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Abstract

This paper concerns the existence of invariant hypersurfaces for higher derivations on varieties over fields of positive characteristic. The classical results about generic non-integrabilty of vector fields due to Jouanolou, Bernstein-Lunts and others fail in positive characteristic, motivating the study of analogous questions for higher derivations. We prove generic non-integrability results for higher derivations on smooth rings, on $\mathbb{P}^n$ and more generally for higher derivations with poles along a divisor on a smooth variety. In the case of iterative higher derivations, we prove a non-integrability result on rings and an integrability result on $\mathbb{P}^n$. Along the way, we prove a Jouanolou-type dichotomy for foliations in positive characteristic.

Contents

1 Introduction

The main aim of this paper is to study generic solvability of higher differential equations. In characteristic 0, there is a large body of theory on differential equations in the context of algebraic geometry. One kind of result is the non-existence of algebraic solutions to generic differential equations, proved in various forms by Jouanolou [7], Bernstein and Lunts [1],[9], Coutinho and Pereira [2] and others. The immediate analogue of this fact, that a *vector field* does not have integral hypersurfaces, is very false in positive characteristic. In fact, work by Pereira [14] implies that a generic vector field always has reduced integral hypersurfaces. This can be seen as an indication that a vector field is not a good notion of differential equation in positive characteristic. Thus, in this paper we investigate integrability properties of higher derivations, and prove a general result on the non-existence of integral hypersurfaces for higher derivations. To do this in greater generality, we define higher derivations on schemes. Various forms of such a notion has occured in the literature, but we do not know of a complete reference. We also define and study higher derivations with poles along a divisor, and construct parameter spaces for higher derivations. Along the way, we prove a quite general Jouanolou-type theorem for foliations in positive characteristic.

Over complex affine spaces, the question whether a generic differential equation has an algebraic solution was solved by Jounaolou. He proved by calculation on a cleverly choosen concrete equation that it does not have any algebraic solution, thus proving non-emptiness of the open set of non-solvable equations. The equation used has solutions over fields of positive characteristic. Later, Bernstein and Luntz gave a different proof as a part of their proof that a generic D-module of a certain kind is non-holonomic. A key idea in their proof is to prove that a sufficiently generic equation cannot have any integral curves with worse singularities than nodal. This is achived by the Poincare-Dulac theorem on local forms of vector fields. Correctly stated, namely, that any vector field is locally equivalent to one with only resonant terms, the Poincare-Dulac theorem is true in positive characteristic. The problem in positive characteristic is that there are canonically resonant terms, namely the p-th powers, which means that generic vector fields cannot be locally linearized, meaning that the Bernstein-Luntz proof will not go through in positive characteristic.

This reflects the fact that the situation is entirely different in positive characteristic. Pereira [14] proves that a derivation D on affine space over a field of positive characteristic almost always has a reduced integral hypersurface, namely, the reduced components of a certain determinant (set-theoretically cutting out the locus of dependence of the derivations D^{p^i}). Hence much more differential equations are non-trivially solvable in positive characteristic. This can be seen as an indication that the notion of a vector field is not the natural, or at least not the only natural, notion of a differential equation in positive characteristic. This is our main motivation for the results in this paper, investigating generic integrability properties of higher derivations.

1.1 Acknowledgments

Section 2 is a reminder of basic definitions about higher derivations on rings. The material in 3.1 is presumably known, but we have not seen it written down, the material in section 3.2 occurs in scattered form in the literature, section 3.3 follows [16] very closely. The material in section 3.4 is original. In section 4, the extension theorem for rings is due to Matsumura, whereas the extension theorems for schemes are original. Section 5 is original. In section 6, Theorem 43 is well-known. Theorem 44 extends results due to Kim, Pereira and others. Propositions 47 and 48 are probably known but we have not found a reference. Section 6.2 is original, as well as the theorems in section 6.3
This report is a slightly revised version of [4].

The author is indepted to Rickard Bögvad for suggesting the problem of studing invariant varieties for higher derivations, for his continous advice and for suggesting numerous improvements, and to Torsten Ekedahl for suggesting the proof of Theorem 61.

1.2 Notation and conventions

All rings and algebras in this paper are commutative, including algebras over sheaves. By a variety we mean an integral, noetherian, separated scheme of finite type over a field. $\mathbb{N}$ denotes the set $\{0, 1, 2, ...\}$ and $\mathbb{N}^+$ denotes the set $\{1, 2, 3, ...\}$. $\operatorname{spf} R[[t]]$ denotes the formal spectrum with respect to the ideal (t), and for rings R, S with maximal ideals m, n respectively $R\hat{\otimes}S$ denotes the formal tensor product $\lim R/m^k \otimes S/n^k$. If X is a variety T_X denotes its tangent sheaf.

2 Higher derivations on rings

Let $k \subseteq R$ be commutative rings. A higher (or Hasse-Schmidt) derivation of length n on R is a sequence of k-linear maps

$$\begin{aligned} D: &\quad (\delta_0, \delta_1, ..., \delta_n) \\ \delta_i: &\quad R \to R \end{aligned}$$

such that:

1. δ_0 is a k-algebra homomorphism,
2. $\delta_m(xy) = \sum_{i+j=m} \delta_i(x)\delta_j(y)$

A higher derivation D of length n can also be seen as a k-algebra homomorphism

$$\begin{aligned} \mathbf{e}^D: &\quad R \to \tfrac{R[[t]]}{t^{n+1}} \\ \mathbf{e}^D(x) = &\quad \sum \delta_i(x) t^i \end{aligned}$$

Conversely, any k-algebra homomorphism $R \to \frac{R[[t]]}{t^{n+1}}$ such that the composition with the map taking t to 0 is a k-algebra homomorphism gives rise to a higher derivation by letting $\delta_n(x)$ be the coefficient of the n:th power term of the image of x. An infinite sequence $(\delta_0, \delta_1, ..., \delta_n, ...)$ satisfying the same relations as above corresponds to a ring homomorphism $R \to R[[t]]$ and is called a higher derivation of infinite length. When $\delta_0 = id_R$ we say the derivation is normal. Denote by $Der_k^n(R)$, with n possibly infinite, the set of normal higher derivations over k of length n on R. We may also define higher derivations taking values in an arbitrary k-algebra A in the same maner. We denote the set of such derivations by $Der_k^n(R, A)$. Note that with this notation $Der_k^n(A, A)$ is not equal to $Der_k^n(R)$.

Example 1. Let k be a field of characteristic 0, R a k-algebra and D a k-derivation on R. Then $(\frac{1}{n!}D^n)_{n \in \mathbb{N}}$ is a higher derivation.

Example 2. Let $k = \frac{\mathbb{Z}}{p\mathbb{Z}}$. On $\mathbb{Z}[x_1, .., x_n]$ we have operators $d_n = \frac{1}{n!}\frac{\partial^n}{\partial x_i^n}$. If f is a representative in $\mathbb{Z}[x_1, ..., x_n]$ for an element $[f] \in k[x_1, ..., x_n]$ we can define $d_n[f]$ as $[d_n f]$, which induces operators d_n on $k[x_1, ..., x_n]$, forming a higher derivation. Note that the operators d_n are differential operators which are not in the subring generated by derivations.

The following proposition gives rise to a third characterisation of higher derivations:

Proposition 3. *Let $D = (d_i)_{i \in \omega}$ be a normal higher derivation on a ring R. Then d_i is a differential operator (in the sense of Grothendieck) on R of order at most i.*

Proof: We show the claim by induction. The case $i = 0$ is clear. For $a \in R$ we have

$$\begin{aligned}
[d_i, a](x) &= d_i(ax) - ad_i(x) \\
&= \sum_{j=0}^{i} d_j(x)d_{i-j}(a) - ad_i(x) \\
&= \sum_{j=0}^{i-1} d_j(x)d_{i-j}(a) \\
&= (\sum_{j=0}^{i-1} d_{i-j}(a)d_j)(x)
\end{aligned}$$

The operator in the last term is by the induction hypothesis a differential operator of order at most $i-1$, so d_i is a differential operator of order at most i. □

Hence a normal higher derivation can also be defined as a sequence of differential operators satisfying Leibniz rule.

Definition 4. A higher derivation is said to be iterative if $d_i \circ d_j = \binom{i+j}{i} d_{i+j}$ for all i, j.

In particular, if (d_i) is an iterative higher derivation on a field of char $k = p$, $i + j = p$ and $i, j \neq 0$, then $d_i \circ d_j = 0$ but the iterativity condition imposes no restrictions on d_p. In terms of the exponential map $\mathbf{e}^D : R \to R[[t]]$, iterativity can be expressed as follows:

$$\begin{array}{ccc} R & \longrightarrow & R[[s]] \\ \downarrow & & \downarrow \\ R[[s+t]] & \longrightarrow & R[[s,t]] \end{array} \tag{2.1}$$

where the map $R \to R[[s+t]]$ is given by $r \to \sum_i d_i(r)(s+t)^i$, the map $R[[s]] \to R[[s,t]]$ is given by $\sum_i r_i s^i \to \sum_i \sum_j d_j(r_i)s^i t^j$, and the map $R[[s+t]] \to R[[s,t]]$ is the inclusion.

3 Higher derivations on schemes

3.1 Higher derivations as sheaf homomorphisms or differential operators

The definitions of higher derivations on rings generalize readily to schemes;

Definition 5. Let $f : X \to B$ be schemes, and let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. A derivation of length n on X relative to B into $\mathcal{A}$ is a sequence of homomorphisms of sheaves of abelian groups $d_i : \mathcal{O}_X \to \mathcal{A}$ satisfying:

1. The composition
$$\mathcal{O}_B \xrightarrow{f^\sharp} f_\star \mathcal{O}_X \xrightarrow{f_\star d_i} f_\star \mathcal{A}$$
is zero.

2. For any open set $U \subseteq X$ and $x, y \in \mathcal{O}_U$ the identity $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ holds.

If $U \subset B$ and $V \subset f^{-1}(U)$ are affine, say $U = \operatorname{spec} k$ and $V = \operatorname{spec} R$, we get ring homomorphisms $d_i : R \to \mathcal{A}(V)$. By taking sections over U the composition
$$k \to R \to \mathcal{A}(V)$$
is zero, so a sequence of endomorphisms of sheaves is a higher derivation iff it is locally a higher derivation in the sense of rings.

As in the case of rings, we say that a higher derivation is normal if its initial term is the identity morphism. The endomorphisms of a normal higher derivation are differential operators on a scheme as well:

Proposition 6. *Let (d_i) be a normal higher derivation. Then d_i is a differential operator of order at most i.*

Proof: An endomorphism is a differential operator iff it is a differential operator locally, and the latter is true by proposition 3. □

3.2 Higher derivations as morphisms of formal schemes

If $X = \operatorname{spec} R$ is an affine algebraic variety, the homomorphism $R \to R[[t]]$ corresponds to a map:

$$\operatorname{spec} R[[t]] \to \operatorname{spec} R = X$$

If $\operatorname{spec} k \to X$ is a point in X, we have a corresponding homomorphosm $R \to k$, which can be extended to a homomorphism $R[[t]] \to k[[t]]$ by letting $t \to t$ and extending linearly. Composing we get a homomorphism $R \to k[[t]]$, which corresponds to a map $\operatorname{spec} k[[t]] \to X$. This is an embedding of the formal affine line into X. Furthermore, if the derivation is normal the base point will be the point $R \to k$ we started with. A normal higher derivation can thus be seen as a field of formal curves, or as a section to the arc space of X (see [16]).

In the case of a normal iterative higher derivation, the diagram 2.1 gives a diagram of affine formal schemes. Since $R[[t]] = R \otimes k[[t]]$ and $k[[s,t]] = k[[s]]\hat{\otimes}k[[t]]$ we have:

$$\begin{array}{ccc} \operatorname{spec} R & \longleftarrow & \operatorname{spec} R \times \operatorname{spf} k[[t]] \\ \uparrow & & \uparrow \\ \operatorname{spec} R \times \operatorname{spf} k[[s+t]] & \longleftarrow & \operatorname{spec} R \times \operatorname{spf} k[[s]] \times \operatorname{spf} k[[t]] \end{array}$$

which together with the normality condition (implies that the identity act trivially) means that we have an action of the formal additive group on $\operatorname{spec} R$.

3.3 Divided differentials

In this section we construct sheaves whose algebra duals are the higher derivations, following Vojtas presentation in [16]

Let $s : k \to R$ be commutative rings and let $A = R[x^{(n)}]_{x \in R, n \in \mathbb{N}^+}$, i.e the polynomial ring in the variables $x^{(n)}$ for all $x \in R$ and $n \in \mathbb{N}$. Define the following subsets of A:

$$\begin{aligned} I_1 &= \{x^{(m)}\}_{x \in im\, s, m \in \mathbb{N}^+} \\ I_2 &= \{(x+y)^{(m)} - x^{(n)} - y^{(m)}\}_{x,y \in R, m \in \mathbb{N}^+} \\ I_3 &= \{(xy)^{(m)} - \sum_{i+j=m} x^{(i)} y^{(j)}\}_{x,y \in R, m \in \mathbb{N}^+} \\ I_n &= \{x^{(m)}\}_{x \in R, m > n} \end{aligned}$$

Let I be the ideal generated by the elements in the sets I_1, I_2, I_3, I_n and let:

$$HS^n_{R/k} = \frac{A}{I}$$

Now the following is evident:

Proposition 7. *The sequence of maps $d_i : R \to HS^n_{R/k}$ defined by $d_i(x) = x^{(i)}$ where $x^{(0)} = x$ is a higher derivation, and any higher derivation D from R to A for a k-algebra A will factor through it. Furthermore, if $A = R$ and $D = (s_i)_i$ is normal there is R-algebra homomorphism $\phi : HS^n_R \to R$ such that $s_i = d_i \circ \phi$.*

Corollary 8. $Hom_{k-alg}(HS^m_{R/k}, A) \cong Der^m(R, A)$

Corollary 9. $Hom_{R-alg}(HS^m_{R/k}, A) \cong Der^m(R)$

To use the divided differentials to construct sheaves of higher derivations on schemes we need to know how they behave under localization:

Proposition 10. *Let S be a multiplicativly closed set in R. Then $HS^n_{S^{-1}R/k} = S^{-1}HS_{R/k}$*

Proof: See [Vojta].

Proposition 11. *Let S be a multiplicatively closed set in k. If the morphism $k \to R$ factors through $S^{-1}k$ then $HS^n_{R/S^{-1}k} = HS^n_{R/k}$.*

Proof: The two rings are generated by the same elements, and the relations in the sets I_2 and I_3 are the same. The only difference comes from the relations in the set I_1. Since for $a, b \in k$ we have

$$0 = a^{(n)} = (\frac{a}{b}b)^{(n)} = \sum (\frac{a}{b})^{(i)} b^{(j)}$$

we can inductively conclude that $(\frac{a}{b})^{(k)} = 0$ for all $a, b \in k$ and $r > 0$ in $HS^n_{R/k}$ as well. □

This makes it possible to patch together the HS-algebras of the local rings of a scheme X over Y to form the HS-sheaf, which will be a sheaf of $\mathcal{O}_X$-algebras.

Proposition 12. *For a scheme $f : X \to Y$ there is a sheaf of $\mathcal{O}_X$-algebras (and hence $f^{-1}\mathcal{O}_Y$-algebras) $HS_{X/Y}$ such that for any affine subscheme* $\operatorname{spec} k$ *of Y and any* $\operatorname{spec} R \subseteq f^{-1}(\operatorname{spec} k)$ *we have* $HS_{X/Y}(\operatorname{spec} R) = HS_{R/k}$

Proof: For affine sets $U \subseteq X$ and $V \subseteq Y$ such that $U \subseteq f^{-1}(V)$ there is a natural candidate for $HS^m_{X/Y}(U)$, namely the set $HS^m_{R/S}$ with $V = \operatorname{spec} R$ and $U = \operatorname{spec} S$. It follows from proposition 10 that it suffices to define $HS^m_{X/Y}$ on any affine open, so we only need to prove that this definition is independent of the affine base chosen. Hence assume that we have two sets $V, V' \subseteq Y$ such that $U \subseteq f^{-1}(V), f^{-1}(V')$. Then $U \subseteq f^{-1}(V \cap V')$, so we may assume $V' \subseteq V$. If $V' = \operatorname{spec} R'$, R' will be a localization of R in some set S. But then by proposition 11 $HS^m_{S/R} = HS^m_{S/R'}$.

Corollary 13. *Let X be scheme over a field k. Then*

$$Der^n_X(U) = Hom_{\mathcal{O}_U-alg}(HS^n_X(U), \mathcal{O}_X(U))$$

is sheaf of sets on X.

We use this as the definition of the sheaf of higher derivations. Note that, as in the ring case, we consider only normal derivations. Summing up, we have:

Proposition 14. *Let X be a scheme over k. The following data are equivalent:*

1. *$D \in \Gamma(X, Der_X^{\infty})$, a global section of the sheaf of higher derivations.*
2. *A morphism of formal schemes $\mathbf{e}^D : \mathrm{spf}\, k[[t]] \times X \to X$ such that the composition with the natural map $X \to \mathrm{spf}\, k[[t]] \times X$ is the identity.*
3. *A sequence of sheaf homomorphisms $d_i : \mathcal{O}_X \to \mathcal{O}_X$ satisfying the Leibniz rule with δ_0 the identity.*
4. *A sequence of differential operators $(\delta_i)_{i\in\mathbb{N}}$ with $\delta_0 = id$, satisfying the Leibniz rule.*

Proof:(Sketch) The equivalence of (3) and (4) is proposition 6. Given a global section D of the sheaf of higher derivations as in (1), i.e a homomorphism of sheaves of algebras $HS^{\infty}(X) \to \mathcal{O}_X$ the composition with the canonical morphisms $d_i : \mathcal{O}_X \to HS^{\infty}(X)$ are easily verified to be endomorphisms satisfying the Leibniz rule using the relations of $HS^{\infty}(X)$. Conversely, proposition 7 yields a morphism $HS^{\infty} \to \mathcal{O}_X$ for any such sequence of endomorphisms. Thus the data of (1) is equivalent to the data of (3). Finally, the data of (2) yields ring homomorphisms $\mathcal{O}_X(U) \to k[[t]]\hat{\otimes}\mathcal{O}_X(U) = \mathcal{O}_X(U)[[t]]$ for any open set, thus yielding ring homomorphisms $d_i : \mathcal{O}_X(U) \to O_X(U)$ which by construction glue to sheaf homomorphisms as in (3). $\square$

3.4 Higher derivations with poles along a divisor

As was seen in the prevoius chapter, a higher derivation can be seen as a sequence of differential operators satisfying the generalized Leibniz rule. We shall use this characterization to define a notion of higher derivations with poles along a divisor.

Let X be a scheme and $\mathcal{L}$ a line bundle on X. Given a commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ we have natural maps

$$\mathcal{A} \otimes \mathcal{L}^{\otimes i} \otimes \mathcal{A} \otimes \mathcal{L}^{\otimes j} \to \mathcal{A} \otimes \mathcal{L}^{\otimes (i+j)}$$

defined on generators by $(a \otimes l_1) \otimes (a' \otimes l_2) = aa' \otimes l_1 \otimes l_2$ with $l_1 \in L^{\otimes i}$ and $l_2 \in L^{\otimes j}$.

Now we can define what it means so satisfy Leibniz rule:

Definition 15. Let X be a scheme and $\mathcal{L}$ a line bundle. The sheaf $Der_X^n \otimes \mathcal{L}$ of higher derivations with poles along $\mathcal{L}$ is the subsheaf of the product sheaf (of sets) $\Pi Diff^i(X, \mathcal{L}^{\otimes i})$ consisting of sequences of sections (s_i) with $s_i \in Diff^i(\mathcal{O}_X, \mathcal{L}^i)$, such that for any sections $s_i, i = 0, ..., n$, open set U, and $x, y \in \mathcal{O}_X(U)$ the Leibniz rule $s_n(U)(xy) = \sum s_i(x)s_j(y)$ is satisfied.

Now assume $\mathcal{L}$ is very ample. Then (see [5])

$$X = \text{proj} \bigoplus_{k \geq 0} \Gamma(X, \mathcal{L}^k) \tag{3.1}$$

Let $S(\Gamma(X,\mathcal{L}))$ denote the symmetric algebra on the vector space $\Gamma(X,\mathcal{L})$. The morphism $S(\Gamma(X,\mathcal{L})) \to \bigoplus_{k\geq} \Gamma(X,\mathcal{L}^k)$ yields a projective embedding of X. Since $\mathcal{L}$ is invertible we can find a section $s \in \mathcal{L}$ generating $\mathcal{L}$ on some open U. Then on U we have $\mathcal{L} = s\mathcal{O}_X$ and $\mathcal{L}^k = s^k\mathcal{O}_X$. Let U_s be the set where s does not vanish. Then on U_s we have by equation 3.1 that:

$$\bigoplus_{k\geq 0} \frac{\Gamma(X,\mathcal{L}^k)}{s^k} = \mathcal{O}_{U_s}(U_s)$$

inducing a filtration on $\mathcal{O}_{U_s}(U_s)$. The isomorphism $\mathit{Diff}^{\,i}(\mathcal{O}_X,\mathcal{L}) \mid_{U_s} \to \mathit{Diff}^{\,i}(\mathcal{O}_{U_s},\mathcal{O}_{U_s})$ given by $d \to \frac{1}{s^i}d$ can be composed with the restriction map yielding maps

$$\theta_k : \mathit{Diff}^{\,i}(\mathcal{O}_X,\mathcal{L}^k) \to \mathit{Diff}^{\,i}(\mathcal{O}_{U_s},\mathcal{O}_{U_s})$$

Proposition 16. *The union of the images of the maps θ_k cover $\mathit{Diff}^{\,i}(\mathcal{O}_{U_s},\mathcal{O}_{U_s})$.*

Proof:
Let I_Δ be the sheaf of ideals corresponding to the diagonal i X and $I_{\Delta'}$ the sheaf of ideals corresponding to the diagonal in U. Since $\mathit{Diff}^{\,i}(\mathcal{O}_X,\mathcal{L}^k) = \mathit{Hom}(\frac{\mathcal{O}_X\times\mathcal{O}_X}{I_\Delta^{i+1}},\mathcal{L}^k)$ we can consider a differential operator d from $\mathcal{O}_X$ to $\mathcal{L}^i$ as a morphism

$$d : \frac{\mathcal{O}_X \times \mathcal{O}_X}{I_\Delta^{i+1}} \to \mathcal{L}^k$$

and a differential operator d' on $\mathcal{O}_U$ as a morphism

$$d' : \frac{\mathcal{O}_U \times \mathcal{O}_U}{I_\Delta^{i+1}} \to O_U$$

.

We have a diagram:

$$\begin{array}{ccc} \frac{\mathcal{O}_X\times\mathcal{O}_X}{I_\Delta^{i+1}} & \overset{d}{\dashrightarrow} & \mathcal{L}^k \\ \rho\downarrow & & \downarrow\rho \\ j_*\frac{\mathcal{O}_U\times\mathcal{O}_U}{I_{\Delta'}^{i+1}} & \underset{d'}{\longrightarrow} & j_*\mathcal{O}_U \end{array}$$

and need to prove that any d' lifts to a d as above. Now d' induces a morphism $\frac{\mathcal{O}_X\otimes\mathcal{O}_X}{I_\Delta} \to \mathcal{O}_U$ by composition with the restriction map ρ. By the preceding lemma we have a filtration of $j_*\mathcal{O}_U = \cup im\,\theta_k$ into submodules, which are coherent since they are images of coherent modules. Since $\frac{\mathcal{O}_X\times\mathcal{O}_X}{I_\Delta^{i+1}}$ is a coherent $\mathcal{O}_X$-module, its image in $j_*\mathcal{O}_U$ is coherent. Thus we can cover X be finitely many open affines on which the image is generated by finitely many sections, hence the image is contained in a finite part of the filtration. Taking the maximal of these parts over the finite set of open affines we see that the image is contained in one of the modules in the filtration. Hence there is an integer k such that the image is contained in $\frac{1}{s^k}\mathcal{L}$, which means it lifts to a morphism

$$d : \frac{\mathcal{O}_X \otimes \mathcal{O}_X}{I_\Delta} \to \mathcal{L}^k$$

which was to be proved. □

Definition 17. Let $\mathcal{L}$ be an invertible sheaf on a scheme X, s a section of $\mathcal{L}$ and U the complement of the zero set of S. For a differential operator D on U we say that D has degree i with respect to $\mathcal{L}$ if i is the least integer such that D lies in the image of θ_i as defined above.

Now, if $U \subseteq \mathbb{A}^n$ is an affine variety we can use the standard embedding $\mathbb{A}^n \to \mathbb{P}^n$, which is unique up to a projective automorphism, to define the projective closure X of U in $\mathbb{P}^n$. Then X is a projective variety with U as an open subset which is the zero set of the pullback $\mathcal{L}$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ along the embedding of X in $\mathbb{P}^n$, meaning it is a very ample divisor.

Definition 18. [1] Let $U \subseteq \mathbb{A}^n$ be an embedded affine variety, and D a differential operator on U. Then we say D has degree i with respect to the embedding if it has degree i in the sense of definition 17 with respect to the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ on its projective closure.

Definition 19. A higher derivation (d_i) on an embedded affine variety $U \subseteq \mathbb{A}^n$ is bounded if the degree of d_i is bounded by i. It is of finite growth if there is a $k \in \mathbb{N}$ such that the degree of d_i is bounded by ki

Lemma 20. *If $U \subseteq \mathbb{A}^n$ is an embedded affine variety, the set of higher derivations of growth factor k is the set of bounded derivations for some embedding $U \subseteq \mathbb{A}^m$.*

Proof: A derivation is of growth k iff it's i:th component is in the image of the restriction morphism from $\mathit{Diff}^i(X, \mathcal{L}^{ik})$, where $\mathcal{L}$ is the sheaf corresponding to the complement of U in its projective closure X. The projective embedding associated with $\mathcal{L}^k$ and a hyperplane extending the complement of $\mathbb{A}^n$ in $\mathbb{P}^n$ yields $\mathcal{L}^k$ as the pullback of $\mathcal{O}(1)$. □

Due to the preceeding lemma, when looking at an affine variety, we shall always consider the set of bounded higher derivations with respect to some embedding.

[1] It would be as natural to define the degree of a differential operator as the least i such that $DV_k \subseteq V_{k+i}$, where V_k is the filtration of $\mathcal{O}_U$ induced by $\mathcal{L}$. It can be shown that this notion of degree is equivalent to the one we give.

Example 21. A higher derivation of finite length trivially has growth factor 0. If δ is a derivation $\delta = \sum P_i \frac{\partial}{\partial x_i}$ on the polynomial ring $k[x_1, ..., x_n]$ over a field in characteristic 0 with the P_i:s polynomial of degree k, the growth factor of $\frac{\delta^i}{i!}$ is k

Example 22. The higher derivation on $\mathbb{A}^1 \subseteq \mathbb{P}^1$ corresponding to the homomorphism $k[x] \to k[x][[t]]$ taking x to $\sum_n x^{n!} t^n$ has infinite growth factor.

Proposition 23. *Let X be scheme, $\mathcal{L}$ a line bundle and U an open set on which $\mathcal{L}$ is trivial. Then the image of the restriction morphism $Der_X^\infty \otimes \mathcal{L} \to Der^\infty(U)$ consists of the bounded higher derivations with respect to $\mathcal{L}$. By taking succesively higher powers of $\mathcal{L}$ we get a filtration of the set of higher derivations on U which may not be exhaustive.*

Proof: By the definitions and example 22. □

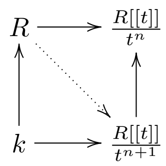
Corollary 24. *Let $U \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be an affine variety. Then the restriction of the sheaf of higher derivation with poles along $\mathcal{O}_{\mathbb{P}^n}(1)$ to $\mathbb{A}^n$ is isomorphic to the subsheaf of $Der^\infty(U)$ consisting of derivations bounded w.r.t $\mathcal{O}_{\mathbb{P}^n}(1)$.*

4 Extensions

A natural question is whether any higher derivation of finite length n can be extended to a higher derivation of length $n+1$. For smooth rings, the answer is affirmative:

Proposition 25. *If R is smooth over a ring k, then any higher derivation of R over k of finite length can be extended to one of infinite length.*

Proof: This amounts to completing the following diagram



which can be done by the smoothness of R since the ideal (t^n) is nilpotent in $\frac{R[[t]]}{t^{n+1}}$. □

Furthermore, fibres of the truncation map have a nice structure:

Lemma 26. *Let $D = (d_i)$ and $D' = (d'_i)$ be two higher derivations of length n such that $d_i = d'_i$ for $i < n$. Then $d_n - d'_n$ is a derivation.*

Proof:

$$\begin{aligned}(d_n - d'_n)(xy) &= \sum_{i+j=n} \left[d_i(x)d_j(y) - d_i(x)d'_j(y)\right] \\ &= xd_n(y) + yd_n(x) - xd'_n(y) + yd'_n(x) \\ &= x(d_n - d'_n)(y) + y(d_n - d'_n)(x)\end{aligned}$$

□

The converse of this lemma is also true:

Lemma 27. *Let $D = (d_i)_0^n$ be a higher derivation and δ a derivation. Then the sequence $(d_i, d_n + \delta)_0^{n-1}$ is a higher derivation.*

Proof: Clearly it suffices to check Leibniz rule for $d_n + \delta$. We have:

$$(d_n + \delta)(xy) = d_n(xy) + \delta(xy) \quad (4.1)$$

$$= \sum_{i=0}^{n} d_i(x)d_j(y) + x\delta x + y\delta y \quad (4.2)$$

$$= x\delta y + xd_n(y) + \sum_{i=1}^{n-1} d_i(x)d_{n-i}(y) + y\delta x + yd_n x \quad (4.3)$$

□

Thus the fibre of the truncation map is a principal homogeneous space under the derivations.

Now let X be a scheme smooth over a field k. There are natural truncation maps $t_n : Der^{n+1}_{X/k} \to Der^n_{X/k}$. The two previous lemmas hold over any open set contained in the preimage of an open set in Y; however the *existence* of a higher derivation of length $n+1$ over a given derivation of length n on an open set U is not guaranteed unless U is affine , so the fibre of the truncation map on an open set U is either a principal homogeneous space of the derivations over U, or empty. Put in other words (see [12]):

Proposition 28. *Let X be a scheme smooth over a field k and let $\mathcal{L}$ be a line bundle. Then $t_n^{-1}(Der^n_{X/k} \otimes \mathcal{L})$ is a torsor under $Der_{X/k} \otimes \mathcal{L}^n$.*

Now let U_i be a covering of X with open affines, and let D be a derivation with poles along L of length n. Then there are derivations D_i defined on U_i of length $n+1$ extending D. On each intersection $U_i \cap U_j$ the morphism δ_{ij} defined as the difference between the $n+1$:st terms of D_i and D_j is an element of $T_X \otimes \mathcal{L}^{n+1}$, which defines a Cech 1-cocycle of the sheaf $T_X \otimes \mathcal{L}^{n+1}$. Clearly, if this cocycle is zero, D can be globally extended to a derivation of length $n+1$. Thus we have:

Proposition 29. *Let X be smooth over a field k and $\mathcal{L}$ a line bundle on X such that $H^1(X, T_X \otimes \mathcal{L}^n) = 0$ for $n \geq 0$. Then any higher derivation with poles along $\mathcal{L}$ extends to an infinite one.*

Example 30. Since the tangent bundle on projective space over a field is ample, the higher derivations $Der^n_{\mathbb{P}^n}$ on projective space always extend to infinite ones.

5 Parameter spaces

From now on we assume all schemes to be over a field, and whenever we say smooth or proper we mean smooth and proper over this field.

To speak about a generic higher derivation on X we want a parameter space for the set of global higher derivation possibly with poles. Since Diff^i_X is coherent, $H^0(X, \mathit{Diff}^i_X \otimes \mathcal{L})$ is finite-dimensional for any ample $\mathcal{L}$. Consider the product of vector spaces

$$S_{X,n,\mathcal{L}} = H^0(X, \mathit{Diff}^0_X(\mathcal{O}_X, \mathcal{L})) \times \cdots \times H^0(X, \mathit{Diff}^n_X(\mathcal{O}_X, \mathcal{L}^n))$$

as an algebraic variety parametrizing the set of sequences of differential operators into the line bundles $\mathcal{L}^{\otimes i}$.

Proposition 31. *The set of higher derivations of length n with poles along $\mathcal{L}$ is a closed subset $\mathbf{Hder}^n_{X,\mathcal{L}}$ of S.*

Proof: For each pair $x, y \in O_X(U)$ the Leibniz rule imposes a closed condition on $S_{n,\mathcal{L}}$, and the set $\mathbf{Hder^n_{X,\mathcal{L}}}$ of higher derivations with poles along $\mathcal{L}$ is the set of points in $S_{n,\mathcal{L}}$ satisfying these conditions for all U and all $x, y \in \mathcal{O}_X(U)$. Since this is an intersection of closed sets, it is a closed set. Thus the set of higher derivations is a closed subset of $S_{X,n,\mathcal{L}}$. $\square$

We shall write $\mathbf{Der}_{X,\mathcal{L}}$ for $\mathbf{Hder^1_{X,\mathcal{L}}}$, i.e, the space of derivations. This is simply the vector space $\Gamma(X, Der(\mathcal{O}_X, \mathcal{L}))$ considered as affine space.

Proposition 32. *Assume X is a smooth scheme, $\mathcal{L}$ a line bundle such that $H^1(X, T_X \otimes \mathcal{L}^n) = 0$ for $0 \leq n \leq n$ Then $\mathbf{Hder}^n_{X,\mathcal{L}}$ is irreducible.*

Proof: First we note that since the fibers of the truncation map

$$\mathbf{Hder^{n+1}_{X,\mathcal{L}}} \to \mathbf{Hder^n_{X,\mathcal{L}}}$$

are principal homogeneous spaces under the group of derivations with poles along $\mathcal{L}$, which is a vector group and hence irreducible the fibres are irreducible as well. Next we have by prop 29 that the condition $H^1(X, T_X \otimes \mathcal{L}^k) = 0$ implies that all higher derivations of length $k-1$ with poles along $\mathcal{L}$ extend. We can now prove irreducibility by induction on the length n of the derivations. The case $n = 0$ is clear. Assume $\mathbf{Hder}^k_{\mathcal{L}}$ is irreducible. If two different fibers in $\mathbf{Hder}^{k+1}_{\mathcal{L}}$ would lie in different irreducible components, their projections on $\mathbf{Hder}^n_{\mathcal{L}}$ must be disjoint and closed. But this space was irreducible, which yields a contradiction. $\square$

Corollary 33. *Let $U \subseteq \mathbb{A}^n$ be an embedded affine variety. Then for any $n \geq 0$ the space of bounded higher derivations of length n is irreducible.*

The set of higher derivations of infinite length can be seen as the limit of the spaces of derivations of finite length. Since the truncation maps are affine with the topology given above, this will actually be a scheme. However, we will not go further into this because our main purpose with parameter spaces is to define genericity, and for infinite derivations this can be defined in terms of genericity of their finite truncations.

We can now make the following definition:

Definition 34. A set of higher derivations of length n with poles along $\mathcal{L}$ is generic if it contains an open, nonempty subset of $\mathbf{Hder}^n_{\mathcal{L}}$. A set of derivations of infinite length is generic if all its sets of trunctions are.

We would also like to have a relative version of genericity in terms of what derivations occur as differences of extensions. By the calculation in the proof of 26, if two global higher derivations with poles along $\mathcal{L}$ agree up to order $n-1$, the difference between the top terms will be in $H^0(X, Der_X(\mathcal{O}_X, \mathcal{L}^n)$. In fact, we have:

Lemma 35. *Let $D \in \mathbf{Hder}^n_{\mathcal{L}}$. Then the maps $t_i : \mathbf{Hder}^n_{\mathcal{L}} \to \mathbf{Der}^n_{\mathcal{L}}$ mapping D' to the difference of the i:th order terms of D and D' are morphisms of algebraic varieties.*

Proof: Evident, for example by choosing a basis for each $\mathbf{H}^0(X, \mathit{Diff}^{\,i}_X \otimes \mathcal{L}^{\otimes i})$. $\square$

Definition 36. An extension sequence is a sequence

$$G = (G_n), G_n \subseteq \mathbf{P}_{H^0(X, Der_X(\mathcal{O}_X, \mathcal{L}^n))}$$

of constructible subsets of the sets of derivations with poles along $\mathcal{L}$.

Definition 37. A set S of higher derivations is G-generic if S contains an open, nonempty subset of $t_i^{-1}(G_i)$ for each i.

6 Invariant varieties

We shall restrict attention to algebraic varieties, by which we mean an integral, noetherian, separated scheme of finite type over an algebraically closed field.

6.1 Derivations

Definition 38. Let R be a k-algebra and D a k-derivation on R. We say that an ideal I in invariant under D if $DI \subseteq I$. We say that an element $f \in R$ is invariant if $f \notin K$ and $D(f) \subseteq (f)$, i.e if there is an element $\lambda \in R$ such that $Df = \lambda f$. If X is an algebraic variety and D a vector field on X we say that a subvariety H is invariant under D if $D \in \mathcal{T}_H$. Equivalently we can say that for every open affine, the ideal defining H in that open set should be invariant under the restriction of D.

Definition 39. Let R be a k-algebra and D a k-derivation. We say D is completely integrable if there is an element $r \in R\backslash(k \cup R^p)$ such that $Dr = 0$. If X is an algebraic variety defined over k we say that a k-derivation D is completely integrable if there is a reduced non-constant section $s \in \mathcal{O}_X$ such that $Ds = 0$.

Definition 40. Let X be an algebraic variety and F a subsheaf of the tangent sheaf T_X. If H is a subvariety of X we say H is invariant under F if its sheaf of ideals is invariant under all elements of F. We say F is completely integrable if the subsheaf $annF$ of $\mathcal{O}_X$ of elements killed by the elements of F properly extends $\mathcal{O}_X^p$

The next proposition says that complete integrability implies that there are plenty of invariant subvarieties.

Proposition 41. *Let D be a vector field on an algebraic variety X. If D is completely integrable, there is a surjective, non-trivial morphism $X \to B$ whose fibers are invariant.*

Proof: It is easy to verify that the set of sections killed by D is a subsheaf $annD$ of rings of $\mathcal{O}_X$. The morphism $f : \operatorname{spec} annD \to X$ corresponds on open affine subsets $U = \operatorname{spec} R$ to the inclusion morphism of the ring of constants into R. If $V \subseteq \operatorname{spec} annD$ is affine, and $U = \operatorname{spec} R$ extends $f^{-1}(V)$, the fiber over a point $p \in V$ is simply Rp, and $Drp = pDr + rDp = pDr \in Rp$, so fibers are invariant. $\square$

Note that in the characteristic zero case the morphism in the proposition above cannot be finite (because the ring of constants is algebraically closed in R), thus inducing a fibration of X into integral subvarieties. In positive characteristic, this is not true, and the morphism above may well be finite and is then inseparable.

In the case of a subsheaf of the tangent sheaf, we have by the above argument:

Proposition 42. *Let F be a completely integrable subsheaf of the tangent sheaf then there is a surjective, non-trivial morphism whose fibers are invariant under F*

The following is a well-known result from the theory of foliations in positive characteristic:

Theorem 43. *(Ekedahl) Let X be an algebraic variety over a field k of characteristic $p > 0$ with $\dim X = n \geq 2$ and let F be a p-closed, involutive subsheaf of the tangent sheaf, with rank $r < n$. Then F is completely integrable. (Much more is true. If we say $\mathcal{F}$ is saturated if $\frac{T_X}{\mathcal{F}}$ is torsion-free, there is a one-to-one correspondence between the set of saturated, p-closed involutive subsheaves and normal varieties between X and X^p)*

Proof: See [13] or [3]. □

In [8], Kim proves a positive characteristic version of Jouanolou's theorem on integrability of Pfaffian equations using the Cartier operator, and in [14] Pereira proves a similiar result for vector fields on affine space over a positive characteristic field. The following proposition generalizes both results to foliations of arbitrary rank on an arbitrary smooth variety.

Theorem 44. *Let X be a smooth algebraic variety over a field k of characteristic $p > 0$. Let $\mathcal{F}$ be a subsheaf of the tangent sheaf. Then $\mathcal{F}$ is either completely integrable or has only finitely many integral hypersurfaces.*

Proof: Assume H is an integral hypersurface. This means that if H is given locally by an equation $f = 0$ and D is a section of F we have $Df = \lambda f$ for some polynomial λ. Thus it is clear that H is also an integral hypersurface of D^{p^s} for any integer s. If D, D' are sections of $\mathcal{F}$ and $Df = \lambda f$, $D'f = \lambda' f$ on some open affine, then

$$\begin{aligned}(DD' - D'D)(f) &= DD'f - D'Df \\ &= D\lambda' f - D'\lambda f \\ &= fD\lambda' + \lambda' Df - fD'\lambda - \lambda D'f \\ &= (D\lambda' + \lambda'\lambda + D'\lambda - \lambda\lambda')f \\ &= (D\lambda' + D'\lambda)f\end{aligned}$$

so any hypersurface invariant under $\mathcal{F}$ is also invariant under the commutator of two sections of $\mathcal{F}$. Thus we may assume that $\mathcal{F}$ is involutive and p-closed.

Since the tangent bundle of H has rank $n - 1$ we see that $\mathcal{F}$ cannot have full rank at the points of H, so H is contained in the support of $\mathcal{T}_X/\mathcal{F}$. If this support is the whole of X, $\mathcal{F}$ has rank less than n, and is then completely integrable by the previous theorem. Otherwise, the support is a proper closed subset, and so has only finitely many irreducible components. □

Pereira [14] has shown a partial converse to the above, namely:

Proposition 45. *Let D be a vector field on $\mathbb{A}^n_k$ with k a field of positive characteristic. Choose coordinates $x_1, ..., x_n$ on $\mathbb{A}^n_k$ and let $DepD$ be the determinant of the vectors D^{p^i} in $Derk[x_1, ..., x_n] = k[\bar{x}]\frac{\partial}{\partial x_1} \oplus \cdots \oplus k[\bar{x}]\frac{\partial}{\partial x_n}$. If $DepD$ is reduced, its factors are invariant hypersurfaces.*

Proof: See [14] □

We shall use the following notion:

Definition 46. If M is a constructible set of derivations, we say that a hypersurface is generically M-stable if it is stable under all derivations in an open non-empty set of M.

We need a slight variant of the theorem that there are no reduced der-stable ideals in a smooth ring:

Proposition 47. *Let X be a non-singular affine variety over a field k. Then there is no hypersurface invariant under all derivations in an open subset of the derivations of bounded degree.*

Proof: Since X is non-singular there is an etale covering $\mathbb{A}_k^n \to X$ for some n. An open subset S in the space of derivations of degree less than d will lift to an open subset S' of the derivations on $\mathbb{A}_k^n$, since derivations lift uniquely along etale morpisms. If H is a hypersurface in X invariant under all derivations in S, i.e $D \in T_H$ for all $D \in S$ then the preimage of H will be invariant under all derivations in S'. Hence it suffices to prove the assertion for affine space. Here, the set of derivation of degree less than d are given by an n-tuple $(f_1, .., f_n)$ of polynomials of degree at most d. For any hypersurface H and any non-singular point p on H the condition that $D' = \sum P_i \frac{\partial}{\partial x_i}$ be tangent to H at p is a closed condition. Since H is non-singular at p there is a tangent v at p not in the tangent space of H at p, so the vector field with constant value v is not tangent to H. Thus the condition is non-empty. Since a reduced hypersurface cannot be singular at all points, there are no common hypersurfaces for a generic set of derivations. □

Proposition 48. *For $n \geq 2$ there is no projective hypersurface invariant under a generic set of vector fields on projective space.*

Proof: By a similiar argument, or use the fact that the vector fields with constant coeffients on some affine chart extend.

6.2 Higher derivations

Given a higher n-derivation $d_1, ..d_n$ on an algebraic variety X we say that a subvariety Y is invariant if its ideal is stable under all the maps d_n.

Theorem 49. *Let X be a proper algebraic variety, $\mathcal{L}$ an invertible sheaf, and G an extension sequence. If for some d, there are no generically G_d-stable hypersurfaces and then a G-generic global higher derivation with poles along $\mathcal{L}$ of length n has no integral hypersurfaces.*

Proof: If $\Gamma(X, Der^n \otimes \mathcal{L}) = \emptyset$ there is nothing to prove, so we assume there is a global section. If the conclusion was false, there would be a set of higher derivations S such that all elements in S have integral hypersurfaces and the intersection of S and the preimages $t_i^{-1}(G_i)$ all contain an open subset of $t_i^{-1}(G_i)$. Pick an arbitrary $D \in S$. The initial term d_0 of D is a derivation so by Theorem 44 there are only finitely many hypersurfaces, say $H_1, .., H_k$ possibly invariant under D. Let U be an open subset of $t_d^{-1}(G_d)$ contained in the intersection $S \cap t_d^{-1}(G_d)$. Any element of U must have some of the H_i:s as invariant hypersurfaces, so if U_i is the set of elements with H_i as an invariant hypersurface,

$U = \cup U_i$. Since U_i closed in U, we must have $U_i = U$ for some i. But then every element of $t_i(U)$ is a difference between two operators with H_i as an invariant hypersurface, so every derivation in $t_i(U)$ has H_i as an invariant hypersurface. Since t_i is one-to-one, its image is open, so there is an open subset of G_d in which all elements have H_i as an invariant hypersurface, making H_i generically G_d-stable contrary to the assumption.

Corollary 50. *Let $U \subseteq \mathbb{A}^n$ be an embedded affine variety. Then for any $n \in \mathbb{N}$ a generic element in the set of bounded higher derivations of length n has no invariant hypersurfaces.*

Proof: X can be embedded in projective space $\mathbb{P}^n$ such that the higher derivations of bounded degree correspond to higher derivations with poles along powers of $\mathcal{O}_{\mathbb{P}^n}(m)$.

Corollary 51. *A generic global higher derivation on $\mathbb{P}^n$ has no invariant hypersurface*

Proof: Let $\mathcal{L}$ be trivial and use the fact that derivations with constant coeffients have no common integral hypersurface.

Corollary 52. *If X is and $\mathcal{L}$ a line bundles such that that there are no hypersurfaces stable under $H^0(X, \mathcal{T}_X \otimes \mathcal{L})$, then a generic global higher derivation with poles along $\mathcal{L}$ has no invariant hypersurface.*

Proof: Set $G_i = T_X \otimes \mathcal{L}^{\otimes i}$ in the theorem.

Corollary 53. *Let X be an algebraic variety over a field k and $\mathcal{L}$ an ample divisor on X. Then for some k and all n, a generic global derivation with poles along $\mathcal{L}^k$ has no invariant hypersurfaces.*

Proof: Pick n such that $\mathcal{L}^n$ is very ample. Then the ordinary derivations with poles along $\mathcal{L}^n$ correspond to derivations on a smooth affine variety of degree bounded by n with respect to the projective embedding associated with L. Picking L suffienctly large, we can make this bound arbitrarily large. On a smooth affine variety there are no hypersurfaces generically invariant under all derivations of sufficiently high degree.

6.3 Iterative higher derivations

A higher derivation is said to be iterative if $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$. It follows that $D_i \circ D_1 = \binom{i+1}{1} D_{i+1}$, and hence that D_i is determined by its predecessors unless p divides i. Also, since $D_i^{p-1} = cD_{ip-i}$ for some c, we have $D_i^p = cD_{ip-i} \circ D_i = c\binom{ip}{i} D_{ip} = 0$

Let $D = (D_1, ...)$ be a higher derivation on R. Then $annD_1$ is a subring of R, which is also annihilated by D_i for $i < p$. The restriction of D_p to $annD_1$ is a derivation, since $D_p(fg) = \sum_{i+j=p} D_i(f)D_j(g) = fD_p(g) + gD_p(f)$. Similarly, $D^{p^{k+1}}$ is a derivation on $annD^{p^k}$, which implies that:

$$D^{p^k} f^{p^{k+1}} = D^{p^k}(f^{p^k})^p = pD^{p^k}(f^{p^k})^{p-1} = 0$$

by the Leibniz rule.

Lemma 54. *Let D be a derivation on a ring R, inducing a derivation $\bar{D}$ on $Q(R)$. Then $Q(annD) \subseteq ann\bar{D}$.*

Proof: Assume $Da = Db = 0$. Then $D(a/b) = (bDa - aDb)/b^2 = 0$.

Proposition 55. *Let k be a field with $\operatorname{char} k = p$, R a ring over k, D a derivation on R and let $annD = \{f \in R \mid Df = 0\}$. Denote by $Q(S)$ the fraction field of S. Then $[Q(annD) : Q(R)] = p$.*

Proof: Using the fact that a differential equation

$$\sum_{i=1}^{n} a_i D^i$$

has at most n solutions linearly independent over the constant field. which is proved using the Wronskian criterion for linear dependence; namely elements $x_1, ..x_n$ are linearly dependent over the constant field of D if the determinant

$$\begin{vmatrix} x_1 & ... & x_n \\ Dx_1 & ... & Dx_n \\ ... & & ... \\ D^n x_n & ... & D^n x_n \end{vmatrix} = 0 \tag{6.1}$$

□ and the fact that R is the set of solutions of the differential equation $D^p x = 0$ over $annD$, we have that $[Q(annD) : Q(R)] \leq p$.□

For any finite-dimensional ring R, $[Q(R^p) : Q(R)] = p^{dimR}$. The theorem above therefore implies that a if $dimR \geq 2$, R^p is a proper subring of $annD_p$. Thus we have proved:

Theorem 56. *Any iterative higher derivation of finite length on a ring R with $dimR \geq 2$ has a non-trivial constant of integration.*

To study fields of constants for infinite iterative derivations, we need a theorem due to Zerla:

Theorem 57. *Let K/k be fields with K finitely generated over k and k countable. Then k is a field of constants of an iterative higher derivation on K iff k is algebraically closed in K and K is separable over k*

Proof: See [17]

Theorem 58. *Let R be a ring finitely generated over a countable field k such that the fraction field of R is separable over k. Then a generic infinite higher derivation on R over k has no non-trivial constant of integration.*

Proof: We need only prove non-emptiness of the set of higher derivation without constants of integration. Any higher derivation on R extends uniquely to a higher derivation on $K = Q(R)$, the fraction field of R. By assumption K is finitely generated and separable over k and k is countable, so by proposition 57 there is a higher derivations D such that k is the field of constants of D, i.e, D has no non-trivial constants of integration. □

According to section 3.2 an infinite iterative higher derivation on a scheme X over a ring k can be seen as an action of the formal additive group $spf k[[t]]$ on X. We shall use this viewpoint to prove that an iterative higher derivation on projective space over a field always has invariant varieties. We need a few lemmas on automorphisms on $\mathbb{P}^n$.

Lemma 59. *Let F be a commutative formal algebraic group, G an algebraic group and $\phi : F \to G$ a morphism of formal groups. Then $\overline{\operatorname{im}}\,\phi$ is a commutative algebraic subgroup of G.*

Proof: Both closedness and commutative are local so we may assume F and G to be affine. Then $F = spf R$ and $G = \operatorname{spec} S$, and we have multiplication morphisms $\mu_R : R \to R\hat{\otimes}R$ and $\mu_S : S \to S \otimes S$. We have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & R \\ \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \mu_R} \\ S \otimes S & \xrightarrow{\phi\hat{\otimes}\phi} & R\hat{\otimes}R \end{array}$$

inducing a diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\phi} & S/\ker\phi & \longrightarrow & R \\ \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \hat{\mu}_R} & & \downarrow{\scriptstyle \mu_R} \\ S \otimes S & \longrightarrow & S/\ker\phi \otimes S/\ker\phi & \longrightarrow & R\hat{\otimes}R \end{array}$$

Now it is easily verified that $\hat{\mu}_R$ defines a commutative algebraic group on $\operatorname{spec} R/\ker\phi$ into which F embeds. Furthermore it is clear that it is the smallest such algebraic group, hence is is the Zariski closure of the image of ϕ. □

Lemma 60. *Any commutative algebraic subgroup G of PGL^n has an invariant flag.*

Proof: Say PGL^n acts on $\mathbb{P}(V)$. Then G lifts to a commutative subgroup G' of GL_n, acting on V. If all elements in G' act by scalar multiplication, G is trivial and we are done. Otherwise, there is an element $g \in G'$ with a proper eigenspace V'. For any $v \in V'$ and $g' \in G'$ we have $gg'v = g'gv = \lambda g'v$ for some constant λ, so $g'v$ is an eigenvector and thus $g'v \in V'$, so V' is G'-invariant.

Repeating the argument yields a complete flag of invariant subspaces. □

Theorem 61. *Any iterative derivation on projective space has an invariant variety.*

Proof: An iterative derivation can be seen as an action of the formal affine group on $\mathbb{P}^n$, and an variety is invariant under the derivation iff it is invariant under this action. Thus we have a morphism of formal groups $\mathrm{spf}\, k[[t]] \to PGL(\mathbb{P}^n)$. By the previous lemmas, the image of this group is commutative and thus has an invariant flag. □

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