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Super-Differential Calculus in F-Spaces

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Abstract *We prove more properties of the new concept of what we have called super-differential (or quasi-differential) . This concept of differential is totally different from the Fréchet and still has the advantage that it is a linear and continuous operator and Gateaux differential on maps between locally bounded F-spaces. It has valuable applications as it is shown here.*

AMS Subject classification: 32K15, 46A16

Keywords: Infinite dimensional F-spaces, Non locally convex spaces, Quasi-Differentials, Schwartz symmetric theorem, Fréchet and Gâteaux differentials.

0.1 Introduction

One of the fundamental concepts of the theory of functions between normed spaces is differentiability, where the differentials of Fréchet and Gâteaux appear to be the most appropriate concepts.

In this paper we focus our study on a new and different concept of differentiability, discovered by the author in 1995, see[1, 6-8]. Herein called *Super-Differentiability* (or Bayoumi Quasi-*Differentiability* or *pq-Differentiability* as before), for maps $f : E \to F$ between locally bounded F-spaces E, F. In fact we become awared that some mathematicians have used the term "Quasi-*Differentiability" in a quite a different way.* Also this name "super differentiability" arises since it is stronger than the Fréchet one and has wider applications, *(*cf. References*)* . From the other side, it has the same advantage of the Fréchet differential, that is, it is a linear and continuous operator.

In section 1, definitions of new concepts are given. Section 2 is devoted to study super-differentiability of multilinear maps and polynomials between locally bounded F-spaces. Further, a generalized Schwartz symmetric theorem for twice super-differentiable maps is given.

In section 3, this theorem is extended to m-super-differentiable maps. We prove in section 4 that a super-differentiable map is Gâteaux differentiable, hence the super-differential is stronger than the classical Gâteaux one. In addition, it is worth pointing out that the super-differentiability and Fréchet one are totally different for general maps between locally bounded F-spaces (see [22]).

Finally it is worthwhile to point out that we have worked with spaces not necessarily locally convex and nothing about holomorphic properties were known before. However we have succeeded to get the simplest and easiest methods to extend the fundamental and central results in functional and complex analysis [1-23].

0.1.1 Super-Differentiable Maps in Locally bounded Spaces

Definition 1 (1995). *Let E and F be p-normed and q- normed spaces, respectively $(0 < p, q \leq 1)$, and U a nonempty open set in E. A mapping $f : U \to F$ is said to be* **m-super-differentiable** *(or* **m-quasi-differentiable** *or* **m-pq-differentiable)** *at $a \in U$, if there exists a linear map $T_a \in L(E, F)$, such that f and the continuous affine linear map $x \in E \to f(a) + T_a(x - a)$ are m-pq-tangent at a, that is,*

$$\lim_{x \to a}(\|f(x) - f(a) - Ta(x - a)\|^p / \|x - a\|^{mq}) = 0. \quad (0.1)$$

and T_a is called the m-differential of f at a and is denoted by $D^m_a f$. Hence, for every $\epsilon > 0$, there exists $\delta > 0$:

$$0 < \|x - a\| < \delta \Rightarrow \|f(x) - f(a) - T_a(x - a)\| \leq \epsilon \|x - a\|^{mq/p}$$

When $m = 1, f$ is called **super-differentiable** (*or* **quasi-differentiable**) *at a and we write $T_a = D_a f$*

Definition 2 *If $f : U \to F$ is m-super-differentiable at each point of U, then f is said to be* **m-super-differentiable** *on U.*

Remark 1 *It is to be noted that if E and F are both p-normed spaces or quasi-normed spaces with the same quasi-norm constants, then the condition (0.1) turns out to be loks like the following classical one :*

$$\lim_{x \to a}(\|f(x) - f(a) - T_a(x - a)\| / \|x - a\|) = 0.$$

0.2 Properties of Super-Differentials

Some properties of the super-differential have appeared in [[1], [2], [3]]

In what follows we explore more properties of the super-differentiability of maps between locally bounded F-spaces.

0.2.1 Super-Differentials of Multilinear Maps

We note that the product rule

$$(fg)' = f'g + fg'$$

can be derived from the following theorem. Observe that the map

$$x \in U \to f(x)g(x) \in \mathbb{K}$$

is a composition of the following two maps: $x \in U \to (f(x), g(x)) \in \mathbb{K}^2$, $(y_1, y_2) \in \mathbb{K}^2 \to y_1 y_2 \in \mathbb{K}$, where the first map taking values in the product space $\mathbb{K}^2$ and the second being a continuous bilinear map.

Definition 3 (**Multilinear Map**) *If E_i 's are different p-normed spaces, $1 \leq i \leq m$, over the same field $\mathbb{K}$, and F is a locally bounded F-space, then the map,*

$$A : E_1 \times \cdots \times E_m \to F$$

is said to be **multilinear** *if it is linear in each variable separately. This, of course, does not imply that A is linear on the product vector space E , that is, A is multilinear if each partial map,*

$$x_i \to A(x_1, \cdots, x_i, \cdots x_m)$$

is a linear map of E_i into F.

If $m = 2$, a multilinear map is called **bilinear**.

In general, a multilinear map on the product of m p_i- normed spaces is called an m-linear map.

Let $L(E_1, \ldots, E_m; F)$ denote the space of continuous m-linear maps A from $E_1 \times \ldots \times E_m$ into F. Then we have the following theorem of super-differential of A.

Theorem 1 *Let $E_1, E_2, \ldots, E_m$ be p_i-normed spaces , $(i = 1, \ldots, m)$ and F be a q-normed space. If $A \in L(E_1, \ldots, E_m; F)$ is a continuous m-linear mapping, then A is super-differentiable at $(a_1, \ldots, a_m) \in L(E_1, \ldots, E_m; F)$. Furthermore*

$$DA(a_1, ..., a_m)(x_1, ..., x_{m)} = \sum_1^m A(a_1, ...a_{i-1}, x_i, a_{i+1,} ..., a_m), \tag{0.2}$$

$$i.e. \quad DA(a) : E_1 \times ... \times E_m \to L(E_1, ...E_m, F)$$

Proof. For $m = 1$, the result is given by $[[6], Theorem\ 3.1]$. If $m = 2$, we claim

$$DA(a_1, a_2)(x_1, x_2) = A(x_1, a_2) + A(a_1, x_2)$$

Since

$$A(x_1, x_2) = A(a_1, a_2) + A(h_1, a_2) + A(a_1, h_2) + A(h_1, h_2)$$

where $h_1 = x_1 - a_1$, $h_2 = x_2 - a_2$, we get

$$\begin{aligned} & \|A(x_1, x_2) - A(a_1, a_2) - A(x_1 - a_1, a_2) - A(a_1, x_2 - a_2)\|^{1/q} \\ \leq \ & \|A\| \|x_1 - a_1\|^{1/p_1} \|x_2 - a_2\|^{1/p_2} \leq \|A\| \|x - a\|^2 \end{aligned}$$

where we used the F-norm $\|x\| = \sup\{\|x_1\|^{1/p_1}, \|x_2\|^{1/p_2}\}$ for $x = (x_1, x_2) \in E_1 \times E_2$.

If we write

$$A_a(x_1, x_2) = A(x_1, a_2) + A(a_1, x_2)$$

then $A_a \in L(E_1 \times E_2; F)$ and

$$\begin{aligned} \|A(x) - A(a) - A_a(x - a)\|^{1/q} & \leq \|A\| \|x - a\|^{1/p_1} \|x_2 - a_2\|^{1/p_2} \\ & \leq \|A\| \|x - a\|^2. \end{aligned}$$

Hence

$$DA(a) = A_a.$$

The proof for the general case can be completed by induction. ■

0.2.2 Super-Differentials of Polynomials

Definition 4 (Homogeneous polynomials) *Let E, F be vector spaces over the same field $I\!K$. A mapping $P : E \to F$ is said to be an* m**-homogeneous polynomial** *or* **a homogeneous polynomial of degree** m *if there exists an m-linear map $A : E^m \to F$ such that*

$$P(x) = Ax^m \tag{0.3}$$

for all $x \in E$.

If $P : E \to F$ is an m-homogeneous polynomial, then $P(rx) = r^m P(x)$ for any $r \in I\!K$

We shall denote by $P_a(^mE; F)$ the vector space of all m**- homogeneous polynomials** of E into F. The suffix 'a' is to indicate that the m- homogeneous polynomials in this space are not necessarily continuous. For convenience, we agree to write $F = P_a(^\circ E, F)$, see [1]and [4] for more details on polynomials.

The following theorem shows that the super-differential of a polynomial is a linear map.

Theorem 2 *If $P \in P(^mE, F)$, a continuous m-homogenous polynomial and $P = \hat{A}$, the corresponding diagonalization defined by a symmetric m-linear map A, then*

$$DP(x) = mAx^{m-1}; \tag{0.4}$$

$$i.e., DP : E \to L(E; F) = P(^1E : F).$$

Proof. We use the multinomial formula which in particular has the following binomial formula

$$A(x + y)^m = A(x + y, \ldots, x + y) = \sum_{i=0}^{m} \binom{m}{i} Ax^{m-i}y^i$$

where $x, y \in E$. Now

$$\lim_{y \to 0} \|A(x + y)^m - Ax^m - mAx^{m-1}y\|^{1/q} / \|y\|^{1/p} = 0$$

and if we compare it with

$$\lim_{y\to 0} \|P(x+y) - P(x) - DP(x)(y)\|^{1/q} / \|y\|^{1/p} = 0$$

we get $DP(x) = mAx^{m-1}$. ■

In this part we introduce the concept of higher order super-differentials for maps between locally bounded spaces, and prove an analogy to the theorem, which states that :

"**All higher super-differentials maps can be considered as symmetric multilinear maps ".**

We shall also introduce the Gâteaux differential and study its relationship with the Super-differential.

0.3 Generalized Schwartz Symmetric Theorems

Let E and F be p-normed and q-normed spaces respectively , $(0 < p, q \leq 1)$, U open subset of E, and $f : U \to F$ super-differentiable. Then

$$Df : U \to L(E, F).$$

Since $L(E, F)$ is a q-normed space, let us define the second super-differential of f after we discuss the following natural isometry.

0.3.1 Natural Isometry $L(E_1, .., E_m; F) \simeq L(E_1; L(E_2, .., E_m; F))$.

When we deal with the differential calculus and other applications we often need to identify

$$L(E_1, \cdots, E_m; F) \quad \text{with} \quad L(E_1, L(E_2, \cdots, E_m; F))$$

from the space of continuous multilinear maps to the space of m- times repeatedly continuous linear maps, under an isometric isomorphism. For simplicity we consider the bilinear case and discuss here such an isometry between $L(E_1, E_2; F)$ and $L(E_1; L(E_2; F))$ since the general case can be studied similarly. Here, as usual, E_i are p_i-normed spaces and F is a q-normed space.

Theorem 3 *Let E_i 's be p_i- normed spaces* $(1 \leq i \leq n)$, *and F be a q-normed space. Let* $\phi : L(E_1, E_2; F) \to L(E_1; L(E_2; F))$ *be defined by*

$$\phi(A)(x)(y) = A(x, y)$$

where $A \in L(E_1, E_2; F)$, $x \in E_1$ *and* $y \in E_2$. *Then ϕ is an isomorphic isometry i.e. ϕ preserves the F-norm.*

Proof. Obviously ϕ is linear , and $\|\phi(A)(x)(y)\| \leq \|A\|^q \|x\|^{q/p_1} \|y\|^{q/p_2}$. Hence for each $x \in E_1$, $\phi(A)(x) \in L(E_2; F)$ and $\|\phi(A)(x)\| \leq \|A\|^q \|x\|^{q/p_1}$; consequently $\varphi(A) \in L(E_1; L(E_2, F)$ and $\|\varphi(A)\| \leq \|A\|$. Therefore φ is a continuous linear map with $\|\varphi\| \leq 1$.

On the other hand, let $\alpha : L(E_1; L(E_2; F)) \to L(E_1, E_2; F)$ be such that

$$\alpha(B)(x, y) = B(x)(y).$$

Then α is linear and $\alpha(B)$ is bilinear for each B in $L(E_1; L(E_2; F))$. Since $\|\alpha(B)(x, y) \| \leq \|B\|^q \|x\|^{q/p_1} \|y\|^{q/p_2}$, $\alpha(B) \in L(E_1, E_2; F)$ and $\|\alpha(B)\| \leq \|B\|$; so $\|\alpha\| \leq 1$ and α is continuous. It is clear that α and ϕ are inverse to each other ;thus,

$$I = \|\alpha \circ \phi\| \leq \|\alpha\| \|\phi\|$$

and we conclude that $\|\alpha\| = \|\phi\| = 1$ since $\|\alpha\| \leq 1$, and $\|\phi\| \leq 1$.

This implies that ϕ is an isomorphic isometry. ■

We can now state the general form of the detected theorem.

Theorem 4 *There exists an isomorphic isometry*

$$L(E_1, \cdots, E_m; F) \simeq L(E_1; L(E_2; L(E_3, \cdots, L(E_m; F) \cdots)))$$

from the space of continuous multilinear maps to the space of m- times repeatedly continuous linear maps. ■

0.3.2 Second Super-Differentials

Definition 5 *Assume that $f : U \to F$ is super-differentiable on U. The function f is said to be* **twice super- differentiable (or quasi-differentiable of order two**) *at $a \in U$ (or on U), when the mapping $Df : U \to L(E; F)$ is super-differentiable at a, (or on U), respectively. In this case*

$$D(Df)(a) \in L(E; L(E; F))$$

is called the **second super-differential** *of* f *on* U *and is denoted by*

$$D^2 f : U \to L(E; L(E; F)). \tag{0.5}$$

This concept of second super-differential, obtained by a simple repetition of differentiation of order one, has the apparent disadvantage that the values of the second super-differentials belongs to the q-normed space $L(E; L(E; F))$. In this form this space appears complicated, since it is a q-normed of continuous linear maps. However, because of its natural isomorphism to $L(^2E; F)$ which follows from Theorem 3, we are able to treat the concept of the second super-differentials as follows :

Consider

$$D^2 f : U \to L(E; L(E; F))$$

and denote by

$$d^2 f(a) \in L(^2E; F)$$

the element of $L(^2E; F)$ that corresponds to $D^2 f$ by the natural isometric isomorphism given in Theorem 3. We note that the relation between $D^2 f(a)$ and $d^2 f(a)$ is characterized by

$$d^2 f(a)(s, t) = D^2 f(a)(s)(t)$$

for any $s, t \in E$. Therefore, the second super-differential

$$D^2 f(a) : A \to L(E; L(E; F)):$$

corresponds to the second super-differential

$$d^2 f : A \to L(^2E; F).$$

Thus, the super-differential of order two now takes values in the q-normed space $L(^2E; F)$. We recall that this is done by identifying $L(E; L(E; F))$ with $L(^2E; F)$, the space of all continuous bilinear mappings of $E \times E$ into F, which appears less complicated .

Remark 2 *In the case of super-differentials of order one , we also use the notation*

$$df(a) = Df(a), \quad df = Df.$$

We point out here that without assuming that f is super-differentiable on the entire subset , we can say more generally that f is **twice super-differentiable** *at the point $a \in V$ if :*

1- f is super-differentiable on a neighborhood of V of a.

2- The mapping $Df : V \to L(E; F)$ is super-differentiable at the point a.

0.3.3 Schwartz Symmetric Theorem for Twice Super-differentiable Maps

In elementary calculus it is known that if $U \subset \mathbb{R}^2$, $f : U \to \mathbb{R}$, $a \in U$, and the partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : U \to \mathbb{R}$$

are differentiable on U and the second derivatives are continuous, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

We will derive in this section a generalization of this symmetric theorem for twice super-differentiable functions of locally bounded F-spaces(see Theorem 8). This can be obtained from the following generalized theorem.

Theorem 5 (Generalized Schwartz Symmetric Theorem*)*

Let $f : U \to \mathbb{R}$ be twice super-differentiable at $a \in U$. Then $d^2 f(a)$ is a symmetric bilinear map, that is, for all $k, h \in E$,

$$d^2 f(a)(h, k) = d^2 f(a)(k, h). \tag{0.6}$$

Proof. Since f is twice super-differentiable at a, there is some $r > 0$ such that $B(a, r) \subset U$ and f is super-differentiable on $B(a, r)$.

Consider the function

$$\triangle (h, k) = f(a + h + k) - f(a + h) - f(a + k) + f(a)$$

where $h, k \in E$ are such that $a + h + k, a + h, a + k$ belong to U. More precisely, we may assume $\| h \| \leq r/2$ and $\| k \| \leq r/2$. We note that

$$\triangle (h, k) = \triangle (k, h)$$

We wish by a suitable process to approximate $d^2f(a)(h,k)$ with $\triangle\ (h,k)$ to show that $d^2f(a)$ is symmetric and give us the desired equality (0.10).

To do this fix h and define $g : B(a, r/2) \to F$ by

$$g(x) = f(x+h) - f(x)$$

Then

$$Dg(x) = Df(x+h) - Df(x) \quad \text{and}$$

$$\triangle\ (h,k) = g(a+k) - g(a).$$

Hence

$$\|\triangle\ (h,k) - d^2f(a)(h)(k)\ \|^{1/q} \leq \sigma \left[\begin{array}{c} \|\ g(a+k) - g(a) - Dg(a)(k)\ \|^{1/q} \\ +\ \|\ Dg(a)(k) - D^2f(a)(h)(k)\ \|^{1/q} \end{array} \right]$$

$$\leq \sigma\ \|\ g(a+k) - g(a) - Dg(a)(k)\ \|^{1/q}$$

$$+\sigma\ \|\ Dg(a) - D^2f(a)(h)\ \|^{1/q} \|\ k\ \|^{1/p} \tag{0.7}$$

for some $\sigma \geq 1$. On the other hand applying [1,Theorem 51, p.95], we get

$$\|\ g(a+k) - g(a) - Dg(a)(k)\ \|^{1/q} \leq \sup \|\ Dg(a+x) - Dg(a)\ \| \|\ (k)\ \|^{1/p}$$

$$\leq \sigma \sup(\ \|\ Dg(a+x) - D^2f(a)(h)\ \|^{1/q}$$

$$+\|\ D^2f(a)(h)\ -\ Dg(a)\ \|)$$

$\|\ k\ \|^{1/p}$

$$\leq 2\sigma\ \|\ k\ \|^{1/p} \sup \|\ Dg(a+x) - D^2f(a)(h)\ \| \tag{0.8}$$

where the suprema are taken over $\|\ x\ \| \leq \|\ k\ \|$. That is for all $x \in A_a^{a+k}$ the arc segment in $B(a, r/2)$. It is proved that the unit ball of every locally bounded p-normed space contains all its arc segment $A_a^b = \left\{(1-t)^{1/p}a + t^{1/p}b; 0 \leq t \leq 1\right\}$, $(0 < p < 1)$; see[1, p.50].

The inequality (0.7) now becomes

$$\|\triangle\ (h,k) - d^2f(a)(h)(k)\ \|^{1/q} \leq 3\sigma\ \|\ k\ \|^{1/p} \sup \|\ Dg(a+x) - D^2f(a)(h)\ \| \tag{0.9}$$

for $\|\ x\ \| \leq \|\ k\ \|$.

We first estimate $\| Dg(a+x) - D^2f(a)(h) \|$. Notice that

$$Dg(a+x) - D^2f(a)(h) = [Df(a+x+h) - Df(a) - D^2f(a)(x+h)]$$
$$- \left[Df(a+x) - Df(a) - D^2f(a)(x)\right] \quad (0.10)$$

Since Df is super-differentiable at a, for any $\epsilon > 0$, there is $0 < 4s < r$, such that

$$\| Df(a+t) - Df(a) - D^2f(a)(h)(t) \|^{1/q} \leq \epsilon \ \| t \|^{1/p} \quad (0.11)$$

for $\| t \| \leq 2s$. Then from (0.10) and (0.11) we have

$$\begin{aligned} \| \quad & Dg(a+x) - D^2f(a)(h) \|^{1/q} \leq \epsilon(\sigma \| x+h \|^{1/p} + \sigma \| x \|^{1/p} \quad (0.12) \\ \leq \quad & \epsilon(\sigma^2 \| h \|^{1/p} + (\sigma + \sigma^2) \| x \|^{1/p}). \quad (0.13) \end{aligned}$$

We can write the inequality (0.13) as

$$\| \triangle\ (h,k) - d^2f(a)(h,k) \|^{1/q} \leq 3\sigma\epsilon(\sigma^2 \| h \|^{1/p} \| k \|^{1/p} + (\sigma^2 + \sigma) \| k \|^{2/p}) \quad (0.14)$$

provided that $\| k \| \leq s$ and $\| h \| \leq s$. Interchanging h and k in (0.14) we obtain

$$\| \triangle\ (k,h) - d^2f(a)(k,h) \|^{1lq} \leq$$
$$3\sigma\epsilon(\sigma^2 \| h \|^{1/p} \| k \|^{1/p} + (\sigma^2 + \sigma) \| h \|^{2/p}) \quad (0.15)$$

Thus it follows from (0.14) and (0.15) that

$$\| d^2f(a)(k,h) - d^2f(a)(h,k) \|^{1/q} \leq$$
$$3\sigma^2\epsilon \left[(2\sigma^2 \| h \|^{1/p} \| k \|^{1/p} + (\sigma^2 + \sigma)(\| h \|^{2/p} \| + \| k \|^{2/p})\right] \quad (0.16)$$

provided that $\| k \|\leq s$ and $\| h \|\leq s$.

We now claim that the inequality (0.15) is true for any h and k in E . In fact, for any h and k in E, choose $\lambda > 0$ such that $\| \lambda k \|\leq s$ and $\| \lambda h \|\leq s$. Replacing h and k by λk and λh we obtain the inequality (0.16) multiplied by $\lambda^{2/p}$ on both sides; hence the result is true for any h and k in E. Since $\epsilon > 0$ was arbitrary in (0.16), we conclude that

$$d^2 f(a)(h,k) = d^2 f(a)(k,h)$$

for any h , k in E, which completes the proof. ■

0.3.4 Higher Super-Differentials

Let $f : U \to F$ be twice super-differentiable. Then we have the second super-differential

$$D^2 : U \to L^1(E; L(E;F)) \text{ and}$$

$$d^2 : U \to L(E;F)$$

By repeating the procedure inductively we can define the mth super-differential of f. Let us first simplify some notations. For $m = 1$, we write

$$L^1(E;F) = L(E;F)$$

and for $m = 2$,

$$L^2(E;F) = L(E; L(E;F)).$$

Inductively we write

$$L^m(E;F) = L(E; L^{m-1}(E;F))$$

For example , if $f : U \to F$ is twice super-differentiable on U, we have

$Df : U \to L^1(E;F) = L(E;F)$ and $D^2 f : U \to L^2(E;F)$.

After defining

$$D^{m-1} f : U \to L^{m-1}(E;F)$$

if $D^{m-1} f$ is super-differentiable at the point $a \in U$, we let

$$D^m f(a) = D(D^{m-1} f)(a) \in L^m(E;F)$$

which will be called the **mthsuper-differential** of f at a .Of course,we also have

$$D^m f : U \to L^m(E; F)$$

if $D^m f$ exists at each point a of U. The identity

$$D^p(D^q f) = (D^m f)$$

is trivially true if $p + q = m$ and $D^m f$ exists. Also, the mthsuper-differential $D^m f$ is linear in the sense that

$$D^m(f + g) = D^m f + D^m g \text{ and } D^m(\lambda f) = \lambda D^m f$$

where $\lambda \in \mathbb{K}$. For convenience, we identify$D^0 f = f$.

For $a \in U$ and $x_1, ..., x_m \in E$, we have

$$D^m f(a)(x_1)...(x_k) \in L^{m-k}(E; F)$$

for $1 \leq k \leq m$. To avoid the cumbersome parentheses, we often write

$$D^m f(a)(x_1, ..., x_k) = D^m f(a)(x_1)...(x_k).$$

Definition 6 *If $D^m f : U \to L^m(E; F)$ is continuous, we say that f is of* **class** C_q^m.

If f is of class C_q^m for every m , f is said to be of **class** C_q^∞; *in this case f is called* **infinitely many times super-differentiable** .

0.3.5 The Super-Differentials $\mathbf{D^m f}$ and $\mathbf{d^m f}$

As in section 3.1 we shall identify the mth super-differential $D^m f$ with the corresponding m-linear map $d^m f$. More explicitly,

Definition 7 *Let*

$$\phi : L(^m E; F) \to L^m(E; F)$$

be the natural isometric isomorphism defined by

$$\phi(A)(x_1)...(x_m) = A(x_1, ..., x_m).$$

For $D^m f : U \to L^m(E;F)$, *we define* $d^m f : U \to L^m(E;F)$ *to be such that*

$$d^m f = \phi^{-1} \ o \ D^m f \ .$$

We call $d^m f$ *also the* **m**th **super-differential** *on* U. *Then it is evident that*

$$d^m f(a)(x_1, ..., x_m) = D^m f(a)(x_1, ..., x_m).$$

We notice that if $m = 1$,

$$df = Df.$$

We should always keep in mind that the mth super-differentials $D^m f$ and $d^m f$ represent two different objects if m$\neq 1$. We will prefer the mth differential $d^m f$ to $D^m f$ because the former is much simpler .

We should also note that although for

$$m = p + q \text{ we have } d^m f(a) = d^p(d^q f)(a)$$

we cannot expect to have $d^m f(a) = d^p d^q f(a)$ since the last two objects are elements of two different spaces .

Theorem 6 *If* $A \in L_s(^m E;F)$ *and* $P = \hat{A} \in P(^m E;F)$, *then*

$$d^k P(x) = \left\{ \begin{array}{ll} m(m-1)...(m-k+1)Ax^{m-k} & 0 \le k \le m \\ 0 & k > m \end{array} \right\}$$

$d^k P : E \to L(^k E;F)$.

Proof. We have shown this result for $k = 1$ in Theorem 2. The rest can be proved by induction.

0.3.6 The m th Super-Differential in Relation to Linear Maps

For the higher super-differential , we have statements similar to those obtained with the first differential in relation to the linear maps, (see [4] Theorem 3).

Notice that if $A \in L(^m E;F)$ and $\phi \in L(F;G)$, then the composite map

$\phi o A \in L(^m E; G)$. Let

$$\phi^* : L(^m E; F) \rightarrow \in L(^m E; G)$$

be given by

$$\phi^*(A) = \phi o A.$$

Then it is immediate that ϕ^* is linear . Since

$$\| \phi o A(x_1, ..., x_m) \|^{1/q} \leq \| \phi \| . \| A \| . \| x_1 \|^{1/p} ... \| x_m \|^{1/p}$$

ϕ^* is also continuous where E is a p-normed space and G is a q-normed space, $(0 < p, q \leq 1)$.

Theorem 7 *For an open subset* U *of* E *, let* $f : U \rightarrow F$ *be* m *times super-differentiable and let* $\phi \in L(F; G)$*. Then for any* $a \in U$ *we have*

$$d^m(\phi o f)(a) = \phi o d^m(f)(a).$$

Proof. Use induction on m. Consider the map

$$x \in U \rightarrow d^{m-1}(\phi o f)(a).$$

By induction we get

$$d^{m-1}(\phi o f)(x) \ = \phi o d^{m-1} f(x).$$

Now the composite map of

$$d^{m-1} f : U \rightarrow L^{m-1}(E; F)$$

and

$$\phi^* : L^{m-1}(E; F) \rightarrow L^{m-1}(E; G)$$

is super-differentiable and

$$d(\phi^* o d^{m-1} f) = \phi^* o d^m f$$

Therefore, for any a,

$$d^m(\phi o f)(a) = \phi o d^m(f)(a)$$

■

0.3.7 Generalized Schwartz Symmetric Theorem of mth super-differentials $d^m f$

We have seen in Theorem 5 that the second super-differential $d^2f(a)$ is a symmetric bilinear map . This result can be extended to the mth super-differential $d^mf(a)$ by induction . We need first the following lemma .

Lemma 8 *Let $f : U \to F$ be $(m-1)$ super-differentiable on U and m super-differentiable at a. For $m \geq 2$ and for any fixed $x_2, ..., x_m \in E$, if $g : U \to F$ is defined by*

$$g(x) = d^{m-1}f(x)(x_2, ..., x_m)$$

then g is super-differentiable at a and

$$dg(a)(x) = d^m f(a)(x, x_2, ..., x_m)$$

for all $x \in E$.

Proof. The map g:$U \to F$ satisfies the composition

$$g = \lambda o \phi^{-1} o D^{m-1} f$$

where

$$\phi : L(E^{m-1}; F) \to L^{m-1}(E; F)$$

is the natural isometric isomorphism and $\lambda : L^{m-1}(E; F) \to F$ is given by

$$\lambda(A) = A(x_2, ..., x_m) \quad \text{for all} \quad A \in L^{m-1}(E; F).$$

λ is clearly continuous and linear . Now

$$Dg = D(\lambda o \phi^{-1} D^{m-1} f) = \lambda o \phi^{-1} o D(D^{m-1} f) = \lambda o \phi^{-1} D^m f$$

Hence , for all $x \in E$, we have

$$\begin{aligned} dg(a)(x) &= Dg(a)(x) = (\lambda o \phi^{-1} o D^m f(a))(x) = \lambda(\phi^{-1} o D^m f(a)(x)) \\ &= \lambda(d^m f(a)(x)) = d^m f(a)(x, x_2, ..., x_m) \end{aligned}$$

which is what set out to prove. ■

Theorem 9 *(***Generalized Schwartz Theorem** *) Let $f : U \to F$ be m-super-differentiable at $a \in U$. Then $d^m f(a)$ is a symmetric m-linear map; i.e.,*

$$d^m f(a) \in L_s(^m E; F).$$

Proof. For $m = 2$, this was proved in Theorem 5. We now proceed by induction on m for $m \geq 3$.

Assume that the theorem has been proved for $m-1$. Then $d^m f(a)$ is the super-differential of the mapping $d^{m-1}f$, which by our assumption exists in a neighborhood V of a . By the induction hypothesis, we can assume

$$d^{m-1}f : V \rightarrow L_s(^mE; F).$$

By the Lemma 8, for fixed $x_3, ..., x_m \in E$, the mapping $g : V \rightarrow F$ defined by

$$g(x) = d^{m-2}f(x)(x_3, ..., x_m)$$

is super-differentiable at a and its super-differential at a is given by

$$dg(x)(y) = d^{m-1}f(x)(y, x_3, ..., x_m)$$

for all $x \in V$ and for all $y \in E$. Similarly, applying the lemma 8 to the map

$$x \rightarrow d^{m-1}f(x)(x_2, ..., x_m)$$

where $x_2, ..., x_m$ are fixed, we obtain

$$d^2g(a)(x_2, x_1) = d^m f(x)(x_1, x_2, ..., x_m) \tag{0.17}$$

for all $x_1 \in E$. From Theorem 9 we have

$$d^2g(a)(x_1, x_2) = d^2g(a)(x_2, x_1)$$

and hence

$$d^m f(a)(x_2, x_{1,}x_3, ..., x_m) = d^m f(a)(x_1, ..., x_m). \tag{0.18}$$

But $d^{m-1}f(a)$ is symmetric and $x_1, ..., x_m$ were fixed but arbitrary, so from (0.21), for $x_1 \in E$, the map

$$d^m f(a)(x_1) : (x_2, ..., x_m) \rightarrow d^m f(a)(x_1, ..., x_m) \tag{0.19}$$

is symmetric. This shows that the multilinear map

$$d^m f(a) : E^m \rightarrow F$$

is symmetric by(0.22) and (0.23) , so the proof is complete .■

0.4 Directional Derivatives

Definition 8 (Gâteaux Differential). *Let U be open in a p-normed space E $(0 < p \leq 1)$. If for a given $a \in U$ and $h \in E$, the limit*

$$\frac{\partial f}{\partial h}(a) = \lim_{\lambda \to 0} \left[\frac{f(a + \lambda h) - f(a)}{\lambda} \right] \tag{0.20}$$

*exists ,where $\lambda \in \mathbb{K}$, then f is said to be "***Gâteaux differentiable at a in the direction of h ";** *sometimes we write $\partial f(h, a)$ instead of $\frac{\partial f}{\partial h}(a)$. Also $\frac{\partial f}{\partial h}(a)$ is called the* **Gâteaux derivative of f at a in the direction h.**

Definition 9 *If f is Gâteaux differentiable at a in any direction, we say that f is "***Gâteaux differentiable at a** *" and the mapping*

$$\partial f(a) : h \in E \to \frac{\partial f}{\partial h}(a) \in F \tag{0.21}$$

*is called the "***Gâteaux differential" or Gâteaux derivative.**

Definition 10 *If f is Gâteaux differentiable at each point of U in the direction h, the mapping*

$$\frac{\partial f}{\partial h} : x \in U \to \frac{\partial f}{\partial h}(x) \in F \tag{0.22}$$

is called the " **derivative of f on U in the direction h ■.**

0.4.1 Gâteaux Differentials

We have remarked that the Gâteaux differentials may not be linear, and Gâteaux differentiability does not necessarily imply the continuity of the function.

In what follows we show that the super-differentiability is stronger than the Gâteaux differentiability (or directional differentiability).

Theorem 10 *Let E and F be p-normed and q-normed spaces respectively $(0 < p, q \leq 1)$. If*

$$f : : E \to F$$

is super-differentiable at $a \in E$, then f is Gâteaux differentiable at a and

$$\frac{\partial f}{\partial h}(a) = Df(a)h, \quad i.e.\ \partial f(a) = Df(a). \tag{0.23}$$

Proof. We have

$$\lim_{x\to 0} \parallel f(a+x) - f(a) - Df(a)x \parallel^{1/q} / \parallel x \parallel^{1/p} = 0.$$

Replace x with λh to get the desired result. In fact

$$\lim_{x\to 0} \parallel f(a+\lambda h) - f(a) - Df(a)(\lambda h) \parallel^{1/q} / \parallel \lambda h \parallel^{1/p}$$

$$= \frac{1}{\parallel h \parallel^{1/p}} \lim_{\lambda\to 0} \parallel \frac{f(a+\lambda h) - f(a)}{\lambda} - \frac{\lambda Df(a)(h)}{\lambda} \parallel^{1/q} = 0;$$

i.e.,

$$\lim_{\lambda\to 0} \left[\frac{f(a+\lambda h) - f(a)}{\lambda}\right] = Df(a)(h)$$

and hence

$$\frac{\partial f}{\partial h}(a) = Df(a)h.$$

■

Corollary 11 *Let* E *and* F *be p-normed and q-normed spaces respectively* $(0 < p, q \leq 1)$*. If* $f : U \to F$ *is super-differentiable at the point* $a \in U$ *, then*

$$\frac{df(a+\lambda h)}{d\lambda}\Big|_{\lambda=0} = \frac{\partial f}{\partial h}(a) \tag{0.24}$$

$$= \partial f(h, a).$$

That is , the usual derivative of $f(a+\lambda h)$ *with respect to* λ *at* $\lambda = 0$ *is equal to the Gâteaux derivative at* a *in the direction* h*.* ■

Proof. Define $g : V \to F$ by $g(\lambda) = f(a+\lambda h)$ where V is a neighborhood of 0 in $\mathbb{K}$. By chain rule we obtain the derivative

$$g'(0) = Df(a)h,$$

since $g'(0) = \lim_{\lambda\to 0}\left[\frac{g(\lambda)-g(0)}{\lambda}\right]$. Hence ,

$$g'(0) = \lim_{\lambda\to 0}\left[\frac{f(a+\lambda h) - f(a)}{\lambda}\right] = \frac{\partial f}{\partial h}(a) = Df(a)(h).$$

■

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