p_1^i)$, we define its *chain code* $c = (c_i)_{i=1,\dots,n}$ by $c_i = f(i) - f(i-1)$, $i = 1, \dots, n$. For the functions we work on, c_i is equal to zero or one. (This definition agrees with the Freeman chain code in this case.)

The simplest straight line segments in the digital plane are the horizontal, diagonal and vertical ones. In the remaining cases the graph contains both horizontal and diagonal steps; we shall call them *constant* and *increasing*, respectively, so in this case we have at least one point preceded by a horizontal interval and followed by a diagonal interval, or conversely.

Definition 2.1. When a graph P is given, we shall say that a digital curve consisting of $m+1$ points, $B = (b^i)_{i=0}^m$, $m \geqslant 2$, is a *boomerang in* P if it consists of a horizontal segment $[b^0, b^k]$, where $0 < k < m$, followed by a diagonal segment $[b^k, b^m]$, or conversely, and if B is maximal with this property. We shall call the horizontal and diagonal segments, $\mathsf{Con}(B)$ and $\mathsf{Inc}(B)$, respectively.

We use $|\mathsf{Con}| = |\mathsf{Con}(B)| = k$ for the number of horizontal intervals in the segment $[b^0, b^k]$, and $|\mathsf{Inc}| = |\mathsf{Inc}(B)| = m-k$ for the number of diagonal intervals in the segment $[b^k, b^m]$, or conversely if the horizontal segment comes last. They are equal to the number of zeros and ones in the related chain code, respectively. We introduce $|B| = k + (m-k) = m$ as the sum of $|\mathsf{Con}(B)|$ and $|\mathsf{Inc}(B)|$. We remark that the boomerangs need not be disjoint and that the last segment a boomerang may be a starting segment of the next boomerang, so the number of boomerangs is equal to the number of vertices.

We thus divide the collection of graphs of monotone functions on bounded intervals into two cases:
(I) Horizontal or diagonal;
(II) All others.

The case (I) is straightforward. We shall now discuss the second type of digital curves.

Definition 2.2. Given any subset P of $\mathbb{R}^2$ we define its *chord set* $\mathsf{chord}(P)$ as the union of all chords, i.e., all segments with endpoints in P, as

$$\mathsf{chord}(P) = \bigcup_{x,y\in P} [x,y] \subseteq \mathbb{R}^2.$$

We also need the *broken line* defined for a finite sequence $P = (p^i)_{i=0}^n$,

$$\mathsf{BL}(P) = \bigcup_{i=0}^{n-1} [p^i, p^{i+1}] \subseteq \mathbb{R}^2.$$

Similarly for an infinite sequence $(p^i)_{i\in\mathbb{N}}$ or $(p^i)_{i\in\mathbb{Z}}$.

Lemma 2.3. *For an 8-connected sequence $P = (p^i)_{i=0}^n$ we have*

$$\mathsf{BL}(P) + B^1_<(0,1) \subseteq \bigcup_{i=0}^{n} \left(\{p^i\} + B^\infty_<(0,1)\right),$$

where $\mathsf{BL}(P) + B^1_<(0,1)$ and $\{p^i\} + B^\infty_<(0,1)$ are the dilations of $\mathsf{BL}(P)$ and $\{p^i\}$ by the open unit ball for the l^1 and l^∞ metric, respectively.

Proof. We can see easily that

$$\begin{aligned} &\mathsf{BL}(P) + B^1_<(0,1) = \bigcup_{i=1}^{n-1} [p^i, p^{i+1}] + B^1_<(0,1) \subseteq \\ &\left(\{p^0, p^n\} + B^1_<(0,1)\right) \cup \left(\bigcup_{i=0}^{n-1} [p^i, p^{i+1}] + \{0\} \times [-1,1]\right). \end{aligned} \tag{2.1}$$

We have

$$\{p^0, p^n\} + B^1_<(0,1) \subseteq \{p^0, p^n\} + B^\infty_<(0,1), \tag{2.2}$$

and

$$\bigcup_{i=0}^{n-1}[p^i, p^{i+1}] + \{0\} \times [-1, 1] \subseteq \bigcup_{i=0}^{n}\{p^i\} + B^\infty_<(0, 1). \tag{2.3}$$

Then (2.2) and (2.3) give the result. ☐

Remark 2.4. In the equation (2.1), if we consider an infinite sequence $P = (p^i)_{i\in\mathbb{Z}}$, we have

$$\bigcup_{i}[p^i, p^{i+1}] + B^1_<(0, 1) \subseteq \bigcup_{i}\left([p^i, p^{i+1}] + \{0\} \times [-1, 1]\right).$$

2.1. Boomerangs and vertical distance

Suppose that $P = (p^i)_{i=0,\dots,n}$ is a sequence of points which has b boomerangs. Let $V = (v^i)_{i=1}^b$ be the sequence of all vertices of the boomerangs of P. We define the vertical distance d_v as $d_\mathrm{v}(x, y) = |x_2 - y_2|$ when $x_1 = y_1$. We shall show a relation between vertical distances and $DSLS_8$ and $DSLS_{\mathrm{Kh}}$.

Theorem 2.5. *Let $P = (p^i)_{i=0}^n$ be an 8-connected sequence of points which is the graph of a function and has b boomerangs. Let $V = (v^i)_{i=1,\dots,b}$ be the sequence of all vertices of its boomerangs. Then $P \in DSLS_8$ if and only if for all $i = 1, \dots, b$ and all real points $a \in \mathsf{chord}(P)$ such that $a_1 = v^i_1$ we have $d_\mathrm{v}(v^i, a) < 1$.*

Proof. Suppose that there is a vertex $v = p^j$ for some $0 < j < n$ and a point $a \in \mathsf{chord}(P)$ with $a_1 = v_1$ such that $d_\mathrm{v}(v, a) \geqslant 1$. We shall show that $P \notin DSLS_8$. Since we have $d_\mathrm{v}(v, a) \geqslant 1$,

$$a \notin \{v\} + B^\infty_<(0, 1). \tag{2.4}$$

Also

$$|a_1 - p^i_1| \geqslant 1 \text{ for } i \neq j. \tag{2.5}$$

Therefore, by (2.4), (2.5), we see that

$$a \notin \{p^i\}_{i=0}^n + B^\infty_<(0, 1),$$

and so $P \notin DSLS_8$.

Conversely, suppose that $P \notin DSLS_8$, so there is a point c and two indices k, l such that $0 \leqslant k < l \leqslant n$ and $c \in [p^k, p^l]$ but $c \notin P + B^\infty_<(0, 1)$. By Lemma 2.3,

$$c \notin \mathsf{BL}(P) + B^1_<(0, 1). \tag{2.6}$$

Define $Q_{k,l} = \mathsf{BL}((p^i)_{i=k,\dots,l})$. Consider the function $F_{k,l}\colon [p_k, p_l] \to \mathbb{R}$ defined by

$$F_{k,l}(x) = d_\mathrm{v}(x, y) \text{ for } y \in Q_{k,l} \text{ with } y_1 = x_1.$$

Consider the point $x \in Q_{k,l}$ with $x_1 = c_1$. By (2.6),

$$d_{\mathrm{v}}(c, x) \geqslant 1.$$

Therefore $F_{k,l}(c) \geqslant 1$. The function $F_{k,l}$ attains its maximum at a point that lies on a vertical line passing through a vertex, so there is a vertex v of the boomerang B such that the function $F_{k,l}$ attains its maximum at the point $a \in [p^k, p^l]$ with $a_1 = v_1$, thus

$$1 \leqslant F_{k,l}(c) \leqslant F_{k,l}(a) = d_{\mathrm{v}}(v, a).$$

This shows that, for the vertex v and a point $a \in \mathsf{chord}(P)$ with same first coordinate as v, we have $d_{\mathrm{v}}(a, v) \geqslant 1$. We are done. □

We shall now study the same result for Khalimsky-connectedness. We consider mixed points $m = (m_1, m_2)$ which lie on P and such that for some vertex $v = (v_1, v_2)$, we have $m_1 = v_1 \pm 1$. In the next theorem we shall show that we have straightness if and only if the vertical distance is less than one at these mixed points.

Theorem 2.6. *Suppose that $P = (p^i)_{i=0}^n$ is a Khalimsky-connected sequence with pure endpoints and let b be the number of its boomerangs. Let M be the set of all mixed points in P. Then $P \in DSLS_{\mathrm{Kh}}$ if and only if for all $m \in M$ and all $a \in \mathsf{chord}(P)$ with $a_1 = m_1$ we have $d_{\mathrm{v}}(m, a) < 1$.*

Proof. Suppose that there exist a mixed point $m = p^j$ for some $0 < j < n$ and a point $a \in \mathsf{chord}(P)$ with $a_1 = m_1$ such that $d_{\mathrm{v}}(m, a) \geqslant 1$, so that

$$a \notin \{m\} + B_{<}^{\delta^\infty}(0, 1), \tag{2.7}$$

where $B_{<}^{\delta^\infty}(0, 1)$ is the open unit ball for the metric δ^∞.
It is clear that

$$\left|a_1 - p_1^{j-2}\right| = 2 \text{ and } \left|a_1 - p_1^{j+2}\right| = 2;$$

so

$$\left|a_1 - p_1^k\right| \geqslant 2 \text{ for } k \geqslant j + 2 \text{ and } k \leqslant j - 2. \tag{2.8}$$

We can see easily also that

$$\left|a_2 - p_2^{j-1}\right| = \left|a_2 - p_2^{j+1}\right| \geqslant 1. \tag{2.9}$$

Therefore, by (2.7), (2.8) and (2.9)

$$a \notin P + B_{<}^{\delta^\infty}(0, 1).$$

Thus $P \notin DSLS_{\mathrm{Kh}}$.

Conversely, suppose that $P \in DSLS_{\mathrm{Kh}}$ so that there is a straight line L with equation $x_2 = \alpha x_1 + \beta$ whose digitization equals the set of points P. Without loss of generality we may assume that $0 < \alpha < 1$. As we saw in remark 1.6, there is a strip

$$S(\alpha, \beta, \rho) = \{x \in \mathbb{R}^2; \alpha x_1 + \beta - \rho(1+\alpha) \leqslant x_2 \leqslant \alpha x_1 + \beta + \rho(1+\alpha)\},$$

with diagonal half-width ρ less than $\frac{1}{2}$, which contains P and also $\mathsf{chord}(P)$. We shall show that the vertical distance between an arbitrary mixed point $m = (m_1, m_2)$ in M and the two boundary lines $S(\alpha, \beta, \rho)$ is less than one, so that the vertical distance between m and all $a \in \mathsf{chord}(P)$ with $a_1 = m_1$ is less than one. Consider a mixed point m in M. Since α is less than 1, the two pure points $p = (m_1 - 1, m_2)$ and $q = (m_1 + 1, m_2)$ belong to P. Let $r \in D(p) \cap L$ where $D(p)$ is the cross defined in subsection 1.3. By the construction of Melin's digitization which we mentioned in subsection 1.3, the distance with l^∞ metric between the pure point p and the line segment L is less than $\frac{1}{2}$, i.e.,

$$d^\infty(p, r) < \tfrac{1}{2}. \tag{2.10}$$

The diagonal half-width of the strip S is less than $\frac{1}{2}$, so the distance with the l^∞ metric between the line segment L and the strip S is less than $\frac{1}{2}$. Thus

$$d^\infty(r, S) < \tfrac{1}{2}. \tag{2.11}$$

By (2.10) and (2.11)

$$d^\infty(p, S) \leqslant d^\infty(p, r) + d^\infty(r, S) < 1. \tag{2.12}$$

In the same way, we have

$$d^\infty(q, S) < 1. \tag{2.13}$$

By (2.12) and (2.13), we conclude that $d_{\mathrm{v}}(m, a) < 1$ for all $a \in S$ with $a_1 = m_1$. □

3. Boomerangs and straightness

We shall now discuss straightness by considering boomerangs and using the conditions on vertical distances in Theorems 2.5 and 2.6. First we just consider one boomerang. In two lemmas we shall find conditions for straightness in the 8-connected case and the Khalimsky case, and then we shall do the same when we have more than one boomerang.

Lemma 3.1. *Let $B = (b^i)_{i=0}^n$ be an 8-connected boomerang. Then the following two properties are equivalent.*
(i) $B \in DSLS_8$;
(ii) If $|\mathsf{Con}(B)| \geqslant 2$, then $|\mathsf{Inc}(B)| = 1$;

Proof. *(i) ⇒ (ii).* Suppose that a boomerang $B \in DSLS_8$ and $|\mathsf{Con}| \geqslant 2$ and $|\mathsf{Inc}| \geqslant 2$. Therefore the vertical distance between the vertex of B and $\mathsf{chord}(B)$ is at least one. Theorem 2.5 now gives a contradiction.

(ii) ⇒ (i). Suppose that $|\mathsf{Inc}| = 1$, and that $|\mathsf{Con}| = m \geqslant 2$. We can check easily the condition Theorem 2.5 and see that $B \in DSLS_8$. □

Lemma 3.2. *Let a boomerang $B = (b^i)_{i=0}^n$ be a Khalimsky-connected set with pure end points. Then the following two properties are equivalent.*
(i) $B \in DSLS_{\text{Kh}}$;
(ii) If $|\mathsf{Con}(B)| \geqslant 4$, then $|\mathsf{Inc}(B)| = 1$.

Proof. *(i) ⇒ (ii).* Suppose that $B \in DSLS_{\text{Kh}}$ and $|\mathsf{Con}| \geqslant 4$ and $|\mathsf{Inc}| \geqslant 2$. By Theorem 2.6, we have contradiction.

(ii) ⇒ (i). Suppose that $|\mathsf{Inc}| = 1$, and $|\mathsf{Con}| \geqslant 4$. We can see easily that the condition in Theorem 2.6 is satisfied, and we are done. □

The two previous Lemmas 3.1 and 3.2 show the relation between the class $DSLS$ and an arbitrary boomerang, but of course there are digital curves such that all its constituent boomerangs satisfy the condition of these lemmas but the curve itself is not in $DSLS$. In order to avoid complicated proofs in Propositions 3.3 and 3.4 and Lemmas 3.6 and 3.7, or a complicated statement in Theorem 3.8, we will consider only concave boomerangs.

Proposition 3.3. *Suppose that $P = (p^i)_{i=0}^n$ is a set of points such that $P \in DSLS_8$ and denote by b the number of concave boomerangs in P. If $|\mathsf{Con}(B_j)| \geqslant 2$ for some j with $1 \leqslant j \leqslant b$, then $|\mathsf{Inc}(B_i)| = 1$ for all i with $1 \leqslant i \leqslant b$.*

Proof. Let $P \in DSLS_8$. Suppose that there exist $1 \leqslant i \leqslant j \leqslant b$ such that $|\mathsf{Inc}(B_i)| \geqslant 2$ and $|\mathsf{Con}(B_j)| \geqslant 2$. We may assume that $|\mathsf{Inc}(B_i)| = 2$, $|\mathsf{Con}(B_j)| = 2$ by passing to subsets and B_i is the closest boomerang to B_j with cardinality of the increasing part not equal to 1. If $i = j$, the result is obvious by Lemma 3.1. For $j - i = 1$, by Lemma 3.1 we must have $|\mathsf{Con}(B_i)| = |\mathsf{Inc}(B_j)| = 1$. By Theorem 2.5 we do not have straightness in this case.

Suppose now that $j - i > 1$. In this case the chain code for P is

$$(1, 1, 0, (1, 0)^t, 1, 0, 0),$$

where $(1, 0)^t$ means that we have t times the subsequence $(1, 0)$. Let $(p^i)_{i=l}^{l+2t+6}$ be the points related to this chain code. The slope of the line segment $\left[p^l, p^{l+2t+6}\right]$ is equal to $\frac{3+t}{6+2t} = \frac{1}{2}$. We can check easily that the vertical distance between the vertex p^{l+2} and the line segment $\left[p^l, p^{l+2t+6}\right]$ is 1. Thus we are done just by considering Theorem 2.5. □

Proposition 3.4. *Suppose that $P = (p^i)_{i=0}^n$ is a Khalimsky-connected sequence with pure endpoints such that $P \in DSLS_{\text{Kh}}$ and denote by b the number of concave boomerangs in P. If $|\mathsf{Con}(B_j)| \geqslant 4$ for some $1 \leqslant j \leqslant b$, then $|\mathsf{Inc}(B_i)| = 1$ for all $1 \leqslant i \leqslant b$.*

Proof. We do as in the proof of Proposition 3.3. Suppose that there exist $1 \leqslant i \leqslant j \leqslant b$ such that $|\mathsf{Inc}(B_i)| \geqslant 2$ and $|\mathsf{Con}(B_j)| \geqslant 4$. We may assume that $|\mathsf{Inc}(B_i)| = 2$, $|\mathsf{Con}(B_j)| = 4$ by passing to subsets. We can assume that B_i is the closest boomerang to B_j with cardinality of the increasing part not equal to 1. For $j-i = 1$, we can find a contradiction as in Proposition 3.3. Finally, we shall show that we do not have straightness when $j - i > 1$. Let $(1, 1, 0, 0, (1, 0, 0)^t, 1, 0, 0, 0, 0)$ be the related chain code for the set of boomerangs $B_i, \dots, B_j$ and $(p^i)_{i=l}^{l+3t+9}$ be the points related to this chain code. The slope of the line segment $[p^l, p^{l+3t+9}]$ is equal to $\frac{3+t}{9+3t} = \frac{1}{3}$. Thus, we can see that the vertical distance between the mixed point p^{l+3} and the line segment $[p^l, p^{l+3t+9}]$ is equal to 1. Therefore, we do not have straightness by Theorem 2.6. □

By Propositions 3.3 and 3.4, there are just two cases when we study straightness. We write them in the following definition.

Definition 3.5. Let $I_i = |\mathsf{Inc}(B_i)|$ and $C_i = |\mathsf{Con}(B_i)|$, where $1 \leqslant i \leqslant b$ and b is the number of boomerangs in P. We shall consider four cases:
(8-a) $I_i = 1$ for all $1 \leqslant i \leqslant b$;
(8-b) $C_i = 1$ for all $1 \leqslant i \leqslant b$;
(Kh-a) $I_i = 1$ for all $1 \leqslant i \leqslant b$;
(Kh-b) $C_i = 2$ for all $1 \leqslant i \leqslant b$.
We shall call P *dominant constant* if it satisfies condition (8-a) in the case of 8-connectedness, and condition (Kh-a) in the case of Khalimsky connectedness, and *dominant increasing* if it satisfies condition (8-b) in the case of 8-connectedness and condition (Kh-b) in the case of Khalimsky connectedness.

If the discrete straight line has slope between 0 and $\frac{1}{2}$, we have dominant constant and for the slope of the line between $\frac{1}{2}$ and 1, we have dominant increasing.

There are some results on the runs of 8-connected digital straight lines that are related to our work. We give a summary of them. Freeman (1970:260) has observed that (except possibly at the beginning and end of the segment) the "successive occurrencies of the element occurring singly are as uniformly spaced as possible."

Rosenfeld (1974) provided a formal proof of these facts for the 8-connected case. We present two propositions, in the 8-connected case and the Khalimsky-connected case with this conclusion. We shall show that we have two possibilities for the number of boomerangs in both cases. This result is similar to Rosenfeld's conclusion in the 8-connected case for runs. We shall use the results of these lemmas in Theorem 3.8, so we write the statements of the two lemmas using boomerangs. To prove these lemmas we shall use Theorems 2.5 and 2.6.

Lemma 3.6. *If $P \in DSLS_8$, then we have at most two possible values for the cardinality of the boomerangs in P, that is, $\big||B_{i+k}| - |B_i|\big| \leqslant 1$ for all $i, k \in \mathbb{N}$.*

Proof. Let P be dominant increasing. To avoid complicated indices and to simplify the construction of the proof, we consider concave boomerangs only. We choose k minimal such that

$$\big||B_{i+j}| - |B_i|\big| = 1 \text{ for } 1 \leqslant j < k,$$

and

$$\big||B_{i+k}| - |B_i|\big| \geqslant 2.$$

Without loss of generality, we may assume that $|B_{i+j}| \geqslant |B_i|$ for $1 \leqslant j \leqslant k$. Thus

$$|B_{i+j}| - |B_i| = 1 \text{ for } 1 \leqslant j < k,$$

and

$$|B_{i+k}| - |B_i| \geqslant 2.$$

Consider now the line segment $[p, q]$ such that p is the starting point of $\mathsf{Con}(B_{i-1})$ and q is the endpoint of $\mathsf{Inc}(B_{i+k})$. This line segment has slope

$$\frac{(k+1)I_i + k - 1 + t}{(k+1)I_i + 2k + t},$$

where

$$t = |B_{i+k}| - |B_i| \geqslant 2 \text{ and } I_i = |\mathsf{Inc}(B_i)|.$$

We can see easily that the vertical distance is at least one at the point $(I_i + 2, I_i)$ (which is the vertex of a convex boomerang). Therefore, we get a contradiction by Theorem 2.5. The proof for dominant constant can be obtained in the same way. $\square$

Lemma 3.7. *If $P \in DSLS_{\mathrm{Kh}}$, then we have two possible values for the cardinality of boomerangs in P, that is, in the dominant increasing case,*

$$\big||B_{i+k}| - |B_i|\big| \leqslant 1 \textit{ for all } k \in \mathbb{N},$$

and in the dominant constant case,

$$\big||B_{i+k}| - |B_i|\big| \leqslant 2 \textit{ for all } k \in \mathbb{N}.$$

Proof. For the dominant increasing, we do as in Lemma 3.6. Here we consider, as in Lemma 3.6, concave boomerangs. We choose k minimal such that

$$|B_{i+j}| - |B_i| = 1 \text{ for } 1 \leqslant j < k,$$

and

$$|B_{i+k}| - |B_i| \geqslant 2.$$

Consider the line segment $[p, q]$ such that p is the starting point of $\mathsf{Con}(B_{i-1})$ and q is the endpoint of $\mathsf{Inc}(B_{i+k})$. This line segment has slope

$$\frac{(k+1)I_i + k - 1 + t}{(k+1)I_i + 3k + t + 1},$$

where

$$t = |B_{i+k}| - |B_i| \geqslant 2 \text{ and } I_i = |\mathsf{Inc}(B_i)|.$$

We can see easily that the vertical distance is at least one at the mixed point $(I_i + 3, I_i)$. Thus, we are done for the dominant increasing case by getting a contradiction with Theorem 2.6.

Suppose now that P is dominant constant. We may choose k minimal such that

$$|B_{i+j}| - |B_i| = 2 \text{ for } 1 \leqslant j < k,$$

and

$$|B_{i+k}| - |B_i| \geqslant 4.$$

Consider the line segment $[p, q]$ where p and q are the start point of $\mathsf{Inc}(B_i)$ and the endpoint of $\mathsf{Con}(B_{i+k})$, respectively. We can easily check that the vertical distance is at least one at the mixed point $(C_i + 3, 2)$, where $C_i = |\mathsf{Con}(B_i)|$. Thus, considering Theorem 2.6, we get a contradiction. □

The conditions in Lemmas 3.6 and 3.7 are necessary but not sufficient for straightness. An example for this claim is the set of 8-connected points with Freeman chain code 11010110101010. These points satisfy the conclusion of Proposition 3.6 but do not have the chord property. In the Khalimsky plane we can see these results in the set of points with Freeman chain code 11001001100100100100.

Hung and Kasvand (1984) introduced a way to find the sufficient condition for straightness in the 8-connected plane. He considered a digital arc as a sequence of two symbols. Then he noted that a segment in a sequence of symbols is a continuous block of symbols of this sequence; the number of symbols in a segment is the length of this segment. All segments having the same length in a sequence were called equal segments. Two equal segments he called uneven if their sums differ by more than 1. He called any two uneven segments an uneven pair. Then he went on to prove that a digital arc has the chord property if and only if there are no uneven segments in its chain code. He named a digital arc straight if and only if for equal segments in this arc, their sums cannot differ by more than 1. Therefore, like the chord property, the absence of uneven segments is one of the most fundamental properties in the structure of a digital straight line.

Bruckstein (1991) presented several interesting self-similarity properties of chain codes of digital straight line. He introduced some transformations given by matrices of determinant ± 1. These matrices belong to the well-known group $GL(2, \mathbb{Z})$. As a result of these transformations, he showed that the new sequence produced by applying these transformation to a sequence of 0 and 1 is the chain code of digital straight line segment if and only if the original sequence is the chain code of a digital straight line segment.

To find a sufficient condition for straightness, we shall define a mapping which transforms certain codes to the set $\{0, 1\}$. Let $\mathscr{B}(P)$ be the collection of all boomerangs in P. By Lemmas 3.6 and 3.7, we have just two possibilities for the values of $|B_i|$. Thus we can define a mapping from the set of Freeman chain code of P to $\{0,1\}$ which maps the boomerangs with greater cardinality to 1 and the other boomerangs to 0. The graph of f is an 8-connected set and by this fact we can see easily that if P is a Khalimsky-connected set, then $f(P)$ will be an 8-connected set and so for investigating the straightness in Khalimsky plane we can go to the 8-connected case. In the following theorem we shall show that $f(P)$ and so the composition of f with itself can give a necessary and sufficient condition for straightness in the 8-connected case and therefore also in the Khalimsky-connected plane.

Theorem 3.8. *We define a function f on a subset of the set $\{0,1\}^{\mathbb{N}}$ of sequences of zeros and ones and with values in the same set: $f(C)$ is defined for those chain codes that represent dominant increasing or dominant constant sequences which arise from sets of boomerangs of at most two different lengths. We define $f(C)$ as the sequence obtained by replacing the chain code of a long concave boomerang by 1 and that of a short concave boomerang by 0. Then*
(I) C is the chain code of an element of $DSLS_8$ if and only if $f(C) \in DSLS_8$, and
(II) C is the chain code of an element of $DSLS_{\mathrm{Kh}}$ if and only if C the chain code of a Khalimsky-connected set and $f(C) \in DSLS_8$.

Remark 3.9. If we compose f with itself and define $f^0(C) = C$, $f^{n+1}(C) = f(f^n(C))$ for $n \in \mathbb{N}$, then $f^n(C)$ belongs to $DSLS_8$ for all $n \in \mathbb{N}$ and all $C \in DSLS_8$, and $f^n(C)$ belongs to $DSLS_8$ for all $n \in \mathbb{N}^* = \mathbb{N} \smallsetminus \{0\}$ for all $C \in DSLS_{\mathrm{Kh}}$.

Proof. We define a transformation which gives us the chain code of $f(C)$. We want to transform a short boomerang to a vector V_1 which comes from the line segment between the starting point and the endpoint of this boomerang. Then, in analogy with short boomerangs we can do the same for a long boomerang and transform it to a vector V_3. We define a grid T which is contained in $\mathbb{R}^2$ and has two linearly independent basis vectors V_1 and V_2, where V_2 is the sum of V_1 and V_3. Therefore

$$T = \{a + x_1V_1 + x_2V_2; x = (x_1, x_2) \in \mathbb{Z}^2\} \text{ with } a = (a_1, a_2) \text{ as origin.}$$

With this transformation, we can map the set $\mathsf{chord}(P)$ into $\mathbb{R}^2$. The image of $x = (x_1, x_2)^{\mathrm{T}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2$ in T is

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{3.1}$$

where

$$A = \begin{pmatrix} 1 & -1 \\ 1-p & p \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 & 1 \\ 1 & -p \end{pmatrix} \tag{3.2}$$

for the set of 8-connected points which is dominant increasing or dominant constant, respectively, and p denotes the cardinality of a short boomerang.

In the same way we define a transformation which gives us the chain code of $f(C)$ in the Khalimsky case. We notice that in the dominant increasing case, the constant part is always 2 and in the dominant constant case, the constant part must be an even number. We can write this transformation in the Khalimsky case using a matrix A defined as follows:

$$A = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 2-p & p \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2}p \end{pmatrix}$$

for the set of points which is dominant increasing or dominant constant, respectively. The number p is the cardinality of the short boomerangs, which is an odd number for the dominant constant case. In both cases we can come back from T to $\mathbb{Z}^2$ as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\left[\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right], \tag{3.3}$$

By the statement of Theorem 3.8, C is dominant increasing or dominant constant. If $f(C)$ is a digital straight line segment, then we have four possibilities in each of the two cases, the 8-connected case and the Khalimsky case. We present them in the following list.

(3.4)

1. C is dominant increasing and $f(C)$ is dominant increasing, so C has dominant long boomerangs;
2. C is dominant constant and $f(C)$ is dominant increasing, so C has dominant long boomerangs;
3. C is dominant increasing and $f(C)$ is dominant constant, so C has dominant short boomerangs;
4. C is dominant constant and $f(C)$ is dominant constant, so C has dominant short boomerangs.

In the case of 8-connectedness, there are no special differences in the proof of the four cases in (3.4), but in the Khalimsky case, we must be careful which possibility we choose to work on, and how we can transform a mixed point to a vertex and vice versa.

Case (I), $\Rightarrow$. Now we shall prove the implication $\Rightarrow$ in case (I). Let $C \in DSLS_8$. If $f(C) \notin DSLS_8$ then we can find a vertex $v = (v_1, v_2)^{\mathrm{T}}$ of a boomerang B such that we have vertical distance at least one at this point. Suppose that this vertical distance is attained between v and the line segment with equation $Y = MX + N$ in T. Thus

$$d_{\mathrm{v}}(v, a) = v_2 - Mv_1 - N \geqslant 1.$$

Since we exclude convex boomerangs in this section, we can find easily the vertical distances without considering the absolute value. We may assume that C is dominant increasing. The transformation of the vertex v into $\mathbb{Z}^2$ is an endpoint of a boomerang in C. Let v' be this image. Thus

$$v' = \begin{pmatrix} p & 1 \\ p-1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} pv_1 + v_2 \\ (p-1)v_1 + v_2 \end{pmatrix}.$$

To find the image of the straight line $Y = MX + N$ in $\mathbb{Z}^2$, we do as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p & 1 \\ p-1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{3.5}$$

By (3.5) and a simple calculation,

$$x = (p + M)X + N,$$
$$y = (p - 1 + M)X + N.$$

This implies

$$y = \frac{M + p - 1}{M + p} x + \frac{N}{M + p}. \tag{3.6}$$

Thus the vertical distance between the line segment with equation (3.6) and the vertex $(pv_1 + v_2 - 1, (p-1)v_1 + v_2)^{\mathrm{T}}$ is

$$\begin{aligned} &(p-1)v_1 + v_2 - \frac{M + p - 1}{M + p}(pv_1 + v_2 - 1) - \frac{N}{M + p} \\ &= \frac{(v_2 - Mv_1 - N) + (M + p - 1)}{M + p} \geqslant 1. \end{aligned}$$

By Theorem 2.5, we can conclude that $C \notin DSLS_8$. That is a contradiction. Therefore, the assertion is proved when C is dominant increasing. The proof is similar for the dominant constant case.

Case (I), $\Leftarrow$. Conversely, we shall now prove the implication $\Leftarrow$ in case (I). Let $f(C)$ be in $DSLS_8$. By the statement of this Theorem, C must be dominant increasing or dominant constant. We have two possibilities for the cardinalities of boomerangs. By (3.4) we have four possibilities and the proof for those we use the same construction. We must consider the matrix for the transformation with the construction of C as dominant increasing or dominant constant. Assume now we are in case 1 in (3.4). Thus the sequences C and $f(C)$ are dominant increasing and C has dominant long boomerangs. Suppose that $C \notin DSLS_8$. Then by Theorem 2.5 we can find a vertex v of a boomerang B such that the vertical distance between this vertex and $\mathsf{chord}(C)$ is at least one. First, we shall show that the maximal vertical distance in C is attained at a vertex v of a long boomerang, where the following boomerang is short. Let $(B_l, \ldots, B_{l+k})$ be the set of all long boomerangs which lie between two short boomerangs and such that there is no short boomerang between them. Consider the line segment $[a, b]$ with equation $y = \alpha x + \beta$ in the $\mathsf{chord}(P)$ such that the maximal vertical distance is attained between this line segment and the vertex v. The point a must be the starting point of a boomerang and b the endpoint of another boomerang. Thus the slope of this line segment is

$$\alpha = \frac{(r + s)p - s}{(r + s)p + r}, \tag{3.7}$$

where r and s are the number of long and short boomerangs, respectively. By a simple calculation, we can see that the condition for the maximal vertical distance to be attained at the vertex of B_{l+k} is

$$\frac{p-1}{p} \leqslant \alpha \leqslant \frac{p}{p+1}. \tag{3.8}$$

We can check that the inequality (3.8) is correct by using (3.7). By the previous discussion, the vertex v must be the vertex of the last boomerang, i.e., B_{l+k}. Since $(B_l, \ldots, B_{l+k})$ are long boomerangs and B_{l+k+1} is a short boomerang, the image of $(B_l, \ldots, B_{l+k}, B_{k+l+1})$ in T is a boomerang with its vertex equal to the image of the endpoint of B_{l+k} in T. By the previous discussion, the maximal vertical distance is attained at the vertex $v = (v_1, v_2)^{\mathrm{T}}$ of the boomerang B_{l+k}. So that the point a with the same first coordinate as v and which lies on the line segment $y = \alpha x + \beta$ satisfies

$$d_{\mathrm{v}}(v, a) = v_2 - \alpha v_1 - \beta \geqslant 1.$$

Since C is dominant increasing, the endpoint of B_{l+k} is $q = (v_1 + 1, v_2)^{\mathrm{T}}$. The image of q in T is

$$q' = \begin{pmatrix} 1 & -1 \\ 1-p & p \end{pmatrix} \begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 + 1 \\ (1-p)(v_1+1) + pv_2 \end{pmatrix},$$

that is, the vertex of the boomerang B in T. The image of a line segment with equation $y = \alpha x + \beta$ in T is

$$Y = \frac{\alpha p - p + 1}{1 - \alpha} X + \frac{\beta}{1-\alpha},$$

so the vertical distance between this line segment and q' is

$$\begin{aligned} &(1-p)(v_1+1) + pv_2 - \frac{\alpha p - p + 1}{1-\alpha}(v_1 - v_2 + 1) - \frac{\beta}{1-\alpha} \\ &= \frac{v_2 - \alpha v_1 - \beta - \alpha}{1-\alpha} \geqslant \frac{1-\alpha}{1-\alpha} = 1. \end{aligned}$$

Finally, by considering Theorem 2.5, we get a contradiction. For case 3 in (3.4), the maximal vertical distance is attained at the vertex of a long boomerang where the following boomerang is short. In case 2 [4] we have maximal vertical distance at the vertex of a long [long] boomerang where the previous boomerang is short [short]. We can prove these facts in the same way as in case 1. The proofs for straightness in these cases are also similar to that of case 1.

Case (II), $\Rightarrow$. We shall now prove the implication $\Rightarrow$ in case (II). Let $C \in DSLS_{\mathrm{Kh}}$. If $f(C) \notin DSLS_8$ then we can find a vertex $v = (v_1, v_2)^{\mathrm{T}}$ of a boomerang B and the line segment with equation $Y = MX + N$ in T such that for the point a which lies on this line segment and has the same first coordinate as v, the vertical distance is at least one. Thus

$$d_{\mathrm{v}}(v, a) = v_2 - Mv_1 - N \geqslant 1.$$

Suppose that C is dominant increasing. The transformation of the vertex v into $\mathbb{Z}^2$ is an endpoint of a boomerang in C. Let v' be this image. Thus

$$v' = \begin{pmatrix} p & 1 \\ p-2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} pv_1 + v_2 \\ (p-2)v_1 + v_2 \end{pmatrix}.$$

The image of the straight line $Y = MX + N$ in $\mathbb{Z}^2$ is

$$y = \frac{M+p-2}{M+p}x + \frac{2N}{M+p}. \tag{3.9}$$

The point $m = (pv_1 + v_2 - 1, (p-2)v_1 + v_2)^{\mathrm{T}}$ is a mixed point in a boomerang in C. Thus the vertical distance between the line segment with equation (3.9) and the mixed point m is

$$\begin{aligned} &(p-2)v_1 + v_2 - \frac{M+p-2}{M+p}(pv_1 + v_2 - 1) - \frac{2N}{M+p} \\ &= \frac{(2v_2 - 2Mv_1 - 2N) + (M+p-2)}{M+p} \geqslant \frac{2+M+p-2}{M+p} = 1. \end{aligned}$$

By Theorem 2.6; $C \notin DSLS_{\mathrm{Kh}}$. That is a contradiction.

Suppose now C is dominant constant. The image of the vertex v in $\mathbb{Z}^2$ is:

$$v' = \begin{pmatrix} p & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} pv_1 + 2v_2 \\ v_1 \end{pmatrix}.$$

Without loss of generality, we can assume that the vertex v is the vertex of a convex boomerang. Thus, the vertical distance between this point and the line segment with equation $Y = MX+N$ is Mv_1+N-v_2, which is at least one. Same as previous discussion, we can find the image of the straight line $Y = MX + N$ in $\mathbb{Z}^2$ as follows:

$$y = \frac{1}{p+2M}x - \frac{2N}{p+2M}. \tag{3.10}$$

The point $\begin{pmatrix} pv_1 + 2v_2 + 2 \\ v_1 + 1 \end{pmatrix}$ is a mixed point in a concave boomerang of C. The vertical distance between this point and the line segment in (3.10) is:

$$\begin{aligned} &(v_1 + 1) - \frac{pv_1 + 2v_2 + 2 - 2N}{p+2M} \\ &= \frac{2Mv_1 + 2N - 2v_2 - 2 + p + 2M}{p+2M} \geqslant \frac{2-2+p+2M}{p+2M} = 1. \end{aligned}$$

Thus, the result in this case is also obvious by a contradiction with Theorem 2.6.

Case (II), $\Leftarrow$. Conversely, we shall now prove the implication $\Leftarrow$ in case (II). Let $f(C) \in DSLS_8$. As in *Case (I),* $\Leftarrow$, we have the four possibilities which were mentioned in (3.4). First, we consider case 1. Thus, C and $f(C)$ are dominant increasing and C has dominant long boomerangs. Suppose that $C \notin DSLS_{\mathrm{Kh}}$. Therefore by Theorem 2.6, there is a mixed point m of a boomerang B such that the vertical distance at this point is at least one. We shall show that the maximal vertical distance in C is attained at a mixed point $m = (m_1, m_2)^{\mathrm{T}}$ of a long boomerang, where the following boomerang is short. Suppose that this maximal vertical distance is attained between the mixed point m and the line segment $[a, b]$ with equation $y = \alpha x + \beta$. We can see easily that we have the maximal vertical distance when a is the starting point of a boomerang and b is the endpoint of another boomerang. The slope of this line segment is

$$\alpha = \frac{(r+s)p - 2s}{(r+s)p + 2r}, \tag{3.11}$$

where r and s are the number of long and short boomerangs, respectively. By a simple calculation, we find that the condition for the maximal vertical distance to be attained at a mixed point of the last long boomerang where the following boomerang is short, is

$$\frac{p-2}{p} \leqslant \alpha \leqslant \frac{p}{p+2}. \tag{3.12}$$

We can see that the inequalities in (3.12) are correct by using (3.11). We can prove in the same way as for the case 3 in (3.4), that the maximal vertical distance is attained at the vertex of a long boomerang where the following boomerang is short. As in the 8-connected case, for case 2 [4], we have maximal vertical distance at the vertex of a long [long] boomerang where the previous boomerang is short [short]. With the same discussion as *Case (I)*, $\Leftarrow$, we must show that the vertical distance at the image of the point $m' = (m_1 + 1, m_2)^{\mathrm{T}}$ in the grid T is at least 1. Let m'' be this image. Thus

$$m'' = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ 2-p & p \end{pmatrix}\begin{pmatrix} m_1 + 1 \\ m_2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} m_1 - m_2 + 1 \\ (2-p)(m_1+1) + pm_2 \end{pmatrix}.$$

The point m'' is a vertex of a boomerang in $f(C)$. The image of a line segment $y = \alpha x + \beta$ in T is

$$Y = \frac{2 - p + p\alpha}{1-\alpha} X + \frac{\beta}{1-\alpha}. \tag{3.13}$$

Therefore, the vertical distance between line segment with equation (3.13) and the vertex m'' is

$$\begin{aligned} &\frac{(2-p)(m_1+1) + pm_2}{2} - \frac{2-p+p\alpha}{1-\alpha}(m_1 - m_2 + 1) - \frac{\beta}{1-\alpha} \\ &= \frac{2m_2 - 2\alpha m_1 - 2\beta + \beta - 2\alpha}{1-\alpha} \geqslant \frac{2 - 2\alpha + \beta}{1-\alpha} = 2 + \frac{\beta}{1-\alpha}. \end{aligned}$$

That is a contradiction. We can prove case 3 in (3.4) in the same way.

Let us now prove case 2 in (3.4). In this case, the maximal vertical distance is attained at the mixed point of a long boomerang where the previous boomerang is short. We consider the image of the point $m' = (m_1 - 2, m_2 - 1)^{\mathrm{T}}$. The image of m' is the vertex of a convex boomerang in $f(C)$. We shall show that the vertical distance at this point is at least one. The image of the point m' can obtain as follows:

$$m'' = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2}p \end{pmatrix} \begin{pmatrix} m_1 - 2 \\ m_2 - 1 \end{pmatrix} = \begin{pmatrix} m_2 - 1 \\ \dfrac{m_1 + p - pm_2 - 2}{2} \end{pmatrix}.$$

The image of a line segment $y = \alpha x + \beta$ in T is

$$Y = \frac{1 - p\alpha}{2\alpha} X - \frac{\beta}{2\alpha}. \tag{3.14}$$

Therefore, the vertical distance between line segment with equation (3.14) and the vertex m'' is

$$\begin{aligned} &\frac{1 - p\alpha}{2\alpha}(m_2 - 1) - \frac{\beta}{2\alpha} - \frac{m_1 + p - pm_2 - 2}{2} \\ &= \frac{m_2 - \alpha m_1 - \beta + 2\alpha - 1}{2\alpha} \geqslant \frac{1 + 2\alpha - 1}{2\alpha} = 1. \end{aligned}$$

Now Theorem 2.5 gives a contradiction. The case 4 in (3.4) can be proved in the same way. □

Remark 3.10. The matrices A in (3.2) have determinant ± 1 so they have inverses with integer entries. The 2×2 matrices with determinant ± 1 (called unimodular matrices) form a linear group $GL(2, \mathbb{Z})$. Bruckstein (1991) introduced such a transformation defined by 2×2 matrices with determinant ± 1. These matrices belong to $GL(2, \mathbb{Z})$ and so have inverses in this group. He wrote that the image of all such transformations will provide chain codes of linearly separable dichotomies if and only if the transformed line induces a linearly separable dichotomy. Using this fact, he noted that all sequence transformations having this property yield chain codes for straight lines if and only if the original chain code is a digitized straight line. In the Khalimsky plane the matrices A have determinant ± 2. Thus, they do not have such properties.

In the next theorem, we shall present another transformation to show the relation between $DSLS_{\mathrm{Kh}}$ and $DSLS_8$.

Theorem 3.11. *We define a function g on a subset of the set $\{0, 1\}^{\mathbb{N}}$ of sequences of zeros and ones and with values in the same set. For a chain code C, $g(C)$ is defined for dominant increasing or dominant constant sequences in the Khalimsky plane. We define $g(C)$ by replacing each pair of zeros by one zero. Then C is the chain code of an element of $DSLS_{\mathrm{Kh}}$ if and only if $g(C) \in DSLS_8$.*

Proof. We define a transformation which gives us the chain codes of $g(C)$. To get the chain code of $g(C)$, we must replace 00 by 0, and 1 by 1. Thus, we define a grid T same as the proof of Theorem 3.8, where

$$V_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \text{ and } V_3 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$

We use the matrix $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ for the equation in (3.1). We can come back from T into $\mathbb{Z}^2$ by using the matrix $A^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ in (3.3).

We shall prove the implication $\Rightarrow$. Let $C \in DSLS_{\text{Kh}}$. Suppose that $g(C) \notin DSLS_8$. Thus, we can find a vertex v such that the vertical distance between this point and the line segment with equation $Y = MX + N$ is at least one. The image of this line in $\mathbb{Z}^2$ is

$$y = \frac{M}{2-M}x + \frac{2N}{2-M}. \tag{3.15}$$

Let v' be the image of the point v in $\mathbb{Z}^2$. Thus,

$$v' = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 - v_2 \\ v_2 \end{pmatrix}.$$

It is clear that v' is a vertex of C. We shall show that the vertical distance between the mixed point $(2v_1 - v_2 + 1, v_2)^{\text{T}}$ and the line segment with equation (3.15) is at least one. This vertical distance is

$$v_2 - \frac{M}{2-M}(2v_1 - v_2 + 1) - \frac{2N}{2-M} = \\ \frac{2(v_2 - Mv_1 - N) - M}{2-M} \geqslant \frac{2-M}{2-M} = 1.$$

This is a contradiction with Theorem 2.6.

We shall now prove the implication $\Leftarrow$. Let $g(C) \in DSLS_8$. If $C \notin DSLS_{\text{Kh}}$, then we can find a mixed point m such that the vertical distance is at least one at this point. Suppose that the maximal vertical distance is attained between the mixed point m and the line segment with equation $y = \alpha x + \beta$. We may assume that $m_2 \geqslant \alpha m_1 + \beta + 1$. The proof for the case $m_2 \leqslant \alpha m_1 + \beta - 1$ is similar. We consider the image of the point $m' = (m_1 - 1, m_2)^{\text{T}}$ in the grid T. Let this image be m''. Thus

$$m'' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 - 1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(m_1 + m_2 - 1) \\ m_2 \end{pmatrix}.$$

The image of the line $y = \alpha x + \beta$ under T is:

$$Y = \frac{2\alpha}{1+\alpha}X + \frac{\beta}{1+\alpha}.$$

Therefore

$$m_2 - \frac{2\alpha}{1+\alpha}\frac{m_1+m_2-1}{2} - \frac{\beta}{1+\alpha}$$
$$= \frac{m_2 - \alpha m_1 - \beta + \alpha}{1+\alpha} \geqslant \frac{1+\alpha}{1+\alpha} = 1.$$

Hence, we get a contradiction with Theorem 2.5. □

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References

Bruckstein, Alfred M. 1991. Self-similarity properties of digitized straight lines. *Contemporary Math* **119**, 1–20.

Freeman, Herbert. 1970. Boundary encoding and processing. In: *Picture Processing and Psychopictorics*, B. S. Lipkin and A. Rosenfeld, Eds., 241–266. New York and London: Academic Press.

Hung, S. H. Y.; Kasvand. T. 1984. On the chord property and its equivalences. In: *Proceedings of the 7th International Conference on Pattern Recognition*, 116–119.

Jamet, Damien; Toutant, Jean-Luc. 2006. On the connectedness of rational arithmetic discrete hyperplanes. In: A. Kuba; L. G. Nyúl; K. Palágyi (Eds.), *Discrete Geometry for Computer Imagery*, Lecture Notes in Computer Science **4245**, 223–234.

Kim, C. E. 1982. On cellular straight line segments. *Computer Graphics and Image Processing* **18**, 369–381.

Kiselman, Christer O. 2004. *Digital geometry and mathematical morphology.* Lecture notes, Uppsala University.

Maloň, Stanislaw; Freeman, Herbert. 1961. On the encoding of arbitrary geometric configurations. *IRE Trans. EC-10*, 260–268.

Melin, Erik. 2005. Digital straight lines in the Khalimsky plane. *Mathematica Scandinavica* **96**, 49–64.

Rosenfeld, Azriel. 1974. Digital straight line segments. *IEEE Transactions on Computers.* C-32, NO 12, 1264–1269.

The number of Khalimsky-continuous functions on intervals

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Abstract

We determine the number of Khalimsky-continuous functions defined on an interval and with values in an interval.

1. Introduction

In this paper we shall determine the number of continuous functions which are defined on an interval of the digital line $\mathbb{Z}$ equipped with the Khalimsky topology and with values in that line. The Khalimsky topology is a topology for which the digital line is connected. We shall begin by recalling the definition and first properties of the Khalimsky topology and then consider Khalimsky-continuous functions. Then in section 2, we consider these functions when they have two points in the codomain. In this section we see a new example of the classical fibonacci sequence. In section 3 and 4, we study the Khalimsky-continuous functions with three or four points in their codomain and as a consequence of these parts we find some new sequences, the asymptotic behavior of which we investigate. Finally, in the section 5 we consider Khalimsky-continuous functions with one fixed endpoint. In this section we get a sequence which has the same recursion relation as the Pell numbers but with different initial values. We also obtain a new example of the Delannoy numbers.

The Khalimsky topology

There are several different ways to introduce the Khalimsky topology on the integers. We present the Khalimsky topology using a topological basis. For every even integer m, the set $\{m-1, m, m+1\}$ is open, and for every odd integer n, the singleton set $\{n\}$ is open. A basis is given by

$$\{\{2n+1\}, \{2n-1, 2n, 2n+1\}; n \in \mathbb{Z}\}.$$

It follows that even points are closed. A digital interval $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$ with the subspace topology is called a *Khalimsky interval,* and a homeomorphic image of a Khalimsky interval into a topological space is called a *Khalimsky arc.* On the digital plane $\mathbb{Z}^2$, the Khalimsky topology is given by the product topology. A point with both coordinates odd is open. If both coordinates are even, the point is closed. These types of points are called *pure.* Points with one even and one odd coordinate are neither open nor closed; these are called *mixed.* Note that a mixed point $m = (m_1, m_2)$ is connected only to its 4-neighbors,

$$(m_1 \pm 1, m_2) \text{ and } (m_1, m_2 \pm 1),$$

whereas a pure point $p = (p_1, p_2)$ is connected to all its 8-neighbors,

$$(p_1 \pm 1, p_2), (p_1, p_2 \pm 1), (p_1 + 1, p_2 \pm 1) \text{ and } (p_1 - 1, p_2 \pm 1).$$

More information on the Khalimsky plane and the Khalimsky topology can be found in Kiselman (2004).

Khalimsky-continuous functions

When we equip $\mathbb{Z}$ with the Khalimsky topology, we may speak of continuous functions $\mathbb{Z} \to \mathbb{Z}$. It is easily proved that a continuous function f is Lipschitz with constant 1. This is however not sufficient for continuity. It is not hard to prove that $f\colon \mathbb{Z} \to \mathbb{Z}$ is continuous if and only if (i) f is Lip-1 and (ii) for every x, $x \not\equiv f(x) \pmod 2$ implies $f(x \pm 1) = f(x)$. For more information see Melin (2005).

Also, we observe that the following functions are continuous:
(1) $\mathbb{Z} \ni x \mapsto a \in \mathbb{Z}$, where a is constant;
(2) $\mathbb{Z} \ni x \mapsto \pm x + c \in \mathbb{Z}$, where c is an even constant;
(3) $\max(f, g)$ and $\min(f, g)$ if f and g are continuous.

Actually every continuous function on a bounded Khalimsky interval can be obtained by a finite succession of the rules (1), (2), (3); Kiselman (2004).

2. Continuous functions with a two-point codomain

We shall first look at the functions which take their values in an interval consisting of two points. It turns out that the number of such functions is given by the Fibonacci sequence.

Theorem 2.1. *Let a_n be the number of Khalimsky-continuous functions $[0, n-1]_\mathbb{Z} \to [0, 1]_\mathbb{Z}$. Then $a_n = F_{n+2}$, where $(F_n)_0^\infty$ is the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geqslant 2$.*

Proof. Let $a_n^i = \operatorname{card}(\{f\colon [0, n-1]_\mathbb{Z} \to [0, 1]_\mathbb{Z}; f(n-1) = i\})$ for $i = 0, 1$, so that

$$a_n = a_n^0 + a_n^1. \tag{2.1}$$

By the definition of the Khalimsky topology, we see that

$$\begin{aligned} a_{2k+1}^0 &= a_{2k}, \qquad k \geqslant 1, \\ a_{2k+1}^1 &= a_{2k}^1, \qquad k \geqslant 1. \end{aligned} \tag{2.2}$$

Moreover,

$$a^1_{2k} = a_{2k-1}, \qquad k \geqslant 1, \tag{2.3}$$
$$a^0_{2k} = a^0_{2k-1}, \qquad k \geqslant 1.$$

Hence, using in turn (2.1), (2.2) and (2.3), we obtain

$$a_{2k+1} = a^0_{2k+1} + a^1_{2k+1} = a_{2k} + a^1_{2k} = a_{2k} + a_{2k-1},$$

which is the Fibonacci relation. Similarly, by using (2.1), (2.3) and (2.2), we get

$$a_{2k} = a^0_{2k} + a^1_{2k} = a^0_{2k-1} + a_{2k-1} = a_{2k-2} + a_{2k-1}.$$

Now we need only observe that $a_1 = 2 = F_3$ and $a_2 = 3 = F_4$ to finish. □

We notice that Theorem 2.1 leads us to a new example of the classical Fibonacci sequence. We list the number a_n of Khalimsky-continuous functions for $n = 1, \ldots, 14$ in the next table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_n	2	3	5	8	13	21	34	55	89	144	233	377	610	987

The asymptotic behavior of the number of continuous functions with a two-point codomain

We consider two frequencies

$$P^0_n = \frac{a^0_n}{a_n},$$

and

$$P^1_n = \frac{a^1_n}{a_n}.$$

By (2.1) we have

$$P^0_n + P^1_n = 1. \tag{2.4}$$

We shall determine these frequencies asymptotically. First, we recall the interesting property of the Fibonacci sequence: the fraction $\frac{F_{n+1}}{F_n}$ tends to α as $n \to \infty$ and where α denotes the Golden Section $\frac{1}{2}(\sqrt{5}+1)$. Therefore $\frac{F_{n+1}}{F_{n-1}}$ tends to α^2. In the following theorem we consider the frequencies for odd and even indices separately.

Theorem 2.2. *Let a_n and a^i_n be as in Theorem 2.1 and define $P^i_n = a^i_n/a_n$ for $i = 0, 1$. Then as $k \to +\infty$ we have*

$$P^0_{2k-1} \to \tfrac{1}{\alpha}, P^0_{2k} \to \tfrac{1}{\alpha^2}$$

and

$$P^1_{2k-1} \to \tfrac{1}{\alpha^2}, P^1_{2k} \to \tfrac{1}{\alpha}$$

where $\alpha = \frac{1}{2}(\sqrt{5}+1)$.

Proof. By (2.3) and (2.1),

$$a^1_{2k} = a^1_{2k-1} + a^0_{2k-1},$$

therefore we obtain another relation between frequencies and the values of a_{2k} and a_{2k-1} as

$$P^1_{2k} a_{2k} = P^1_{2k-1} a_{2k-1} + P^0_{2k-1} a_{2k-1}. \tag{2.5}$$

Then using (2.4) lead us to

$$P^1_{2k} a_{2k} = a_{2k-1}.$$

Thus,

$$P^1_{2k} = \frac{a_{2k-1}}{a_{2k}} \to \frac{1}{\alpha} \text{ as } k \to +\infty.$$

By Theorem 2.1,

$$a_{2k} - P^0_{2k} a_{2k} = a_{2k-1},$$

so

$$P^0_{2k} = \frac{a_{2k} - a_{2k-1}}{a_{2k}}. \tag{2.6}$$

By using (2.1), (2.3) and (2.2) we have

$$a_{2k} - a_{2k-1} = a_{2k-2}, \tag{2.7}$$

thus by (2.6) and (2.7) we have

$$P^0_{2k} = \frac{a_{2k-2}}{a_{2k}},$$

and so

$$P^0_{2k} \to \frac{1}{\alpha^2} \text{ as } k \to +\infty.$$

As before, we can find

$$P^0_{2k+1} = \frac{a_{2k}}{a_{2k+1}},$$

implying that

$$P^0_{2k+1} \to \frac{1}{\alpha} \text{ as } k \to +\infty.$$

Also,

$$P^1_{2k+1} = \frac{a_{2k-1}}{a_{2k+1}},$$

which implies that

$$P^1_{2k+1} \to \frac{1}{\alpha^2} \text{ as } k \to +\infty.$$

□

In the next table we can see the values of a^0_n, a^1_n, a_n, P^0_n and P^1_n for $n = 6, \ldots, 13$.

n	6	7	8	9	10	11	12	13
a_n^0	5	13	13	34	34	89	89	233
a_n^1	8	8	21	21	55	55	144	144
a_n	13	21	34	55	89	144	233	377
P_n^0	0.3846	0.6190	0.3824	0.6182	0.3820	06181	0.382	0.618
P_n^1	0.6154	0.381	0.6176	0.382	0.618	0.382	0.618	0.382

3. Continuous functions with a three-point codomain

We sum up the results for functions with up to three values.

Theorem 3.1. *Let b_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbb{Z}} \to [0, 2]_{\mathbb{Z}}$. Then $b_1 = 3$, $b_2 = 5$, and*

$$\begin{aligned} b_{2k} &= b_{2k-1} + b_{2k-2} + b_{2k-3} = 2b_{2k-2} + 3b_{2k-3}, && k \geqslant 2, \\ b_{2k-1} &= b_{2k-2} + 2b_{2k-3}, && k \geqslant 2. \end{aligned} \tag{3.1}$$

Proof. Let $b_n^i = \operatorname{card}(\{f\colon [0, n-1]_{\mathbb{Z}} \to [0, 2]_{\mathbb{Z}}; f(n-1) = i\})$ for $i = 0, 1, 2$. Therefore it is clear that

$$b_n = b_n^0 + b_n^1 + b_n^2. \tag{3.2}$$

From the properties of the Khalimsky topology we see that

$$\begin{aligned} b_{2k}^0 &= b_{2k-1}^0, && k \geqslant 1, \\ b_{2k}^1 &= b_{2k-1}^0 + b_{2k-1}^1 + b_{2k-1}^2, && k \geqslant 1, \\ b_{2k}^2 &= b_{2k-1}^2, && k \geqslant 1. \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} b_{2k-1}^0 &= b_{2k-2}^0 + b_{2k-2}^1, && k \geqslant 2, \\ b_{2k-1}^1 &= b_{2k-2}^1, && k \geqslant 2, \\ b_{2k-1}^2 &= b_{2k-2}^2 + b_{2k-2}^1, && k \geqslant 2. \end{aligned} \tag{3.4}$$

We assume that $n = 2k-1$ in equation (3.2) and then using in turn (3.4) and (3.3) we obtain the equalities

$$b_{2k-1} = b_{2k-2} + 2b_{2k-2}^1 = b_{2k-2} + 2b_{2k-3}. \tag{3.5}$$

Now we need to do the same for $n = 2k$ in equation (3.2) and then using in turn (3.3) and (3.4) we obtain

$$b_{2k} = b_{2k-1} + b_{2k-1}^0 + b_{2k-1}^2 = b_{2k-1} + b_{2k-2} + b_{2k-2}^1. \tag{3.6}$$

Now if we use equation (3.3) in (3.6) we can see the result for b_{2k}, i.e.,

$$b_{2k} = b_{2k-1} + b_{2k-2} + b_{2k-3}. \tag{3.7}$$

The another result for b_{2k} will be obvious if we put equation (3.5) into equation (3.7), i.e.,

$$b_{2k} = b_{2k-1} + b_{2k-2} + b_{2k-3} = b_{2k-2} + 2b_{2k-3} + b_{2k-2} + b_{2k-3} = 2b_{2k-2} + 3b_{2k-3}.$$

□

The Jacobsthal sequence is defined by $J_n = J_{n-1} + 2J_{n-2}$ with $J_1 = 0$ and $J_2 = 1$ (the sequence number A001045 in Sloane's On-line Encyclopedia of Integer Sequences), and the Tribonacci sequence is defined by the formula $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with initial values $1, 1, 1$ (sequence number A000213), so by Theorem (3.1) we see that b_n is a mixture between the Tribonacci and Jacobsthal sequences.

We give below the sequence (b_n) for $n = 1, \dots, 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
b_n	3	5	11	19	41	71	153	265	571	989	2131	3691

The asymptotic behavior of the number of continuous functions with a three-point codomain

We shall now determine how the number of continuous functions grows with the number of points in the domain.

Theorem 3.2. *Let b_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbb{Z}} \to [0, 2]_{\mathbb{Z}}$. Then there is a sequence (t_n) tending to a positive limit $t = \frac{1}{2} + \frac{1}{6}\sqrt{3} \approx 0.788675$ as $k \to +\infty$ and such that*

$$\begin{aligned} b_{2k} &= t_{2k}\sqrt{3}\left(2+\sqrt{3}\right)^k, && k \geqslant 2, \\ b_{2k-1} &= t_{2k-1}\left(2+\sqrt{3}\right)^k, && k \geqslant 2. \end{aligned} \tag{3.8}$$

Proof. We define a sequence (t_n) by the following equations,

$$\begin{aligned} t_{2k} &= b_{2k}\theta^{-1}\gamma^{-k}, && k \geqslant 2, \\ t_{2k-1} &= b_{2k-1}\gamma^{-k}, && k \geqslant 2. \end{aligned} \tag{3.9}$$

Thus, using (3.9) and (3.1) we get

$$\begin{aligned} t_{2k} &= 2\gamma^{-1}t_{2k-2} + 3\gamma^{-1}\theta^{-1}t_{2k-3}, && k \geqslant 2, \\ t_{2k-1} &= \theta\gamma^{-1}t_{2k-2} + 2\gamma^{-1}t_{2k-3}, && k \geqslant 2. \end{aligned} \tag{3.10}$$

With equation (3.10) we have the following equation for all $\theta, \gamma > 0$,

$$t_{2k} - t_{2k-1} = (2\gamma^{-1} - \theta\gamma^{-1})t_{2k-2} + (3\gamma^{-1}\theta^{-1} - 2\gamma^{-1})t_{2k-3}. \tag{3.11}$$

While this formula is true for all values of γ and θ, it is of interest mainly when the two coefficients in equation (3.11) sum up to zero. We therefore define γ and θ so that $2\gamma^{-1} - \theta\gamma^{-1} + 3\gamma^{-1}\theta^{-1} - 2\gamma^{-1} = 0$. This implies $\theta = \sqrt{3}$.

Next we consider the equation for $t_{2k+1} - t_{2k}$,

$$t_{2k+1} - t_{2k} = \theta\gamma^{-1}t_{2k} + 2\gamma^{-1}t_{2k-1} - t_{2k} = (\theta\gamma^{-1} - 1)t_{2k} + 2\gamma^{-1}t_{2k-1}. \tag{3.12}$$

In the same way we consider the special case of equation (3.12) when the coefficients have zero sum, and therefore we get $\gamma = 2 + \theta = 2 + \sqrt{3}$. By using induction in equation (3.11) we have,

$$t_{2k} - t_{2k-1} = \left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{k-1} (t_2 - t_1), \tag{3.13}$$

and for equation (3.12),

$$t_{2k+1} - t_{2k} = \left(\frac{-2}{2+\sqrt{3}}\right)(t_{2k} - t_{2k-1}) = \left(\frac{-2}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{k-1} (t_2 - t_1). \tag{3.14}$$

Since $\left|\frac{2-\sqrt{3}}{2+\sqrt{3}}\right| < 1$, equation (3.13) and (3.14) lead us to the same limit t, $0 < t < +\infty$ for the sequence (t_n) as k tends to infinity.

To determine the limit t, we shall use matrices, inspired by the treatment in Cull et al. (2005:16).

Formula (3.1) can be written in matrix form

$$X_n = AX_{n-1} \text{ where } X_n = \begin{pmatrix} b_{2n} \\ b_{2n-1} \end{pmatrix} \text{ and } A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

With initial condition $X_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ we have $X_n = A^{n-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. The matrix A has characteristic polynomial

$$\mathrm{ch}_A(x) = \det\begin{pmatrix} 2-x & 3 \\ 1 & 2-x \end{pmatrix} = (2-x)^2 - 3$$

and has distinct eigenvalues, $\lambda_1 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$, and this implies that A is diagonalizable. With a simple computation, we can see that $A = PDP^{-1}$, where

$$D = \begin{pmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{pmatrix}, \quad P = \begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix}, \text{ and } P^{-1} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & \sqrt{3} \\ -1 & \sqrt{3} \end{pmatrix}.$$

Therefore

$$\begin{aligned} X_n &= A^{n-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = PD^{n-1}P^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} \\ &= \frac{1}{2\sqrt{3}}\begin{pmatrix} (5\sqrt{3}+9)(2+\sqrt{3})^{n-1} + (5\sqrt{3}-9)(2-\sqrt{3})^{n-1} \\ (5+3\sqrt{3})(2+\sqrt{3})^{n-1} + (-5+3\sqrt{3})(2-\sqrt{3})^{n-1} \end{pmatrix}, \end{aligned}$$

so

$$b_{2n} = \frac{1}{2\sqrt{3}}\left((5\sqrt{3}+9)(2+\sqrt{3})^{n-1} + (5\sqrt{3}-9)(2-\sqrt{3})^{n-1}\right). \tag{3.15}$$

Inserting the values already found for θ and γ into (3.9) we obtain

$$t_{2n} = \frac{1}{6}\left((5\sqrt{3}+9)(2+\sqrt{3})^{-1} + (5\sqrt{3}-9)\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{n-1}(2+\sqrt{3})^{-1}\right),$$

proving that t_{2n} tends to $\frac{1}{2}+\frac{1}{6}\sqrt{3} \approx 0.7886751$ and so t_n converges to this number. $\square$

Proposition 3.3. *Let b_n be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbb{Z}} \to [0,2]_{\mathbb{Z}}$, let b_n^i be the number of Khalimsky-continuous functions $[0, n-1]_{\mathbb{Z}} \to [0,2]_{\mathbb{Z}}$ satisfying $f(n-1) = i$ for $i = 0, 1, 2$, and define $P_n^i = b_n^i / b_n$ for $i = 0, 1, 2$. As k tends to infinity,*

$$P_{2k}^2 = P_{2k}^0 \to \tfrac{1}{2} - \tfrac{1}{6}\sqrt{3}, \quad P_{2k-1}^2 = P_{2k-1}^0 \to \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2},$$

also

$$P_{2k}^1 \to \frac{1}{\sqrt{3}}, \quad P_{2k-1}^1 \to 2 - \sqrt{3}.$$

Proof. Using the Khalimsky topology we have

$$\begin{aligned} b_{2k}^0 &= b_{2k-1}^0, & k \geqslant 2, \\ b_{2k}^1 &= 2b_{2k-1}^0 + b_{2k-1}^1, & k \geqslant 2, \\ b_{2k}^2 &= b_{2k-1}^0, & k \geqslant 2, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} b_{2k-1}^0 &= b_{2k-2}^0 + b_{2k-2}^1, & k \geqslant 2, \\ b_{2k-1}^1 &= b_{2k-2}^1, & k \geqslant 2, \\ b_{2k-1}^2 &= b_{2k-2}^0 + b_{2k-2}^1, & k \geqslant 2. \end{aligned} \tag{3.17}$$

Let

$$P_n^i = \frac{b_n^i}{b_n} \text{ for } i = 0, 1, 2.$$

Also we can see easily that $P_n^0 = P_n^2$, so by using (3.2) we get

$$2P_n^0 + P_n^1 = 1. \tag{3.18}$$

It is obvious that the frequencies for odd and even indices are different but there is a relation between them. We shall study them separately. By (3.16),

$$\begin{cases} P_{2k}^0 b_{2k} = P_{2k-1}^0 b_{2k-1}, \\ (1 - 2P_{2k}^0) b_{2k} = 2P_{2k-1}^0 b_{2k-1} + (1 - 2P_{2k-1}^0) b_{2k-1}. \end{cases} \tag{3.19}$$

We solve the equation (3.19) and obtain

$$P_{2k}^0 = \frac{b_{2k} - b_{2k-1}}{2b_{2k}} \text{ and } P_{2k-1}^0 = \frac{b_{2k} - b_{2k-1}}{2b_{2k-1}}.$$

Therefore by Theorem (3.2) we see that, as $k \to \infty$,

$$P_{2k}^0 \to \frac{\theta - 1}{2\theta} = \frac{1}{2} - \frac{1}{6}\sqrt{3},$$

and

$$P_{2k-1}^0 \to \frac{\theta - 1}{2} = \frac{1}{2}\sqrt{3} - \frac{1}{2}.$$

Also by using (3.18) and a simple calculation,

$$P_{2k}^1 \to \frac{1}{\sqrt{3}} \text{ and } P_{2k-1}^1 \to 2 - \sqrt{3}.$$

□

In the following table we can see some values of P_n^i for $i = 0, 1, 2$.

n	6	7	8	9	10	11	12
b_n^0	15	56	56	209	209	780	780
b_n^1	41	41	153	153	571	571	2131
b_n^2	15	56	56	209	209	780	780
b_n	71	153	265	571	989	2131	3691
P_n^0	0.2113	0.36601	0.21132	0.36602	0.21132	0.36602	0.21132
P_n^1	0.57746	0.26797	0.57736	0.26795	0.57735	0.26795	0.57735
P_n^2	0.2113	0.36601	0.21132	0.36602	0.21132	0.36602	0.21132

4. Continuous functions with a four-point codomain

Theorem 4.1. *Let c_n be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbb{Z}} \to [0, 3]_{\mathbb{Z}}$ and let c_n^i be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbb{Z}} \to [0, 3]_{\mathbb{Z}}$ such that $f(n-1) = i$ for $i = 0, 1, 2, 3$. Then $c_1^1 = c_1^2 = 1$, $c_2 = 7$, $c_3 = 15$ and*

$$c_n = c_{n-1} + 2c_{n-2} + c_{n-3}^1 + c_{n-3}^2. \tag{4.1}$$

Formula (4.1) together with formulas (4.3) and (4.4) below determine the c_n.

Proof. We have by definition

$$c_n = c_n^0 + c_n^1 + c_n^2 + c_n^3. \tag{4.2}$$

Using properties of the Khalimsky topology, we see that

$$\begin{aligned} c_{2k+1}^0 &= c_{2k}^0 + c_{2k}^1, & k \geqslant 1, \\ c_{2k+1}^1 &= c_{2k}^1, & k \geqslant 1, \\ c_{2k+1}^2 &= c_{2k}^1 + c_{2k}^2 + c_{2k}^3, & k \geqslant 1, \\ c_{2k+1}^3 &= c_{2k}^3, & k \geqslant 1, \end{aligned} \tag{4.3}$$

and

$$c^0_{2k} = c^0_{2k-1}, \qquad k \geqslant 1,$$
$$c^1_{2k} = c^0_{2k-1} + c^1_{2k-1} + c^2_{2k-1}, \quad k \geqslant 1,$$
$$c^2_{2k} = c^2_{2k-1}, \qquad k \geqslant 1,$$
$$c^3_{2k} = c^2_{2k-1} + c^3_{2k-1}, \qquad k \geqslant 1. \tag{4.4}$$

If we insert (4.3) into (4.2), we get

$$c_{2k+1} = c_{2k} + 2c^1_{2k} + c^3_{2k}. \tag{4.5}$$

By using (4.4), we have

$$2c^1_{2k} + c^3_{2k} = 2c_{2k-1} + c^2_{2k-1} - c^3_{2k-1}. \tag{4.6}$$

But the equations in (4.3) give us

$$c^2_{2k-1} = c^1_{2k-2} + c^2_{2k-2} + c^3_{2k-2},$$
$$c^3_{2k-1} = c^3_{2k-2}. \tag{4.7}$$

Now, we need just to consider the equations (4.5), (4.6) and (4.7) to have the result for odd n, $n = 2k+1$. Next we proceed in the same way for $n = 2k$. Using properites of the Khalimsky topology we see that if we add equation (4.4) to equation (4.2), we see that

$$c_{2k} = c_{2k-1} + 2c^2_{2k-1} + c^0_{2k-1}. \tag{4.8}$$

Therefore, by (4.3) we have

$$2c^2_{2k-1} + c^0_{2k-1} = 2c_{2k-2} + c^1_{2k-2} - c^0_{2k-2}. \tag{4.9}$$

Also, (4.4) gives us

$$c^1_{2k-2} = c^0_{2k-3} + c^1_{2k-3} + c^2_{2k-3},$$
$$c^0_{2k-2} = c^0_{2k-3}. \tag{4.10}$$

We insert (4.10) and (4.9) into (4.8) to get the result for even n. □

We present in the following table the sequence with four values in the codomain and $n \leqslant 10$ points in the domain.

n	1	2	3	4	5	6	7	8	9	10
c_n	4	7	15	31	65	136	285	597	1251	2621

The asymptotic behavior of the number of continuous functions with a four-point codomain

Theorem 4.2. *Let c_n^i be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \to [0,3]_{\mathbb{Z}}$ such that $f(n-1) = i$ for $i = 0,1,2,3$, and let c_n be their sum. Then*

$$\frac{c_n^1 + c_n^2}{c_{n-1}^1 + c_{n-1}^2}, \quad \frac{c_n^0 + c_n^3}{c_{n-1}^0 + c_{n-1}^3} \text{ as well as } \frac{c_n}{c_{n-1}} \text{ tend to}$$

$$\tfrac{1}{2}\sqrt{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}} \approx 2.095293985.$$

Proof. Let us fix a positive number γ (to be determined later) and define sequence t_n^i for $i = 0, \ldots, 3$ by the following equation

$$c_n^i = t_n^i \gamma^n. \tag{4.11}$$

Let

$$t_n = t_n^0 + t_n^1 + t_n^2 + t_n^3 \tag{4.12}$$

Then (4.3) and (4.11) yield

$$\begin{aligned} t_{2k+1}^0 &= \gamma^{-1}(t_{2k}^0 + t_{2k}^1), \\ t_{2k+1}^1 &= \gamma^{-1} t_{2k}^1, \\ t_{2k+1}^2 &= \gamma^{-1}(t_{2k}^1 + t_{2k}^2 + t_{2k}^3), \\ t_{2k+1}^3 &= \gamma^{-1} t_{2k}^3. \end{aligned} \tag{4.13}$$

By (4.4) and (4.11) we get

$$\begin{aligned} t_{2k}^0 &= \gamma^{-1} t_{2k-1}^0, \\ t_{2k}^1 &= \gamma^{-1}(t_{2k-1}^0 + t_{2k-1}^1 + t_{2k-1}^2), \\ t_{2k}^2 &= \gamma^{-1} t_{2k-1}^2, \\ t_{2k}^3 &= \gamma^{-1}(t_{2k-1}^2 + t_{2k-1}^3). \end{aligned} \tag{4.14}$$

We now define a sequence (X_n) as follows.

$$X_n = \begin{pmatrix} t_n^0 \\ t_n^1 \\ t_n^2 \\ t_n^3 \end{pmatrix}, \tag{4.15}$$

and introduce the two matrices

$$A_{2k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_{2k-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.16}$$

By using (4.13), (4.14), (4.15) and (4.16) we can see easily that

$$X_n = \gamma^{-1} A_n X_{n-1} \quad \text{for} \quad n \geqslant 2. \tag{4.17}$$

Let B be equal to $A_{2k+1}A_{2k}$, which is independent of k. Then

$$B = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is symmetric, so there exist a diagonal matrix D whose diagonal entries are the eigenvalues of B and a matrix P such that each column of P is an eigenvector of B and $B = PDP^{\mathrm{T}}$. The columns of P form an orthogonal set, so $PP^{\mathrm{T}} = P^{\mathrm{T}}P$. We shall now determine the eigenvalues and eigenvectors of the matrix B. It has the following characteristic function.

$$\det(B - xI) = x^4 - 7x^3 + 13x^2 - 7x + 1. \tag{4.18}$$

The symmetry of the coefficients in this equation implies that if λ is eigenvalue then also $\frac{1}{\lambda}$ is an eigenvalue. Thus we can find the four eigenvalues of equation (4.18) by putting $\alpha = \lambda_0 + \frac{1}{\lambda_0}$ and $\beta = \lambda_1 + \frac{1}{\lambda_1}$. Then we get $\alpha + \beta = 7$ and $\alpha\beta = 11$, so $\alpha = \frac{7+\sqrt{5}}{2}$ and $\beta = \frac{7-\sqrt{5}}{2}$ and therefore

$$\begin{aligned} \lambda_0 &= \tfrac{7+\sqrt{5}-\sqrt{38+14\sqrt{5}}}{4} \quad \text{and} \quad \lambda_3 = 1/\lambda_0 = \tfrac{7+\sqrt{5}+\sqrt{38+14\sqrt{5}}}{4}, \\ \lambda_1 &= \tfrac{7+\sqrt{5}-\sqrt{38-14\sqrt{5}}}{4} \quad \text{and} \quad \lambda_2 = 1/\lambda_1 = \tfrac{7+\sqrt{5}+\sqrt{38-14\sqrt{5}}}{4}. \end{aligned} \tag{4.19}$$

Let $P = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 \end{pmatrix}$, where P_i is an eigenvector with respect to the eigenvalue λ_i for $i = 0, \dots, 3$. Therefore $BP_i = \lambda_i P_i$. Now we shall solve the following equation system.

$$\begin{cases} 2x + y + z = \lambda x, \\ x + y + z = \lambda y, \\ x + y + 3z + t = \lambda z, \\ z + t = \lambda t, \end{cases} \tag{4.20}$$

where λ is equal to one of the eigenvalues λ_i, and where $P_i = \begin{pmatrix} x & y & z & t \end{pmatrix}^{\mathrm{T}}$ for $i = 0, \dots, 3$. Therefore

$$y = \frac{\lambda - 1}{\lambda} x, \quad z = \frac{\lambda^2 - 3\lambda + 1}{\lambda} x, \quad t = \frac{\lambda^2 - 3\lambda + 1}{\lambda(\lambda - 1)} x.$$

We choose for convenience $x = \lambda(\lambda - 1)$; thus

$$y = (\lambda - 1)^2, \quad z = (\lambda^2 - 3\lambda + 1)(\lambda - 1), \quad t = \lambda^2 - 3\lambda + 1.$$

Let from now on $\lambda = \lambda_3$ and $(x, y, z, t)^{\mathrm{T}}$ be the eigenvectors related to λ_3. Since we need to consider B^k as $k \to \infty$, we need not consider the powers of λ_i for $i = 0, 1, 2$. Hence, the powers of B that we need to consider are

$$B^k = \begin{pmatrix} \lambda^k x^2 & \lambda^k xy & \lambda^k xz & \lambda^k xt \\ \lambda^k xy & \lambda^k y^2 & \lambda^k yz & \lambda^k yt \\ \lambda^k xz & \lambda^k yz & \lambda^k z^2 & \lambda^k zt \\ \lambda^k xt & \lambda^k yt & \lambda^k zt & \lambda^k t^2 \end{pmatrix}.$$

Equation (4.17) and the previous calculation lead us to

$$X_{2k-1} = (\gamma^{-2}\lambda)^{k-3} \begin{pmatrix} x^2 t_5^0 + xyt_5^1 + xzt_5^2 + xtt_5^3 \\ xyt_5^0 + y^2 t_5^1 + zyt_5^2 + tyt_5^3 \\ xzt_5^0 + yzt_5^1 + z^2 t_5^2 + tzt_5^3 \\ xtt_5^0 + ytt_5^1 + ztt_5^2 + t^2 t_5^3 \end{pmatrix}. \tag{4.21}$$

Let $\alpha = xt_5^0 + yt_5^1 + zt_5^2 + tt_5^3$. Thus by (4.14) and (4.21)

$$\begin{aligned} t_{2k}^1 &= \gamma^{-1}(t_{2k-1}^0 + t_{2k-1}^1 + t_{2k-1}^2) = (\gamma^{-2}\lambda)^{(k-3)}\gamma^{-1}(x+y+z)\alpha \\ t_{2k-1}^2 &= (\gamma^{-2}\lambda)^{(k-3)} z\alpha. \end{aligned}$$

We now define $\gamma = \sqrt{\lambda}$, the positive square root of the largest eigenvalue, and find that $(\gamma^{-2}\lambda)^{k-3}$ tends to 1 as $k \to +\infty$. We claim that $\gamma^{-1}(x+y+z) = z$ or equivalently that

$$0 = \gamma^{-1}(x+y+z) - z = x\left[\gamma^{-1}\left(1 + \frac{\lambda-1}{\lambda} + \frac{\lambda^2 - 3\lambda + 1}{\lambda}\right) - \frac{\lambda^2 - 3\lambda + 1}{\lambda}\right].$$

We need to show that

$$\gamma^{-1}(\lambda - 1)\lambda - (\lambda^2 - 3\lambda + 1) = 0.$$

Since λ is the largest root of equation (4.18), we obtain

$$\begin{aligned} 0 &= \lambda^4 - 7\lambda^3 + 13\lambda^2 - 7\lambda + 1 \\ &= \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda + 1 - \lambda^3 + 2\lambda^2 - \lambda \\ &= (\lambda^2 - 3\lambda + 1)^2 - \lambda(\lambda - 1)^2. \end{aligned} \tag{4.22}$$

The equations in (4.22) imply

$$\lambda = \frac{\lambda^2(\lambda-1)^2}{(\lambda^2 - 3\lambda + 1)^2}.$$

Therefore

$$\gamma^{-1} = \frac{\lambda^2 - 3\lambda + 1}{\lambda(\lambda - 1)}.$$

This proves our claim. Hence the sequences t^1_{2k} and t^2_{2k-1} have the same formula, and therefore they tend to the same limit $z\alpha$ as $k \to \infty$. Similarly, we can prove the corresponding result for some other sequences as follows:

$$\begin{aligned} t^1_{2k} &= t^2_{2k-1} \to z\alpha \text{ as } k \to \infty, \\ t^2_{2k} &= t^1_{2k-1} \to y\alpha \text{ as } k \to \infty, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} t^3_{2k} &= t^0_{2k-1} \to x\alpha \text{ as } k \to \infty, \\ t^0_{2k} &= t^3_{2k-1} \to t\alpha \text{ as } k \to \infty. \end{aligned} \tag{4.24}$$

If we sum the two limits in (4.23) we obtain

$$(t^1_n + t^2_n) \text{ tends to } (y+z)\alpha \text{ as } n \to \infty. \tag{4.25}$$

Analogously, (4.24) shows that

$$(t^0_n + t^3_n) \text{ tends to } (x+t)\alpha \text{ as } n \to \infty. \tag{4.26}$$

We now easily conclude that the sum of these two sequences, i.e., (t_n), converges to $(x+y+z+t)\alpha$. Since the sequence $(t^1_n + t^2_n)$ converges, we see easily that

$$\frac{c^1_n + c^2_n}{c^1_{n-1} + c^2_{n-1}} \to \gamma \text{ as } n \to \infty,$$

and also the convergence of the sequence $(t^0_n + t^3_n)$ leads us to

$$\frac{c^0_n + c^3_n}{c^0_{n-1} + c^3_{n-1}} \to \gamma \text{ as } n \to +\infty.$$

We have the same result for c_n/c_{n-1} because as we found, the sequence (t_n) converges to some real number, so

$$\frac{c_n}{c_{n-1}} \to \gamma \text{ as } n \to +\infty.$$

□

We shall now investigate frequencies in the case of a four-point codomain.

Proposition 4.3. *Let c_n be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \to [0,3]_{\mathbb{Z}}$ and let c^i_n be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \to [0,3]_{\mathbb{Z}}$ such that $f(n-1) = i$ for $i = 0,1,2,3$. If $p^i_n = c^i_n/c_n$ for $i = 0,1,2,3$, then*

$$\begin{aligned} &P^3_{2k} \text{ and } P^0_{2k-1} \to \tfrac{x}{x+y+z+t} \approx 0.258582; \\ &P^2_{2k} \text{ and } P^1_{2k-1} \to \tfrac{y}{x+y+z+t} \approx 0.199679; \\ &P^1_{2k} \text{ and } P^2_{2k-1} \to \tfrac{z}{x+y+z+t} \approx 0.418335; \\ &P^0_{2k} \text{ and } P^3_{2k-1} \to \tfrac{t}{x+y+z+t} \approx 0.123402. \end{aligned} \tag{4.27}$$

as $k \to \infty$, where x, y, z, t are the numbers which were defined in the proof of Theorem 4.2. As a consequence, if we add these numbers two and two, the different parities play no role, and we obtain

$$\begin{aligned} P_n^1 + P_n^2 &\to \tfrac{y+z}{x+y+z+t} \approx 0.618014 \\ P_n^0 + P_n^3 &\to \tfrac{x+t}{x+y+z+t} \approx 0.381984 \end{aligned} \tag{4.28}$$

as n tends to infinity.

Proof. By the proof of Theorem 4.2 we know that the sequence (t_n) convergence to number $(x+y+z+t)\alpha$. This fact and (4.23) imply that

$$\begin{aligned} P_{2k}^2 \text{ and } P_{2k-1}^1 &\to \tfrac{y}{x+y+z+t} \approx 0.199679 \text{ as } k \to \infty; \\ P_{2k}^1 \text{ and } P_{2k-1}^2 &\to \tfrac{z}{x+y+z+t} \approx 0.418335 \text{ as } k \to \infty. \end{aligned} \tag{4.29}$$

Analogously, by using (4.24) we conclude that

$$\begin{aligned} P_{2k}^3 \text{ and } P_{2k-1}^0 &\to \tfrac{x}{x+y+z+t} \approx 0.258582 \text{ as } k \to \infty; \\ P_{2k}^0 \text{ and } P_{2k-1}^3 &\to \tfrac{t}{x+y+z+t} \approx 0.123402 \text{ as } k \to \infty. \end{aligned} \tag{4.30}$$

It is obvious that if we sum up the limits in (4.29) we obtain

$$P_n^1 + P_n^2 \to \frac{y+z}{x+y+z+t} \approx 0.618014 \text{ as } n \to \infty,$$

similarly if we sum the limits in (4.30) we have

$$P_n^0 + P_n^3 \to \frac{x+t}{x+y+z+t} \approx 0.381984 \text{ as } n \to \infty.$$

□

In the next table we can see the values of P_n^i for $i = 0, \ldots, 3$ and the sums of some of the frequencies.

n	6	7	8	9	10
c_n^0	17	74	74	324	324
c_n^1	57	57	250	250	1097
c_n^2	27	119	119	523	523
c_n^3	35	35	154	154	677
c_n	136	285	597	1251	2621
P_n^0	0.125	0.259649	0.123953	0.258992	0.123616
P_n^1	0.419117	0.2	0.418760	0.199840	0.418542
P_n^2	0.1985294	0.4175439	0.19933	0.4180655	0.1995422
P_n^3	0.2573529	0.122807	0.2579564	0.1231015	0.2582984
$P_n^0 + P_n^3$	0.3823529	0.382456	0.3819094	0.3820935	0.3819144
$P_n^1 + P_n^2$	0.6176464	0.6175439	0.61809	0.6179055	0.6180842

5. Continuous functions with one fixed endpoint

Theorem 5.1. *Let y_n be the number of Khalimsky-continuous functions $f\colon [0,n-1]_{\mathbb{Z}} \mapsto \mathbb{Z}$ such that $f(0)=0$. Then*

$$y_n = 2y_{n-1} + y_{n-2} \quad \text{for } n \geqslant 3. \tag{5.1}$$

Proof. Let y_n^i be the number of Khalimsky-continuous function $f\colon [0,n-1]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0)=0$ and $f(n-1)=i$. We have $y_n = \sum_{i=-(n-1)}^{n-1} y_n^i$, but with Khalimsky topology we can conclude that we have symmetry for y_n^i, that is, $y_n^i = y_n^{-i}$ for $i=1,\ldots,n-1$. Therefore we can consider another formulation for y_n, i.e.,

$$y_n = y_n^0 + 2\sum_{i=1}^{n-1} y_n^i. \tag{5.2}$$

Moreover, using properties of the Khalimsky topology, we see that

$$y_{2k}^i = \begin{cases} y_{2k-1}^{i-1} + y_{2k-1}^i + y_{2k-1}^{i+1}, & i = 2t-1 \text{ for } t=1,\ldots,k-1,\\ y_{2k-1}^i, & i=2t \text{ for } t=1,\ldots,k-1,\\ y_{2k-1}^{2k-2}, & i=2k-1,\\ y_{2k-1}^0, & i=0, \end{cases} \tag{5.3}$$

and

$$y_{2k+1}^i = \begin{cases} y_{2k}^{i-1} + y_{2k}^i + y_{2k}^{i+1}, & i = 2t \text{ for } t=1,\ldots,k-1,\\ y_{2k}^i, & i=2t-1 \text{ for } t=1,\ldots,k,\\ y_{2k}^{2k-1}, & i=2k,\\ y_{2k}^0 + 2y_{2k}^1, & i=0. \end{cases} \tag{5.4}$$

We shall show the formula for $n=2k$ and for $n=2k+1$ we can have the result in the same way,

$$y_{2k} = y_{2k}^0 + 2\sum_{i=1}^{2k-1} y_{2k}^i = y_{2k}^0 + 2y_{2k}^{2k-1} + 2\sum_{t=1}^{k-1} y_{2k}^{2t} + 2\sum_{t=1}^{k-1} y_{2k}^{2t-1}. \tag{5.5}$$

Equation (5.5) comes from (5.2) and the simple separation of odd and even indices. Plugging equations (5.3) into (5.5) gives us

$$y_{2k} = y_{2k-1}^0 + 2y_{2k-1}^{2k-2} + 2\sum_{t=1}^{k-1} y_{2k-1}^{2t} + 2\sum_{t=1}^{k-1}\left(y_{2k-1}^{2t-2} + y_{2k-1}^{2t-1} + y_{2k-1}^{2t}\right),$$

and then with a simple calculation,

$$\begin{aligned} y_{2k} = \; & y_{2k-1}^0 + 2y_{2k-1}^{2k-2} + 2\sum_{t=1}^{k-1} y_{2k-1}^{2t} + \\ & 2\sum_{t=1}^{k-1} y_{2k-1}^{2t-2} + 2\sum_{t=1}^{k-1} y_{2k-1}^{2t-1} + 2\sum_{t=1}^{k-1} y_{2k-1}^{2t}. \end{aligned} \tag{5.6}$$

We have

$$2\sum_{t=1}^{k-1} y_{2k-1}^{2t-2} = 2y_{2k-1}^{0} + 2\sum_{t=2}^{k-1} y_{2k-1}^{2t-2} = 2y_{2k-1}^{0} + 2\sum_{t=1}^{k-2} y_{2k-1}^{2t}. \tag{5.7}$$

Therefore, by putting (5.7) in (5.6) and using (5.2);

$$y_{2k} = 2y_{2k-1} + 2y_{2k-1}^{2k-2} + y_{2k-1}^{0} + 2\sum_{t=1}^{k-2} y_{2k-1}^{2t} - 2\sum_{t=1}^{k-1} y_{2k-1}^{2t-1}. \tag{5.8}$$

Pluggin (5.4) into (5.8) gives us

$$\begin{aligned} y_{2k} = {} & 2y_{2k-1} + 2y_{2k-2}^{2k-3} + y_{2k-2}^{0} + 2y_{2k-2}^{1} \\ & +2\sum_{t=1}^{k-2} y_{2k-2}^{2t-1} + 2\sum_{t=1}^{k-2} y_{2k-2}^{2t} + 2\sum_{t=1}^{k-2} y_{2k-2}^{2t+1} - 2\sum_{t=1}^{k-1} y_{2k-2}^{2t-1}. \end{aligned} \tag{5.9}$$

By a simple calculation we have the two followings equations,

$$2y_{2k-2}^{2k-3} + 2\sum_{t=1}^{k-2} y_{2k-2}^{2t-1} - 2\sum_{t=1}^{k-1} y_{2k-2}^{2t-1} = 0, \tag{5.10}$$

and

$$2y_{2k-2}^{1} + 2\sum_{t=1}^{k-2} y_{2k-2}^{2t+1} = 2\sum_{t=1}^{k-1} y_{2k-2}^{2t-1}. \tag{5.11}$$

Finally, by putting (5.10) and (5.11) into (5.9) and by using (5.2), we obtain the desired formula. □

The sequence in Theorem 5.1 is a well-known sequence, and appears as sequence number A078057 in Sloane's Encyclopedia. It is given by the explicit formula $y_n = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]$. Actually y_n has the same recursion formula as the Pell numbers P_n, but with different initial values. The sequence (P_n) is defined as

$$P_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ 2P_{n-1} + P_{n-2}, & n \geqslant 2. \end{cases}$$

The reader can find more information about this sequence in item (A000129) of the encyclopedia. Now we shall study the asymptotic behavior of this sequence as we did for earlier sequences. In the next theorem we shall show that y_n tends to the Silver Ratio $1+\sqrt{2}$ as n tends to infinity.

Theorem 5.2. *Let y_n be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbb{Z}} \mapsto \mathbb{Z}$ such that $f(0) = 0$. Then $y_{n+1}/y_n \to 1+\sqrt{2}$ as $n \to \infty$.*

Proof. We define the sequence (t_n) by the equation $t_n = y_n\gamma^{-n}$ for $n \geqslant 1$. By using (5.1) we have

$$\gamma^2 t_n - 2\gamma t_{n-1} - t_{n-2} = 0,$$

thus

$$t_n - t_{n-1} = (\frac{2}{\gamma} - 1)t_{n-1} + \frac{1}{\gamma^2}t_{n-2}. \tag{5.12}$$

We are interested in having the sum of the two coefficients in (5.12) to be zero. Hence, we conclude that γ is the positive solution of the equation $\gamma^2 - 2\gamma - 1 = 0$. Thus, $\gamma = 1 + \sqrt{2}$.

$$|t_n - t_{n-1}| = \gamma^{-2}|t_{n-1} - t_{n-2}| = \gamma^{-2(n-2)}|t_2 - t_1|.$$

The sequence (t_n) is a Cauchy sequence and hence it converges. Thus

$$\frac{y_n}{y_{n-1}} = \frac{t_n}{t_{n-1}}\gamma \to 1 + \sqrt{2} \text{ as } n \to \infty.$$

□

The following table shows the values of y_n^i and y_n for $1 \leqslant n \leqslant 10$.

9										1
8									1	1
7								1	1	17
6							1	1	15	15
5						1	1	13	13	113
4					1	1	11	11	85	85
3				1	1	9	9	61	61	377
2			1	1	7	7	41	41	231	231
1		1	1	5	5	25	25	129	129	681
0	1	1	3	3	13	13	63	63	321	321
−1		1	1	5	5	25	25	129	129	681
−2			1	1	7	7	41	41	231	231
−3				1	1	9	9	61	61	377
−4					1	1	11	11	85	85
−5						1	1	13	13	113
−6							1	1	15	15
−7								1	1	17
−8									1	1
−9										1
y_n	1	3	7	17	41	99	239	577	1393	3363

There is a nice relation between the Delannoy numbers and the number of Khalimsky-continuous functions with a fixed point $f(0) = 0$. The Delannoy numbers were introduced by Henri Delannoy (1895). The Delannoy array $d_{j,k}$ is

$$d_{j,k} = d_{j-1,k} + d_{j,k-1} + d_{j-1,k-1},$$

with conditions $d_{0,0} = 1$ and $d_{j,k} = 0$ for $j < 0$ or $k < 0$. The numbers $(d_{i,i})_{i \geqslant 0} = 1, 3, 13, 63, 321, 1683, 8989, 48639, \dots$ (A001850) are known as the central Delannoy numbers. We give the Delannoy numbers in the following table:

6	1	13	85	377	1289	3653	8989
5	1	11	61	231	681	1683	3653
4	1	9	41	129	321	681	1289
3	1	7	25	63	129	231	377
2	1	5	13	25	41	61	85
1	1	3	5	7	9	11	13
0	1	1	1	1	1	1	1
	0	1	2	3	4	5	6

There are connections between many matematical problems and the Delannoy numbers. Sulanke (2003) listed 29 different contexts where the central Delannoy numbers appear. A classical example is the number of lattice paths from $(0, 0)$ to (n, n) using the steps $(0, 1)$, $(1, 0)$, and $(1, 1)$. From this path model one can obtain a combinatorial proof that, for $n \geqslant 0$,

$$d_{n,n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n+i}{i}.$$

In the next theorem we can see the 30th example of Delannoy numbers.

Theorem 5.3. *Let y_n^i be the number of Khalimsky-continuous functions $f\colon [0, n-1]_{\mathbb{Z}} \mapsto \mathbb{Z}$ such that $f(0) = 0$ and $f(n-1) = i$. Then $y_n^i = d_{r,s}$ for $r = \frac{1}{2}(n-1-i)$ and $s = \frac{1}{2}(n-1+i)$ where $n-1+i \in 2\mathbb{Z}$.*

Proof. We shall use induction to prove the result. It is easy to see that $y_1^0 = 1 = d_{0,0}$, $y_2^1 = 1 = d_{0,1}$, $y_2^{-1} = 1 = d_{1,0}$ and $y_3^0 = 3 = d_{1,1}$. Suppose that the formula is true for $t < 2k$. We shall show that the result is true for $t = 2k$. The proof for $t = 2k+1$ can be done in the same way. We consider i such that $2k-1+i \in 2\mathbb{Z}$; hence i is odd number. For an even number i the proof is simple because by (5.3), $y_{2k}^i = y_{2k-1}^i$ and so it is true by induction. By (5.3)

$$y_{2k}^i = y_{2k-1}^{i-1} + y_{2k-2}^i + y_{2k-1}^{i+1}. \tag{5.13}$$

By the statement we have

$$y_{2k-1}^{i-1} + y_{2k-2}^i + y_{2k-1}^{i+1} = d_{r,s-1} + d_{r-1,s-1} + d_{r-1,s}, \tag{5.14}$$

where

$$\frac{2k-1-i}{2} = r \text{ and } \frac{2k-1+i}{2} = s \tag{5.15}$$

Thus by (5.13), (5.14) and (5.15), we get the result. □

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References

Cull, Paul; Flahive, Mary; Robsen, Robby. 2005. *Difference Equations. From Rabbits to Chaos.* Springer. Xiii + 392 pp.

Delannoy, H[enri]. 1895. Emploi de l'échiquier pour la résolution de certains problèmes de probabilité. *Comptes rendus du congrès annuel de l'Association Française pour l'avancement des sciences* **24**, Bordeaux, 70–90.

Kiselman, Christer O. 2004. *Digital geometry and mathematical morphology.* Lecture notes, Uppsala University.

Melin, Erik. 2005. Extension of continuous functions in digital spaces with the Khalimsky topology. *Topology Appl.* 153, 52–65.

Sloane, Neil J. A. *The On-Line Encyclopedia of Integer Sequences.*

Sulanke, Robert A. 2003. Objects counted by the central Delannoy numbers. *J. Int. Sequences.* Vol. 6, Article 03.1.5.