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Series and Integrals in Several and Infinitely Many Complex Variables

Lisa Nilsson

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Department of Mathematics
Stockholm University

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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Series and Integrals in Several and Infinitely Many Complex Variables

Lisa Nilsson

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Contents

0.1 General introduction

0.1.1 Outline of the thesis

This thesis consists of two separate parts.

The first part, Chapter 1, deals with the theory of amoebas and coamoebas and their connection to hypergeometric functions. In the introduction we give some motivation for this study by discussing Gauss' classical hypergeometric function and describing how this function can be related to some combinatorics of very simple polytopes and amoebas. We then treat the general several variable case for hypergeometric series in Section 1.3. We prove a Proposition stating the exact number of series generated by a certain simplex in the GKZ-method. We also give a Theorem defining the exact convergence domains for these series. In Section 1.5 we discuss the *Mellin-Barnes* integral representation of hypergeometric functions and describe the exact convergence domains for these integrals. In Section 1.6 we do some more explicit calculations of series, integrals, amoebas and coamoebas.

Chapter 2 deals with infinite dimensional complex analysis, and the possibility of obtaining integral representations formulas valid on an infinite dimensional space. We first give some background information about holomorphic functions in infinitely many variables, topologies, fully nuclear spaces, and Gaussian (pro)measures. We then in Section 2.6 show that we can extend an integral representation formula known for holomorphic functions in finite dimensions in Fischer-Fock space to the case of entire functions of exponential type on infinite dimensional spaces that are fully nuclear with a basis.

0.1.2 Notation and prerequisites

We will use the following standard notation: By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ we denote respectively, the set of natural numbers, the set of integers, the set of real numbers and the set of complex numbers, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathbb{N}^{\mathbb{N}}$ is the set of n-tuples of natural numbers. The space of Laurent polynomials with monomials in some generating set A will be denoted by $\mathbb{C}^A$. $\mathcal{A}_f$ is the amoeba of the polynomial f, $\mathcal{A}'_f$ is the coamoeba of f, and $\mathcal{N}_f$ the Newton polytope of f. We will use Pochammer symbol $(\alpha)_k = \Gamma(\alpha + k)\Gamma(\alpha)$ where $\Gamma(z)$ is the classical Gamma-function. E_A will denote the generalized resultant called the principal A-determinant.

We will denote by E a locally convex Hausdorff space, where E^* is the algebraic dual space and E' is the topological dual. E'_β is the strong dual, i.e. the dual space equipped with the strong topology β.

0.1.3 Acknowledgments

I would like to thank my supervisor Mikael Passare for his support and inspiration. The first part of this thesis is based on joint work with Mikael Passare and August Tsikh, with whom I also had many interesting discussions.

I would also like to thank Seán Dineen for his hospitality during my time at University College Dublin, and for our joint work on which Chapter 2 of this thesis is based.

Chapter 1

Hypergeometric series and integrals

Summary

We give a brief and accessible introduction to the general theory of A-hypergeometric functions, and we prove two new theorems on the domains of convergence for$A$-hypergeometric series and for the associated Mellin-Barnes type integrals. The exact convergence domains are described in terms of the (co-)amoebas of the corresponding principal A-determinants.

1.1 Introduction

The classical Gauss hypergeometric function

$$F(z) = {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}$$

is a solution to the second order differential equation

$$z(z-1)\, F'' + \big[(1+\alpha+\beta)z - \gamma\big]\, F' + \alpha\beta\, F = 0\,.$$

It is an important function in many contexts in mathematics and mathematical physics. Several well known functions in mathematical physics such as the Legendre polynomials, the Chebyshev polynomials, the Jacobi polynomials, the Rieman P-series and complete elliptic integrals can all be expressed in terms of ${}_2F_1$.

There are numerous generalizations of the Gauss functions, also to the case of several variables. This has, among others, been done by Jacob Horn , who in

1889 gave the following definition of a hypergeometric function in two variables: The double power series

$$\sum_{m,n=0}^{\infty} A_{m,n} x^m y^n$$

is hypergeometric if the quotients $A_{m+1,n}/A_{m,n}$ and $A_{m,n+1}/A_{m,n}$ are rational functions of the indices m and n. He also constructed a list of the 34 distinct convergent series of order 2 and their convergence domains, se [5] or [3]. The order of a series is the highest degree of the denominator and numerator of $A_{m+1,n}/A_{m,n}$ and $A_{m,n+1}/A_{m,n}$. We will in this paper consider generalizations of the hypergeometric functions to n variables consistent with the definition used by Horn.

In a seminal series of papers during the 1980's Gelfand, Kapranov, Zelevinsky (GKZ) and their collaborators developed a new fruitful approach to the general theory of hypergeometric functions. It has connections to toric geometry, combinatorics of polytopes and a number of other fields.

We will assume the values of the parameters α, β, γ to be generic, that is, not integer valued or differing from each other by integer values. This means we can without restriction use formulas such as the *Second Functional Equation* for manipulating the Γ functions that occur. We remind you that the *Second Functional Equation* is given by

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin \pi z}.$$

Using this will enable us to move the Γ-functions between numerator and denominator.

The basic idea of the GKZ-approach is to cleverly introduce extra variables, one for each parameter in the hypergeometric series. Their main observation was that the new function (of many variables) thus obtained will satisfy a very simple (binomial) system of differential equations with constant coefficients.

Let us illustrate the GKZ-method in the case of the classical gauss function F:

The function

$$\Phi(a,b,c,d) = \frac{a^{-\alpha} b^{-\beta} c^{-1+\gamma}}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\gamma)} F\Big(\frac{cd}{ab}\Big)$$

can also be written (as a more universal Γ-series)

$$\Phi(a,b,c,d) = \sum_{\nu \in \mathbf{N}} \frac{a^{-\alpha-\nu}}{(-\alpha-\nu)!} \frac{b^{-\beta-\nu}}{(-\beta-\nu)!} \frac{c^{-1+\gamma+\nu}}{(-1+\gamma+\nu)!} \frac{d^{\nu}}{\nu!} \,,$$

and therefore obviously satisfies the differential equation $\Phi_{ab} = \Phi_{cd}$.

The fact that Φ is essentially a one-variable function is reflected by three supplementary homogeneity equations:

$$\left\{\begin{array}{rcl} a\Phi_a + b\Phi_b + c\Phi_c + d\Phi_d & = & (-\alpha - \beta - 1 + \gamma)\Phi\,, \\ a\Phi_a + d\Phi_d & = & -\alpha\Phi\,, \\ b\Phi_b + d\Phi_d & = & -\alpha\Phi\,. \end{array}\right.$$

More generally, any shifted Γ-series formally will satisfy these kind of equations and it is not dependent of the representation used. By a shifted Γ-series we mean a series obtained by replacing the indice ν by a linear translation of it. The reason is the symmetric form of the terms over which we are summing.

Notice that $\Phi(1,1,1,z) = C\,F(z)$ for some constant C.

It is natural to make a shift by $-1+\gamma$ and thus to consider also the new function

$$\Phi'(a,b,c,d) = \sum_{\nu\in\mathbf{N}} \frac{a^{-\alpha-1+\gamma-\nu}}{(-\alpha-1+\gamma-\nu)!}\frac{b^{-\beta-1+\gamma-\nu}}{(-\beta-1+\gamma-\nu)!}\frac{c^{\nu}}{\nu!}\frac{d^{1-\gamma+\nu}}{(1-\gamma+\nu)!}\,,$$

which satisfies *the same* equations.

One then has $\Phi'(1,1,1,z) = CF'(z)$, where

$$F'(z) = z^{1-\gamma}{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; z)\,.$$

Again we can in Φ make the shift $\nu \mapsto 1-\beta-\nu$ which gives the series

$$\Phi''(a,b,c,d) = \sum_{\nu\in\mathbf{N}} \frac{a^{-\alpha+\beta+\nu}}{(-\alpha+\beta+\nu)!}\frac{b^{\nu}}{\nu!}\frac{c^{-1+\gamma-\beta-\nu}}{(-1+\gamma-\beta-\nu)!}\frac{d^{-\beta-\nu}}{(-\beta-\nu)!}\,,$$

where $\Phi''(1,1,1,z) = C'F''(z)$ for some constant C' and

$$F''(z) = z^{-\beta}{}_2F_1(\alpha-\beta, 2-\beta, 1-\gamma+\beta, z^{-1}).$$

A final shift in Φ with $\nu \mapsto -\alpha-\nu$ gives

$$\Phi'''(a,b,c,d) = \sum_{\nu\in\mathbf{N}} \frac{a^{\nu}}{\nu!}\frac{b^{-\beta+\alpha+\nu}}{(-\beta+\alpha+\nu)!}\frac{c^{-1+\gamma-\alpha-\nu}}{(-1+\gamma-\alpha-\nu)!}\frac{d^{-\alpha-\nu}}{(-\alpha-\nu)!}\,.$$

Letting a, b and c equal to 1 gives $\Phi''(1,1,1,z) = C''F'''(z)$, for some constant C''' where we see that

$$F'''(z) = z^{-\alpha}{}_2F_1(2-\alpha, \beta-\alpha, 1-\gamma+\alpha, z^{-1}).$$

According to the theory of GKZ we can relate some combinatorics to these series. We consider the configuration of the four points $\{(0,0),(1,0),(0,1),(1,1)\} \in \mathbb{Z}^2$ and form the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

of the corresponding projective vectors whose columns are the points above. If the matrix B is the integer basis of the kernel of the linear mapping $A : \mathbb{Z}^4 \to \mathbb{Z}^3$, then B is the following matrix

$$B = \pm \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that the entries in the B-matrix equal the coefficients for ν in Φ and Φ', whereas the coefficients for ν in Φ'' and Φ''' coincides with the entries of the negative B-matrix. We now draw the convex hull of $\{(0,0),(1,0),(0,1),(1,1)\} \in \mathbb{Z}^2$. This is called the Newton polygon of the polynomial f where f is the polynomial with exponent vectors described by $\{(0,0),(1,0),(0,1),(1,1)\}$, i.e. $f = a + bx + cy + dxy$.

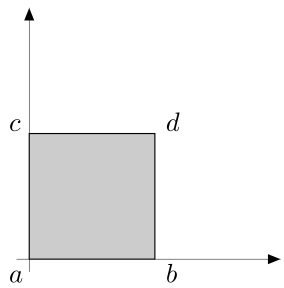

FIGURE 1. *Newton polygon of the polynomial f.*

There are two different triangulations of the Newton polygon.

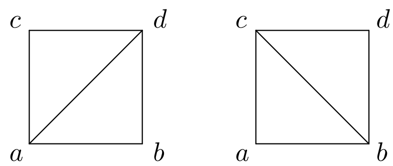

FIGURE 2. *The triangulations of the Newton polygon.*

We will see that choosing one of the four simplices in Figure 2 and letting the complex variables related to the vertices of this simplex equal to 1 will correspond to choosing exactly one of the four shifted Γ-series ϕ_1, ϕ_2, ϕ_3 and ϕ_4. The principal A-determinant of the matrix A, which is the discriminant of the polynomial f, is

$$E_A(a, b, c, d) = abcd(-ab + cd).$$

It is homogeneous of degree 6, and if we take the dehomogenized polynomial, letting three of the variables equal to 1, this gives us the one-variable polynomials $E_A(1, 1, 1, d) = d(-1 + d)$, $E_A(1, 1, c, 1) = c(-1 + c)$ and so on.

The amoeba of the algebraic set $\{E_A(1, 1, 1, x) = 0\}$ is defined to be its image under the mapping $\mathrm{Log} : x \mapsto \log |x|$. Taking the amoeba of $E_A(1, 1, 1, x)$, $E_A(1, 1, x, 1)$, $E_A(1, x, 1, 1)$, $E_A(x, 1, 1, 1)$ will in each case give simply the set $\{x = 0\}$ in logarithmic coordinates.

FIGURE 3. *The amoeba of the principal A-determinant $E_A(x)$.*

Obviously this amoeba has two connected components in its complement. It turns out that the convergence domain for our four series will coincide with these complement components. Φ and Φ' will converge in the component $\{x < 0\}$ and Φ'' and Φ''' in the component $\{x > 0\}$. In fact the convergence domain of each series will correspond to that particular complement component which contains the normal cone at the selected vertex of the Newton polygon of the principal A-determinant polynomial.

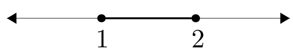

FIGURE 4. *The Secondary polytope of f and the normal cones at the vertices.*

The function ${}_2F_1$ may also be represented as the following integral that was introduced by E. W. Barnes in 1908

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s ds, \quad (1.1) $$

where $|arg(-z)| < \pi$ and where the path of integration is indented if necessary in such a manner as to separate the poles at $s = 0, 1, 2, \ldots$ from the poles at $s = -\alpha - n, \quad s = -\beta - n \quad (n = 0, 1, 2, \ldots)$ of the integrand. It is always possible to find such a path of integration provided that both α and β is different from $0, -1, -2, \ldots$

The proof of (1.1) follows by classical calculus of residues as the sum of residues of the integrand at the poles $s = 0, 1, 2, \ldots$.

Now calculating the coamoeba i.e. the image of the mapping $\mathrm{Arg} : x \mapsto \arg(x)$, of the principal A-determinant $E_A(x)$ we get the set $\{x = 0\}$.

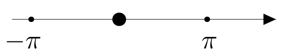

FIGURE 5. *The coamoeba of the A-discriminant $E_A(x)$.*

Note that the argument is periodic, meaning $-\pi$ is identified with π, so the coamoeba in Figure 5 has in fact one complement component. The convergence domain of the integral is easily seen to coincide with the complement of the coamoeba under the inverse logarithmic mapping.

In this paper we will give a thorough study of this situation generalized to hypergeometric series in several variables.

1.2 Basic definitions

We start by introducing some background information and give some notation and definitions that will be needed througout Chapter 1. Terminology and notation of less overall importance will be defined later on in the specific context in which it is used.

We let f be a (Laurent) polynomial in n variables, $f(x_1, x_2, \ldots, x_n) = \sum_{\omega \in \mathbb{Z}^k} a_\omega x^\omega$. An important characteristic of f is its Newton polytope of f, $\mathcal{N}_f$ defined as follows

Definition 1 *The Newton polytope $\mathcal{N}_f$ is the convex hull in $\mathbb{R}^n$ of the set $\{\omega : a_\omega \neq 0\}$.*

We denote by $\mathcal{Z}_f$ the hypersurface determined by the equation $f = 0$, and introduce the logarithmic mapping $(\mathbb{C}^*)^n \to \mathbb{R}^n$ given by

$$\mathrm{Log} : (x_1, x_2, \ldots, x_n) \mapsto (\log|x_1|, \log|x_2|, \ldots, \log|x_n|).$$

Using this mapping we define the amoeba $\mathcal{A}_f$ of the polynomial f.

Definition 2 $$\mathcal{A}_f := \mathrm{Log}(\mathcal{Z}_f)$$

All components in the complement of the amoeba will be convex. Furthermore if we consider the Laurent expansions of the rational function $1/f$ these will be in bijective correspondence with the amoeba complement components. In fact the same can be said about the vertices of the Newton polytope, these are also in bijective correspondence with the amoeba complement components. This should motivate that these are all indeed very natural geometric constructions to make.

Similarily we define the coamoeba $\mathcal{A}'_f$ using the argument mapping $\mathrm{Arg} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by

$$\mathrm{Arg} : (x_1, x_2, \ldots, x_n) \mapsto (\arg|x_1|, \arg|x_2|, \ldots, \arg|x_n|).$$

Definition 3 $$\mathcal{A}'_f := \mathrm{Arg}(\mathcal{Z}_f)$$

In order to study the singularities of the function f we introduce a generalised version of the ordinary resultant R_A (see [4]), which we call the principal A-determinant. We define this in terms of R_A as follows

Definition 4 *Let $A \subset \mathbb{Z}^{n-1}$ be a finite subset which affinely generates $\mathbb{Z}^{n-1}$ over $\mathbb{Z}$. For any $f = f(x_1, \ldots x_{n-1}) \in \mathbb{C}^A$, where $\mathbb{C}^A$ denotes the space of Laurent polynomials with monomials from A, we define the principal A-determinant,*

$$E_A(f) = R_A\Big(x_1 \frac{\partial f}{\partial x_1}, \ldots, x_{n-1} \frac{\partial f}{\partial x_{n-1}}, f\Big).$$

Note that this make sense since all $x_i(\partial f/\partial x_i)$ belongs to $\mathbb{C}^A$. Furthermore E_A is clearly a polynomial function in coefficients of f. We will in this paper be concerned primarily with amoebas of such principal A-determinant polynomials. These have some pleasant properties such as always being solid, a result by Passare, Tsikh and Sadykov [7].

1.3 A-hypergeometric series

Consider a finite set $\mathfrak{A} = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^{n-1}$ and compose the following $n \times N$-matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_N \end{pmatrix},$$

where the α_i:s are column vectors with $n-1$ entries. Assume that the columns of A generate the lattice $\mathbb{Z}^n$. We shall denote by B the Gale transform for A, that is, the $N \times (N-n)$-matrix which annihilates the matrix A, in the sense that $AB = 0$, and contains a unit matrix E_{N-n} of size $N-n$. More precisely, there should be a subset $J \subset \{1, \ldots, N\}$ with $|J| = N-n$ such that the rows in B with numbers from J together form a unit matrix. Notice that J also gives a subset of $N-n$ columns from A. We shall write $I = \{1, \ldots, N\} \setminus J$, so that $|I| = n$. We then have a collection $\{\alpha_i\}_{i \in I}$ of points in $\mathbb{Z}^{n-1}$, and this collection determines a simplex $\sigma = \sigma_I$ of full dimension $n-1$ with vertices in the points α_i, $i \in I$. This simplex σ_I lies in the polytope Q which is the convex hull (in $\mathbb{R}^{n-1}$) of the original set $\mathfrak{A}$. Let us now associate with each $\alpha_i \in \mathfrak{A}$ a complex variable a_i, $i = 1, \ldots, N$. To simplify the notation we shall assume that $I = (1, 2, \ldots, n)$.

Definition 5 *A general A-hypergeometric function we define, for every Gale transform B of the matrix A, or equivalently, for every $(n-1)$-dimensional simplex $\sigma \subset Q$ with vertices in $\mathfrak{A}$, to be the following power series*

$$\phi(a) = \phi_B(a) = \sum_{k \in \mathbb{N}^{N-n}} \frac{a^{\gamma + \langle B, k \rangle}}{\prod_{j=1}^{n} \Gamma(\gamma_j + \langle B_j, k \rangle + 1)\, k!}, \tag{1}$$

where the B_j denote the rows in the matrix B, and $k! = k_1! \cdots k_{N-n}!$.

Using the notation $a = (a', a'') \in (\mathbb{C} \setminus \{0\})^n \times (\mathbb{C} \setminus \{0\})^{N-n}$ we see that the numerator in (1) can be written $(a')^{\langle B', k \rangle} \cdot (a'')^k$, since the matrix B is of the form $(B' \,|\, E_{N-n})^{\mathrm{tr}}$. Note that the series (1) is a Puiseux series (since B' is a rational matrix) multiplied by the monomial $(a')^\gamma$. We shall assume that the exponent vector $\gamma \in \mathbb{C}^n$ is in general position.

In fact, for fixed γ the expression (1) may represent several series (corresponding to different branches of a multivalued function), and we will prove below that the number of different series obtained is equal to the normalized volume Δ of the simplex σ_I.

Proposition 1 *Let A be a matrix of size $n \times N$, $n \le N$ with integer entries. Then the following claims are equivalent:*
(i) the column span of A is the entire lattice $\mathbb{Z}^n$;
(ii) the maximal minors of A are relatively prime;
(iii) there is a unimodular matrix M of size $N \times N$ such that $AM = (E_n|0)$ (the unity matrix of size $n \times n$ enlarged by adding zeros to a $n \times N$-matrix);
(iv) there is a completion $\widetilde{A}$ which is unimodular, i.e. we can enlarge A to a $N \times N$ integer square matrix $\widetilde{A}$ with $\det \widetilde{A} = 1$.

Proof $(i) \Rightarrow (ii)$ Given the condition(i), there exists an integer $N \times n$-matrix C with the property that AC is equal to the unit matrix E_n of size $n \times n$. From this it follows that all the maximal minors of the matrix A are relatively prime, since by the well known Binet-Cauchy formula ([1]) the determinant of AC equals the sum of the maximal minors of A multiplied with the corresponding minors of C.

$(ii) \Rightarrow (iii)$ By the invariant factor theorem ([1]) it follows that there exist unimodular integer matrices D and F of size n and N respectively, such that $DAF = (\delta|0)$ where δ is a diagonal $n \times n$-matrix with integers $\epsilon_1, \ldots, \epsilon_n$ on the diagonal, and 0 is the zero-matix of size $n \times (N-n)$. From the representation $A = D^{-1}(\delta|0)F^{-1}$ it is now easy to see that in fact $\delta = E_n$, because some ϵ_j being different from 1 would contradict the fact that the maximal minors of A are relatively prime.

$(iii) \Rightarrow (iv)$ By (iii) we have

$$A = D^{-1}(E_n|0)F^{-1} = (D^{-1}|0)F^{-1},$$

and the desired enlargement of A may be taken to be

$$\widetilde{A} = \left(\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & E_{N-n} \end{array} \right) F^{-1}.$$

$(iv) \Rightarrow (i)$ This is obvious.

□

Proposition 2 *For a given $n \times N$-matrix A whose columns constitute a basis of $\mathbb{Z}^n$, and a chosen simplex $\sigma \subset Q$ of normalized volume Δ, there will be precisely Δ distinct A-hypergeometric series, which are linearly independent.*

Proof Let A' denote the $n \times n$-submatrix in A with columns corresponding to the simplex σ. By the asssumption made before Definition 1 this means that A' consists of the n first columns in A. So we can write A in the block form $(A' \,|\, A'')$ with A'' being the $n \times (N-n)$-matrix of the last $N-n$ columns in A. Along with the annihilator B that was introduced above, we shall also consider an integer valued annihilator R, which is also a matrix of size $N - n \times N$, and whose columns constitute a basis for the relation lattice between the columns of the original matrix A. We write also this matrix R in block form

$$R = \left(\begin{array}{c} R' \\ \hline R'' \end{array} \right) .$$

Now since the columns of A generate $\mathbb{Z}^n$ it follows from Proposition (1) that there is a unimodular completion $\widetilde{A}$ of A, which, recalling that $A = (A' \,|\, A'')$, we write in the block form

$$\widetilde{A} = \left(\begin{array}{c|c} A' & A'' \\ \hline * & * \end{array} \right) .$$

Let us consider the corresponding block composition of the inverse matrix :

$$\widetilde{A}^{-1} = \left(\begin{array}{c|c} * & R' \\ \hline * & R'' \end{array} \right) ,$$

with R' and R'' of size $n \times (N-n)$ and $(N-n) \times (N-n)$ respectively. According to Jacobi's formula ([1]) one has $\det A' / \det R'' = (\det \widetilde{A})^{N-n-1} = \pm 1$ and hence $|\det R''| = |\det A'|$. Since $|\det A'|$ equals the normalized volume Δ of the simplex spanned by the vectors in A', we conclude that

$$\Delta = |\det A'| = |\det R''| .$$

In the lattice $\mathbb{Z}^N$ we consider the sublattice $L = A^{-1}(0)$ whose rank is $N-n$. It is clear that it is generated by the columns of the matrix R. The columns of the matrix B generate another lattice M of rank $N-n$, which contains L as a sublattice. By the invariant factor theorem there are unimodular matrices X and Y, such that the new bases for the lattices L and M given by

$$\widetilde{R} = R \cdot X , \quad \widetilde{B} = B \cdot Y$$

have the property that the basis $\widetilde{R}$ is expressed in the basis $\widetilde{B}$ by means of a diagonal integer matrix:

$$\widetilde{R} = \widetilde{B} \cdot \begin{pmatrix} \delta_1 & & \\ & & \\ & & \delta_{N-n} \end{pmatrix} , \qquad \delta_j \in \mathbb{Z} .$$

In other words, if $\tilde{r}^1, \ldots \tilde{r}^{N-n}$ denote the column vectors of $\widetilde{R}$, then

$$M = \left\{ \frac{\tilde{r}^1}{\delta_1} s_1 + \ldots + \frac{\tilde{r}^{N-n}}{\delta_{N-n}} s_{N-n} \right\}_{s \in \mathbb{Z}^{N-n}} .$$

From this it is easily seen that the series (1) can be re-written in the powers

$$\left(a^{\tilde{r}^1/\delta_1} \right) s_1 \cdots \left(a^{\tilde{r}^{N-n}/\delta_{N-n}} \right) s_{N-n} .$$

Clearly, the index $M : L$ is equal to $\Delta = |\det R''| = |\delta_1 \cdots \delta_{N-n}|$ and by choosing various radicals $(a^{\tilde{r}^j})^{1/\delta_j}$ we obtain Δ different, and linearly independent series.

1.4 Domains of convergence for A-hypergeometric series

We want to construct a triangulation of $(Q, \mathfrak{A})$, i.e. a triangulation on Q with the set of vertices on $\mathfrak{A}$. We do this in the following way. Take any function

$\psi : \mathfrak{A} \to \mathbb{R}$ and consider in the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ the union of the vertical half-lines

$$\{(\omega, y) \in \mathfrak{A} \times \mathbb{R} : y \leq \psi(\omega)\}.$$

Let $\mathcal{G}_\psi$ be the convex hull of all these half-lines. This is an unbounded polyhedron projecting onto Q. The faces of G_ψ which do not contain vertical half-lines(i.e. are bounded) form the bounded part of the boundary of G_ψ, which we call the upper boundary of G_ψ. Clearly the upper boundary projects bijectively onto Q. If the function ψ is chosen to be generic enough, then all the bounded faces of G_ψ are simplices and therefore their projections to Q form a triangulation of $(Q, \mathfrak{A})$.

Let T be an arbitrary triangulation of $(Q, \mathfrak{A})$, and let $\psi : A \to \mathbb{R}$ be any function. Then there is a unique T-piecewise-linear function $g_{\psi,T} : Q \to \mathbb{R}$ such that $g_{\psi,T}(\omega) = \psi(\omega)$ when ω is a vertex of the triangulation T. The function $g_{\psi,T}$ is obtained by affine interpolation of ψ inside each simplex. Note that the values of ψ at points that are not vertices of any simplex of T does not affect the function $g_{\psi,T}$.

Definition 6 *Let T be a triangulation of $(Q, \mathfrak{A})$. For each simplex σ of this triangulation we shall denote by $C(\sigma)$ the cone in $\mathbb{R}^{\mathfrak{A}}$ consisting of functions $\psi : \mathfrak{A} \to \mathbb{R}$ with the following properties:*
(a) the function $g_{\psi,T} : Q \to \mathbb{R}$ is concave.
(b) for any $\omega \in \mathfrak{A}$ which is not a vertex of the simplex σ in the triangulation T, we have $g_{\psi,T}(\omega) \geq \psi(\omega)$.

Now, let $\mathfrak{A} \subset \mathbb{Z}^{n-1}$ be a finite subset, and Q the convex hull of $\mathfrak{A}$ as before. We assume that $\dim(Q) = n - 1$. Fix a translation invariant volume form Vol on $\mathbb{R}^n$. Let T be a triangulation of (Q, A). By the characteristic function of T we shall mean the function $\varphi_T(\omega) : A \to \mathbb{R}$ defined as follows:

$$\varphi_T(\omega) = \sum_{\delta : \omega \in Vert(\delta)} \mathrm{Vol}(\delta),$$

where the summation is over all (maximal) simplices δ of T for which ω is a vertex. In particular, $\varphi_T(\omega) = 0$ if ω is not a vertex of any simplex of T. Let $\mathbb{R}^A$ denote the space of all functions $A \to \mathbb{R}$.

Definition 7 *The secondary polytope $\Sigma(\mathfrak{A})$ is the convex hull in the space $\mathbb{R}^{\mathfrak{A}}$ of the vectors φ_T for all the triangulations T of $(Q, \mathfrak{A})$.*

The normal cone to $\Sigma(\mathfrak{A})$ at every φ_T will be called $N_{\varphi_T}\Sigma(\mathfrak{A})$ and consists of all linear forms ψ on $\mathbb{R}^{\mathfrak{A}}$ such that

$$\psi(\varphi_T) = \max_{\varphi \in \Sigma(A)} \psi(\varphi).$$

The point φ_T is a vertex of $\Sigma(A)$ if and only if the interior of this cone is non-empty. The union of the normal cones $N_{\varphi_{T_1}}\Sigma(A), \ldots, N_{\varphi_{T_k}}\Sigma(A)$ where

$T_1, \ldots, T_k$ are all the triangulations of $(Q, \mathfrak{A})$ that contains the simplex σ will be called the normal cone $N_{\varphi_\sigma}\Sigma(\mathfrak{A})$.

Let $C^{\mathfrak{A}}$ be the space of Laurent polynomials $f(x_1, \ldots, x_n) = \sum_{\omega \in \mathfrak{A}} a_\omega x^\omega$ with monomials from $\mathfrak{A}$. Now for $f(x_1, \ldots, x_n) \in C^{\mathfrak{A}}$ we have the principal A-determinant

$$E_A(a) = R_A\Big(x_1\frac{\partial f}{\partial x_1}, \ldots, x_{n-1}\frac{\partial f}{\partial x_{n-1}}, f\Big),$$

where R_A is the so called A-resultant of polynomials $x_1\frac{\partial f}{\partial x_1}, \ldots, x_{n-1}\frac{\partial f}{\partial x_{n-1}}, f$ with supports in $\mathfrak{A}$, compare [4]. Note that $E_A(a)$ is a polynomial in the coefficients of f. Using this construction we formulate the following theorem.

Theorem 1 *The domain of convergence D_σ of the series $\phi(1, a'')$ in (1) is a complete Reinhardt domain with the property that the corresponding convex domain $Log(D_\sigma)$ contains all the connected components of the amoeba complement $\mathbb{R}^n \setminus \mathcal{A}_\sigma$, where A_σ is the amoeba of $E_A(1, a'')$, that are associated with the triangulations of (Q, A) that contain the simplex σ, i.e. are associated with a certain vertex in the secondary polytope $\Sigma(A)$, while it is disjoint from all the other components.*

Proof: We know from the implicit function theorem that D_σ is not empty, and Abel's lemma tells us that whenever a point a belongs to D_σ, then so does the full polydisc centered at a. Therefore D_σ is indeed a complete Reinhardt domain , and the corresponding domain $\mathrm{Log}(D_\sigma)$ will contain the negative orthant $-\mathbb{R}_+^{n-1}$ in the corresponding cone $C(\sigma)$.

In fact, we will show that $C(\sigma)$ is the negative orthant. This we can see by letting the function $\psi = 0$ on all the points a_j in the simplex σ. This corresponds to chosing exactly this simplex σ. (We could choose ψ equal to anything at the points in σ.) Now $C(\sigma)$ consists of functions $\psi : A \to \mathbb{R}$ such that $g_{\psi,T}$ is concave and $g_{\psi,T}(\omega) \leq \psi(\omega)$ for all ω which are not vertices in σ. Hence $\psi(\omega) \leq 0$ for all ω and all functons ψ, and thus $C(\sigma)$ is equal to the negative orthant $\mathbb{R}_+^{n-1}$.

Let E be a connected component of $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ that intersects the domain $\mathrm{Log}(D_\sigma)$. Then we claim that we actually have an inclusion $E \subset \mathrm{Log}(D_\sigma)$.

Accepting this, it follows, from what we have prooved so far, that the domain $\mathrm{Log}(D_\sigma)$ cannot intersect any component of the amoeba complement $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ whose cone $\mathcal{C}(\sigma)$ is not in the negative orthant. On the other hand every connected component of $\mathbb{R}^{n-1} \setminus \mathcal{A}_\sigma$ with the corresponding cone $C(\sigma)$ contained in $\mathbb{R}_+^{n-1}$ will necessarily intersect, and hence be contained in , the domain $\mathrm{Log}(D_\sigma)$. The following proposition therefore suffices to make the proof of Theorem 1 complete.

□

Proposition 3 *The normal cone $N_{\varphi_T}\Sigma(\mathfrak{A})$ at a vertex of the secondary polytope $\Sigma(A)$ is contained in the negative orthant $-\mathbb{R}_+^{n-1}$ if and only if the corresponding triangulation of Q contains the simplex σ. In fact the union of such normal cones $N_{\varphi_\sigma}\Sigma(A)$ is equal to $-\mathbb{R}_+^{n-1}$.*

Proof: We will prove this proposition by proving that the normal cone $N_{\varphi_\sigma}\Sigma(A)$ coincides with the cone $C(\sigma)$. We get at once from the definitions of φ_T and $g_{\psi,T}$, and the fact that the integral of an affine-linear function g over a simplex σ is equal to the arithmetic mean of values of g at the vertices of σ times the volume of σ, the following:

$$(\psi, \varphi_T) = n \int_Q g_{\psi,T}(x)dx. \tag{1.2}$$

We now fix $\psi \in \mathbb{R}^A$. The upper boundary of G_ψ can be regarded as the graph of a piecewise-linear function $g_\psi : Q \to \mathbb{R}$.

$$g_\psi(x) = \max\{y : (x, y) \in G_\psi\}.$$

We can furthermore state about the function g_ψ the following:
(a) g_ψ is concave.
(b) For any triangulation T of (Q, A) we have $g_\psi(x) \geq g_{\psi,T}(x), \forall x \in Q$.
(c)We have

$$\max_{\varphi\in\Sigma(A)} (\psi, \varphi) = n \int_Q g_\psi(x)dx. \tag{1.3}$$

(a) follows by construction. To varify (b), it suffices to consider x varying in some fixed simplex σ of T. By definition, $g_{\psi,T}$ is affine-linear over σ and $g_\psi(\omega) \geq \psi(\omega) = g_{\psi,T}$ for any vertex $\omega \in \sigma$. Hence the inequality is valid over σ. The maximum in (1.3) can be taken over the set of the φ_T for all triangulations T of (Q, A), since $\Sigma(A)$ is defined as the convex hull of these φ_T. Hence part (b) together with (1.2) imply that the left hand side of (1.3) is greater than or equal to the right hand side. To show the equality, it suffices to exhibit a triangulation T for which $g_\psi = g_{\psi,T}$. To do this, we consider the projections of the bounded faces of the polyhedron G_ψ into Q. These are polytopes with vertices in A. Take a generic ψ' close to ψ. Then the bounded faces of the polyhedron $\mathcal{G}'_\psi$ give a triangulation T of (Q, A) which induces a triangulation of each of the above polytopes. Hence g_ψ is T-piecewise-linear and coincides with $g_{\psi,T}$. This proves (1.3). Hence the cones coincide.

$\square$

Remark. Theorem 1 was proven for the special case $n = 2$ in [9].

Remark. The discriminant set is not always a hypersurface set. There are cases of higher codimension. For example whenn $n \geq 2$ for any general linear polynomial $a_0+a_1z_1+\cdots+a_nz_n$ we get the discriminant set $\{a_1 = \cdots = a_n = 0\}$, hich is not a hypersurface set. Another example is when $p = g(z_1, \ldots, z_{n-1}) + az_n$ where g is a non trivial polynomial. Here the discriminant set is of type $\{a = 0, \Delta' = 0\}$, where Δ' is the discriminant of g with respect to variables $z_1, \ldots, z_n$.

1.5 The Mellin-Barnes integral

By the multiple Mellin-Barnes integral we mean the integral

$$\Phi_\delta(t) = \frac{1}{(2\pi i)^n}\int_{\delta+i\mathbb{R}^n} \frac{\prod_{j=1}^p \Gamma(\langle A_j, z\rangle + c_j)}{\prod_{k=1}^q \Gamma(\langle B_k, z\rangle + d_k)} t_1^{-z_1}\cdots t_n^{-z_n}dz, \tag{1.4}$$

where all parameters $A_j, B_k \in \mathbb{R}^n$, $c_j, d_k \in \mathbb{R}$ are real and $dz = dz_1 \dots dz_n$. The vector $\delta \in \mathbb{R}^n$ is chosen so that the integration subspace $\delta + i\mathbb{R}^n$ is disjoint from the poles of the gamma-functions in the numerator.

For brevity we rewrite (1.4) as

$$\Phi_\delta(t) = \frac{1}{(2\pi i)^n}\int_{\delta+i\mathbb{R}^n} F(z)t^{-z}dz, \tag{1.5}$$

denoting by $F(z)$ the ratio of the products of gamma-functions, and t^{-z} denotes the product $t_1^{-z_1}\cdots t_n^{-z_n}$. We suppose that the variable t varies in the complex torus $\mathbb{T}^n = (\mathbb{C}\setminus 0)^n$ and that

$$t_\nu^{-z_\nu} = e^{-z_\nu \log t_\nu}, \quad |\arg t_\nu| < \pi,$$

and introduce the following notations:

$$x_\nu = \mathrm{Re} z_\nu, \quad y_\nu = \mathrm{Im} z_\nu, \quad \nu = 1,\dots,n.$$

Let x and y be the vectors in $\mathbb{R}^n$ with coordinates x_ν and y_ν, correspondingly. Denote by $V_1,\dots,V_{p+q}$ all the hyperplanes $\langle A_j, y\rangle = 0$, $\langle B_k, y\rangle = 0$ in $\mathbb{R}^n$. By all nonempty intersections $V_{i_1}\cap\cdots\cap V_{i_s}$, these hyperplanes define a conic polyhedron which we will denote by K. Let $v_1,\dots,v_d$ be the unit vectors of the one-dimensional cones of K. Finally denote $\theta = \arg t = (\arg t_1,\dots,\arg t_n)$, so one has for any $t\in\mathbb{T}^n$ the representation $t = (z,\theta)\in\mathbb{R}_+^n\times T^n$, where T^n is the real n-dimensional torus.

Theorem 2 *The convergence domain of the Mellin-Barnes integral (1.4) is equal to $\mathbb{R}_+^n \times P^\circ$ with P° being the interior of the polytope*

$$P = \{\theta\in T^n : |\langle v_\nu, \theta\rangle| \le \frac{\pi}{2} g(v_\nu), \quad l = 1,\dots,d\}$$

where

$$g(y) = \sum_{j=1}^p |\langle A_j, y\rangle| - \sum_{k=1}^q |\langle B_k, y\rangle|.$$

Proof: Since the asymptotic equality $|y|^{x-1/2} \sim (|y|+1)^{x-1/2}$ as $|y| \to \infty$ is valid for every fixed $x \in \mathbb{R}$, Stirling's formula implies that there are constants c_1 and c_2 such that

$$C_1(|y|+1)^{x-1/2}e^{-\frac{\pi}{2}|y|} \leq |\Gamma(x+iy)| \leq C_2(|y|+1)^{x-1/2}e^{-\frac{\pi}{2}|y|}, \qquad (1.6)$$

where $x \in K \subset \mathbb{R} \setminus \{0,-1,-2,\dots\}$ (K is a compact set), $y \in \mathbb{R}$, and the constants C_1 and C_2 depend only on the choice of K. Using (1.6), and our notation $y_\nu = \mathrm{Im} z_j$, we can make the following estimate for the integrand in (1.5):

$$|F(z)t^{-z}| \leq \mathrm{const}\frac{\prod_{j=1}^{p}\tau_j}{\prod_{k=1}^{q}\xi_k}\exp\left\{|\langle y,\theta\rangle| - \frac{\pi}{2}g(y)\right\} \qquad (1.7)$$

where

$$\tau_j = (|\langle A_j, y\rangle|+1)^{\langle A_j,x\rangle+c_j-1/2}, \quad \xi_k = (|\langle B_k, y\rangle|+1)^{\langle B_k,x\rangle+d_k-1/2},$$

and $g(y)$ was defined above. Moreover, (1.7) holds for all $y \in \mathbb{R}^n$ and all x in compact subsets of $\mathbb{R}^n$ disjoint from the polar hyperplanes

$$\{\langle A_j, x\rangle + c_j = -\nu\}, \quad \{\langle B_k, x\rangle + d_k = -\nu\}, \quad \nu = 0,1,2,\dots;$$

in particular, (1.7) is valid for $x = \delta$. It follows from (1.7) that, for each $\theta = \arg t$, provided on $\mathbb{R}^n \setminus \{0\}$ the inequality

$$|\langle y,\theta\rangle| < \frac{\pi}{2}\left(\sum_{j=1}^{m}|\langle A_j,y\rangle| - \sum_{k=1}^{p}|\langle B_k,y\rangle|\right), \qquad (1.8)$$

the integrand in (1.5) decreases exponentially as $\|y\| \to \infty$. Therefore the integral (1.5) converges absolutely. By homogeneity, (1.8) is valid for all $y \in \mathbb{R}^n \setminus \{0\}$ whenever it holds for y on the sphere $\{y : \|y\| = 1\}$. It means that this integral converges for all $\theta = \arg t$ from the intersection of strips:

$$U := \bigcap_{\|y\|=1}\{\theta : |\langle y,\theta\rangle| < \frac{\pi}{2}g(y)\}.$$

Furthermore, it is clear that the integral (1.5) does not converge for $\theta = \arg t$ outside of the closure of U, since the estimate in (1.6) implies not only (1.7), but for some other constant the reversed inequality:

$$|F(z)t^{-z}| \geq \mathrm{const}\frac{\prod_{j=1}^{p}\tau_j}{\prod_{k=1}^{q}\xi_k}\exp\left\{|\langle y,\theta\rangle| - \frac{\pi}{2}g(y)\right\}. \qquad (1.9)$$

Thus if $\theta \in T^n \setminus \overline{U}$ we have an inequality $|\langle\theta,y\rangle| > \frac{\pi}{2}g(y)$ for some y on the sphere $\|y\| = 1$. By (1.9) it means that the integrand will have a positive exponent, and therefore is not integrable.

Hence the domain of convergence U consists of such t for which $\theta = \arg t$ satisfy for $y \neq 0$ the inequality (1.8), i.e. $|\langle\theta,y\rangle| < \frac{\pi}{2}g(y)$. Of course, $U \subset P^\circ$.

Let us explain why any point $\theta \in P^\circ$ belongs to U. Indeed $g(y)$ is a piece-wise linear function whose graph has corners only over the hyperplanes $\langle A_j, y\rangle = 0$ and $\langle B_k, y\rangle = 0$. Correspondingly, the function $\Psi_\theta(y) := \frac{|\langle\theta,y\rangle|}{g(y)}$ is a piece-wise fractional linear function with respect to the variable y. Consequently all extremal points of the function $\widetilde{\Psi}_\theta(y) = \Psi_\theta(y)\Big|_{\|y\|=1}$ lie on the vertices of the polyhedron $P \cap \{\|y\| = 1\}$. Therefore $\theta \in P^\circ$ implies $\theta \in U$.

$\square$

Remark. Some partial results on the domains of convergence for integral (1.4) were obtained in [10].

1.6 Examples

We will consider the polytope Q which is the convex hull of the set $\{\alpha_i, i = 1, \ldots, 5\}$ in $\mathbb{R}^2$, where $\alpha_1 = (0,0)', \alpha_2 = (1,0)', \alpha_3 = (0,1), \alpha_4 = (1,1)', \alpha_5 = (0,2)'$, that is the Newton polytope of the polynomial $f = a+bx+cy+dxy+ey^2$ in two variables.

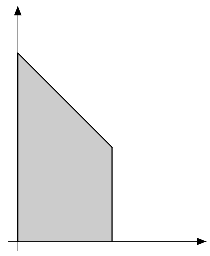

FIGURE 6 . *Newton polytope of* $f = a + bx + cy + dxy + ey^2$.

There will be 5 different coherent triangulations of the polytope Q where some simplices will occur once, and others in several of the triangulations. In total we will get 9 different simplices.

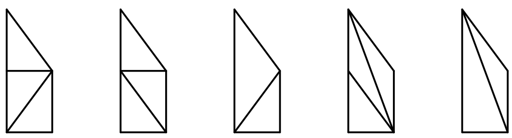

FIGURE 7. *The* 5 *triangulations of* Q.

According to the theory described in Section 1.3 each simplex of normalized volume will correspond to one hypergeometric series, while a simplex of larger volume will correspond to several series. Some of these series will also be representable as integrals. We can to the polytope Q relate several different A-matrices depending on how we order the points α_i. We will to the points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 relate the variables a, b, c, d, e in this order. In deciding the order of the α_i:s we also make the selection of the particular simplex who's vertice variables are set to one, and hence determine the variables in the hypergeometric series, all according to the description in the beginning of Section 1.3. This is the procedier used in the first section of examples.

In the second section we will instead select a particular amoeba, keep the variables fixed, and relate all the series from the first section of examples to this amoeba.

1.6.1 Taylor series

We formulate the A-matrix

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{pmatrix}.$$

Now the first three α_i:s, which we will denote by $\{\alpha, \beta, \gamma\}$, in this case $\{\alpha_2, \alpha_3, \alpha_4\}$, determines the simplex σ_I, that is the simplex with corners in $(1,0), (0,1)$ and $(1,1)$.

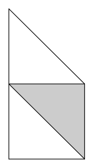

FIGURE 8. *The simplex selected in* A_1 .

We calculate the principal A-determinant for this matrix, or rather the principal A-determinant of the polynomial $f = e + ax + by + cxy + dy^2$.

$$E_{A_1}(a,b,c,d,e) = abcde(b^2 - 4de)(abc - a^2d + c^2e),$$

where a, b, c, d, e is the complex variables that we associate with the vertices in Q.

We determine a B-matrix corresponding to our matrix A_1.

$$B_1 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that there is a large number of A-matrices that could generate the same B-matrix, and that the B-matrix is uniquely determined by the choice of A and a the choice of simplex only up to the order of the columns. This gives the following hypergeometric series

$$\phi_1 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha+m-n} b^{\beta-m-n} c^{\gamma-m+n}}{\Gamma(\alpha+m-n+1)\Gamma(\beta-m+1)\Gamma(\gamma-m+n+1)m!n!} d^m c^n.$$

This can be reformulated into the classical form used by Horn [5], as follows

$$\phi_1 = C \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{-m+n}(\beta')_{m+n}(\gamma')_{m-n}}{m!n!}(s_1)^m(t_1)^n,$$

where $s_1 = ab^{-1}c^{-1}d$ and $t_1 = a^{-1}b^{-1}ce$, the constant $C = -\pi^{-3}a^\alpha b^\beta c^\gamma \Gamma(-\alpha)\Gamma(-\beta)\Gamma(-\gamma)$, and we use the notation $(\alpha)_k := \Gamma(\alpha+k)\Gamma(\alpha)$, and $\alpha' = -\alpha$.

Now taking the complex hull of α, β, γ we get a simplex of normalized volume 1. We let the complex variables a, b, c correlated to these points, be equal to 1, which gives us a hypergeometric series in 2 complex variables. We draw the Newton polytope of the principal A-determinant $E_{A_1}(1,1,1,d,e) = de(1-4de)(1-d-e)$, this is also called the secondary polytope of f.

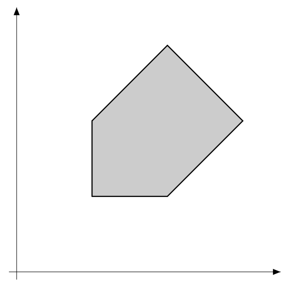

FIGURE 9. *Secondary polytope of* $f = e + ax + by + cxy + dy^2$.

According to the theory of GKZ there is a bijective correspondence between the coherent triangulations and the vertex monomials in the principal A-determinant. The vertices of each non unit variable will be a vertice in one or more simlices in each triangulation. This bijection is described by for each variable calculating this number of simplices, multiplying it with the normalized volume of the simplex, and put the resulting number as the vector of exponents applied to our vector of variables. That is, in our case we get that the triangulation in Figure 6 will be associated with the term d^1e^1 by this bijection, which is the furthest down to the left point in the secondary polytope. We draw the polytope again, together with the normal cone at this particular point.

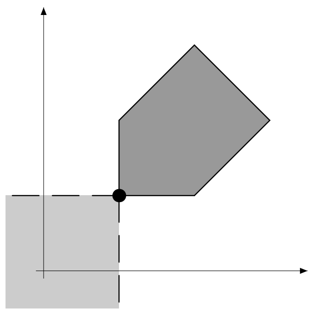

Figure 10 . *Secondary polytope of and normal cone at* $(1,1)$.

We draw the amoeba of the dehomogenized principal A-determinant $E_{A_1} = de(1-4de)(1-d-e)$.

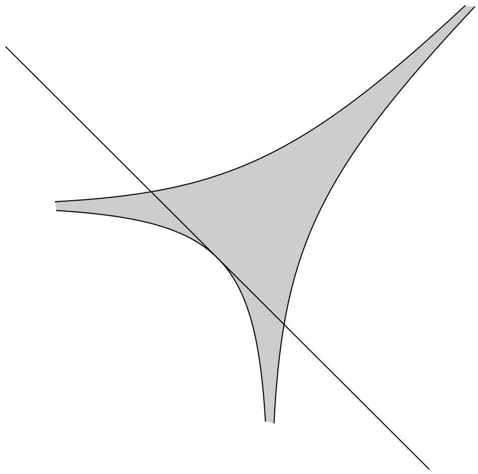

Figure 11. *Amoeba of the principal* A*-determinant* E_{A_1}.

The statement of Theorem 1 is that the convergence domain of the series ϕ_1 is exactly the amoeba complement component in which we can fit the normal cone in Figure 10. We highlight the convergence domain (or its image under the logatrithmic mapping) in the following picture.

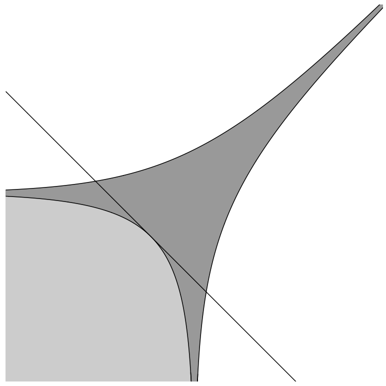

FIGURE 12. *Convergence domain of ϕ_1.*

That the convergence domain in this simple case should be in the third quadrant is of course natural since the series ϕ_1 can be recognized as a Taylor series.

Permutating A_1 we get a new matrix A_2 and hence a new choice of simplex. Note that our polytope Q is the same as before. We let A_2 be as follows.

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 0 \end{pmatrix}.$$

This time our selected simplex appears in two triangulations.

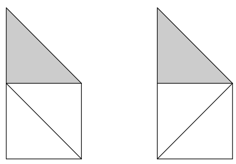

FIGURE 13. *The two triangulations in which the simplex occurs.*

We choose the B-matrix

$$B_2 = \begin{pmatrix} -2 & -1 \\ 0 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding series is

$$\phi_2 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha-2m-n} b^{\beta-n} c^{\gamma+m+n}}{\Gamma(\alpha-2m-n+1)\Gamma(\beta-n+1)\Gamma(\gamma+m+n+1)} d^m e^n,$$

or,

$$\phi_2 = c \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{2m+n}(\beta')_n}{(\gamma')_{m+n} m! n!} s_2^m t_2^n,$$

where $s_2 = a^{-2}cd$, and $t_2 = a^{-1}b^{-1}ce$.

The principal A-determinant is

$$E_{A_2}(a,b,c,d,e) = abcde(4cd - a^2)(b^2 d - abe + ce^2),$$

and dehomogenized

$$E_{A_2}(1,1,1,d,e) = de(4d-1)(d-e+e^2) = 4d^3 - 4d^2e^2 + 4d^2e^3 - d^2e + de^2 - de^3.$$

We draw the Newton polytope of $E_{A_2}(1,1,1,d,e)$ with the normal cones in the vertices in bijective correspondence with the two triangulations in Figure 13.

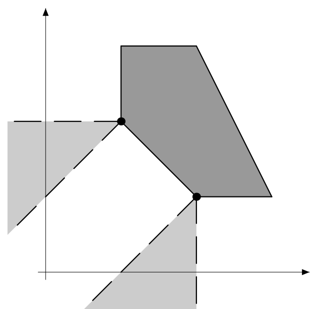

FIGURE 14. *Secondary polytope and normal cones.*

We draw the amoeba of the principal A-determinant $E_{A_2}(1,1,1,d,e)$.

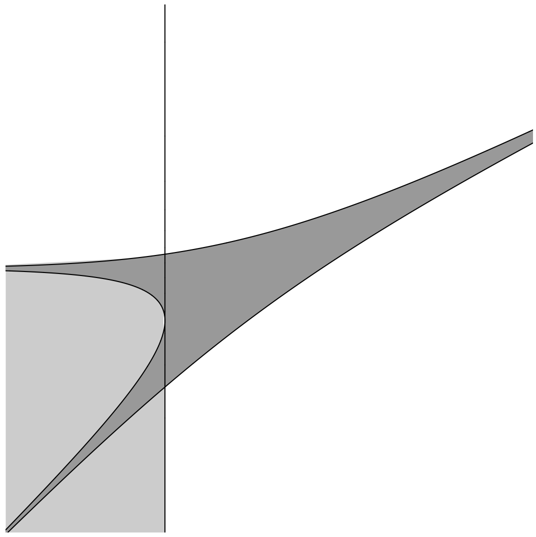

FIGURE 15. *Amoeba of the principal A-determinant E_{A_2}.*

We see that the two normal cones fit into the two amoeba complement components in the third quadrant. Hence according to Theorem 1 the convergence domain of ϕ_2 will be the domain $\mathrm{Log}^{-1}(D)$ where D is the convex domain that contains both the complement components in the third quadrants in Figure 15.

In this case we can obtain an analytic continuation of the series ϕ_2 by means of a Mellin-Barnes integral, using residue calculus:

$$\sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{2m+n}(\beta')_n}{(\gamma')_{m+n} m!n!}(s_2)^m(t_2)^n = \frac{\Gamma(\gamma')}{\Gamma(\alpha')\Gamma(\beta')} \sum_{\mathbb{Z}_+^2} \frac{\Gamma(\alpha'+2m+n)\Gamma(\beta'+n)}{\Gamma(\gamma'+m+n)m!n!} s_2^m t_2^n$$

$$= c \sum_{m,n\geq 0} res_{z=(-m,-n)} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma(\alpha'-2z_1-z_2)\Gamma(\beta'-z_2)}{\Gamma(\gamma'-z_1-z_2)}(-s_2)^{-z_1}(-t_2)^{-z_2}$$

$$= \frac{c}{(2\pi i)^2}\int_{\delta+i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma(\alpha'-2z_1-z_2)\Gamma(\beta'-z_2)}{\Gamma(\gamma'-z_1-z_2)}(-s_2)^{-z_1}(-t_2)^{-z_2}dz$$

Hence we have the integral representation of the hypergeometric function as follows

$$\Phi_\delta(\omega) \tag{1.10}$$
$$= \frac{K}{(2\pi i)^2}\int_{\delta+i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma(-2z_1-z_2+\alpha')\Gamma(-z_2+\beta')}{\Gamma(-z_1-z_2+\gamma')}(-\omega_1)^{-z_1}(-\omega_2)^{-z_2}dz.$$

Here $K = -\Gamma(\gamma')a^\alpha b^\beta c^\gamma \Gamma(-\alpha)\Gamma(-\beta)\Gamma(-\gamma)/(\Gamma(\alpha')\Gamma(\beta')\pi^3)$, and $\omega = (\omega_1, \omega_2) = (a^{-2}cd, a^{-1}b^{-1}ce)$. We assume that $\alpha', \beta' > 0$ are real, positive and that $\delta' = (\delta_1, \delta_2) \in \mathbb{R}^n$ is any point of the polygon $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, 0 <$

$x_2 < \beta, 2x_1 + x_2 < \alpha\}$. For any other choice of parameters $\alpha', \beta' \in \mathbb{C}$ one has to make a deformation of the contour $\delta + i\mathbb{R}^2$.

We draw the coamoeba of the principal A-determinant with the variables corresponding to the points α, β, γ set to 1, $E_{A_2}(1,1,1,d,e)$.

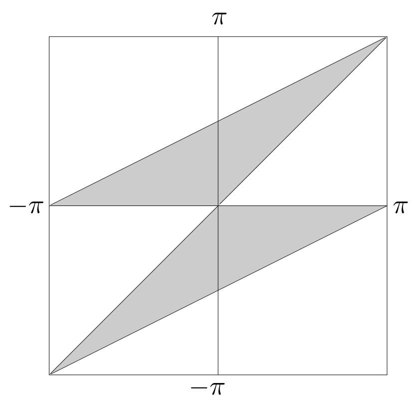

FIGURE 16. *Coamoeba of the A-discriminant E_{A_2}.*

Note that there is really just two complement components of the coamoeba in Figure 16, since the argument is periodic. We draw the coamoeba again and mark the complement component that corresponds to the convergence domain of the integral (1.10). To make this clear we draw the shifted coamoeba.

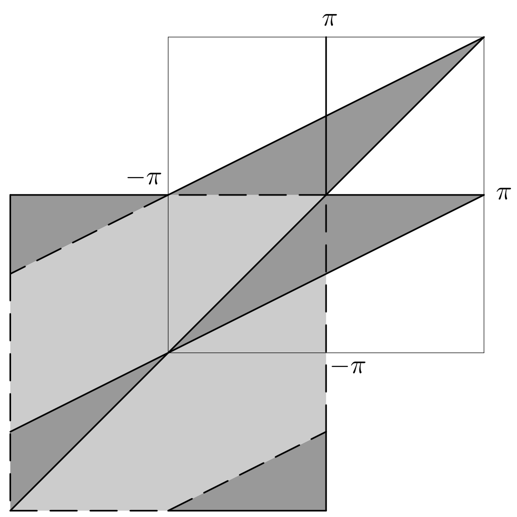

FIGURE 17. *Convergence domain of the integral (1.10).*

We permutate the A-matrix again and consider the simplex with corners $(0,0),(0,2)$ and $(1,0)$.

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

We select a B-matrix

$$B_3 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 0 & -2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This simplex has normalized volume 2.

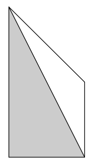

Figure 18. *Simplex of volume* 2.

Since the chosen simplex has volume two we get two series, her given in both our notations, namely

$$\phi_3 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha-m/2-n/2} b^{\beta-m/2+n/2} c^{\gamma-n}}{\Gamma(\alpha-m/2-n/2+1)\Gamma(\beta-m/2+n/2+1)\Gamma(\gamma-n+1)} d^m e^n$$
$$= \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{1}{2}m+\frac{1}{2}n} (\beta')_{\frac{1}{2}m-\frac{1}{2}n} (\gamma')_n}{m!n!} s_3^m t_3^n,$$

and

$$\phi_3' = \sum_{\mathbb{Z}_+^2} (-1)^{m+n} \frac{a^{\alpha-m/2-n/2} b^{\beta-m/2+n/2} c^{\gamma-n}}{\Gamma(\alpha-m/2-n/2+1)\Gamma(\beta-m/2+n/2+1)\Gamma(\gamma-n+1)} d^m e^n$$
$$= \sum (-1)^{m+n} \frac{(\alpha')_{\frac{1}{2}m+\frac{1}{2}n} (\beta')_{\frac{1}{2}m-\frac{1}{2}n} (\gamma')_n}{m!n!} s_3^m t_3^n,$$

where $s_3 = d/\sqrt{ab}$ and $t_3 = (\sqrt{ab}e)/ac$

We have the principal A-determinant

$$E_{A_3}(a,b,c,d,e) = abce(4ab - d^2)(e^2b - cde + ac^2).$$

We draw the secondary polytope of the principal A-determinant with the variables corresponding to the points α, β, γ set to 1,

$$E_{A_3}(1,1,1,d,e) = e(4-d^2)(e^2 - de + 1) = 4e^3 - 4de^2 + 4e - d^2e^3 - d^3e^2 - d^2e.$$

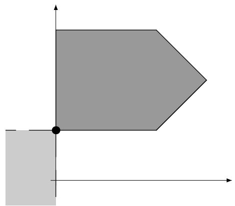

FIGURE 19. *Secondary polytope and normal cone.*

We draw the amoeba of the principal A-determinant with the variables corresponding to the points α, β, γ set to 1, $E_{A_3}(1,1,1,d,e)$.

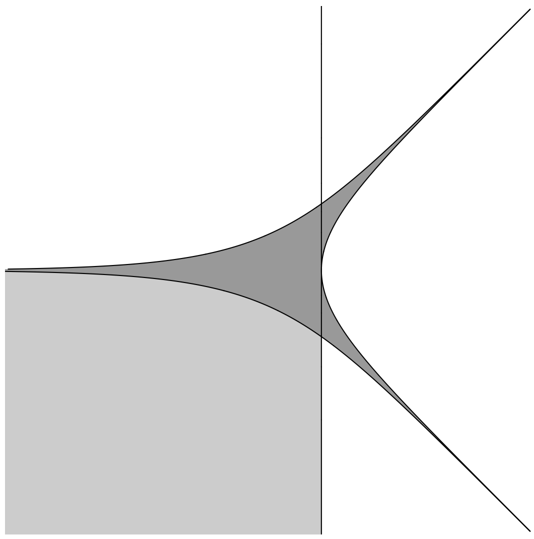

FIGURE 20. *Amoeba of the principal A-determinant E_{A_3}.*

Hence both series ϕ_3 and ϕ_3' converges in the domain corresponding to the complement component in the third quadrant in the picture.

So we have seen how rearranging the points α_i corresponds to choosing a certain simplex in the polytope Q. To complete this example we will give the hypergeometric functions and draw the amoebas for the remaining 6 simplices (there are 9 in total). In the figure below we have numbered the simplices in the same order as we have numbered the series and the amoebas.

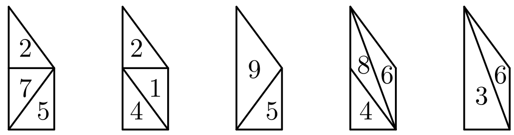

FIGURE 21. *The five different triangulations of the polytope Q.*

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

We select a B-matrix

$$B_4 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have the principal A-determinant

$$E_{A_4}(a,b,c,d,e) = abcde(4ae - c^2)(ad^2 - cbd + b^2e),$$

and the same dehomogenized becomes

$$E_{A_4}(1,1,1,d,e) = de(4e-1)(d^2-d+e) = 4d^3e^2 - 4d^2e^2 + 4de^3 - d^3e + d^2e - de^2.$$

The generated series is

$$\phi_4 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha+m+n}b^{\beta-m}c^{\gamma-m-2n}}{\Gamma(\alpha+m+n+1)\Gamma(\beta-m+1)\Gamma(\gamma-m-2n+1)} d^m e^n$$
$$= \sum_{\mathbb{Z}_+^2} \frac{(\beta')_m(\gamma')_{m+2n}}{(\alpha')_{m+n}m!n!} s_4^m t_4^n.$$

where $s_4 = ab^{-1}c^{-1}d$ and $t_4 = ac^{-2}e$.

We draw the amoeba of the principal A-determinant $E_{A_4}(1,1,1,c,d)$. The series converges in the convex set corresponding to the two complement components in the third quadrant (under the inverse logarithmic mapping).

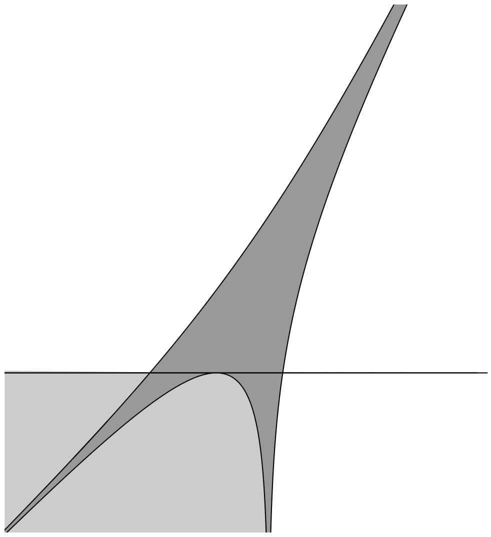

FIGURE 22. *Amoeba of the principal A-determinant E_{A_4}.*

We obtain an analytic continuation of the series ϕ_4 as a integral $\Phi_{\sigma_4}(t)$ and draw the coamoeba.

$$\Phi_{\delta_4}(\omega)$$
$$= \frac{c}{(2\pi i)^2} \int_{\delta + i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma(-z_1+\beta')\Gamma(-z_1-2z_2+\gamma')}{\Gamma(-z_1-z_2+\alpha')} (-\omega_1)^{-z_1}(-\omega_2)^{-z_2} dz.$$

Here $c = \Gamma(\gamma')/\Gamma(\alpha')\Gamma(\beta')$, and $\omega = (\omega_1, \omega_2) = (a^{-2}cd, a^{-1}b^{-1}ce)$. We assume that $\alpha', \beta' > 0$ are real, positive and that $\delta' = (\delta_1, \delta_2) \in \mathbb{R}^n$ is any point of the polygon $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, 0 < x_2 < \beta, 2x_1 + x_2 < \alpha\}$. For any other choice of parameters $\alpha', \beta' \in \mathbb{C}$ one has to make a deformation of the contour $\delta + i\mathbb{R}^2$.

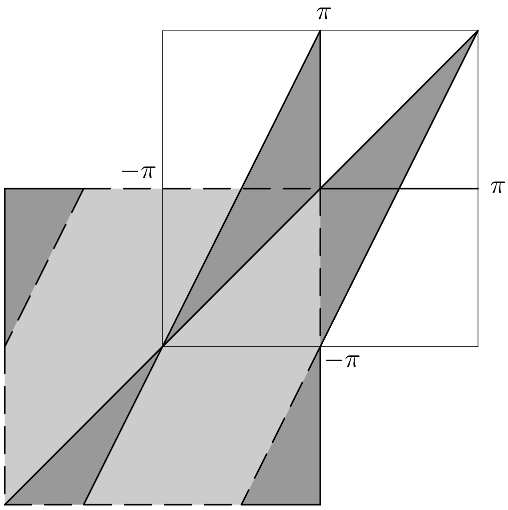

FIGURE 23. *Coamoeba of the principal A-determinant E_{A_4}.*

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

We select a B-matrix

$$B_5 = \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the series

$$\phi_5 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha+2m+n} b^{\beta-m-n} c^{\gamma-2m-n}}{\Gamma(\alpha+2m+n+1)\Gamma(\beta-m-n+1)\Gamma(\gamma-2m-n+1)} d^m e^n$$

$$= \sum_{\mathbb{Z}_+^2} \frac{(\beta')_{m+n}(\gamma')_{2m+n}}{(\alpha')_{2m+n} m! n!} s_6^m t_6^n,$$

where $s_6 = a^{-1}b^{-1}cd$ and $t_6 = ac^{-2}e$.

The principal A-determinant is given by

$$E_{A_5}(a,b,c,d,e) = abcde(4bd - e^2)(bc^2 - ace + a^2 d).4$$

We draw the correlated amoeba and coamoeba of the dehomogenized principal A-determinant,

$$E_{A_5}(1,1,1,d,e) = de(4d-e^2)(1-e+d) = 4d^2e-4d^2e^2+4d^3e-de^3+de^4-d^2e^3.$$

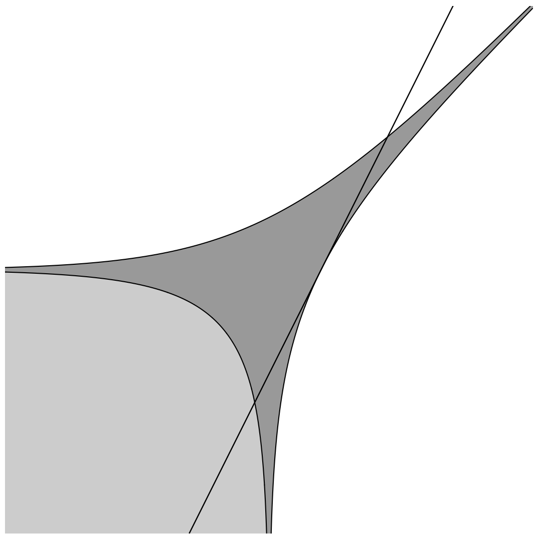

FIGURE 24. *Amoeba of the principal A-determinant E_{A_5}.*

$$A_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

We select a B-matrix

$$B_6 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the series

$$\phi_6 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha+m+n}b^{\beta-m}c^{\gamma-m-2n}}{\Gamma(\alpha+m+n+1)\Gamma(\beta-m+1)\Gamma(\gamma-m+2n+1)} d^m e^n$$

$$= \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{m+2n}(\gamma')_{m+n}}{(\beta')_{m+2n}m!n!} s_5^m t_5^n,$$

where $s_5 = a^{-1}bc^{-1}d$ and $t_5 = a^{-2}b^2c^{-1}e$.

We get the principal A-determinant

$$E_{A_6}(a,b,c,d,e) = abce(4ce-d^2)(b^2e-abd+a^2c).$$

We draw the correlated amoeba and coamoeba of the dehomogenized A-determinant

$$E_{A_6}(1,1,1,d,e) = e(4e-d^2)(e-d+1) = 4e^3-4de^2+4e^2-d^2e^2+d^3e-d^2e.$$

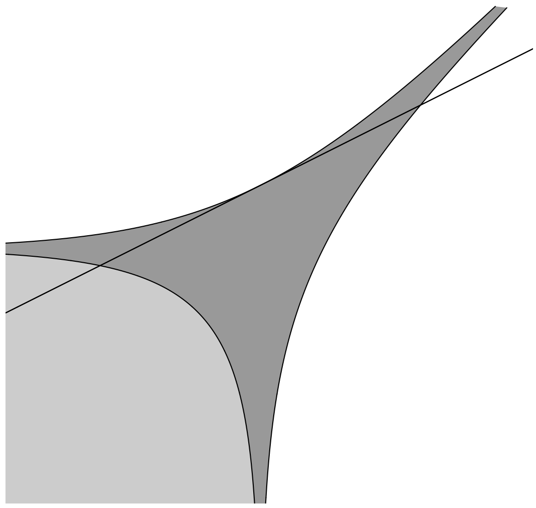

FIGURE 25. *Amoeba of the principal A-determinant E_{A_6}.*

$$A_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}.$$

We select a B-matrix

$$B_7 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \\ 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the series

$$\phi_7 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha-m+n}b^{\beta-m}c^{\gamma+m-2n}}{\Gamma(\alpha-m+n+1)\Gamma(\beta-m+1)\Gamma(\gamma+m-2n+1)} d^m e^n$$

$$= \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{m-n}(\beta')_m(\gamma')_{-m+2n}}{m!n!} s_7^m t_7^n.$$

where $s_7 = a^{-1}b^{-1}cd$ and $t_7 = ac^{-2}e$.

We have the principal A-determinant

$$E_{A_7} = abcde(4ae - c^2)(ab^2 - bcd + d^2e).$$

We draw the correlated amoeba and coamoeba of the principal A-determinant with the variables corresponding to the points α, β, γ set to 1,

$$E_{A_7}(1,1,1,c,d) = de(4e-1)(1-d+d^2e) = 4de^2-4d^2e^2+4d^3e^3-de+d^2e-d^3e^2.$$

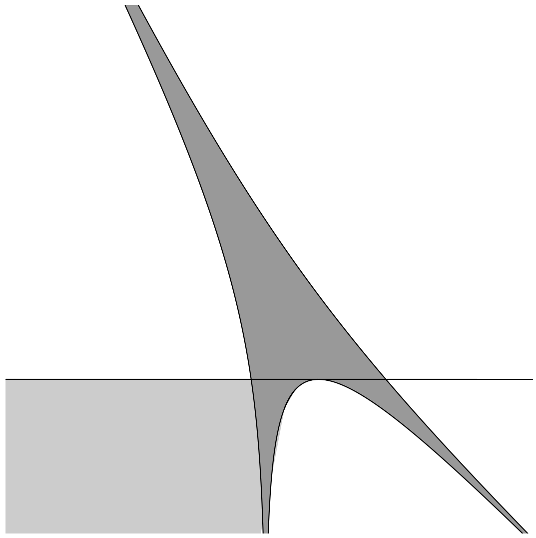

FIGURE 26. *Amoeba of the principal A-determinant E_{A_7}.*

$$A_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We select a B-matrix

$$B_8 = \begin{pmatrix} 1 & -1 \\ -2 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the series

$$\phi_8 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha+m-n} b^{\beta-2m+n} c^{\gamma-n}}{\Gamma(\alpha+m-n+1)\Gamma(\beta-m+1)\Gamma(\gamma-n+1)} d^m e^n$$

$$= \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{-m+n}(\beta')_{2m-n}(\gamma')_n}{m!n!} s_8^m t_8^n,$$

where $s_8 = ab^{-2}d$ and $t_8 = a^{-1}bc^{-1}e$.

We have the principal A-determinant

$$E_{A_8} = abcde(4ad - b^2)(de^2 - bce + ac^2).$$

We draw the correlated amoeba and coamoeba of the principal A-determinant with the variables corresponding to the points α, β, γ set to 1,

$$E_{A_8}(1, 1, 1, d, e) = de(4d - 1)(de^2 - e + 1).$$

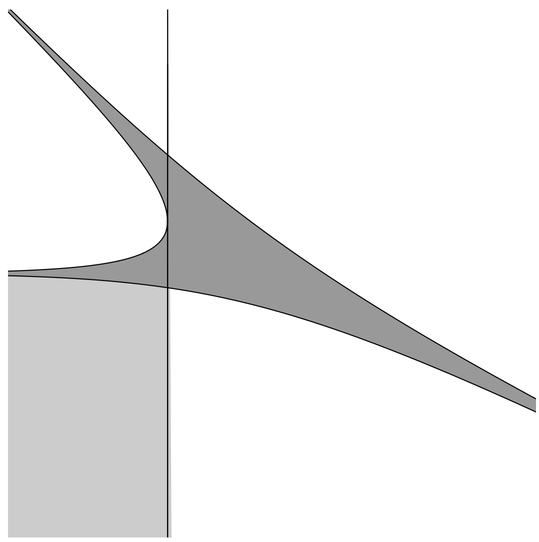

FIGURE 27. *Amoeba of the principal A-determinant E_{A_8}.*

We permutate the vectors in A one last time and get

$$A_9 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

We select a B-matrix

$$B_9 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -2 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This gives the series

$$\phi_9 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha-\frac{m}{2}-\frac{n}{2}} b^{\beta+\frac{m}{2}-\frac{n}{2}} c^{\gamma-m}}{\Gamma(\alpha-\frac{m}{2}-\frac{n}{2}+1)\Gamma(\beta+\frac{m}{2}-\frac{n}{2}+1)\Gamma(\gamma-m+1)} d^m e^n$$
$$= \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{m}{2}+\frac{n}{2}}(\beta')_{-\frac{m}{2}+\frac{n}{2}}(\gamma')_n}{m!n!} s_9^m t_9^n.$$

and

$$\phi_9' = \sum_{\mathbb{Z}_+^2} (-1)^{m+n} \frac{a^{\alpha-\frac{m}{2}-\frac{n}{2}} b^{\beta+\frac{m}{2}-\frac{n}{2}} c^{\gamma-m}}{\Gamma(\alpha-\frac{m}{2}-\frac{n}{2}+1)\Gamma(\beta+\frac{m}{2}-\frac{n}{2}+1)\Gamma(\gamma-m+1)} d^m e^n$$
$$= \sum_{\mathbb{Z}_+^2} (-1)^{m+n} \frac{(\alpha')_{\frac{m}{2}+\frac{n}{2}}(\beta')_{-\frac{m}{2}+\frac{n}{2}}(\gamma')_n}{m!n!} s_9^m t_9^n.$$

where $s_9 = s_9' = \sqrt{bd^2}/\sqrt{ac^2}$ and $t_9 = t_9' = e/\sqrt{ab}$. We have the principal A-determinant

$$E_{A_9} = abcd(4ab - e^2)(ac^2 - cde + bd^2).$$

We draw the correlated amoeba and coamoeba of the A-discriminant with the variables corresponding to the points α, β, γ set to 1,

$$E_{A_9}(1,1,1,d,e) = d(4-e^2)(1-de+d^2) = 4d - 4d^2e + 4d^3 - de^2 + d^2e^3 - d^3e^2.$$

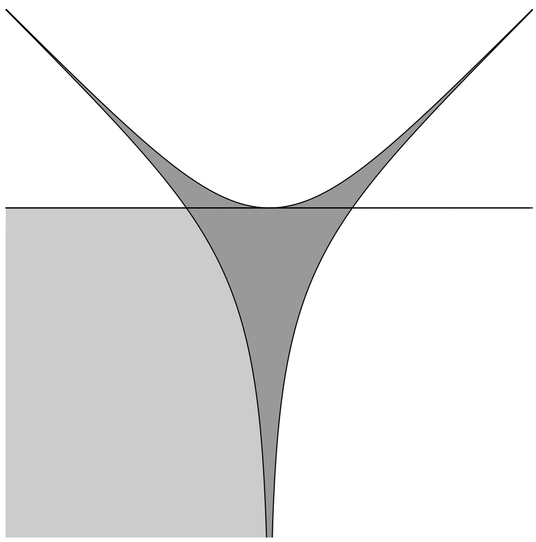

FIGURE 28. *Amoeba of the principal A-determinant E_{A_9}.*

1.6.2 Laurent-Puiseux series

Let us again consider a particular amoeba, for example the one generated by the A-discriminant E_{A_2}, that we picture again below in Figure 11. In this case we had the polynomial $f = d + ex + ay + bxy + cy^2$. We draw again the Newton polytope of this polynomial and write out explicitely to each triangulation the vertex monomial in the principal A determinant that the triangulation is bijectively linked to.

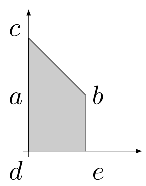

FIGURE 29. *The Newton polytope and the complex variables a, b, c, d, e.*

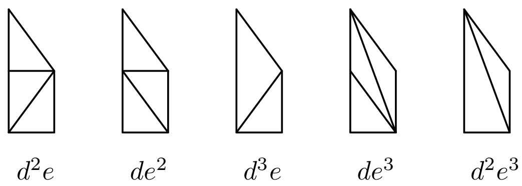

FIGURE 30. *The bijective correspondence between the triangulations of the Newton polytope and the vertex monomials in the principal A-determinant E_{A_2}.*

We draw the secondary polytope, with all its integer points and all the normal cones at the vortex points.

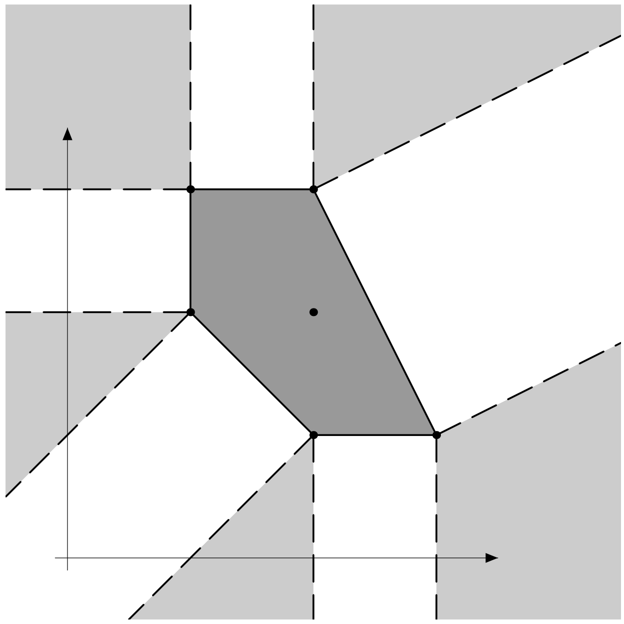

FIGURE 31. *Secondary polytope for Q.*

As we have seen, each hypergeometric series is associated with a certain simplex occuring in one or several of the triangulations of the polytope Q. The conclusion of Theorem 1 is that the convergence domain for a hypergoemetric series is, under the invers logaritmic mapping, the components of the amoeba complement that contain the normal cones at the vertices in the secondary polytope, that are associated with the triangulations that contain this simplex. We illustrate this by the picture in Figure 32.

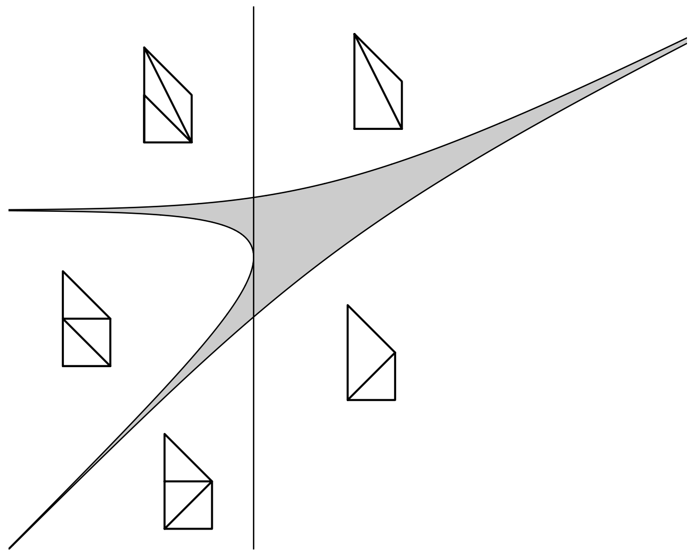

FIGURE 32. *Amoeba and the triangulations of Q placed in the correlated components of the amoeba complement.*

We list the Laurent-Puiseux series below. The convergence domain of the series ξ_i is the complement componenent related to the simplex numbered i under the invers logarithmic mapping. See Figure 32 and Figure 21.

$$\xi_1 = \sum_{\mathbb{Z}_+^2} \frac{a^{\beta-m-n}}{\Gamma(\beta-m+1)} \frac{b^{\gamma-m+n}}{\Gamma(\gamma-m+n+1)} \frac{c^m}{m!} \frac{d^n}{n!} \frac{e^{\alpha+m-n}}{\Gamma(\alpha+m-n+1)}.$$

$$\xi_1 = C \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{-m+n}(\beta')_{m+n}(\gamma')_{m-n}}{m!n!} s_1^m t_1^n,$$

where $s_1 = a^{-1}b^{-1}ce$ and $t_1 = a^{-1}bde^{-1}$, and the constant $C = -\pi^{-3}a^\alpha b^\beta c^\gamma \Gamma(-\alpha)\Gamma(-\beta)\Gamma(-\gamma)$.

$$\xi_2 = \sum_{\mathbb{Z}_+^2} \frac{a^{\alpha-2m-n}}{\Gamma(\alpha-2m-n+1)} \frac{b^{\beta-n}}{\Gamma(\beta-n+1)} \frac{c^{\gamma+m+n}}{\Gamma(\gamma+m+n+1)} d^m e^n,$$

or,

$$\xi_2 = K \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{2m+n}(\beta')_n}{(\gamma')_{m+n} m!n!} s_2^m t_2^n,$$

where $s_2 = a^{-2}cd$, and $t_2 = a^{-1}b^{-1}ce$.

$$\xi_3 = \sum_{\mathbb{Z}_+^2} \frac{a^m}{m!}\frac{b^n}{n!}\frac{c^{\alpha-m/2-n/2}}{\Gamma(\alpha-m/2-n/2+1)}\frac{d^{\beta-m/2+n/2}}{\Gamma(\beta-m/2+n/2+1)}\frac{e^{\gamma-n}}{\Gamma(\gamma+n+1)}$$
$$= K\sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{1}{2}m+\frac{1}{2}n}(\beta')_{\frac{1}{2}m-\frac{1}{2}n}(\gamma')_n}{m!n!} s_3^m t_3^n,$$

$$\xi_3' = \sum_{\mathbb{Z}_+^2} (-1)^{m+n}\frac{a^m}{m!}\frac{b^n}{n!}\frac{c^{\alpha-m/2-n/2}}{\Gamma(\alpha-m/2-n/2+1)}\frac{d^{\beta-m/2+n/2}}{\Gamma(\beta-m/2+n/2+1)}\frac{e^{\gamma-n}}{\Gamma(\gamma+n+1)}$$
$$= K\sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{1}{2}m+\frac{1}{2}n}(\beta')_{\frac{1}{2}m-\frac{1}{2}n}(\gamma')_n}{m!n!} s_3^m t_3^n,$$

where $s_3 = a/\sqrt{cd}$ and $t_3 = (b\sqrt{d})/(e\sqrt{c})$.

$$\xi_4 = \sum_{\mathbb{Z}_+^2} \frac{a^{\gamma-m-2n}}{\Gamma(\gamma-m+2n+1)}\frac{b^m}{m!}\frac{c^n}{n!}\frac{d^{\alpha+m+n}}{\Gamma(\alpha+m+n+1)}\frac{e^{\beta-m}}{\Gamma(\beta-m+1)}$$
$$= K\sum_{\mathbb{Z}_+^2} \frac{(\beta')_m(\gamma')_{m+2n}}{(\alpha')_{m+n}m!n!} s_4^m t_4^n,$$

where $s_4 = bda^{-1}e^{-1}d$ and $t_4 = a^{-2}cd$.

$$\xi_5 = \sum_{\mathbb{Z}_+^2} \frac{a^m}{m!}\frac{b^{\beta-m-n}}{\Gamma(\beta-m-n+1)}\frac{c^{\gamma-2m-n}}{\Gamma(\gamma-2m-n+1)}\frac{d^n}{n!}\frac{e^{\alpha+2m+n}}{\Gamma(\alpha+2m+n+1)}$$
$$= K\sum_{\mathbb{Z}_+^2} \frac{(\beta')_{m+n}(\gamma')_{2m+n}}{(\alpha')_{2m+n}m!n!} s_5^m t_5^n,$$

where $s_5 = ab^{-1}c^{-2}e^2$ and $t_5 = b^{-1}c^{-1}de$.

$$\xi_6 = \sum_{\mathbb{Z}_+^2} \frac{a^n}{n!}\frac{b^{\gamma-m-2n}}{\Gamma(\gamma-m+2n+1)}\frac{c^m}{m!}\frac{d^{\beta-m}}{\Gamma(\beta-m+1)}\frac{e^{\alpha+m+n}}{\Gamma(\alpha+m+n+1)}$$
$$= K\sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{m+2n}(\gamma')_{m+n}}{(\beta')_{m+2n}m!n!} s_6^m t_6^n,$$

where $s_6 = b^{-1}cd^{-1}e$ and $t_6 = ab^{-2}e$.

$$\xi_7 = \sum_{\mathbb{Z}_+^2} \frac{a^{\gamma+m-2n}}{\Gamma(\gamma+m-2n+1)} \frac{b^{\beta-m}}{\Gamma(\beta-m+1)} \frac{c^n}{n!} \frac{d^{\alpha-m+n}}{\Gamma(\alpha-m+n+1)} \frac{e^m}{m!}$$
$$= K \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{m-n}(\beta')_m(\gamma')_{-m+2n}}{m!n!} s_7^m t_7^n,$$

where $s_7 = ab^{-1}d^{-1}e$ and $t_7 = a^{-2}cd$.

$$\xi_8 = \sum_{\mathbb{Z}_+^2} \frac{a^{\beta-2m+n}}{\Gamma(\beta-m+1)} \frac{b^n}{n!} \frac{c^{\alpha+m-n}}{\Gamma(\alpha+m-n+1)} \frac{d^m}{m!} \frac{e^{\gamma-n}}{\Gamma(\gamma-n+1)}$$
$$= K \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{-m+n}(\beta')_{2m-n}(\gamma')_n}{m!n!} s_8^m t_8^n,$$

where $s_8 = a^{-2}cd$ and $t_8 = abc^{-1}d^{-1}$.

$$\xi_9 = \sum_{\mathbb{Z}_+^2} \frac{a^n}{n!} \frac{b^{\gamma-m}}{\Gamma(\gamma-m+1)} \frac{c^{\beta+\frac{m}{2}-\frac{n}{2}}}{\Gamma(\beta+\frac{m}{2}-\frac{n}{2}+1)} \frac{d^{\alpha-\frac{m}{2}-\frac{n}{2}}}{\Gamma(\alpha-\frac{m}{2}-\frac{n}{2}+1)} \frac{e^m}{m!}$$
$$= K \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{m}{2}+\frac{n}{2}}(\beta')_{-\frac{m}{2}+\frac{n}{2}}(\gamma')_n}{m!n!} s_9^m t_9^n$$

and

$$\xi_9' = \sum_{\mathbb{Z}_+^2} (-1)^{m+n} \frac{a^n}{n!} \frac{b^{\gamma-m}}{\Gamma(\gamma-m+1)} \frac{c^{\beta+\frac{m}{2}-\frac{n}{2}}}{\Gamma(\beta+\frac{m}{2}-\frac{n}{2}+1)} \frac{d^{\alpha-\frac{m}{2}-\frac{n}{2}}}{\Gamma(\alpha-\frac{m}{2}-\frac{n}{2}+1)} \frac{e^m}{m!}$$
$$= K \sum_{\mathbb{Z}_+^2} \frac{(\alpha')_{\frac{m}{2}+\frac{n}{2}}(\beta')_{-\frac{m}{2}+\frac{n}{2}}(\gamma')_n}{m!n!} s_9^m t_9^n,$$

where $s_9 = s_9' = \sqrt{c}e/(\sqrt{d}b)$ and $t_9 = t_9' = a/\sqrt{cd}$, and K is some constant term.

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Chapter 2

Integral representation of holomorphic mappings on fully nuclear spaces

Summary

We obtain the following integral representation

$$f(z) = \int_{E'_\beta} e^{\langle z,w\rangle} \cdot (f \circ D)(w) d\mu_\gamma(w)$$

for all $z \in E$, a fully nuclear space with basis, where η and γ belong to E'_β, $\eta/\gamma \in \ell_1$, f is a holomorphic function of η-exponential type on E, μ_γ is a Gaussian measure on E'_β, and D is a densely defined diagonal mapping from E'_β into E.

2.1 Introduction

The Cauchy integral representation formula has no true analogue in infinite dimensions. We will find in this paper a generalization of a related formula to certain infinite dimensional spaces.
It is known that for functions in Fischer-Fock space of entire functions

$$\{f \in H(\mathbb{C}^n); \int_{\mathbb{C}^n} |f|^2 d\mu < \infty\},$$

where $d\mu = \pi^{-n}e^{-|z|^2}d\lambda$ and λ denotes Lebesgue measure, one has the integral representation formula

$$f(z) = \pi^{-n}\int_{\mathbb{C}^n} f(w)e^{z\cdot\bar{w}-|w|^2}d\lambda = \pi^{-n}\int_{\mathbb{C}^n} e^{\langle z,w\rangle}f(w)d\mu.$$

Clearly, entire functions of exponential type, i.e. satisfying

$$|f(z)| \leq Ae^{B||z||}$$

for constants A and B and some norm $||\cdot||$, belong to Fischer-Fock space. We will see that for holomorphic functions of η-exponential type this integral representation is possible to extend to being valid in infinite dimensional spaces of a certain kind; the so called fully nuclear spaces with a basis.

The same formula has been generalized to Banach spaces by Pinasco and Zalduendo [6, 5], using Gaussian measures and abstract Wiener space extensions of Hilbert spaces. The approach used in this paper will be much inspired by the one in [6, 5] although in dealing with fully nuclear spaces we will need a different method for extending Gaussian promeasures to a measure on infinite dimensional space. The necessary method is found in Minlo's Fourier transformation characterization of Gaussian measures, see [1].

2.2 Holomorphic functions on infinite dimensional space

We first define the concept of holomorphic functions in an infinite dimensional space.

Definition 8 *A function $f : U \subset E \to F$ where U is a finitely open subset of a vector space E over $\mathbb{C}$ and F is locally convex space, is Gâteaux or $\mathcal{G}$-holomorphic if for each $\xi \in U, \eta \in E$ and $\phi \in F'$ the $\mathbb{C}$-valued function of one complex variable*

$$\lambda \to \phi \circ f(\xi + \lambda\eta)$$

is holomorphic on some neighborhood of 0 in $\mathbb{C}$. We let $H_G(U;F)$ denote the set of all $\mathcal{G}$-holomorphic mappings from U into F and write $H_G(U)$ in place of $H_G(U;\mathbb{C})$

The concept of Gâteaux-holomorphic functions will not always be enough to satisfy our needs. Note that in the definition above we did not use any locally convex structure on the domain. Doing so now, we define holomorphic, or Fréchet-holomorphic functions.

Definition 9 *If E and F are locally convex spaces over $\mathbb{C}$, and U is an open subset of E then $f : U \to F$ is holomorphic if $f \in H_G(U; F)$ and f is continuous. We let $H(U; F)$ denote the set of all holomorphic mappings from U into F and write $H(U)$ in place of $H(U; \mathbb{C})$*

In other words a function f is holomorphic if it is continuous and its restriction to each finite dimensional subspace is holomorphic in the traditional sense.

In this paper we will exclusively deal with entire function $f : E \to \mathbb{C}$ in which case we can somewhat simplify the more general notation above. The function ϕ in Definition 8 will then be superfluous.

2.3 Topology on spaces of holomorphic mappings

Throughout this paper by a locally convex space is meant a topological vector space over $\mathbb{R}$ that is Hausdorff and locally convex. The topological dual of a locally convex space E will be denoted by E', and the algebraic dual by E^*. Furthermore we denote by $E'_\beta = (E', \beta(E', E))$ the *strong dual*, the dual equipped with the strong topology β.

Definition 10 *Let U denote an open subset of a locally convex space E and let F be a locally convex space. The compact open topology (the topology of uniform convergence on the compact subsets of U) is the locally convex topology generated by the semi-norms*

$$p_{\alpha,K}(f) := ||f||_{\alpha,K} = \sup_{x \in K} \alpha(f(x))$$

where K ranges over the compact subsets of U and α over the continuous semi-norms on F. We denote this topology by τ_0.

Definition 11 *Let U be an open subset of a locally convex space E and let F be a normed linear space. A seminorm p on $H(U; F)$ is carried by the compact subset K of U if for every open set V, $K \subset V \subset U$, there exists $c(V) > 0$ such that*

$$p(f) \leq c(V)||f||_V$$

for all $f \in H(U; F)$. The τ_ω topology on $H(U; F)$ is the topology generated by the semi-norms carried by the compact subsets of U.

Definition 12 *Let U be an open subset of a locally convex space E and let F denote a normed linear space. A semi-norm P on $H(U; F)$ is τ_δ continuous if for each increasing countable open cover of U, $(V_n)_{n=1}^{\infty}$, there exists a positive integer n_0 and $c > 0$ such that*

$$p(f) \leq c||f||_{V_{n_0}}$$

for every $f \in H(U;F)$. The τ_δ topology on $H(U;F)$ is the locally convex topology generated by the τ_δ continuous semi-norms.

Now if E and F are locally convex spaces and U an open subset of E, we have on $H(U;F)$
$$\tau_0 \leq \tau_\omega \leq \tau_\delta.$$

Definition 13 *Let U denote an open subset of a locally convex space E and let F be a complete locally convex space. The β-topology on $H(U;F)$ is the topology of uniform convergence on the bounded subsets of $G(U) = \{\phi \in H(U)^* : \phi \quad \text{is } \tau_0 \text{ continuous on the locally bounded subsets of} \quad H(U)\}$.*

Clearly
$$\tau_0 \leq \beta \leq \tau_\delta.$$
Again these definitions are given in a more general context than what will be used in this paper, since we mainly deal with entire functions on a locally convex space into $\mathbb{C}$. Furthermore we will mainly be concerned here with the topologies used on the dual of a fully nuclear space. The topologies τ_δ and τ_σ is only used on $\mathcal{H}(U;F)$. Conveniently, since we are dealing with fully nuclear spaces where a closed bounded set is compact, the topology τ_0 and the strong topology will coincide. We will further discuss the topologies on fully nuclear spaces with a basis in the next section and also continuously specify what topology we consider.

2.4 Nuclear and fully nuclear spaces

Nuclear mappings were first considered by R. Schatten and J.Von Neumann in investigating the question of which continuous linear mappings of a Hilbert space determine a meaningful trace. The extension of these ideas to Banach spaces led A. Grothendieck to defining the concept of nuclear mappings. He also later introduced the notion of nuclear locally convex spaces.

Definition 14 *A barrelled locally convex space is a space satisfying one of the following equivalent conditions.*
(i) If W is a closed convex balanced absorbing (if $x \in E$ there exists $\lambda > 0$ such that $\lambda x \in W$) subset of E then W is a neighborhood of zero.
(ii) All lower semi-continuous semi-norms on E are continuous,
(iii) The point-wise bounded subsets of E' are locally bounded or equicontinuous.

Fréchet spaces are barrelled (but not all metrizable locally convex spaces). Arbitrary inductive limits of barrelled spaces are barrelled.
In particular $(\mathcal{P}(^nE,F),\tau_w)$, the space of all n-homogeneous polynomials equipped with the carried topology, is barrelled when E is any locally convex space and F is a Banach space.

Definition 15 *An infrabarrelled locally convex space is a space satisfying one of the following equivalent conditions.*
(i) *Every closed convex balanced subset of* E *which absorbs bounded subset is a neighborhood of zero.*
(ii) *Strongly bounded, i.e.* $\beta(E', E)$*-bounded, subsets of* E' *are equicontinuous.*

Barrelled spaces are infrabarrelled.

Definition 16 *Let* E *and* F *be locally convex spaces.*

a) $L \in \mathcal{L}(^nE; F)$ *is called a nuclear* n*-linear mapping from* E *into* F *if there exist a convex balanced zero neighborhood* U *in* E*, a bounded subset* B *of* F*,* $(\lambda_k)_{k=1}^{\infty} \in l_1$ *and sequences* $\phi_{i,k}^{\infty}, \quad i = 1, \ldots, n$ *and* $(y_k)_{k=1}^{\infty}$ *where* $(\phi_{i,k}) \in U'$ *for all* i *and* k *and* $y_k \in B$ *for all* k *such that* $L(x_1, \ldots, x_n) = \sum_{k=1}^{\infty} \lambda_k \phi_{1,k}(x_1) \ldots \phi_{n,k}(x_n) y_k$ *for all* $(x_1, \ldots, x_n) \in E^n$.

We let $\mathcal{L}_N(^nE, F)$ *denote the space of all nuclear* n*-linear mappings from* E *into* F.

b) $P \in \mathcal{P}(^nE, F)$ *is called a nuclear* n *homogeneous polynomial if there exists a convex balanced zero neighborhood* U *in* E*, a bounded subset* B *of* F*,* $(\lambda_k)_{k=1}^{\infty} \in l_1$ *and sequences* $(\phi_k)^{\infty} \subseteq U'$ *and* $(y_k)_{k=1}^{\infty} \subseteq B$ *such that* $P(x) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x) y_k$ *for every* $x \in E^n$.

Taking $n = 1$ in Definition 16 a) we obtain the definition of nuclear linear mapping between locally convex spaces.

Definition 17 *A locally convex space is nuclear if* $\mathcal{L}(E; F) = \mathcal{L}_N(E; F)$ *for every locally convex space* F*, where* $\mathcal{L}(E; F)$ *are all the continuous linear mappings from* E *to* F*, and* $\mathcal{L}_N(E; F)$ *are all the nuclear linear mappings from* E *to* F.

Definition 18 *A locally convex space* E *is fully nuclear if* E *and* E'_β *are both complete infrabarrelled nuclear spaces.*

Definition 19 *A sequence of subspaces* $\{E_n\}_n$ *of a locally convex space* E *is a Schauder composition of* E *if*
(a) *for each* x *in* E *there exists a unique sequence of vectors* $(x_n)_n$*,* $x_n \in E_n$ *for all* n*, such that*

$$x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n.$$

(b) *the projections* $(u_n)_{n=1}^{\infty}$ *defined by*

$$u_m(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{m} x_n$$

are continuous.

A Schauder decomposition $\{E_n\}_n$ of E is an absolute decomposition if for each $p \in \mathrm{cs}(E)$

$$q(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{\infty} p(x_n)$$

defines a semi-norm on E. If each E_n is one-dimensional and $\{E_n\}_n$ is a Schauder decomposition of E we say that E has a Schauder basis. In this case any sequence $(x_n)_n, x_n \neq 0$ and $x_n \in E_n$ is a Schauder basis for E.

Let E be a fully nuclear space with a Schauder basis $(e_n)_n$. Then E can in fact be identified with a sequence space $\Lambda(P)$ where P is a collection of weights (non-negative sequences), i.e.

$$\Lambda(P) = \Big\{(x_n)_n \in \mathbb{C}^n : \sum_{n=1}^{\infty} |x_n|\alpha_n < \infty \quad \text{for all} \quad (\alpha_n)_n \in P\Big\}.$$

The topology of $\Lambda(P)$ is generated by the semi-norms

$$||(x_n)_n||_{\alpha,1} := \sum_{n=1}^{\infty} |x_n|\alpha_n,$$

where $(x_n)_n \in \Lambda(P)$ and $\alpha := (\alpha_n)_n \in P$. In identifying E with $\Lambda(P)$ we may take P to be $\Big\{(p(e_n))_n\Big\}_{p\in cs(E)}$. According to the *Grothendieck-Pietsch* criterion for nuclearity $\Lambda(P)$ is nuclear if and only if for each $(\alpha_n)_n \in P$ there exists a $(\beta_n)_n$ such that $\beta_n \geq \alpha_n$ for all n and $\sum_{n,\alpha_n\neq 0} \frac{\alpha_n}{\beta_n} < \infty$. When $\Lambda(P)$ is nuclear its topology is generated by the semi-norms

$$||(x_n)_n||_{\alpha,\infty} := \sup_n |x_n|\alpha_n.$$

Subsets of $\Lambda(P)$ of the form

$$\{(z_n)_n \in \Lambda(P) : \sup_n |z_n|\beta_n < 1|\}$$

or

$$\{(z_n)_n \in \Lambda(P) : \sup_n |z_n|\beta_n \leq 1|\}$$

is called polydiscs. If $E \cong \Lambda(P)$ is a fully nuclear space with basis then the dual (also a fully nuclear space with basis) E'_β is identified with $\Lambda(P')$ where $P' = \{(|x_n|)_n\}_{(x_n)_n\in\Lambda(P)}$.

The most important spaces in the category of nuclear locally convex spaces are spaces of infinitely differentiable functions. In fact when A. Grothendieck first introduced the concept of nuclear spaces this originated from the study of the spaces $\mathcal{E}$ and $\mathcal{D}$, see the examples below.

Some examples of fully nuclear spaces with basis are the following. The topology considered is the usual topology of uniform convergence on compact

sets.
Fréchet nuclear spaces with basis and their duals are fully nuclear spaces
1) $\mathcal{H}(\mathbb{C}^n)$, i.e. the space of entire functions on $\mathbb{C}^n$.
As we have mentioned fully nuclear spaces are reflexive, hence another example would be the dual space: $\mathcal{H}'(\mathbb{C}^n)$, the space of analytic functionals.
2) $\mathcal{S}(\mathbb{R}^n)$, the space of all rapidly decreasing complex-valued $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$.
$\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.
3) $\mathcal{E}(\mathbb{R}^n)$, the space of all complex-valued C^∞ functions on $\mathbb{R}^n$.
$\mathcal{E}'(\mathbb{R}^n)$, the space of distributions with compact support.
4) $\mathcal{D}(\mathbb{R}^n)$,and the space of all complex-valued C^∞ functions with compact support.
$\mathcal{D}'(\mathbb{R}^n)$, the space of distributions.
Furthermore the set of fully nuclear spaces are also closed under countable products and direct sums. Here we get of course the complex analogue of the real spaces above by complexifying them, i.e. $\mathcal{S}(\mathbb{R}^n) + i\mathcal{S}(\mathbb{R}^n)$ etc.
Whereas the most important non-Banach spaces that arise naturally are fully nuclear spaces, note that the only nuclear Banach spaces are finite dimensional spaces.

2.5 Promeasures and measures on a locally convex space

Lebesgue measure plays a fundamental role in integration theory in $\mathbb{R}^n$. Recall that this is uniquely defined by the following conditions:
a) It assigns finite values to bounded Borel sets and positive numbers to nonempty open sets.
b) It is translation invariant.
In trying to apply Lebesgue measure to infinite dimensional spaces we will immediately face some difficulties. The Borel field on $\mathbb{R}^n$ is the σ-field generated by all open (or closed) subsets of $\mathbb{R}^n$. In going to infinite dimensions we will find it necessary to instead make use of a smaller σ-field and we will therefore introduce the σ-field of cylindrical Borel sets. Still, restricting ourselves to this σ-algebra, it can be shown for any measure that *b*) will fail to be satisfied. Hence extending Legesgue measure to infinite dimensional spaces fails.

Given E a locally convex space we let $F(E)$ denote the set of all closed subspaces of E of finite codimension. Hence $V \in F(E)$ if V is a closed subspace of E and $E = V \oplus \tilde{V}$ for some finite dimensional subspace $\tilde{V}$ of E. $\tilde{V}$ is not unique but the dimension of any such $\tilde{V}$ is uniquely determined. If $V \in F(E)$ then E/V is finite dimensional and isomorphic to $\tilde{V}$. Let $p_V : E \to E/V$ denote the canonical mapping into the quotient space. The Borel sets in E/V, $\mathcal{B}(E/V)$ are generated by the open subsets of the finite dimensional space E/V.

Definition 20 *The collection of all $(p_V^{-1}(B))_{V\in F(E),B\in\mathcal{B}(E/V)}$ are called the cylindrical Borel subsets of E. We denote this collection by $\mathcal{B}_{\mathcal{C}}(E)$.*

We state the following:

1) $\mathcal{B}_{\mathcal{C}}(E)$ is a field of subsets of E.

2) $\mathcal{B}_{\mathcal{C}}(E)$ is a σ-field $\leftrightarrow \dim(E) < \infty$.

We place a partial order on $F(E)$ by using set inclusion. If V and W are two elements in $F(E)$ such that $W \subset W$, we denote by p_{VW} the mapping of E/W into E/V deduced from the identity mapping of E by passage to the quotients. We have the following diagram.

$$\begin{tikzcd}
E \arrow[r, "p_W"] \arrow[dr, "p_V"'] & E/W \arrow[d, "p_{VW}"] \\
 & E/V
\end{tikzcd}$$

By an inverse system of topological spaces we mean a family (T_i, p_{ij}) indexed by the nonempty set I equipped with the preorder relation $i \le j$, where T_i is a topological space and p_{ij} is a continuous mapping of T_j into T_i for $i \le j$. Let $\Gamma = (T_i, p_{ij})$ be an inverse system of topological spaces indexed by I. One calls inverse system of measures on Γ a family $(\mu_i)_{i\in I}$ where μ_i is a bounded measure on T_i for all $i \in T_i$ for all $i \in I$, and where $\mu_i = p_{ij}(\mu_j)$ for $i \le j$.
With this notation the family $\mathcal{L}(E) = (E/V, p_{VW})$ is an inverse system of locally convex spaces, indexed by $F(E)$. We call it the *inverse system of finite-dimensional quotients of E.*

Definition 21 *One calls a promeasure on E every inverse system of measures on the inverse system of finite dimensional quotients of E.*

Now let λ be a bounded measure on E. For every $V \in F(E)$, let us denote by $\tilde{\lambda}_V$ the image of V under the canonical mapping p_V of E onto E/V. It is easy to see that the family $\tilde{\lambda} = (\tilde{\lambda}_V)_{V\in F(E)}$ is a promeasure on E. We say that $\tilde{\lambda}$ is the promeasure associated with the measure λ. Also λ and $\tilde{\lambda}$ has the same mass.

Let E be a locally convex space and $\mu = (\mu_V)_{V\in F(E)}$ a promeasure on E. For every continuous linear form x' on E we denote by $\mu_{x'}$ the measure on $\mathbb{R}$ that is the image under x' of the promeasure μ on E. The Fourier transform of μ is the function $\mathcal{F}\mu$ on E' defined by

$$(\mathcal{F}\mu)(x') = \int_{\mathbb{R}} e^{it} d\mu_{x'}(t).$$

Let λ be a bounded measure on E. The Fourier transform of λ is the function on E' defined by

$$(\mathcal{F}\lambda)(x') = \int_E e^{i\langle x,x'\rangle} d\lambda(x).$$

Let μ be the promeasure associated with λ. For every $x' \in E$ the measure $\mu_{x'}$ on $\mathbb{R}$ is the image under $x' : E \to \mathbb{R}$ of the measure λ on E. We deduces immediately that $\mathcal{F}\mu = \mathcal{F}\lambda$.

Proposition 4 *Let E be a locally convex space . For every positive quadratic form Q on E' there exists one and only one promeasure Γ_Q on E such that $\mathcal{F}\Gamma_Q = e^{-Q/2}$. The total mass of Γ_Q is equal to 1.*

Definition 22 *Let E be a locally convex space. For every positive quadratic form Q on E', the promeasure on E whose Fourier transform is equal to $\mathrm{e}^{-Q/2}$ is called the Gaussian promeasure on E whose Fourier transform with variance Q, and is denoted Γ_Q. A promeasure μ on E is said to be Gaussian if there exists a positive quadratic form Q on E' such that $\mu = \Gamma_Q$*

Theorem 3 *(Minlo's) Let E be a fully nuclear space with basis. If $(\mu_V)_{v\in F(E)}$ is a promeasure on E then $(\mu_V)_{v\in F(E)}$ is a measure if and only if $\mathcal{F}(\mu_V)_{V\in F(E)}$ is continuous.*

2.6 The integral formula

We let

$$\langle \sum_{n=1}^{\infty} z_n e_n, \sum_{n=1}^{\infty} w_n e'_n \rangle = \sum_{n=1}^{\infty} z_n \bar{w}_n$$

denote the dual pairing between E and E'_β. We may also use this dual pairing to define a fundamental system of semi-norms on E; if $\sum_{n=1}^{\infty} \beta_n e'_n \in E'_\beta$ then $\sum_{n=1}^{\infty} z_n e_n \in E \longrightarrow \sum_{n=1}^{\infty} |\beta_n z_n|$ defines a continuous semi-norm on E and if we consider all such semi-norms we obtain a fundamental system of semi-norms. We shall always suppose that E is a complex space and denote by $E_\mathbb{R}$ the space E with its underlying real structure. If we let $e_{n*} = ie_n$, where $i = \sqrt{-1}$, then it is easily seen that $(e_n, e_{m*})_{n,m=1}^{\infty}$ is an absolute basis for $E_\mathbb{R}$. Moreover, the complexification of $E_\mathbb{R}$ is isomorphic to $E \times E$. If $\sum_{n=1}^{\infty} z_n e_n \in E$ then there exists $\sum_{n=1}^{\infty} z'_n e_n \in E$ such that $(|z_n|/|z'_n|)_{n=1}^{\infty} \in \ell_1$.

By [3] the monomials form an absolute basis for $\mathcal{H}(E)$, the entire functions on E, with respect to any of the usual topologies, including the compact open topology τ_0, whenever E is a fully nuclear space with basis. If $\theta := (\theta_n)_{n=1}^{\infty}$ and $z := (z_n)_{n=1}^{\infty}$ are sequences of complex numbers we let

$$\|z\|_\theta = \sum_{n=1}^{\infty} |z_n| \cdot |\theta_n| = \|\theta\|_z.$$

We refer to [3] for further detail on fully nuclear spaces with basis and the theory of holomorphic functions on these spaces.

A holomorphic mapping $f \in \mathcal{H}(E)$ which satisfies any of the equivalent conditions in the following proposition is said to be of *η-exponential type*.

Proposition 5 *If E is a fully nuclear space with basis, $\eta_n \geq 0$ for all n, and $\sum_{n=1}^{\infty} \eta_n e'_n \in E'_\beta$ then the following are equivalent conditions on $f \in \mathcal{H}(E)$:*

(a) $f(z) := \sum_{m\in\mathbb{N}^{(\mathbb{N})}} a_m z^m \in \mathcal{H}(E)$ satisfies $|a_m| \leq ab^{|m|}\eta^m/m!$ for some $a, b > 0$ and all $m \in \mathbb{N}^{(\mathbb{N})}$,

(b) $f = \sum_{n=0}^{\infty} P_n$ where P_n is a continuous n-homogeneous polynomial for each n and $|P_n(z)| \leq cd^n\|z\|_\eta^n/n!$ for all n and all $z \in E$ and some $c, d > 0$,

(c) $|f(z)| \leq Ae^{B\|z\|_\eta}$ for all $z \in E$ where A, B are positive constants.

Proof. If (a) holds then for all n

$$\begin{aligned} |P_n(z)| &\leq \sum_{m\in\mathbb{N}^{(\mathbb{N})},|m|=n} |a_m| \cdot |z^m| \\ &\leq ab^{|m|} \sum_{m\in\mathbb{N}^{(\mathbb{N})},|m|=n} \eta^m |z^m|/m! \\ &= ab^n\|z\|_\eta^n/n!. \end{aligned}$$

and (a) implies (b). If (b) holds then

$$|f(z)| \leq \sum_{n=0}^{\infty} |P_n(z)| \leq c\sum_{n=0}^{\infty} \frac{d^n\|z\|_\eta^n}{n!} \leq ce^{d\|z\|_\eta}$$

and (b) implies (c).

Suppose (c) holds. For any sequence of positive scalars $(\alpha_i)_{i=1}^\infty$ the set $D := \{(z_i)_{i=1}^n : |z_i| = \alpha_i/\eta_i, i = 1, \ldots, n\}$ is the distinguished boundary of the compact polydisc $\{(z_i)_{i=1}^n : |z_i| \leq \alpha_i/\eta_i, i = 1, \ldots, n\}$ and

$$\sup\{\|z\|_\eta : z \in D\} = \sup\{\sum_{i=1}^n |z_i|\eta_i : |z_i| = \alpha_i/\eta_i\} = \sum_{i=1}^n \alpha_i.$$

If $m = (m_1, \ldots, m_n) \in \mathbb{N}^{(\mathbb{N})}$ then, using Cauchy estimates, we have

$$\frac{\alpha^m}{\eta^m}|a_m| = \|a_m z^m\|_{\{z\in D\}} \leq \|f(z)\|_{\{z\in D\}} \leq Ae^{B\sup\{\|z\|_\eta : z\in D\}} \leq Ae^{B\sum_{i=1}^n \alpha_i}.$$

Hence

$$|a_m| \leq A\eta^m \inf_{\alpha_i>0}(\prod_{1\leq i\leq n} \frac{e^{B\alpha_i}}{\alpha_i^{m_i}})$$

and on letting $\alpha_i = m_i/B$ we obtain

$$|a_m| \le A\eta^m \prod_{1\le i\le n} \frac{(Be)^{m_i}}{m_i^{m_i}} \le A\eta^m \prod_{1\le i\le n} \frac{(Be)^{m_i}}{m_i!} \le A(Be)^{|m|}\eta^m/m!.$$

Hence (c) implies (a) and this completes the proof.

$\square$

Let E denote a fully nuclear space with basis over $\mathbb{C}$. We identify, $we'_j := (x+iy)e'_j \in E'_\beta$ with $xe'_j + ye'_{j*} \in (E'_\beta)_\mathbb{R}$ and $ze_j := (u+iv)e_j \in E$ with $ue_j + v_je_{j*} \in E_\mathbb{R}$ for x, y, u and v in $\mathbb{R}$. If $Q : E_\mathbb{R} \longrightarrow \mathbb{R}$ is a positive, that is $Q(x) \ge 0$ for all $x \in E_\mathbb{R}$, continuous quadratic form then by a result of Minlo, see Proposition 4, p.76, and Corollaire p.92 in [1], there is a Gaussian probability measure μ on E'_β such that for all $z \in E$

$$[\mathcal{F}(\mu)](z) := \int_{E'_\beta} e^{i\langle z,w\rangle} d\mu(w) = e^{-Q(z)/2}.$$

The mapping $\mathcal{F}(\mu)$ is called the Fourier transform of μ.

Lemma 1 *If $(e_n)_{n=1}^\infty$ is an absolute basis for the fully nuclear space E and $(\gamma_n)_{n=1}^\infty$ is a sequence of non-negative scalars such that $\sum_{n=1}^\infty \gamma_n e'_n \in E'_\beta$ then $Q(\sum_{n=1}^\infty x_n e_n + \sum_{m=1}^\infty y_m e_{m*}) = \sum_{n=1}^\infty \gamma_n^2(x_n^2 + y_n^2)$ defines a continuous positive quadratic form on $E_\mathbb{R}$. Moreover, if μ_γ is the Gaussian measure on $(E_\mathbb{R})'_\beta$ such that $\mathcal{F}(\mu_\gamma) = e^{-Q/2}$ then for any cylindrical Borel set B with base in the subspace of $(E_\mathbb{R})'_\beta$ spanned by $(e'_n, e'_{m*})_{n,m=1}^k$ we have*

$$\begin{aligned} \mu_\gamma(B) &= \frac{1}{\pi^k(\gamma_1\cdots\gamma_k)^2} \int_B e^{-\sum_{n=1}^k \frac{x_n^2+y_n^2}{\gamma_n^2}} dx_1dy_1\cdots dx_kdy_k \\ &= \frac{i^k}{(2\pi)^k(\gamma_1\cdots\gamma_k)^2} \int_B e^{-\sum_{n=1}^k \frac{|w_n|^2}{\gamma_n^2}} dw_1d\overline{w}_1\cdots dw_kd\overline{w}_k. \end{aligned}$$

Proof. Since $\gamma_n \ge 0$ for all n, Q is a positive quadratic form. If $\sum_{n=1}^\infty \gamma_n e'_n \in E'_\beta$ then, by nuclearity, we can find a sequence of positive real numbers $(\delta_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty 1/\delta_n < \infty$ and $\sum_{n=1}^\infty \delta_n\gamma_n e'_n \in E'_\beta$. By duality $V := \{\sum_{n=1}^\infty z_n e_n \in E : \sup_n |\delta_n\gamma_n z_n| \le 1\}$ is a neighborhood of zero in E. Since

$$\sum_{n=1}^\infty \sup\{\gamma_n^2(x_n^2+y_n^2) : \sum_{n=1}^\infty x_n e_n \in kV, \sum_{m=1}^\infty y_m e_{m*} \in kV\} \le 2k^2 \sum_{n=1}^\infty \delta_n^{-2} < \infty$$

for all positive integers k, Q is the limit, uniformly over kV, of the sequence of continuous positive quadratic forms

$$Q_l(\sum_{n=1}^\infty x_n e_n + \sum_{m=1}^\infty y_m e_{m*}) := \sum_{n=1}^l \gamma_n^2(x_n^2+y_n^2).$$

Hence Q is continuous and by the result of Minlos, quoted above, there exists a Gaussian probability measure μ such that $\mathcal{F}(\mu) = e^{-Q/2}$. The proof is completed by noting that $dxdy = idzd\overline{z}/2$.

$\square$

With the above notation we have the following result.

Lemma 2 *Let $\sum_{n=1}^{\infty}\gamma_n e'_n \in E'_\beta$ where $\gamma_n \geq 0$ for all n. Let $(\theta_n)_{n=1}^{\infty}$ denote a sequence of real numbers and suppose $1 \leq p < \infty$. If $\sum_{n=1}^{\infty}|\theta_n|\gamma_n < \infty$ then the mapping $E'_\beta \ni \sum_{n=1}^{\infty} w_n e'_n \longmapsto e^{\|w\|_\theta}$ belongs to $\mathcal{L}^p(\mu_\gamma)$.*

Proof. We have $|e^{\|w\|_\theta}|^p = e^{p\sum_{n=1}^{\infty}|\theta_n|\cdot|w_n|}$. If $g_k : E'_\beta \ni w = \sum_{n=1}^{\infty} w_n e'_n \longmapsto e^{p\sum_{n=1}^{k}|\theta_n|\cdot|w_n|}$ then the sequence $(g_k)_{k=1}^{\infty}$ is increasing and it suffices by the Monotone Convergence Theorem to show $\lim_{k\to\infty}\int_{E'_\beta} g_k d\mu_\gamma$ is finite. On letting $q = p/2$ we have

$$\begin{aligned}
\int_{E'_\beta} g_k(w)d\mu_\gamma(w) &= \frac{i^k}{(2\pi)^k(\gamma_1\cdots\gamma_k)^2}\int_{\mathbb{C}^k} e^{p\sum_{n=1}^{k}|\theta_n|\cdot|w_n|}e^{-\sum_{n=1}^{k}\frac{|w_n|^2}{\gamma_n^2}}dw_1 d\overline{w}_1\cdots dw_k d\overline{w}_k \\
&= \prod_{n=1}^{k}\frac{i}{2\pi\gamma_n^2}\int_{\mathbb{C}} e^{p|\theta_n|\cdot|w_n| - \frac{|w_n|^2}{\gamma_n^2}}dw_n d\overline{w}_n \\
&= \prod_{n=1}^{k}\frac{ie^{q^2|\theta_n|^2\gamma_n^2}}{2\pi\gamma_n^2}\int_{\mathbb{C}} e^{-\frac{1}{\gamma_n^2}\left(|w_n| - q|\theta_n|\gamma_n^2\right)^2}dw_n d\overline{w}_n \\
&= e^{\sum_{n=1}^{k}q^2|\theta_n|^2\gamma_n^2}\prod_{n=1}^{k}\frac{1}{\gamma_n^2}\int_0^\infty 2se^{-\frac{1}{\gamma_n^2}\left(s-q|\theta_n|\gamma_n^2\right)^2}ds \\
&= e^{\sum_{n=1}^{k}q^2|\theta_n|^2\gamma_n^2}\prod_{n=1}^{k}\int_0^\infty 2re^{-\left(r-q|\theta_n|\gamma_n\right)^2}dr.
\end{aligned}$$

We now consider the integral

$$f(\alpha) := \int_0^\infty 2re^{-(r-\alpha)^2}dr$$

as a function of $\alpha \geq 0$ on the interval $[0,1]$. We have $f(0) = 1$ and, by the Mean Value Theorem, for $0 < \alpha < 1$ and $r > 0$,

$$\left|\frac{e^{-(r-\alpha)^2} - e^{-r^2}}{(r-\alpha) - r}\right| = 2r_\alpha e^{-r_\alpha^2}$$

where $r - \alpha < r_\alpha < r$. Hence

$$|e^{-(r-\alpha)^2} - e^{-r^2}| \leq 2\alpha g(r) := \begin{cases} 2\alpha & \text{if } 0 \leq r \leq 1 \\ 2\alpha r e^{-(r-1)^2} & \text{if } r > 1 \end{cases}$$

and

$$|f(\alpha)-f(0)| \quad \le \quad \int_0^\infty 2r\left|e^{-(r-\alpha)^2}-e^{-r^2}\right|dr \le 4\alpha\int_0^\infty rg(r)dr.$$

As the function $r \longmapsto rg(r)$ is integrable, the constant $C := 4\int_0^\infty rg(r)dr$ is finite and $|f(\alpha)-f(0)| \le C\alpha$ for $0 \le \alpha < 1$. This implies

$$\sum_{n=1}^\infty |f(q|\theta_n|\gamma_n)-f(0)| \le Cq\sum_{n=1}^\infty |\theta_n|\gamma_n < \infty$$

and hence $\prod_{n=1}^\infty \big(1+(f(q|\theta_n|\gamma_n)-f(0))\big) < \infty$. We now have

$$\begin{aligned}
\lim_{k\to\infty}\int_{E'_\beta} g_k d\mu_\gamma &= e^{\sum_{n=1}^\infty q^2|\theta_n|^2\gamma_n^2}\prod_{n=1}^\infty\int_0^\infty 2re^{-\left(r-q|\theta_n|\gamma_n\right)^2}dr \\
&\le e^{\sum_{n=1}^\infty q^2|\theta_n|^2\gamma_n^2}\prod_{n=1}^\infty f(q|\theta_n|\gamma_n) \\
&< \infty.
\end{aligned}$$

This completes the proof.

□

We shall also need the following result (see Lemma 2.1 in [6]).

Lemma 3
$$\int_{E'_\beta} w^m\overline{w}^{m'}d\mu_\gamma = \begin{cases} m!\gamma^{2m} & \textit{if } m=m' \\ 0 & \textit{otherwise} \end{cases}$$

Proof. If $m=(m_1,\ldots,m_k)$ and $m'=(m'_1,\ldots,m'_k)$ then

$$\begin{aligned}
\int_{E'_\beta} w^m\overline{w}^{m'}d\mu_\gamma &= \frac{i^k}{(2\pi)^k(\gamma_1\cdots\gamma_k)^2}\int_{\mathbb{C}^k} w^m\overline{w}^{m'}e^{-\sum_{n=1}^k\frac{|w_n|^2}{\gamma_n^2}}dw_1d\overline{w}_1\cdots dw_kd\overline{w}_k \\
&= \prod_{n=1}^k\frac{i}{2\pi\gamma_n^2}\int_{\mathbb{C}} w^{m_n}\overline{w}^{m'_n}e^{-\frac{|w|^2}{\gamma_n^2}}dwd\overline{w}
\end{aligned}$$

and this is 0 if $m \neq m'$. If $m = m'$ then

$$\begin{aligned}
\int_{E'_\beta} w^m \overline{w}^{m'} d\mu_\gamma &= \prod_{n=1}^{k} \frac{i}{2\pi\gamma_n^2} \int_{\mathbb{C}} |w|^{2m_n} e^{-\frac{|w|^2}{\gamma_n^2}} dw d\overline{w} \\
&= \prod_{n=1}^{k} \frac{1}{\pi\gamma_n^2} \int_{\mathbb{R}^2} (x^2+y^2)^{m_n} e^{-\frac{x^2+y^2}{\gamma_n^2}} dxdy \\
&= \prod_{n=1}^{k} \frac{2}{\gamma_n^2} \int_0^\infty r^{2m_n} e^{-\frac{r^2}{\gamma_n^2}} r dr \\
&= \prod_{n=1}^{k} \gamma^{2m_n} \int_0^\infty u^{m_n} e^{-u} du \\
&= \gamma^{2m} m!
\end{aligned}$$

□

The following is the main result in this paper.

Proposition 6 *Let E denote a fully nuclear space with basis $(e_n)_{n=1}^\infty$, strong dual E' and dual basis $(e'_n)_{n=1}^\infty$. Let $\sum_{n=1}^\infty \eta_n e'_n$ belong to E'_β, $\eta_n \geq 0$ for all n, and suppose $f \in \mathcal{H}(E)$ is of η-exponential type. Let $\sum_{n=1}^\infty \gamma_n e'_n \in E'_\beta$, $\gamma_n \geq 0$ all n, be chosen so that $\sum_{n=1}^\infty \eta_n/\gamma_n < \infty$. If $D : E'_\beta \longrightarrow E$ denotes the densely defined linear operator $D(\sum_{n=1}^\infty w_n e'_n) := \sum_{n=1}^\infty \gamma_n^{-2} w_n e_n$ and μ_γ denotes the measure on E'_β that we have previously associated with γ then*

$$f(z) = \int_{E'_\beta} e^{\langle z, w\rangle} \cdot (f \circ D)(w) d\mu_\gamma(w) \tag{2.1}$$

for all $z \in E$.

Proof. Let $f(z) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m z^m$ for all $z \in E$. By our hypothesis there exist $a > 0$ and $b > 0$ such that $|a_m| \leq a b^{|m|} \eta^m / m!$ for all $m \in \mathbb{N}^{(\mathbb{N})}$. Since

$$\begin{aligned}
\sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m| \cdot |(D(w))^m| &\leq a \sum_{m \in \mathbb{N}^{(\mathbb{N})}} b^{|m|} \eta^m (|w|\gamma^{-2})^m / m! \qquad (2.2) \\
&= a \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{b^{|m|}(|w|\eta\gamma^{-2})^m}{m!} \\
&= a e^{b\|w\|_{\eta\gamma^{-2}}}
\end{aligned}$$

and

$$\sum_{n=1}^\infty \left(\frac{\eta_n}{\gamma_n^2}\right)\gamma_n = \sum_{n=1}^\infty \frac{\eta_n}{\gamma_n} < \infty.$$

Lemma(2) implies that the mapping

$$f \circ D : E'_\beta \longrightarrow \mathbb{C}$$
$$w \longmapsto \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m \cdot (D(w))^m$$

converges pointwise to an element in $\mathcal{L}^p(\mu_\gamma)$ for all $p \geq 1$.

If $|z| := \sum_{n=1}^{\infty} |z_n| e_n \in E$ and $w := \sum_{n=1}^{\infty} w_n e'_n \in E'_\beta$ then

$$\left| e^{\sum_{n=1}^{\infty} z_n \cdot \overline{w}_n} \right| \leq e^{\sum_{n=1}^{\infty} |z_n| \cdot |w_n|} = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{|z|^m \cdot |w|^m}{m!} = e^{\|w\|_{|z|}}. \tag{2.3}$$

and, since $\sum_{n=1}^{\infty} |z_n| \cdot \gamma_n < \infty$, Lemma 2 implies that the mapping $w \in E'_\beta \longrightarrow e^{\sum_{n=1}^{\infty} |z_n||\overline{w}_n|}$ belongs to $\mathcal{L}^p(\mu_\gamma)$ for all $p, 1 \leq p < \infty$.

By Hölder's inequality the mapping

$$E'_\beta \ni w \longmapsto e^{\langle z, w \rangle} \cdot (f \circ D)(w)$$

is integrable and, by (2.2) and (2.3), we may integrate term by term to obtain, using Lemma 3,

$$\begin{aligned}
\int_{E'_\beta} e^{\langle z,w \rangle} \cdot (f \circ D)(w) d\mu_\gamma(w) &= \int_{E'_\beta} \sum_{m', m \in \mathbb{N}^{(\mathbb{N})}} \frac{z^{m'} \overline{w}^{m'}}{m'!} a_m (D(w))^m d\mu_\gamma \\
&= \sum_{m', m \in \mathbb{N}^{(\mathbb{N})}} \frac{a_m z^{m'}}{m'!} \cdot \gamma^{-2m} \int_{E'_\beta} w^m \overline{w}^{m'} d\mu_\gamma \\
&= \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{a_m z^m}{m!} \cdot \gamma^{-2m} m! \gamma^{2m} \\
&= \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m z^m \\
&= f(z).
\end{aligned}$$

This proves (1) and completes the proof.

$\square$

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