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On the Module of Derivations Preserving a Monomial Ideal

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Let I be a monomial ideal in a polynomial ring $\mathbf{A} = \mathbf{k}[x_1, \cdots, x_n]$ over a field $\mathbf{k}$ of characteristic 0 and $\mathfrak{m}$ be the graded maximal ideal of $\mathbf{A}$. An explicit geometric description of the structure of the module $T_{\mathbf{A}/\mathbf{k}}(I)$ of $\mathbf{k}$-derivations of $\mathbf{A}$ preserving I is given, building on [1]. If $\nabla_{\mathbf{A}}$ denotes the $\mathbf{A}$-algebra generated by the operators $x_1\partial_{x_1}, ..., x_n\partial_{x_n}$, we note that any $\nabla_{\mathbf{A}}$-submodule of $\mathbf{A}$, which is the same as a monomial ideal, is cyclic. We show that $T_{\mathbf{A}/\mathbf{k}}(I)$ preserves the integral closure of a monomial ideal and also, in the two variables case, that it preserves the multiplier ideals of I. Let $\Delta_{\mathbf{A}}(I)$ denote the $\mathbf{A}$-algebra generated by $T_{\mathbf{A}/\mathbf{k}}(I)$. We study the problem of finding minimal sets of monomial generators of the $\Delta_{\mathbf{A}}(I)$-module I, and determine the canonical cyclic generators of I. If I is $\mathfrak{m}$-primary and satisfies a certain condition, we prove $\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(\mathbf{A}/I^l) = \dim_{\mathbf{k}}(\mathbf{A}/I^l)$.

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1 Introduction

Differential operators are fundamental tools in many areas of mathematics. Unfortunately, their use in commutative algebra has often been ineffective due to the difficulty of computing rings of differential operators in general. Different attempts have been made remedy this situation [1,4,5,9,10,16,17, 19]. In this thesis we will take one small step in the study of such applications to monomial ideal theory.

Throughout this thesis $\mathbf{k}$ denotes a field of characteristic 0 and $\mathbf{A}$ is the polynomial ring $\mathbf{k}[x_1,\dots,x_n]$. The ring $\mathcal{D}_{\mathbf{A}}$ of differential operators on $\mathbf{A}$, the Weyl Algebra, is

$$\mathcal{D}_{\mathbf{A}} = \cup_{l=0}^{\infty}\mathcal{D}_{\mathbf{A}}^{l},$$

where $\mathcal{D}_{\mathbf{A}}^{0} = \mathbf{A}$ and for $l \geq 1$,

$$\mathcal{D}_{\mathbf{A}}^{l} = \{\delta \in \mathrm{End}_k(\mathbf{A}) \mid [\delta, \mathcal{D}_{\mathbf{A}}^{0}] \in \mathcal{D}_{\mathbf{A}}^{l-1}\}$$

where for $\delta_1, \delta_2 \in \mathcal{D}_{\mathbf{A}}$, $[\delta_1,\delta_2] = \delta_1\delta_2 - \delta_2\delta_1$ is the commutator. The set of all $\mathbf{k}$-*linear derivations on* $\mathbf{A}$ is

$$T_{\mathbf{A}/\mathbf{k}} = \{\delta \in \mathrm{End}_k(\mathbf{A}) \mid \delta(ab) = a\delta(b) + b\delta(a);\ \forall a, b \in \mathbf{A}\}.$$

This is clearly an $\mathbf{A}$-submodule of $\mathcal{D}_{\mathbf{A}}$ which moreover is a Lie algebra. For an ideal I of $\mathbf{A}$, the module of I-*preserving* $\mathbf{k}$-derivations is

$$T_{\mathbf{A}/\mathbf{k}}(I) = \{\delta \in T_{\mathbf{A}/\mathbf{k}} \mid \delta(I) \subseteq I\}.$$

This is also a Lie algebra. The subalgebra of $\mathcal{D}_{\mathbf{A}}$ generated by $\mathbf{A}$ and $T_{\mathbf{A}/\mathbf{k}}(I)$ is denoted by $\Delta_{\mathbf{A}/\mathbf{k}}(I)$. This is a subalgebra of the *idealizer* $\mathcal{D}_{\mathbf{A}}(I) = \{\delta \in \mathcal{D}_{\mathbf{A}} \mid \delta(I) \subseteq I\}$ of I. Put $\nabla_{x_j} = x_j\partial_{x_j}$ and let $\nabla_{\mathbf{A}} \subset \mathcal{D}_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra generated by the ∇_{x_j}, $j = 1, ..., n$.

We can now outline the content of this thesis. In section 2 we recall some generalities concerning the rings $\mathcal{D}_{\mathbf{A}}$ and $\Delta_{\mathbf{A}/\mathbf{k}}(I)$. In Section 3, we consider the module of derivations preserving a monomial ideal I of $\mathbf{A}$. The main result of P. Brumatti and A. Simis [1] is a precise description of $T_{\mathbf{A}/\mathbf{k}}(I)$. It states that a derivation $\delta = \sum_{j=1}^{n} f_j\partial_{x_j}$ of $\mathbf{A}$ preserves I if and only if $f_j \in [I : [I : x_j]]$ for all $j = 1, ..., n$. We refine their result by describing the ideals $[I : [I : x_j]]$ in a geometric way as follows. Let H_j be the coordinate hyperplane $x_j = 0$ in the real space $\mathbb{R}^n$ and $e_j = (0, ..., 0, 1, 0, ..., 0)$ be the standard basis element of $\mathbb{R}^n$. Let I be a monomial ideal of $\mathbf{A}$ having $\{X^{\alpha_1}, ..., X^{\alpha_t}\}$ as its unique minimal set of monomial generators where we write X^{α_i} for a monomial $x_1^{a_{i1}}...x_n^{a_{in}}$, $i = 1,...,t$, and put $\exp(I) = \{\alpha \in \mathbb{Z}^n_{\geq 0} \mid X^\alpha \in I\}$. Then we prove:

(1) If $H_j \cap \exp(I) = \emptyset$, then $[I : [I : x_j]] = (x_j)$, for $j = 1,...,n$.

(2) If $H_j \cap \exp(I) \neq \emptyset$, then $[I : [I : x_j]] = (x_j) + I_{x_j}$, where $j = 1,...,n$ and

$$I_{x_j} := \bigcap_{i=1}^{t} \frac{(X^{\alpha_i - a_{ij}e_j}) \cap (\{X^{\alpha_l - a_{lj}e_j} : a_{lj} < a_{ij}\}_{l=1}^{t})}{X^{\alpha_i - a_{ij}e_j}}.$$

(3)

$$T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_{x_1}, ..., \nabla_{x_n}) + \sum I_{x_j}\partial_{x_j}$$

where the sum runs through all $j = 1,...,n$ for which $H_j \cap \exp(I) \neq \emptyset$.

Note that $H_j \cap \exp(I) = \emptyset$ if and only if $I = x_j I'$ for some monomial ideal I'. This description of $T_{\mathbf{A}/\mathbf{k}}(I)$ is basic to all our results.

For a (monomial) ideal I in $\mathbf{A}$, $\bar{I}$ denotes the integral closure and $\tilde{I}$ the Ratliff-Rush closure of I. For a rational number $r > 0$ let $\mathcal{I}(r \cdot I)$ denotes the multiplier ideal of I (see section 4 for the precise definitions). In section 4 we prove the following inclusions for monomial ideals:

(1) $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(I^l)$ for all $l \in \mathbb{Z}_{>0}$ where equality holds if $I = [I^l : I^{l-1}]$,

(2) $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\tilde{I})$,

(3) $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\bar{I})$,

(4) $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\mathcal{I}(r \cdot I))$.

Here the inclusions in (1) and (2) are elementary and the argument applies to any ideal, while (3) can be proven using the blow-up of I [10], moreover, a proof of (4) is indicated in [loc. cit]. Here a simple and direct proof of (3) is given for monomial ideals. The inclusion (4) is proved in the two variable case $\mathbf{A} = \mathbf{k}[x, y]$. The proof, which is based on Howald's description of $\mathcal{I}(r \cdot I)$ in [6], is perhaps the most technical part of the thesis.

Section 5 deals with *differential reductions of a monomial ideal* I, that is an ideal J contained in I such that $\Delta_{\mathbf{A}}(I) \cdot J = I$. Alternatively, we are looking for generators of I as $\Delta_{\mathbf{A}/\mathbf{k}}(I)$-module. We are in particular interested in minimal differential reductions, i.e. a minimal set of generators. We note that monomial ideals, which are the same as $\nabla_{\mathbf{A}}$-modules, are always cyclic; this is surely known, but we have included a proof for the lack of suitable reference. A procedure for computing the unique *minimal monomial differential reduction of* I is given. Assume that J is the minimal monomial differential reduction of I generated by the unique set of monomials $\{X^{\alpha_1}, ..., X^{\alpha_t}\}$. Following from the result in section 3, we prove that any minimal differential reduction of I is a principal ideal generated by a polynomial of the form,

$$f(X) = \sum_{i=1}^{t} k_i X^{\alpha_i} \text{ where } k_1, ..., k_t \in \mathbf{k} \setminus \{0\}.$$

Note that any cyclic generator f of the $\Delta_{\mathbf{A}}(I)$-module I will have at least t non-zero monomial terms.

Denote the graded maximal ideal of $\mathbf{A}$ by $\mathfrak{m}$. Section 6 deals with the length of the $\Delta_{\mathbf{A}}(I)$-module $\mathbf{A}/I^l$ where I is an $\mathfrak{m}$-primary monomial ideal. It is easy to see that

$$\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(\mathbf{A}/I^l) \leq \dim_{\mathbf{k}}(\mathbf{A}/I^l).$$

We prove that equality holds if $x_i \notin I_{x_j}$ for all $i, j = 1, ..., n$. Moreover, if $I = \mathfrak{m}^d$ for an integer $d > 0$, then

$$\mathfrak{l}_{\Delta_{\mathbf{A}}(\mathfrak{m})}(\mathbf{A}/I^l) = dl,$$

hence in this case we do not get equality.

2 On the Ring of Differential Operators

Definitions and properties stated in this section can be found in e.g. [9,12].

2.1 Definition and Properties of $\mathcal{D}_A$

Let k be a commutative domain containing the field $\mathbb{Q}$ of rational numbers and A be a commutative k-algebra of finite type. Every element $a \in A$ defines a k-linear endomorphism $\phi_a : A \to A$ by $\phi_a(x) = ax$. For any $\delta \in \mathrm{End}_k(A)$, we write $a\delta$ and δa for $\phi_a\delta$ and $\delta\phi_a$ respectively. The ring $\mathcal{D}_A$ of differential operators on A is defined by

$$\mathcal{D}_A = \cup_{l=0}^{\infty} \mathcal{D}_A^l,$$

where $\mathcal{D}_A^0 = A$ and for $l \geq 1$, define

$$\mathcal{D}_A^l = \{\delta \in \mathrm{End}_k(A) \mid [\delta, \mathcal{D}_A^0] \in \mathcal{D}_A^{l-1}\}$$

where for $\delta_1, \delta_2 \in \mathcal{D}_A$, $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$ is the usual commutator. This definition gives the structure of a filtered ring, since

$$\mathcal{D}_A^l \subseteq \mathcal{D}_A^{l+1}, \text{ and } \mathcal{D}_A^l \mathcal{D}_A^r \subseteq \mathcal{D}_A^{l+r} \text{ for all } l, r \geq 0.$$

The *order* of an element $\delta \in \mathcal{D}_A$ is the least non-negative integer l such that $\delta \in \mathcal{D}^l_A$. Clearly $\mathcal{D}_A$ is an A-module with the action defined in a natural way. The set of all *k-linear derivations on A* is defined by

$$T_{A/k} = \{\delta \in \mathrm{End}_k(A) \mid \delta(ab) = a\delta(b) + b\delta(a);\ \forall a, b \in A\}.$$

This too is an A-submodule of $\mathcal{D}_A$, for if $\delta \in T_{A/k}$ and $a \in A$, then $[\delta, a] = \delta(a)$. It is also a Lie-algebra with the Lie bracket as the above commutator.

For an ideal I of A, the set of *I-preserving k-derivations* is

$$T_{A/k}(I) = \{\delta \in T_{A/k} \mid \delta(I) \subseteq I\}.$$

This is a submodule of $T_{A/k}$ and we have $I \cdot T_{A/k} \subseteq T_{A/k}(I)$. The *idealizer of I* is the subalgebra $\mathcal{D}_A(I) = \{\delta \in \mathcal{D}_A \mid \delta(I) \subseteq I\} \subseteq \mathcal{D}_A$. The subalgebra of $\mathcal{D}_A$ generated by A and $T_{A/k}(I)$ is denoted by $\Delta_A(I)$. The filtration of $\mathcal{D}_A$ induces a filtration on $\Delta_A(I)$ and $\mathcal{D}_A(I)$:

$$\Delta^l_A(I) = \mathcal{D}^l_A \cap \Delta_A(I) \subseteq \mathcal{D}^l_A(I) = \mathcal{D}^l_A \cap \mathcal{D}_A(I) \subseteq \mathcal{D}^l_A.$$

2.2 The Ring of Differential Operators on $\mathbf{A} = \mathbf{k}[x_1, ..., x_n]$

In this sub-section we recall few properties of terms defined above when $A = \mathbf{A} = \mathbf{k}[x_1, ..., x_n]$. Then $\mathcal{D}_{\mathbf{A}}$ is the *Weyl Algebra*, that is the free associative algebra

$$\mathbf{k} < x_1, x_2, ..., x_n, \partial_{x_1}, ..., \partial_{x_n} >$$

modulo the relations $x_i x_j - x_j x_i = \partial_{x_i}\partial_{x_j} - \partial_{x_j}\partial_{x_i} = 0$ and $\partial_{x_i} x_j - x_j \partial_{x_i} = \delta_{ij}$ where δ_{ij} is the Kronecker delta and we write ∂_{x_j} in place of $\frac{\partial}{\partial x_j}$. For $\alpha = (a_1, a_2, ..., a_n) \in \mathbf{Z}^n_{\geq 0}$, denote the differential operator $\frac{\partial^{a_1}}{\partial x_1^{a_1}} \frac{\partial^{a_2}}{\partial x_2^{a_2}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}$ simply by ∂^α. The non-commutative $\mathbf{k}$-algebra $\mathcal{D}_{\mathbf{A}}$ has the $\mathbf{k}$-basis

$$\{X^\alpha \partial^\beta \mid \quad \beta, \alpha \in \mathbf{Z}^n_{\geq 0}\}.$$

It follows that $\mathcal{D}_{\mathbf{A}}$ is provided with a $\mathbb{Z}^n_{\geq 0}$-grading, where the degree of an operator $X^\beta \partial^\alpha$ is $\beta - \alpha$. It also follows that $T_{\mathbf{A}/\mathbf{k}} \cong \oplus_{j=1}^n \mathbf{A}\partial_{x_j}$ is a free $\mathbf{A}$-submodule and that the $\mathbf{k}$-algebra $\mathcal{D}_{\mathbf{A}}$ is generated by $\mathbf{A}$ and $T_{\mathbf{A}/\mathbf{k}}$.

If I is an ideal and $\mathbf{B} = \mathbf{A}/I$, then we have an isomorphism

$$T_{\mathbf{B}/\mathbf{k}} \cong T_{\mathbf{A}/\mathbf{k}}(I)/I \cdot T_{\mathbf{A}/\mathbf{k}},$$

see e.g. [1, 17]. The idealizer $\mathcal{D}_{\mathbf{A}}(I)$ is the largest subalgebra of $\mathcal{D}_{\mathbf{A}}$ containing $I \cdot \mathcal{D}_{\mathbf{A}}$ as a two sided ideal, and we have an isomorphism $\mathcal{D}_{\mathbf{B}} \cong \mathcal{D}_{\mathbf{A}}(I)/I \cdot \mathcal{D}_{\mathbf{A}}$, see e.g. [16, Theorem 1.6].

Let $I = (X^{\alpha_1}, ..., X^{\alpha_t})$ be a monomial ideal. In [19], the following assertions are proven:

(1) $\cap_{i=1}^{t}\mathcal{D}_{\mathbf{A}}(X^{\alpha_i}\mathbf{A})$ is Noetherian,

(2) $\mathcal{D}_{\mathbf{A}}(I)$ and $\mathcal{D}_{\mathbf{A}/I}$ are right Noetherian,

(3) $\mathcal{D}_{\mathbf{A}}(I)$ is left Noetherian if and only if I is principal.

If I is a squarefree monomial ideal, then the structure of $\mathcal{D}_{\mathbf{A}}(I)$ and $\mathcal{D}_{\mathbf{A}/I}$ are given in [5, 18] as

$$\mathcal{D}_{\mathbf{A}}(I) = \bigoplus_{\alpha}[I : [I : X^{\alpha}]]\partial^{\alpha} \quad \text{and} \quad \mathcal{D}_{\mathbf{A}/I} = \bigoplus_{\alpha}\frac{[I : [I : X^{\alpha}]]}{I}\partial^{\alpha}$$

with examples showing that these descriptions need not hold if I is not a squarefree. Note also that $\Delta_{\mathbf{A}}(I)$ is a finitely generated algebra but $\mathcal{D}_{\mathbf{A}}(I)$ is far from being finitely generated, [18, Example 4.3]. This is a reason to study $\Delta_{\mathbf{A}}(I)$-modules instead of $\mathcal{D}_{\mathbf{A}}(I)$-modules.

Lemma 2.1. *If I is a monomial ideal, then $\nabla_{\mathbf{A}} \subseteq \Delta_{\mathbf{A}}(I)$ and equality holds if $\sqrt{I} = (x_1x_2\cdots x_n)$.*

Proof. Clearly $\nabla_{x_i} \in \Delta_{\mathbf{A}}(I)$. Assume that $\sqrt{I} = (x_1x_2 \cdot x_n)$ and $X^{\alpha}\partial^{\beta} \in \Delta_{\mathbf{A}}(I)$ where $\alpha = (a_1, ..., a_n)$ and $\beta = (b_1, ..., b_n)$. It suffices to show that $\alpha - \beta \in \mathbb{Z}^n_{\geq 0}$. Assume that there exists j such that $a_j - b_j < 0$. Let X^{θ} be a monomial where $\theta = (c_1, ..., c_n)$, $c_j > 0$ is the minimum exponent of x_j such that $X^{\theta} \in I$. Then $X^{\alpha}\partial^{\beta}(x_j^{b_j-a_j-1}X^{\theta}) = k_0\frac{X^{\alpha+\theta-\beta-e_j}}{x_j^{a_j-b_j}} \notin I$, for some $k_0 \in \mathbf{k}$, since the exponent of x_j in this monomial is $c_j - 1$. Hence $\nabla_{\mathbf{A}} \subseteq \Delta_{\mathbf{A}}(I) \subseteq \nabla_{\mathbf{A}}$. $\square$

3 The Module of Derivations Preserving a Monomial Ideal

Let I be a monomial ideal in A. We frequently use the fact that there exists a unique minimal set of monomials $\{X^{\alpha_1}, X^{\alpha_2}, ..., X^{\alpha_t}\}$ generating I. To make our work easier we denote $X^{\alpha_i} = x_1^{a_{i1}}x_2^{a_{i2}}...x_n^{a_{in}}$ for each $i = 1, ..., t$. We alternatively denote a monomial by either X^{β} or $x_1^{b_1}x_2^{b_2}...x_n^{b_n}$ and the total degree of $X^{\beta} = x_1^{b_1}x_2^{b_2}...x_n^{b_n}$ by $|\beta| = b_1 + b_2 + ... + b_n$. To every monomial ideal I, the set

$$\exp(I) = \{\beta \mid X^{\beta} \in I\ \} \ \subseteq \ \mathbf{Z}^n_{\geq 0}. \tag{3.1}$$

associates the set of multi-degrees of monomials which belong I with points in $\mathbb{Z}^n_{\geq 0}$. Note that each monomial term of a polynomial $f \in I$ is a product of some X^{α_i}, $(i = 1, ..., t)$ and some other monomial term, hence a polynomial belongs to I if and only if each of its monomial terms belongs to I.

For any $j = 1, ..., n$, we denote the coordinate hyperplane of $\mathbb{R}^n$ by $H_j = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n_{\geq 0} \mid x_j = 0\}$ and $\{e_1, ..., e_n\}$ is the standard basis of the real vector space $\mathbb{R}^n$. For a monomial X^β we define $\mathrm{supp}(X^\beta) = \{x_j \mid x_j \text{ divides } X^\beta\}$. The following lemma suggests that for a monomial ideal I, to study the module structure and certain properties of $T_{\mathbf{A}/\mathbf{k}}(I)$, it suffices to work with derivations of the form $X^{\beta_j}\partial_{x_j}$ for some monomial $X^{\beta_j} \in \mathbf{A}$ and $j = 1, ..., n$. It is a precise formulation of a part of [1, Theorem 2.2.1].

Lemma 3.1. *Let I be a monomial ideal, for $j = 1, ..., n$ define polynomials $f_j = \sum_{\theta_j \in \Omega_j} k_{\theta_j} X^{\theta_j} \in \mathbf{A}$, where $k_{\theta_j} \neq 0$, as a sum of distinct monomial terms. Then a derivation $\delta = \sum_{j=1}^n f_j \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$ if and only if $X^{\theta_j}\partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$ for all $j = 1, ..., n$ and for all multi-degrees $\theta_j \in \Omega_j$.*

Proof. The converse is easy to see. Assume $\delta = \sum_{j=1}^n f_j \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$. Then for any $g \in I$, we have $\delta(g) = \sum_{j=1}^n f_j \partial_{x_j}(g) \in I$. In particular for any monomial $X^\alpha \in I$ one has $\sum_{j=1}^n f_j \partial_{x_j}(X^\alpha) \in I$. Now assume that there exists $\theta_j \in \Omega_j$ and a monomial $X^\alpha \in I$ such that $X^{\theta_j}\partial_{x_j}(X^\alpha) \notin I$. Then write

$$\delta(X^\alpha) = \sum_{j=1}^n \sum_{\theta'_j \in \Omega'_j} k_{\theta'_j} X^{\theta'_j} \frac{X^\alpha}{x_j} + \sum_{j=1}^n \sum_{\theta''_j \in \Omega''_j} k_{\theta''_j} X^{\theta''_j} \frac{X^\alpha}{x_j} \in I.$$

Here $\frac{X^\alpha}{x_j}$ is defined to be 0 when $\partial_{x_j}(X^\alpha) = 0$. In the first double-sum we have collected all monomial terms which belong to I and in the second we have all monomial terms which may not belong to I. Since $X^{\theta'_j}\frac{X^\alpha}{x_j} \in I$ we have $X^{\theta'_j}\partial_{x_j}(X^\alpha) \in I$ for all $\theta'_j \in \Omega'_j$. Since also the second double-sum belongs to I but by assumption none of its monomial terms belongs to I, it follows that

$$\sum_{j=1}^n \sum_{\theta''_j \in \Omega''_j} k_{\theta''_j} X^{\theta''_j} \frac{X^\alpha}{x_j} = 0.$$

Since for each $j = 1, ..., n$, $\frac{X^\alpha}{x_j}$ is a fixed monomial multiplying different monomial terms of f_j, the above equation implies

$$\sum_{j=1}^n \sum_{\theta''_j \in \Omega''_j} k_{\theta''_j} \frac{X^{\theta''_j}}{x_j} = 0.$$

This equation holds if and only if $x_j \in \mathrm{supp}(X^{\theta''_j})$. Hence $X^{\theta''_j}\partial_{x_j}(X^\alpha) = X^{\theta''_j - e_j}\nabla_{x_j}(X^\alpha) \in I$ for all $\theta''_j \in \Omega''_j$ resulting in a contradiction. $\square$

The following proposition is another version of a result due to P. Brumatti and A. Simis [1, Theorem 2.2.1]. It gives us a more geometric structure of $[I : [I : x_j]]$. In particular, if $H_j \cap \exp(I) \neq \emptyset$, then the ideal $I_{x_j} (\neq 0)$ has an important role in studying modules over $\Delta_{\mathbf{A}}(I)$ in the remaining sections of the thesis. Note that $H_j \cap \exp(I) = \emptyset$ if and only if $I = x_j I'$ where $I' = [I : x_j]$.

Proposition 3.2. *Let $I \subseteq \mathbf{A}$ be a monomial ideal.*

(1) If $H_j \cap \exp(I) = \emptyset$, then $[I : [I : x_j]] = (x_j)$, for $j = 1, ..., n$.

(2) If $H_j \cap \exp(I) \neq \emptyset$, then $[I : [I : x_j]] = (x_j) + I_{x_j}$, where $j = 1, ..., n$ and

$$I_{x_j} = \bigcap_{i=1}^{t} \frac{(X^{\alpha_i - a_{ij}e_j}) \cap (\{X^{\alpha_l - a_{lj}e_j} : a_{lj} < a_{ij}\}_{l=1}^{t})}{X^{\alpha_i - a_{ij}e_j}}. \tag{3.2}$$

(3)

$$T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_{x_1}, ..., \nabla_{x_n}) + \sum I_{x_j}\partial_{x_j}$$

and the sum runs through all $j = 1, ..., n$ for which $H_j \cap \exp(I) \neq \emptyset$.

Before proving the proposition, we note that if $H_j \cap \exp(I) = \emptyset$, then $I_{x_j} = (0)$, and if $H_j \cap \exp(I) \neq \emptyset$, then the generators of the ideal I_{x_j} are monomials in the polynomial ring $k[x_1, ..., \hat{x}_j, ..., x_n]$. This is because of the following: In the former case, consider a point $\beta = (b_1, ..., b_n) \in \exp(I)$ such that $b_j \leq a_{ij}$ for all $i = 1, ..., t$. Now for any monomial $0 \neq X^\theta \in A$ with $x_j \notin \operatorname{supp}(X^\theta)$ we have $X^\theta \partial_{x_j}(X^\beta) = b_j X^{\beta+\theta-e_j} \notin I$, implying that the k-derivation $X^\theta \partial_{x_j} \notin T_{\mathbf{A}/\mathbf{k}}(I)$. In the later case, it could be seen from the definition of I_{x_j} that for any monomial generator X^β of I_{x_j}, one has $x_j \notin \operatorname{supp}(X^\beta)$.

Proof. (1): Note that $[I : x_j] = (X^{\alpha_1 - e_j}, X^{\alpha_2 - e_j}, ..., X^{\alpha_t - e_j})$. Since $H_j \cap \exp(I) = \emptyset$, each of $X^{\alpha_i - e_j} \in \mathbf{A}$ are monomials $i = 1, ..., t$; and $x_j[I : x_j] \subseteq I \Rightarrow (x_j) \subseteq [I : [I : x_j]]$. To show that equality holds, consider $\beta = (b_1, ..., b_n) \in \exp(I)$ such that $b_j \leq a_{ij}$ for all $i = 1, ..., t$. Then $\beta - e_j \in \exp(I : x_j) \backslash \exp(I)$. Therefore, for any monomial $0 \neq X^\theta \in A$ with $x_j \notin \operatorname{supp}(X^\theta)$ (i.e. $x^\theta \notin (x_j)$), one has $X^\theta x^\beta \notin I$, for otherwise it will contradict the minimality of a_j. Thus, $X^\beta \notin [I : [I : x_j]]$.
(2): Clearly $(x_j) \subseteq [I : [I : x_j]]$. By construction of I_{x_j} a monomial $X^\theta \in I_{x_j}$ if and only if $\theta + \beta - e_j \in \exp(I)$ for all $X^\beta \in I$ with $x_j \in \operatorname{supp}(X^\beta)$ and this is equivalent to $\theta \in \exp([I : [I : x_j]])$, hence $(x_j) + I_{x_j} \subseteq [I : [I : x_j]]$. Conversely, If $X^\theta \in [I : [I : x_j]]$ where $x_j \in \operatorname{supp}(X^\theta)$, then $X^\theta \in (x_j)$ and if $x_j \notin \operatorname{supp}(X^\theta)$, then by definition $\theta \in \exp([I : [I : x_j]])$ if and only if $\theta + \beta - e_j \in \exp(I)$ for all $X^\beta \in I$ implying that $X^\theta \in I_{x_j}$.
(3): Assume that $X^\beta \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$. If $x_j \in \operatorname{supp}(X^\beta)$, then $X^\beta \partial_{x_j} \in (\nabla_{x_1}, ..., \nabla_{x_n})$. And if $x_j \notin \operatorname{supp}(X^\beta)$, then $X^\beta X^{\alpha_{i'} - e_j} \in I$ for every $i' = 1, ..., t$. In particular, there exists an $i \in \{1, 2, ..., t\}$ having the property that a_{ij} is the maximum of all a_{lj} 's, $l = 1, ..., t$, such that $a_{ij} > a_{lj}$ and that $X^\beta X^{\alpha_i - e_j} = X^{\theta_l} \cdot X^{\alpha_l}$ for some monomial X^{θ_l}. Thus $X^\beta = \frac{X^{\alpha_l - a_{lj}e_j}}{X^{\alpha_i - a_{ij}e_j}} X^{\theta_{l'}}$ for some monomial $X^{\theta_{l'}}$. On the other hand assume that $X^\beta \in (x_j) + I_{x_j}$ is a monomial. If $x_j \in \operatorname{supp}(X^\beta)$, then obviously $X^\beta \partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(I)$. Suppose

otherwise, then $X^\beta \in I_{x_j}$ which implies $X^\beta = X^{\alpha_{l'} - a_{l'j} e_j} \cdot X^{\theta_{l'}}$ for every $l' = 1, ..., t$, and for some monomial $X^{\theta_{l'}}$. Now for a monomial generator X^{α_i} of I, if $x_j \notin \operatorname{supp}(X^{\alpha_i})$, then $X^\beta \partial_{x_j}(X^{\alpha_i}) = 0 \in I$; and if $x_j \in X^{\alpha_j}$, then choose the maximum l such that $a_{ij} > a_{lj}$. Finally $X^\beta \partial_{x_j}(X^{\alpha_i}) = X^\beta X^{\alpha_i - e_j} = X^{\alpha_l - a_{lj} e_j} X^{\alpha_i - e_j} X^{\theta_{l'}} = X^{\alpha_l} \cdot X^{\alpha_i - (a_{lj}+1)e_j} \cdot X^{\theta_{l'}} \in I$. □

It is interesting to see the structure of $T_{\mathbf{A}/\mathbf{k}}(I)$ in the two variables case. Assume that $I = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, ..., x^{a_t}y^{b_t}) \subseteq \mathbf{A} = \mathbf{k}[x,y]$ is a monomial ideal, with $a_i < a_{i+1}$ and $b_i > b_{i+1}$ for each $i = 1, 2, ..., t-1$. Then define the *width $w(I)_x$ and $w(I)_y$ of* I in the direction of x and y respectively by

$$w(I)_x = \max\{a_{i+1} - a_i\}_{i=1}^{t} \quad \text{and } w(I)_y = \max\{b_i - b_{i+1}\}_{i=1}^{t-1}.$$

It is easy to see from (3.2) and the remark which follows that $I_x = (y^{w(I)_y})$, when $H_x \cap \exp(I) \neq \emptyset$. Similarly $I_y = (x^{w(I)_x})$ when $H_y \cap \exp(I) \neq \emptyset$. This gives the following corollary.

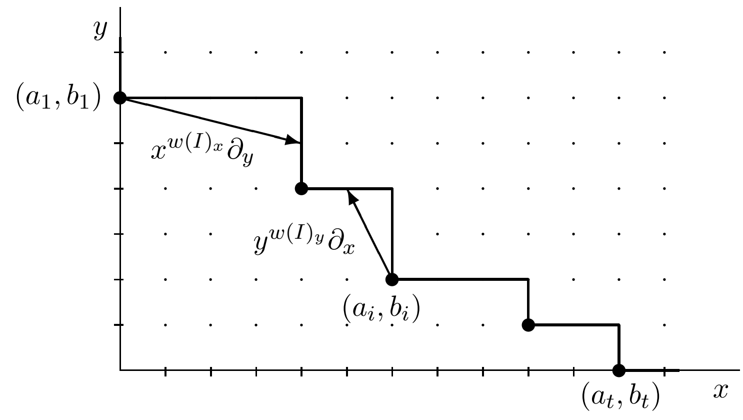
Corollary 3.3. *Assume that $\mathbf{A} = \mathbf{k}[x,y]$. Consider a monomial ideal $I = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, ..., x^{a_t}y^{b_t}) \subseteq \mathbf{A}$, with $a_i < a_{i+1}$ and $b_i > b_{i+1}$ for each $i = 1, 2, ..., t$.*

(1) If $\sqrt{I} = (xy)$, then $T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_x, \nabla_y)$.

(2) If I is (x)-primary, then $T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_x, \nabla_y, y^{w(I)_y}\partial_x)$.

(3) If I is (y)-primary, then $T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_x, \nabla_y, x^{w(I)_x}\partial_y)$.

(4) If I is (x,y)- primary , then $T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_x, \nabla_y, y^{w(I)_y}\partial_x, x^{w(I)_x}\partial_y)$.



At this point we give some guideline how to compute $T_{\mathbf{A}/\mathbf{k}}(I)$, where $I = (X^{\alpha_1}, ..., X^{\alpha_t})$, $X^{\alpha_i} = x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}$ for $i = 1, ..., t$. Since for any monomial $X^\beta \in I$ we have $\nabla_{x_j}(X^\beta) = x_j \partial_{x_j}(X^\beta) = k_0 X^\beta$ if $x_j \in \operatorname{supp}(X^\beta)$ and 0 otherwise, we get (x_j). The "non-trivial" set of derivations (*i.e.* the ones whose coefficients are in I_{x_j}), however, occur if there exists $x_j \notin \operatorname{supp}(X^{\alpha_i})$ for some $i = 1, ..., t$. Let x_j be such a variable. Induce the ordering of the integers $0 = a_{1j} \leq a_{2j} \leq ... \leq a_{tj}$ into "ordering of the set $\{X^{\alpha_i}\}_{i=1}^t$" of monomials as $X^{\alpha_1} \prec X^{\alpha_2} \prec \cdots \prec X^{\alpha_t}$. Take X^{α_i} with $a_{ij} > 0$, and look for all monomials X^β, with $x_j \notin \operatorname{supp}(X^\beta)$ such that $X^\beta \partial_{x_j}(X^{\alpha_i}) = X^{\beta+\alpha_i-e_j} \in I$. This is equivalent to collecting all $\mathbb{Z}$-linearly independent vectors β on the hyperplane plane $x_j = a_{ij} - 1$ with initial point $\alpha_i - e_j$ and terminal point at the boundary of $\exp((\{X^{\alpha_l} \mid a_{ij} > a_{lj}\}_{l=1}^t)) \cap (x_j = a_{ij} - 1)$.

Example 3.4. Consider the ideal $I = (x^4, x^2y^3, xy^4z, z^2) \in \mathbf{A} = \mathbb{Q}[x, y, z]$. Then all $I_x, I_y, I_z \neq (0)$ since $x \notin \operatorname{supp}(z^2)$ and $y, z \notin \operatorname{supp}(x^3)$. We compute I_z and leave I_x and I_y to the reader. Consider the ordering $x^4 \prec x^2y^3 \prec xy^4z \prec z^2$. Since ∂_z annihilates the first two monomials, there is nothing to be done. On the plane $z = 1 - 1 = 0$, $(1, 0, 0)$ is the only vector we need which is associated with xy^4z; and on the plane $z = 2 - 1 = 1$, $(1, 4, 0), (2, 3, 0), (4, 0, 0)$ are the vectors associated with z^2. Thus $I_z = (x) \cap (xy^4, x^2y^3, x^4) = (xy^4, x^2y^3, x^4)$. Then $T_{\mathbf{A}/\mathbb{Q}}(I) = (\nabla_x, \nabla_y, \nabla_z) + [(z) \cap (z^2, yz) \cap (z^2, y^3z)]\partial_x + [(x^2, z^2) \cap (x, z)]\partial_y + [(x) \cap (x^4, xy^4, x^2y^3)]\partial_z = (\nabla_x, \nabla_y, \nabla_z) + (y^3z, z^2)\partial_x + (x^2, z^2)\partial_y + (x^4, xy^4, x^2y^3)\partial_z$.

Example 3.5. Consider $I = (y^8, x^2y^6, x^5y^4, x^7y^2, x^8y, x^{12}) \subseteq \mathbb{Q}[x, y]$. Then $w(I)_x = 4$, $w(I)_y = 2$ and $T_{\mathbf{A}/\mathbf{k}}(I) = (\nabla_x, \nabla_y) + (y^2)\partial_x + (x^4)\partial_y$.

Proposition 3.6. *A proper ideal I of $\mathbf{A}$ is a $\nabla_{\mathbf{A}}$-submodule if and only if it is a monomial ideal. Moreover, I is a cyclic $\nabla_{\mathbf{A}}$-module.*

Consider the finite dimensional commutative Lie-algebra $\mathcal{G}$ generated by the ∇_{x_j}. The fact that monomial ideals in $\mathbf{A}$ are the same as $\nabla_{\mathbf{A}}$-submodules is due to the fact that $\mathbf{A}$ is a semi-simple $\mathcal{G}$-module. The argument below shows that all the monomial terms of a polynomial in I also belong to I.

Proof. Clearly, any monomial ideal is a $\nabla_{\mathbf{A}}$-module. Now assume that I is a proper $\nabla_{\mathbf{A}}$-submodule of $\mathbf{A}$.

Claim 1. We show that any polynomial $f \in I$ does not contain a constant term. Suppose that $f = \sum_{i=1}^l k_{\theta_i} X^{\theta_i} + c_1 \in I$ for some non-zero $c_1 \in \mathbf{k}$, $k_{\theta_i} \neq 0$ and $\theta_i = (b_{i1}, ..., b_{in})$, $i = 1, ..., l$. First we prove by induction on l that every monomial term of f could be written as $X^{\theta_i} = \delta_i(f)$ for some $\delta_i \in \nabla_{\mathbf{A}}$.

Since each monomial term in f has different multi-degree, the rank of the matrix

$$\begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \end{pmatrix}$$

is 2 implying that there exists $r < s = 1, ..., n$ such that $\mathcal{A}_{rs} = b_{1r}b_{2s} - b_{2r}b_{1s} \neq 0$. Assume that $b_{1r}, b_{1s} \neq 0$ for otherwise since one of them, say b_{1r} is non-zero, we have $\nabla_{x_s}(f) \in I$. Hence by induction assumption, $X^{\theta_i} =$

$\delta_i \nabla_{x_s}(f)$ for some $\delta_i \in \nabla_{\mathbf{A}}$ for all $i = 2, ..., l$.
Now acting ∇_{x_r} and ∇_{x_s} on f we obtain,

$$\nabla_{x_r}(f) = \sum_{i=1}^{l} k_{\theta_i} b_{ir} X^{\theta_i} \quad \textit{and} \quad \nabla_{x_s}(f) = \sum_{i=1}^{l} k_{\theta_i} b_{is} X^{\theta_i}. \tag{3.3}$$

Solving the above two equations simultaneously to eliminate x^{θ_1} gives:

$$[b_{1s}\nabla_{x_r} - b_{1r}\nabla x_s](f) = \sum_{i=2,}^{l} k_{\theta_i}(b_{1s}b_{ir} - b_{1r}b_{is})X^{\theta_i} \in I. \tag{3.4}$$

By induction assumption we obtain $X^{\theta_i} = \delta_i(f)$ for some $\delta_i \in \nabla_{\mathbf{A}}$, $i = 2, ..., l$. Putting

$$\delta_1 = \frac{1}{k_{\theta_1}}[b_{1s}\nabla_{x_r} - b_{1r}\nabla_{x_s} - (\sum_{i=2}^{l} k_{\theta_i}(b_{1s}b_{ir} - b_{1r}b_{is})\delta_i)] \in \nabla_{\mathbf{A}}$$

one obtains, $X^{\theta_1} = \delta_1(f)$. Hence $c_1 = f - \sum_{i=1}^{l} k_{\theta_{i'}} \delta_i(f) \in I \Rightarrow I = \mathbf{A}$. This contradicts the assumption that I is a proper ideal of $\mathbf{A}$.

Claim 2. We prove that I is a monomial ideal. Now assume that I is generated by a minimal set $\{f_1, ..., f_t\}$ of polynomials with $f_i = \sum_{\alpha_i \in \Omega_i} k_{\alpha_i} X^{\alpha_i}$, $k_{\alpha_i} \neq 0$ is an expression of each polynomial f_i as a $\mathbf{k}$-linear combination of distinct monomials in a unique way. Let I' be the monomial ideal generated by the set $\{X^\alpha \mid \alpha \in \cup_{i=1}^{t} \Omega_i\}$. It suffices to show that $I = I'$. From the discussion above, one have $I' \subseteq I$. Conversely,

$$\alpha_i \in \Omega_i \Rightarrow X^{\alpha_i} \in I' \text{ for all } \alpha_i \in \Omega_i \Rightarrow f_i \in I' \Rightarrow I \subseteq I'.$$

Hence $I = I'$ and I is a monomial ideal.

Finally assume $X^{\beta_1}, ..., X^{\beta_r}$ is the unique minimal set of generators of I and consider the polynomial $f(X) = \sum_{i=1}^{r} X^{\beta_i} \in I$. Then there exists $\delta_i \in \nabla_{\mathbf{A}}$ such that $X^{\beta_i} = \delta(f) \Rightarrow \nabla_{\mathbf{A}}(f) = I$. This completes the proof. $\square$

4 Preservation of Associated Ideals

Let I be any ideal of a commutative Noetherian k-algebra A. Then I is a $\Delta_A(I)$-submodule of A. It is natural to ask if certain natural operations on I results in new $\Delta_A(I)$-modules. We will investigate this question in the case of monomial ideals, in relation to Ratliff-Rush closure, integral closure and the formation of multiplier ideals.

4.1 Ratliff-Rush Closure of an Ideal

Let A be a commutative Noetherian ring and I be an ideal of A. There exist ideals $\tilde{I}$ which are maximal with respect to the property that $\tilde{I}^l = I^l$ for large l. Ratliff and Rush proved [13, Theorem 2.1], that if I is a regular ideal (*i.e.* it contains a non-zero divisor), then there exists a unique largest such ideal. It can be expressed as

$$\tilde{I} = \cup_{l=1}^{\infty}[I^{l+1} : I^l] = [I^{L+1} : I^L] \text{ for } L >> 0.$$

When I is not regular, the result fails [14]. If I is an $\mathfrak{m}$-primary ideal of a local ring $(A, \mathfrak{m}, k)$, then the ideal $\tilde{I}$ can also be characterized as the unique largest ideal containing I having the same Hilbert polynomial as I. The ideal $\tilde{I}$ is *the Ratliff-Rush closure of* I. If I is a regular ideal such that $I = \tilde{I}$, then I is a *Ratliff-Rush closed ideal.* Results about these ideals can be found in [3, 13, 14, 20].

Proposition 4.1. *Let A be a commutative k-algebra of finite type and I be an ideal of A.*

(1) $T_{A/k}(I) \subseteq T_{A/k}(I^l)$ for any integer $l > 0$. Moreover, Equality holds if $I = [I^l : I^{l-1}]$.

(2) $T_{A/k}(I) \subseteq T_{A/k}(\tilde{I})$.

Proof. (1) The first part is clear. To prove the second part, if $\delta \in T_{A/k}(I^l)$ then for every $a \in I$, we have $\delta(a^l) = la^{l-1}\delta(a) \in I^l$ hence $\delta(a) \in [I^l : I^{l-1}] = I$. (2) Consider $\delta \in T_{A/k}(I)$ and $a \in A$ such that $aI^L \subseteq I^{L+1}$ for $L >> 0$, then $ab \in I^{L+1}$ for some $b \in I^L$ and by (1) $\delta(ab) \in I^{L+1}$, hence $b\delta(a) = \delta(ab) - a\delta(b) \in I^{L+1}$. □

If I is a monomial ideal of $\mathbf{A} = \mathbf{k}[x, y]$ and $l > 0$ is an integer, it is not obvious how $w(I^l)_x$ and $w(I^l)_y$ depend on l.

Corollary 4.2. *If $I \subseteq \mathbf{k}[x, y]$ is a monomial ideal, then $w(I)_x \geq w(I^l)_x$ and $w(I)_y \geq w(I^l)_y$ and equality holds when $I = [I^l : I^{l-1}]$.*

Proof. This follows from Proposition 4.1 (1) and Corollary 3.3. □

4.2 Integral Closure of an Ideal

Given a commutative Noetherian ring A and an ideal I of A. An element $x \in A$ is said to be *integral over* I if x satisfies the equation

(4.1) $\quad x^d + a_1x^{d-1} + ... + a_{d-1}x + a_d = 0$ where $a_i \in I^i$ and $i = 0, 1, ..., d$.

The set of all elements in A which are integral over I, denoted by $\bar{I}$, is *the integral closure of* I. Moreover, I is *integrally closed* if $\bar{I} = I$. The following lemma is Proposition 2.1.2 in [21].

Lemma 4.3. *Let I be a monomial ideal in $\mathbf{A}$. Then $\bar{I}$ is also a monomial ideal. More precisely, a monomial X^β is in $\bar{I}$ if and only if $(X^\beta)^l \in I^l$ for some integer $l > 0$.*

Proof. For the proof that $\bar{I}$ is monomial ideal, see [loc.cit]. We prove only the second statement.Consider the equation of integral dependence $(X^\beta)^d + f_1(X^\beta)^{d-1} + f_2(X^\beta)^{d-2} + ... + f_d = 0$, where the f_i's are polynomials in I^i. Since each I^i is a monomial ideal, considering the multi-degree $\deg((X^\beta)^d)$, we obtain an equation

$$(X^\beta)^d + k_1 X^{\theta_1}(X^\beta)^{d-1} + ... + k_d X^{\theta_d} = 0$$

for some monomial $X^{\theta_i} \in I^i$, $i = 1, ..., d$ and some $k_1, ..., k_d \in \mathbf{k}$. Some coefficient k_l must be different from 0, thus $(X^\beta)^d = k_0 k_l X^{\theta_{d-l}}(X^\beta)^l$ with $X^{\theta_{d-l}} \in I^{d-l}$ and $k_0 \in \mathbf{k}$, so $(X^\beta)^l \in I^l$. Conversely, $x^l - (X^\beta)^l = 0$ gives us the equation of the integral dependence of X^β. ☐

If I is a monomial ideal, the *Newton Polytope of I* denoted $\mathrm{conv}(I)$ consists of all points in $\mathbb{R}^n$ belonging to the convex hull of $\exp(I)$ [2, 7, 8, 21, 22].

In general it is difficult to use the integral dependence (4.1) to prove that $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\bar{I})$. In [10] the inclusion is proved using the blow-up of I. Hence

$$T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\tilde{I}) \subseteq T_{\mathbf{A}/\mathbf{k}}(\bar{I}) \subseteq T_{\mathbf{A}/\mathbf{k}}(\sqrt{I}) \tag{4.2}$$

where the last inclusion was first noted in [15]. We prove these inclusions for monomial ideals in a direct and elementary way.

Proposition 4.4. *The inclusions in (4.2) hold for monomial ideals.*

Proof. The first inclusion follows from Proposition 4.1. To prove the second, consider a monomial ideal J such that $I \subseteq J \subseteq \bar{I}$. Let $\delta = X^{\beta_j}\partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(J)$, and $X^\theta \in \bar{I} = \bar{J}$ be a monomial with $x_j \in \mathrm{supp}(X^\theta)$. Then $(X^\theta)^l \in J^l$ for some $l > 0$. By Proposition 4.1,

$$\delta \in T_{\mathbf{A}/\mathbf{k}}(J^l) \Rightarrow \delta((X^\theta)^l) = l(X^\theta)^{l-1}\delta(X^\theta) \in J^l.$$

Clearly $\delta((X^\theta)^l)$ is a monomial. We have

$$\begin{aligned} X^{\beta_j}\partial_{x_j}[l(X^\theta)^{l-1}X^{\beta_j}\partial_{x_j}(X^\theta)] &= \delta[l(X^\theta)^{l-1}\delta(X^\theta)] \\ &= l(l-1)(X^\theta)^{l-2}(\delta(X^\theta))^2 + l(X^\theta)^{l-1}\delta^2(X^\theta) \end{aligned}$$

which is a monomial in J^l split into the sum of two monomials of the same multi-degree. It follows that $l(l-1)(X^\theta)^{l-2}(\delta(X^\theta))^2 \in J^l$. Similarly,

$$\delta[l(l-1)(X^\theta)^{l-2}(\delta(X^\theta))^2] = l(l-1)(l-2)(X^\theta)^{l-3}(\delta(X^\theta)^3) + 2l(l-1)\delta(X^\theta)\delta^2(X^\theta)$$

is a monomial in J^l, hence $l(l-1)(l-2)(X^\theta)^{l-3}(\delta(X^\theta))^3 \in J^l$. Continuing this computation, we get $l!(\delta(X^\theta))^l \in J^l \Rightarrow (\delta(X^\theta))^l \in J^l$. Hence $\delta(X^\theta) \in \bar{J} = \bar{I}$, as required. To prove the last inclusion, assume that $\delta = X^{\beta_j}\partial_{x_j} \in T_{\mathbf{A}/\mathbf{k}}(\bar{I})$, and $X^\theta \in \sqrt{I}$ be any monomial with $x_j \in \operatorname{supp}(X^\theta)$. Then $(X^\theta)^l \in I \subseteq \bar{I}$ for some $l > 0$ we have $\delta((X^\theta)^l) \in \bar{I}$. A similar computation as above shows that $(\delta(X^\theta))^l \in \bar{I}$. Hence $\delta(X^\theta) \in \sqrt{\bar{I}} = \sqrt{I}$. ☐

Remark 4.5. The inclusions in (4.2) can be strict. For $T_{\mathbf{A}/\mathbf{k}}(I) \subsetneq T_{\mathbf{A}/\mathbf{k}}(\tilde{I})$ see [10]. For $T_{\mathbf{A}/\mathbf{k}}(\tilde{I}) \subsetneq T_{\mathbf{A}/\mathbf{k}}(\bar{I})$, consider the ideal $I = (y^8, x^3y^5, x^7y, x^8) \subseteq \mathbb{Q}[x,y]$. Then $\tilde{I} = I + (x^6y^2)$ (see [20]), and $\bar{I} = (x,y)^8$. Thus $y\partial_x, x\partial_y \in T_{\mathbf{A}/\mathbf{k}}(\bar{I}) \setminus T_{\mathbf{A}/\mathbf{k}}(\tilde{I})$. For the ideal $I = (x^2, y^4) \subseteq \mathbb{Q}[x,y]$, we have $\sqrt{I} = (x,y)$ and $\bar{I} = (x^2, xy^2, y^4)$. Then $y\partial_x, x\partial_y \in T_{\mathbf{A}/\mathbf{k}}(\sqrt{I}) \setminus T_{\mathbf{A}/\mathbf{k}}(\bar{I})$. This last inclusion fails if $\operatorname{Char}(\mathbf{A}) = p > 0$. Indeed, if $I = \bar{I} = (x^py^p)$, then $\sqrt{I} = (xy)$ and $\partial_x, \partial_y \in T_{\mathbf{A}/\mathbf{k}}(\bar{I}) \setminus T_{\mathbf{A}/\mathbf{k}}(\sqrt{I})$.

Remark 4.6. Assume that $\operatorname{Char}\mathbf{k} = p > 0$. The second and the third inclusions in (4.2) hold if the generators of I satisfy the following conditions:

1. I is generated by monomials whose exponents are not divisible by p,
2. for $\bar{I} = (X^{\alpha_1}, X^{\alpha_2}, ..., X^{\alpha_t})$, and $l_i = min\{l \in \mathbb{Z}_{>0} \mid (X^{\alpha_i})^{l_i} \in I^{l_i}\}$, $(1 \le i \le t)$, then $l_i!$ is not divisible by p.

Let I be a monomial ideal and $J = \tilde{I}$ or $J = \bar{I}$. The following example shows that $T_{\mathbf{A}/\mathbf{k}}(I) = T_{\mathbf{A}/\mathbf{k}}(J)$ does not imply $I = J$.

Example 4.7. Consider an ideal $I = (x^2y^{12}, x^4y^{10}, x^7y^7, x^9y^5) \subseteq \mathbf{A} = \mathbb{Q}[x,y]$. Then $\tilde{I} = (x^2y^{12}, x^4y^{10}, x^6y^9, x^7y^7, x^9y^5)$, $\bar{I} = x^2y^5(x,y)^7$, $\sqrt{I} = (xy)$ and $I \subset \tilde{I} \subset \bar{I} \subset \sqrt{I}$. By Corollary 3.3 $T_{\mathbf{A}/\mathbb{Q}}(I) = T_{\mathbf{A}/\mathbb{Q}}(\tilde{I}) = T_{\mathbf{A}/\mathbb{Q}}(\bar{I}) = T_{\mathbf{A}/\mathbb{Q}}(\sqrt{I}) = (\nabla_x, \nabla_y)$.

4.3 Multiplier Ideals of a Monomial Ideal

Given a smooth complex variety X and an ideal I of the structure sheaf $\mathcal{O}_X$, one can attach to I a collection of multiplier ideals $\mathcal{I}(r \cdot I)$ depending on a rational parameter $r > 0$. These ideals and the vanishing theorems they satisfy have found interesting applications in recent years. For a formal definition and properties of multiplier ideals, see [11]. Although multiplier ideals enjoy excellent formal properties, they are hard to compute in general. An important exception is for the case of a monomial ideal, whose multiplier ideals are described by a simple combinatorial formula by Howald [6].

Consider the standard topology on $\mathbb{R}^n$. Given a set of lattice points $P \subseteq \mathbb{Z}^n_{\ge 0} \subseteq \mathbb{R}^n_{\ge 0}$, $\operatorname{Int}(P)$ denote the interior of P with respect to this topology, and for any rational number $r > 0$ define

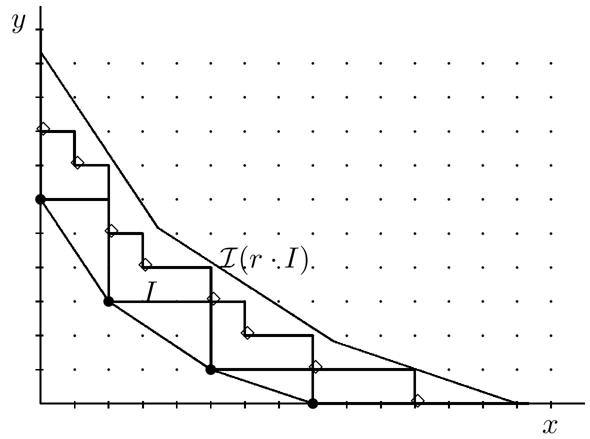
$$r \cdot P = \{r\alpha \mid \alpha \in P\}.$$

Given a monomial ideal I, we regard the Newton polytope $\mathrm{conv}(I)$ as a subset of the real vector space $\mathbb{R}^n$.

Theorem 4.8. *(Howald [6]) Let I be a monomial ideal, and let $conv(I)$ be its Newton Polytope. Then $\mathcal{I}(r \cdot I)$ is a monomial ideal containing the following set of monomials:*

$$\{X^\alpha \mid \alpha + (1, 1, ..., 1) \in Int(r \cdot conv(I) \cap \mathbb{Z}^n_{\geq 0})\}.$$

Example 4.9. The figure below shows the graphical description of $I = (y^6, x^2y^3, x^5y, x^8)$ and $\mathcal{I}(r \cdot I) = (y^8, xy^7, x^2y^5, x^3y^4, x^5y^3, x^6y^2, x^8y, x^{10})$, where $r = \frac{31}{18}$.



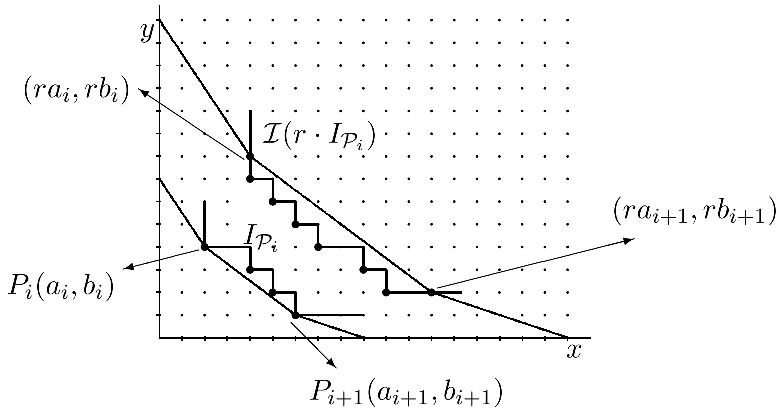
Proposition 4.10. *If $I \subseteq \mathbf{A} = \mathbf{k}[x, y]$ is a monomial ideal and $r \geq 0$ is a rational number, then $T_{\mathbf{A}/\mathbf{k}}(I) \subseteq T_{\mathbf{A}/\mathbf{k}}(\mathcal{I}(r \cdot I))$.*

Proof. If $\sqrt{I} = (xy)$, then by Corollary 3.3 $T_{\mathbf{A}/\mathbf{k}}(I) = \mathbf{A}\nabla_x + \mathbf{A}\nabla_y$, so the assertion easily follows from the fact that $\mathcal{I}(r \cdot I)$ is a monomial ideal. Assume that $\sqrt{I} = (x, y)$. By Corollary 3.3 it suffices to prove

$$w(I)_x \geq w(\mathcal{I}(r \cdot I))_x \text{ and } w(I)_y \geq w(\mathcal{I}(r \cdot I))_y. \tag{4.3}$$

Since $\mathrm{conv}(I) = \mathrm{conv}(\bar{I})$ implies $\mathcal{I}(r \cdot I) = \mathcal{I}(r \cdot \bar{I})$, one can assume that I is integrally closed. Let $P_0(a_0, b_0), P_1(a_1, b_1), ..., P_t(a_t, b_t), P_{t+1}(a_{t+1}, b_{t+1}) \in \mathbb{Z}^2_{\geq 0}$ where $0 = a_0 < a_1 < ... < a_t < a_{t+1}$ and $b_0 > b_2 > ... > b_t > b_{t+1} = 0$ be coordinates of vertices of $\mathrm{conv}(I)$. For each $i = 1, ..., t + 1$, let $\mathcal{P}_i$ be the convex region in $\mathbb{R}^2_{\geq 0}$ bounded by the lines $x \geq a_i$, $y \geq b_{i+1}$ and $y \geq m_i(x - a_i) + b_i$, where $m_i = \frac{b_{i+1} - b_i}{a_{i+1} - a_i} < 0$ (see the figure below). This definition gives us t convex subsets $\mathcal{P}_i$ of $\mathrm{conv}(I)$ corresponding to the different segments $\overline{P_iP_{i+1}}$. Let $I_{\mathcal{P}_i}$ be a monomial ideal generated by

$$\{x^a y^b \mid (a, b) \in \mathcal{P}_i \cap \mathbb{Z}^2_{\geq 0}\}.$$



By definition $\exp(I_{\mathcal{P}_i}) = \mathrm{conv}(I_{\mathcal{P}_i}) \cap \mathbb{Z}^2_{\geq 0} = \mathcal{P}_i \cap \mathbb{Z}^2_{\geq 0}$, consequently $I_{\mathcal{P}_i}$ is integrally closed. Moreover, one can easily see that

$$(a, b) \in \mathrm{conv}(I) \iff a \geq a_i,\ b \geq b_{i+1} \text{ and } b \geq m_i(a - a_i) + b_i$$

for some $i = 0, ..., t$. Thus $\mathrm{conv}(I) = \cup_{i=0}^t \mathrm{conv}(I_{\mathcal{P}_i}) = \cup_{i=0}^t \mathcal{P}_i$, and since I is integrally closed, it follows that $I = \sum_{i=0}^t I_{\mathcal{P}_i}$. By the definition of $I_{\mathcal{P}_i}$, it is easy to see that

$$\begin{aligned} w(I)_x &= \max\{w(I_{\mathcal{P}_i})_x\}_{i=0}^t \quad \text{and} \\ w(I)_y &= \max\{w(I_{\mathcal{P}_i})_y\}_{i=0}^t. \end{aligned} \tag{4.4}$$

The set

$$r \cdot \mathrm{conv}(I) = \{(ra, rb) \mid (a, b) \in \mathrm{conv}(I)\}$$

is an open convex region having boundary vertices $(ra_0, rb_0), ..., (ra_{t+1}, rb_{t+1})$, where for each $i = 0, ..., t$, the segments containing the pair $[(a_i, b_i), (a_{i+1}, b_{i+1})]$ and $[(ra_i, rb_i), (ra_{i+1}, rb_{i+1})]$, respectively, are parallel with slope m_i. By a similar agrement as above, one has $r \cdot \mathrm{conv}(I) = \cup_{i=0}^t r \cdot \mathrm{conv}(I_{\mathcal{P}_i})$, and since $\mathcal{I}(r \cdot I)$ is integrally closed, $\mathcal{I}(r \cdot I) = \sum_{i=1}^t \mathcal{I}(r \cdot I_{\mathcal{P}_i})$. Again it is clear that

$$\begin{aligned} w(\mathcal{I}(r \cdot I))_x &= \max\{w(\mathcal{I}(r \cdot I_{\mathcal{P}_i}))_x\}_{i=0}^t \quad \text{and} \\ w(\mathcal{I}(r \cdot I))_y &= \max\{w(\mathcal{I}(r \cdot I_{\mathcal{P}_i}))_y\}_{i=0}^t. \end{aligned} \tag{4.5}$$

Because of (4.4) and (4.5) it follows that (4.3) holds if we prove

$$w(I_{\mathcal{P}_i})_x \geq w(\mathcal{I}(r \cdot I_{\mathcal{P}_i}))_x \quad \text{and} \quad w(I_{\mathcal{P}_i})_y \geq w(\mathcal{I}(r \cdot I_{\mathcal{P}_i}))_y$$

for all $i = 0, ..., t$. Therefore the proof will be complete with the lemma below. $\square$

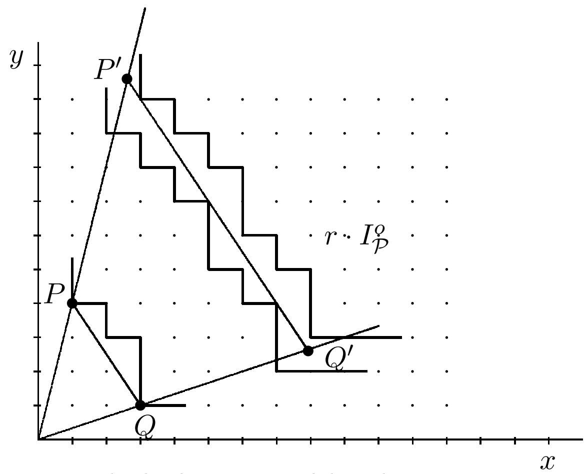
Lemma 4.11. *Let* $(c,d),(e,f)\in\mathbb{Z}^2_{\geq 0}$ *such that* $c<e, d>f$, $m=\frac{f-d}{e-c}<0$, $\mathcal{P}\subseteq\mathbb{R}^2_{\geq 0}$ *be a convex region bounded by the lines* $x\geq c$, $y\geq f$ *and* $y\geq m(x-c)+d$, *and* $I_{\mathcal{P}}\subseteq\mathbf{k}[x,y]$ *be a monomial ideal generated by the set*

$$\{x^py^q \mid (p,q)\in\mathcal{P}\cap\mathbb{Z}^2_{\geq 0}\}.$$

Then

$$w(I_{\mathcal{P}})_x\geq w(\mathcal{I}(r\cdot I_{\mathcal{P}}))_x \quad \textit{and} \quad w(I_{\mathcal{P}})_y\geq w(\mathcal{I}(r\cdot I_{\mathcal{P}})_y).$$

Proof. Let $\{x^{c_1}y^{d_1},...,x^{c_s}y^{d_s}\}$ where $c=c_1<...<c_s=e$ and $d=d_1>d_2>...>d_s=f$ be the unique minimal set of generators of $I_{\mathcal{P}}$. Since $I_{\mathcal{P}}$ is integrally closed, either $c_{i+1}-c_i=1$ or $d_i-d_{i+1}=1$ [21]. Moreover, for each $i=1,...,t$ one has $0\leq d_i-(m(c_i-c)+c)<1$. Let $P=(c,d)$ and $Q=(e,f)$ and the equation of the line joining them is $y=m(x-c)+d$(see the figure below). Let P' and Q' be the points with coordinates (rc,rd) and (re,rf) respectively and the equation of the line joining them is $y=m(x-rc)+rd$.



Let $r\cdot I^o_{\mathcal{P}}$ be the monomial ideal generated by the set

$$\{x^py^q \mid (p,q)\in \mathrm{Int}(r\cdot \text{ conv } (I_{\mathcal{P}}))\cap\mathbb{Z}^2_{\geq 0}\}.$$

According to Theorem 4.8, $x^{p-1}y^{q-1}$ belongs to $\mathcal{I}(r\cdot I)$ if and only if x^py^q belongs to $r\cdot I^o_{\mathcal{P}}$, and x^py^q belongs to the minimal set of generators of $r\cdot I^o_{\mathcal{P}}$ if and only if

$$0<q-(m(p-rc)+rd)\leq 1. \tag{4.6}$$

Hence

$$w(\mathcal{I}(r\cdot I))_x=w(r\cdot I^o_{\mathcal{P}})_x \quad \text{and} \quad w(\mathcal{I}(r\cdot I))_y=w(r\cdot I^o_{\mathcal{P}})_y; \tag{4.7}$$

therefore it suffices to prove

$$w(I_{\mathcal{P}})_x\geq w(r\cdot I^o_{\mathcal{P}})_x \text{ and } w(I_{\mathcal{P}})_y\geq w(r\cdot I^o_{\mathcal{P}})_y.$$

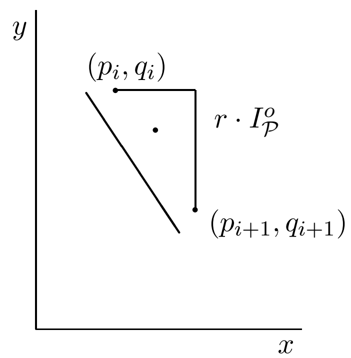
The proof of these inequalities is divided into three cases depending on the slope m. It is clear from the definitions of slope and the ideal $I_{\mathcal{P}}$ that

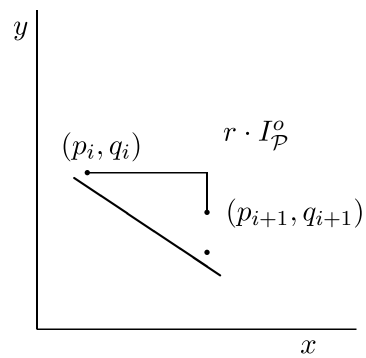
$$(4.8)\qquad \begin{aligned} w(I_{\mathcal{P}})_x &= \begin{cases} \frac{1}{|m|} & \text{if } \frac{1}{m} \in \mathbb{Z} \\ \lfloor \frac{1}{|m|} \rfloor + 1 & \text{if } \frac{1}{m} \notin \mathbb{Z} \end{cases} \\ w(I_{\mathcal{P}})_y &= \begin{cases} |m| & \text{if } m \in \mathbb{Z} \\ \lfloor |m| \rfloor + 1 & \text{if } m \notin \mathbb{Z} \end{cases} \end{aligned}$$

Case 1: $m < -1$. We show that $w(I_{\mathcal{P}})_x = w(r \cdot I^o_{\mathcal{P}})_x = 1$. It is evident that $w(I_{\mathcal{P}})_x = 1$. Assume on the contrary that $w(r \cdot I^o_{\mathcal{P}})_x > 1$. Let $x^{p_i}y^{q_i}$ and $x^{p_{i+1}}y^{q_{i+1}}$ be monomials in the minimal set of generators of $r \cdot I^o_{\mathcal{P}}$ such that $w(r \cdot I^o_{\mathcal{P}})_x = p_{i+1} - p_i$. Then

$$p_{i+1} - p_i > 1 \Rightarrow p_i = p_{i+1} - L_x - 1$$

for some $L_x \in \mathbb{Z}_{>0}$. First we show that $q_i - q_{i+1} = 1$. If we assume otherwise, we will show that the lattice point $(p_{i+1} - 1, q_i - 1)$ belongs to $\exp(r \cdot I^o_{\mathcal{P}})$, contradicting the assumption $w(r \cdot I^o_{\mathcal{P}})_x = p_{i+1} - p_i$ (see the left figure below).





To see this, by (4.6)

$$\begin{aligned} 0 < & q_i - (m(p_i - rc) + rd) \\ &= q_i - (m(p_{i+1} - L_x - 1 - rc) + rd) \\ &= q_i - (m(p_{i+1} - 1) - rc) + rd) + mL_x \\ &= (q_i - 1) - (m(p_{i+1} - 1 - rc) + rd) + mL_x + 1 \\ \iff \quad & 0 < -(mL_x + 1) < (q_i - 1) - (m(p_{i+1} - 1 - rc) + rd), \end{aligned}$$

where the first inequality in the last line follows since $m < -1$. Hence

$(p_{i+1}-1, q_i-1)$ belongs to $\exp(r \cdot I^o_{\mathcal{P}})$. Next again by (4.6),

$$\begin{aligned} & 0 < q_i - (m(p_i - rc) + rd) \\ & \quad = q_{i+1} + 1 - (m(p_{i+1} - 1 - L_x - rc) + rd) \\ & \quad = q_{i+1} - (m(p_{i+1} - rc) + rd) + (m + mL_x + 1) \\ \Longleftrightarrow \quad & -(m + mL_x + 1) < q_{i+1} - (m(p_{i+1} - rc) + rd). \end{aligned}$$

Since $m+mL_x+1 < -1$, this contradicts (4.6) (see the right diagram above). Therefore $L_x = 0$ and $w(r \cdot I^o_{\mathcal{P}})_x = p_{i+1} - p_i = 1$.

To show $w(r \cdot I^o_{\mathcal{P}})_y \leq w(I_{\mathcal{P}})_y$: Suppose, on the contrary, that $w(r \cdot I^o_{\mathcal{P}})_y > w(I_{\mathcal{P}})_y$. Let $x^{p_j}y^{q_j}, x^{p_{j+1}}y^{q_{j+1}}$ be monomials in the minimal set of generators of $r \cdot I^o_{\mathcal{P}}$ such that $w(r \cdot I^o_{\mathcal{P}})_y = q_j - q_{j+1} = w(I_{\mathcal{P}})_y + L_y$ for some integer $L_y > 0$. By (4.6) we have $q_j - (m(p_j - rc) + rd) < 1$ and by (4.8) we have $1 \leq m + w(I_{\mathcal{P}})_y + L_y$. Combining these inequalities we obtain

$$q_j - (m(p_j - rc) + rd) < m + w(I_{\mathcal{P}})_y + L_y \Rightarrow q_{j+1} - (m(p_{j+1} - rc) + rd) < 0$$

which results in a contradiction, since $(p_{j+1}, q_{j+1}) \in \exp(r \cdot I^o_{\mathcal{P}})$. Therefore, $L_y = 0$, and hence $w(r \cdot I^o_{\mathcal{P}})_y \leq w(I_{\mathcal{P}})_y$.

Case 2: $m = -1$. One can easily prove that $w(I_{\mathcal{P}})_x = w(I_{\mathcal{P}})_y = w(r \cdot I^o_{\mathcal{P}})_x = w(r \cdot I^o_{\mathcal{P}})_y = 1$. Clearly we have $c + d = e + f$, thus the minimal generating set of $I_{\mathcal{P}}$ contains elements of the form $x^c y^d, x^{c+1}y^{d-1}, ..., x^{c+v}y^{d-v}$, where $e = c + v$, $f = d - v$, $v \in \mathbb{Z}_{>0}$, and they all belong to the line PQ. The result follows from the fact that the lines PQ and $P'Q'$ are parallel.

Case 3: $-1 < m < 0$. This follows by symmetry from case 1. □

5 Differential Reduction

An ideal I of a k-algebra A can be considered as a $\Delta_A(I)$-module. By Proposition 3.6, any $\nabla_{\mathbf{A}}$-submodule I of a $\mathbf{A}$ is a monomial ideal. Moreover, I is a cyclic $\nabla_{\mathbf{A}}$-module. In this section we construct a principal ideal $J = (f)$ of $\mathbf{A}$ such that $J \subseteq I$ and $\Delta_{\mathbf{A}}(I) \cdot J = I$.

If $J \subseteq I$ are ideals of a k-algebra A, then $\Delta^1_A(I) \cdot J$ is an ideal of A lying between J and I. Similarly, $\Delta^l_A(I) \cdot J \subseteq \Delta^{l+1}_A(I) \cdot J \subseteq I$. Thus we have the following increasing sequence of ideals of A:

$$(5.1) \qquad \Delta^0_A(I) \cdot J \subseteq \Delta^1_A(I) \cdot J \subseteq \Delta^2_A(I) \cdot J \subseteq \cdots \subseteq \Delta^l_A(I) \cdot J \subseteq \cdots \subseteq I.$$

By the Noetherian property, there exists an integer $L \geq 0$ such that $\Delta^L_A(I) \cdot J = \Delta_A(I) \cdot J$.

If $J \subseteq I$ be ideals of A, then J is a *differential reduction of* I if $\Delta_A(I) \cdot J = I$. A differential reduction of I is called a *minimal differential reduction* if it is minimal with respect to inclusion.

If a differential reduction is generated by monomials, then it is *monomial differential reduction.* A minimal monomial differential reduction of I is a monomial ideal which is minimal among the monomial differential reductions of I.

Lemma 5.1. *If $J \subseteq I \subseteq \mathbf{A}$ are monomial ideals, then so is $\Delta_{\mathbf{A}}(I) \cdot J$.*

Proof. By induction it suffices to prove that $\Delta^1_{\mathbf{A}}(I) \cdot J$ is a monomial ideal. By Proposition 3.2 we have $T_{\mathbf{A}/\mathbf{k}}(I) = \sum_{j=1}^{n}((x_j) + I_{x_j})\partial_{x_j}$, where $I_{x_j} = (0)$ if $H_{x_j} \cap \exp(I) = \emptyset$. Assume that $\Omega = \{i_1, ..., i_r\} \subseteq \{1, ..., n\}$ such that $H_{x_{i_j}} \cap \exp(I) \neq \emptyset$ for all $j = 1, ..., r$. Then $T_{\mathbf{A}/\mathbf{k}}(I) \cdot J = \sum_{j=1}^{n}((x_j) + I_{x_j})\partial_{x_j}(J) = J + \sum_{i_j \in \Omega} x_{i_j}\partial_{x_{i_j}}(J) + \sum_{i_j \in \Omega}(I_{x_j})\partial_{x_j}(J)$. We can see that each of the ideals in the summand are monomial. □

Remark 5.2. If I is a monomial ideal, then $T_{\mathbf{A}/\mathbf{k}}(I) \cdot I = I$ since $\nabla_{x_j} \in T_{\mathbf{A}}(I)$ for all $j = 1, ..., n$. Moreover, if $\sqrt{I} = (x_1 x_2 \cdots x_n)$, then I is the only monomial differential reduction of itself (*Lem.* 2.1).

Lemma 5.3. *A monomial ideal $I = (X^{\alpha_1}, ..., X^{\alpha_t})$ in $\mathbf{A}$ has a proper monomial differential reduction if there exists a derivation $\delta = X^{\beta}\partial_{x_j}$, where $X^{\beta} \in I_{x_j}$ for some $j = 1, ..., n$ and $i_1, i_2 = 1, ..., t$ with $i_1 \neq i_2$, such that $X^{\beta}X^{\alpha_{i_1}} = x_j X^{\alpha_{i_2}}$.*

Proof. If $X^{\beta}X^{\alpha_{i_1}} = x_j X^{\alpha_{i_2}}$, then $\delta(X^{\alpha_{i_1}}) = k_0 X^{\alpha_{i_2}}$ for some $k_o \in \mathbf{k}$. The ideal $J = (\{X^{\alpha_1}, ..., X^{\alpha_t}\} \setminus \{X^{\alpha_{i_2}}\})$ is a proper monomial differential reduction of I. □

Proposition 5.4. *A monomial ideal in $\mathbf{A}$ has a unique minimal monomial differential reduction.*

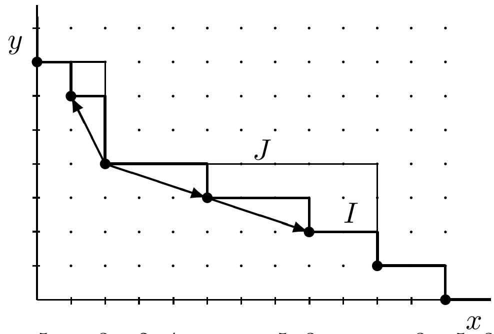
Proof. Let $I = (X^{\alpha_1}, ..., X^{\alpha_t}) \subseteq \mathbf{A}$ be a monomial ideal. If $\sqrt{I} = (X)$, then I is the only monomial differential reduction of itself (c.f Remark 5.2). If there is no derivation $\delta = X^{\beta}\partial_{x_j}$ where $X^{\beta} \in I_{x_j}$ and $i_1, i_2 = 1, ..., t$ with $i_1 \neq i_2$, such that $X^{\beta}X^{\alpha_{i_1}} = x_j X^{\alpha_{i_2}}$, then we assert that I is a minimal monomial differential reduction of itself. Supposing otherwise, we first define the monomial ideal $J_1 = (\{X^{\alpha_1}, ..., X^{\alpha_t}\} \setminus \{X^{\alpha_{i_2}}\})$. By Lemma 5.3, $J_1 \subseteq I$ is a monomial differential reduction. By induction define a monomial ideal J_{l+1} generated by all monomials of J_l except those monomial generators X^{β_2} of J_l such that $X^{\beta}\partial_{x_j}(X^{\beta_1}) = X^{\beta_2}$ for some monomial $X^{\beta} \in I_{x_j}$ and some monomial generator X^{β_1} of J_l. This gives a decreasing sequence

$$\cdots \subseteq J_{l+1} \subseteq J_l \subseteq \cdots \subseteq J_1 \subseteq I.$$

This sequence terminates after a finite number of steps since I is generated by a finite number of monomials. Now consider the monomial ideal $J = \cap_{l=1}^{\infty} J_l = J_{l_o}$ for some l_o having the unique minimal set of generators $X^{\beta_1}, ..., X^{\beta_r}$. For any $j = 1, ..., n$ by construction of J one has

$X^{\theta}X^{\beta_{i'_1}} \neq x_j X^{\beta_{i'_2}}$ for each $X^{\theta} \in I_{x_j}$ and $i'_1 \neq i'_2 = 1, ..., r$ implying that if $J' = (\{X^{\beta_1}, ..., X^{\beta_r}\} \setminus \{X^{\beta_{i'}}\})$ then $X^{\beta_{i'}} \notin \Delta_{\mathbf{A}}(I) \cdot J' \subseteq \Delta_{\mathbf{A}}(I) \cdot J = I$ for all $i' = 1, ..., r$. Hence J is a minimal monomial differential reduction. Uniqueness follows from the construction. $\square$

Example 5.5. Consider the monomial ideal $I = (y^7, xy^6, x^2y^4, x^5y^3, x^8y^2, x^{10}y, x^{12}) \subseteq \mathbb{Q}[x, y]$. Then $T_{\mathbb{Q}[x,y]/\mathbb{Q}}(I) = (x, y^2)\partial_x, +(x^3, y)\partial_y$ (see the diagram below).



Since $y^2(x^2y^4) = x(xy^5)$, $x^3(x^2y^4) = y(x^5y^3)$ and $x^3(x^5y^3) = y(x^8y^2)$; the minimal monomial differential reduction of I is $J = (y^7, x^2y^4, x^{10}y, x^{12})$.

Proposition 5.6. *Let I be a monomial ideal in a $\mathbf{A}$, and J be the unique minimal monomial differential reduction of I having monomial generators $X^{\alpha_1}, ..., X^{\alpha_t}$. Then for $k_1, ..., k_t \in \mathbf{k} \setminus \{0\}$ the polynomial*

$$f(X) = \sum_{i=1}^{t} k_i X^{\alpha_i}$$

is a cyclic generator of the $\Delta_{\mathbf{A}}(I)$-module I. Moreover such polynomials are the cyclic generators with the fewest possible non-zero monomial terms.

Proof. By Proposition 3.6, for each $i = 1, ..., t$ there exists $\delta_i \in \nabla_{\mathbf{A}} \subseteq \Delta_{\mathbf{A}}(I)$ such that $X^{\alpha_i} = \delta_i(f)$. Thus $I \subseteq \Delta_{\mathbf{A}}(I)(f) \subseteq I$. Minimality of the number of terms in f follows from the minimality of the generators $\{X^{\alpha_1}, ..., X^{\alpha_t}\}$ of J. $\square$

Example 5.7. Consider the monomial ideal $I = (y^7, x^2y^5, x^5y^4, x^7y^2, x^8)$. Then $T_{\mathbf{A}/\mathbf{k}}(I) = (x, y^2)\partial_x + (y, x^3)\partial_y$ and $J = (y^7, x^2y^5, x^8)$ is the minimal monomial differential reduction of I. Moreover, if $f = x^8 + x^2y^5 + y^7$, then $\Delta_{\mathbf{A}}(I)(f) = I$, since $x^8 = \frac{1}{320}(5\nabla_x^2 - 2\nabla_x\nabla_y)(f)$, $y^7 = \frac{-1}{98}(5\nabla_x\nabla_y - 2\nabla_y^2)(f)$, and $x^2y^5 = (\frac{1}{2}\nabla_x - \frac{8}{320}(5\nabla_x^2 - 2\nabla_x\nabla_y))(f)$.

Proposition 5.8. *Let $\mathfrak{m}$ denote the graded maximal ideal and put $I = \mathfrak{m}^d$ for an integer $d > 0$. Then any monomial ideal $J = (X^{\alpha_1}, ..., X^{\alpha_t})$ in $\mathbf{A}$ with $|\alpha_1| = ... = |\alpha_t| = d$ is a differential reduction of I. Moreover, every principal ideal generated by a monomial X^{α}, where $|\alpha| = d$, is a minimal differential reduction of I.*

Proof. By Proposition 3.2, we have $T_{\mathbf{A}/\mathbf{k}}(I) = \oplus_{j=1}^{n}\mathfrak{m}\partial_{x_j}$. For any $i, j = 1, ..., n$ and $l = 1, ..., t$, if $x_j \in \operatorname{supp}(X^{\alpha_l})$, then $x_i\partial_{x_j}(X^{\alpha_l})$ is a monomial of total degree d. Thus $\Delta^1_{\mathbf{A}}(I) \cdot J$ is an ideal generated by monomials of total degree d. By induction, $\Delta_{\mathbf{A}}(I)\cdot J \subseteq \mathfrak{m}^d$. Conversely, if $X^\beta = x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} \notin I$ with $b_1 + b_2 + ... + b_n = d$, then $b_j - a_{lj} \neq 0$ for some $j = 1, ..., n$ and any $l = 1, ..., t$. Now fix such an $l = 1, ..., t$ and define δ_i by

$$(5.2) \qquad \delta_i = \begin{cases} x_i^{|b_i - a_{li}|} & if \quad b_i - a_{li} > 0 \\ \frac{\partial^{|b_i - a_{li}|}}{\partial^{|b_i - a_{li}|}x_i} & if \quad b_i - a_{li} < 0. \end{cases} \qquad i = 1, ..., n.$$

Without loss of generality, assume that $\delta_1, ..., \delta_{i'} \in \mathbf{A}$, and $\delta_{i'+1}, ..., \delta_n \in \mathbf{k}[\partial_{x_1}, ..., \partial_{x_n}]$, otherwise permute them so that the first set of operators are monomials and the last are differential operators. Since $|\beta| = d$, the total degree of the monomial $\delta_1...\delta_{i'}$ is the same as the order of $\delta_{i'+1}...\delta_n$ and if $X^\alpha \in I$, the total degree of $\delta_1...\delta_n(X^\alpha) = |\alpha| \geq d$ implying that $\delta = \delta_1...\delta_n \in \Delta_{\mathbf{A}}(I)$, and $\delta(X^{\alpha_l}) = k_0X^\beta$ for some $k_0 \in \mathbf{k}$. This concludes the proof. □

Let J be a monomial ideal and I be either $\bar{J}$ or $\tilde{J}$. It is known that $\Delta_{\mathbf{A}}(I)\cdot J \subseteq I$. One might ask when equality holds. The following examples show that this is not always the case.

Example 5.9. Let $I \subseteq \mathbb{Q}[x_1, ..., x_n]$ be a monomial ideal. If $\exp(I) \cap (\cup_{i=1}^n H_i) = \emptyset$, then $I = \Delta^l_{\mathbb{Q}[x_1...,x_n]}(\bar{I}) \cdot I$ for all $l \geq 1$. Consequently, if $I \neq \bar{I}$ then $\Delta_{\mathbb{Q}[x_1,...,x_n]}(\bar{I}) \cdot I \neq \bar{I}$. We also have a similar result for an (x, y)-primary ideal I. Consider $I = (y^7, x^4y^2, x^6y, x^8) \subseteq \mathbb{Q}[x, y]$. Then $\bar{I} = (y^7, xy^6, x^2y^5, x^3y^4, x^4y^2, x^6y, x^8)$ and $T_{\mathbf{A}/\mathbb{Q}}(\bar{I}) = (x, y^2)\partial_x + (x^2, y)\partial_y$. Here the sequence (5.1) terminates to $\Delta^2_{\mathbb{Q}[x,y]}(\bar{I})\cdot I = (y^7, x^2y^6, x^3y^4, x^4y^2, x^6y, x^8) \neq \bar{I}$. Thus $\Delta_{\mathbf{A}}(\bar{I}) \cdot I \subsetneq \bar{I}$.

Example 5.10. Consider $I = (y^8, x^3y^5, x^7y, x^8) \subseteq \mathbb{Q}[x, y]$. Then $\tilde{I} = I + (x^6y^2)$ and $\Delta^l_{\mathbb{Q}[x,y]}(\tilde{I}) \cdot I = I + (x^4y^4)$ for all $l \geq 1$. We can see that $x^6y^2 \in \tilde{I}\backslash\Delta^l_{\mathbb{Q}[x,y]}(\tilde{I})\cdot I$ for all l. Note in this case that $\bar{I} = (x, y)^8$ and $\Delta_{\mathbb{Q}[x,y]}(\bar{I})\cdot I = \bar{I}$.

6 Length of the $\Delta_A(I)$-Module $\mathbf{A}/I^{l+1}$

In this section $\mathfrak{m}$ denotes the graded maximal ideal of $\mathbf{A}$. We study the length of $\Delta_{\mathbf{A}}(I)$-modules $\mathbf{A}/I^{l+1}$, where $l > 0$ is an integer and I an $\mathfrak{m}$-primary monomial ideal.

Proposition 6.1. *Let I be a monomial $\mathfrak{m}$-primary ideal.*

(1) If $I = \mathfrak{m}^d$ where $d \geq 1$ is an integer, then $\mathfrak{l}_{\Delta_{\mathbf{A}}(\mathfrak{m})}(I^l/I^{l+1}) = d$ for any $l \geq 0$.

(2) If $x_i \notin I_{x_j}$ for all $i, j = 1, ..., n$, then $\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(I^l/I^{l+1}) = \dim_{\mathbf{k}}(I^l/I^{l+1})$ for each integer $l \geq 0$.

Remark 6.2. The condition $x_i \notin I_{x_j}$ for all $i, j = 1, ..., n$ in the proposition implies that all degree-zero elements of $\Delta_{\mathbf{A}}(I)$ belong to $\nabla_{\mathbf{A}}$. To see this, let $\delta = X^\alpha \partial^\beta \in \Delta_{\mathbf{A}}(I)$, $X^\theta \in I$ and $k_0 \in \mathbf{k}$. Then

$$\delta(X^\theta) = k_0 X^\theta \iff \alpha + \theta - \beta = \theta \iff \alpha - \beta = 0.$$

We need two lemmas for the proof of Proposition 6.1.

Lemma 6.3. *Let I be a monomial ideal in $\mathbf{A}$. Then for any integer $l \geq 0$, a $\Delta_{\mathbf{A}}(I)$-submodule of I^l/I^{l+1} has the form J/I^{l+1} where J is a monomial ideal of $\mathbf{A}$ such that $J \subseteq I^{l+1}$.*

Proof. Since $\mathbf{A} \subseteq \Delta_{\mathbf{A}}(I)$, it is clear that J is an ideal of $\mathbf{A}$ containing I^{l+1}. Now assume that $f = \sum_\alpha k_\alpha X^\alpha \in J \subseteq I^l$. Then $X^\alpha \in I^l \subseteq I$ for all α and thus $X^\alpha \partial^\alpha \in \Delta_A(I)$. Now let α_1 be a multi-degree of terms of f such that $|\alpha_1|$ is maximum total degree among the monomials that appear in f. Then $X^{\alpha_1} = k_0 X^{\alpha_1} \partial^{\alpha_1}(f) \in J$, for some $k_0 \in k$. Now consider $f(X) - X^{\alpha_1} \in J$ and apply the same procedure. We eventually obtain all the monomials of f and consequently have the result. □

Lemma 6.4. *Let I be an $\mathfrak{m}$-primary monomial ideal and $l \geq 0$ be an integer.*

(1) There exists a monomial $X^\alpha \in I^l \backslash I^{l+1}$ such that $x_j X^\alpha \in I^{l+1}$ for every $j = 1, ..., n$.

(2) For every monomial X^α satisfying the condition in (1) and a $\delta \in \Delta_{\mathbf{A}}(I)$, either $\delta(X^\alpha) \in I^{l+1}$ or $x_j \delta(X^\alpha) \in I^{l+1}$ for all $j = 1, ..., n$.

Proof. (1): Clearly there exists the smallest integer $d > 0$ such that $\mathfrak{m}^d(I^l/I^{l+1}) = 0$. Then the monomial representatives of $\mathfrak{m}^{d-1}(I^l/I^{l+1})$ in I^l gives the required monomial X^α. (2): Let $\delta = X^{\beta_i} \partial_{x_i} \in T_{\mathbf{A}/\mathbf{k}}(I)$ where $X^{\beta_i} \in (x_i) + I_{x_i}$. If $\delta(X^\alpha) \in I^{l+1}$, there is nothing to prove. Assume that $\delta(X^\alpha) \notin I^{l+1}$. For any $j = 1, ..., n$

$$x_j \delta(X^\alpha) = \delta(x_j X^\alpha) - \delta(x_j) X^\alpha = \delta(x_j X^\alpha) - k_0 X^{\beta_i} X^\alpha \partial_{x_i}(x_j).$$

By Proposition 4.1 (1) we have $\delta(x_j X^\alpha) \in I^{l+1}$, and since $\mathrm{supp}(X^{\beta_i}) \neq \emptyset$, by the assumption on X^α in (1), one has $k_0 X^{\beta_i} X^\alpha \partial_{x_l}(x_j) \in I^{l+1}$. Hence $x_j \delta(X^\alpha) \in I^{l+1}$. □

Note that $\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(I^l/I^{l+1}) \leq \dim_{\mathbf{k}}(I^l/I^{l+1})$ where equality holds if every composition factor of the module I^l/I^{l+1} is a one-dimensional $\mathbf{k}$-vector space. The following example shows that we can obtain strict inequality if the condition in (1) of Proposition 6.1 is not satisfied.

Example 6.5. Consider $I = (x^5, z^4, yz^3, z^2y^2, y^3z, y^4)$ in $\mathbf{A} = \mathbb{Q}[x, y, z]$. Clearly $I \neq \mathfrak{m}^d$ for any $d \geq 1$. Now $\Delta_{\mathbf{A}}(I)$ is generated by $\mathbf{A}$ and $T_{\mathbf{A}/\mathbb{Q}}(I) = [(x) + (y, z)^4]\partial_x + [x^5, y, z]\partial_y + [x^5, y, z]\partial_z$. Then $J_1 = \Delta_{\mathbf{A}}(I)(x^4z^7) + I^2 = (x^4z^7, x^4yz^6, x^4y^2z^5, x^4y^3z^4) + I^2$. Hence $\dim_{\mathbb{Q}}(J_1/I^2) = 4$. On the other hand, $y\partial_z(J_1\backslash I^2) = z\partial_y(J_1\backslash I^2) = J_1\backslash I^2$, that is J_1/I^2 is a simple $\Delta_A(I)$-submodule of I/I^2.

Proof of Proposition 6.1. (1): By Proposition 5.8, for any $r = 0, 1, ..., d$, if a proper $\Delta_{\mathbf{A}}(I)$-module J_r containing I^{l+1} contains a monomial X^α with $|\alpha| = dl + r$, then all monomials with total degree $dl + r$ also belong to J_r. By Lemma 6.3, each J_r is a monomial ideal. Thus a composition series of I^l/I^{l+1} will be:

$$J_d/I^{l+1} \subseteq J_{d-1}/I^{l+1} \subseteq J_{d-2}/I^{l+1} \subseteq \cdots \subseteq J_1/I^{l+1} \subseteq J_0/I^{l+1}$$

where $J_r = (\{X^\alpha \mid |\alpha| = dl + r\})$ for $r = 0, 1..., d$.
(2): Let Ω_1 be the set of all monomials in $I^l \setminus I^{l+1}$ satisfying the condition in (1) of Lemma 6.4. By Remark 6.2 and the condition in (2) of Lemma 6.4, there exists a monomial $X^{\alpha_1} \in \Omega_1$ such that $\Delta_{\mathbf{A}}(I) \cdot (X^{\alpha_1}) = (X^{\alpha_1}) + I^{l+1}$. Define $J_1 = (X^{\alpha_1}) + I^{l+1}$. Then $J_1/I^{l+1} = \mathbf{k}X^{\alpha_1}(\mod I^{l+1})$ is a simple $\Delta_{\mathbf{A}}(I)$-submodule. Let Ω_2 be the set of all monomials in $I^l \setminus J_1$ satisfying the condition in (1) of Lemma 6.4. Similarly there exists a monomial $X^{\alpha_2} \in \Omega_2$ such that $\Delta_{\mathbf{A}}(I) \cdot (X^{\alpha_2}) = (X^{\alpha_2}) + J_1$. Define $J_2 = (X^{\alpha_1}, X^{\alpha_2}) + I^{l+1}$. Then $J_2/J_1 = \mathbf{k}X^{\alpha_2}(\mod J_1)$ is a simple $\Delta_{\mathbf{A}}(I)$-submodule. Continuing this construction gives a composition series of I^l/I^{l+1}. This gives a bijection between the set of $\mathbf{k}$-linearly independent monomials in $I^l \setminus I^{l+1}$ and the number of steps in a composition series of the module I^l/I^{l+1}. Hence $\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(I^l/I^{l+1}) = \dim_{\mathbf{k}}(I^l/I^{l+1})$. $\square$

We have a more general condition on I in two variables to get the maximal possible length.

Proposition 6.6. *Let I be an $\mathfrak{m}$-primary monomial ideal of $\mathbf{A} = \mathbf{k}[x, y]$ such that $I \neq \mathfrak{m}^d$ for every integer $d \geq 1$. Then*

$$\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(I^l/I^{l+1}) = \dim_{\mathbf{k}}(I^l/I^{l+1}).$$

Proof. Let $I = (x^{a_0}y^{b_0}, x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t})$ where $0 = a_0 < a_1 < ... < a_t$ and $b_0 > b_1 > ... > b_t = 0$. Given $I^d \neq \mathfrak{m}$ for all $d \in \mathbb{Z}_{\geq 0}$ we have $w(I)_x > 1$ or $w(I)_y > 1$; say $w(I)_x > 1$.

Let $d_1 \geq 0$ be the smallest integer such that $x^{c_1}y^{d_1} \in I^l \setminus I^{l+1}$ satisfies (1) of Lemma 6.4 and $y^{w(I)_y}\partial_x(x^{c_1}y^{d_1}) \in I^{l+1}$. The existence of such an integer follows from Lemma 6.4 (2). Define

$$J_1 = \Delta_{\mathbf{A}}(I) \cdot (x^{c_1}y^{d_1}) + I^{l+1} = (x^{c_1}y^{d_1}) + I^{l+1}.$$

By construction, $J_1/I^{l+1} = \mathbf{k}x^{c_1}y^{d_1}(\mod I^{l+1})$ is simple since it is a one-dimensional $\mathbf{k}$-vector space. Let $d_2 \geq 0$ be the smallest integer such that $x^{c_2}y^{d_2} \in I^l \backslash J_1$ satisfies Lemma 6.4 (1) and $y^{w(I)_y}\partial_x(x^{c_2}y^{d_2}) \in J_1$. Then

$$J_2 = \Delta_{\mathbf{A}}(I) \cdot (x^{c_2}y^{d_2}) + J_2 = (x^{c_1}y^{d_1}, x^{c_2}y^{d_2}) + I^{l+1}$$

is a $\Delta_{\mathbf{A}}(I)$-submodule of I^l/I^{l+1} and $J_2/J_1 = \mathbf{k}x^{c_2}y^{d_2}(\mod J_1)$ is one-dimensional $\mathbf{k}$-vector space. This construction terminates since $\dim_{\mathbf{k}}(I^l/I^{l+1}) < \infty$. Moreover, each $\mathbf{k}$-linearly independent monomial in $I^l \backslash I^{l+1}$ corresponds to a $\Delta_{\mathbf{A}}(I)$-submodule of I^l/I^{l+1} which occurs in a composition series of I^l/I^{l+1}. $\square$

A Question: It is easy to see that for an $\mathfrak{m}$-primary monomial ideal I one has

$$\mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(\mathbf{A}/I^l) = \sum_{i=1}^{l} \mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(I^{i-1}/I^i).$$

Recall that $\mathfrak{l}_{\mathbf{A}}(\mathbf{A}/I^l)$ is a polynomial in l for $l \gg 0$ (the Hilbert Polynomial). One may ask if the function $l \mapsto \mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(\mathbf{A}/I^l)$ is a polynomial in l for $l \gg 0$. This clearly is true if I satisfies the condition in Proposition 6.1, but in general we only have

$$dl \leq \mathfrak{l}_{\Delta_{\mathbf{A}}(I)}(\mathbf{A}/I^l) \leq \dim_{\mathbf{k}}(\mathbf{A}/I^l),$$

where d is the minimum total degree of a monomial generator in I.

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