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THEORY OF POLYNOMIALS IN F-SPACES

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Abstract. Theory of polynomials plalys fundamental roles in most branches of Mathematical Sciences and their applications. In this paper we build up a new theory of polynomials in spaces which are not necessarily locally convex, and give some of their applications .

1. Introduction and Notation

During our work towards more foundations of Complex Analysis in non locally convex spaces we found ourself in a great need to a theory of polynomials between F-spaces, that is, without convexity condition (cf. references). For theory of polynomials in Banach spaces several mathematicians have worked with it, see for example$[17-25]$.

Multilinear maps play a fundamental role in our study of polynomials and other topics like Diœerential maps and Holomorphy. In [15] we present the basic facts about multilinear maps of p-Banach spaces and give conditions for a multilinear map to be continuous. We have also introduced the p-Banach space of all continuous multilinear maps ($0 < p \leq 1$). By a p-Banach space we mean a complete p-normed space. A vector space with a p-norm is called a p-normed space.

Our approach in this paper for the study of polynomials is the multilinear mappings between spaces which are not necessarily locally convex. Continuous polynomial mappings with respect to the given topologies on non locally convex spaces play a fundamental role in the study of Complex and Functional Analysis. For example polynomials are used to approximate all holomorphic mappings.They represent the simplest holomorphic functions. We deøne a homogenous polynomial on a p-normed space as the restriction of a symmetric multilinear map to be the diagonal one. In this paper we give several necessary and suŒcient conditions for a polynomial to be continuous (Sections 2,3).

A generalized universal constant appears relating the continuous m-homogeneous polynomials and the corresponding symmetric m-linear maps between locally bounded spaces (Section 3). For normed spaces it is due to Nachbin [26]. Further we have paid attention to study the

Banach-Steinhaus theorem for polynomials between locally bounded F-spaces (Section 4).

1.1. Symmetric Multilinear Maps. In this section we study the space of symmetric multilinear maps between Locally bounded Spaces. A locally bounded space is deøned by a p-norrm $(1 \geq p > 0)$, (see [?]), and it is called a p-normed space.

A p-norm on a vector space E over $\mathbb{K}$ is a mapping $||.||$ from E to $\mathbb{R}_+$ satisfying

(i) $||x|| = 0$ if and only if $x=0$

(ii) $||\lambda x|| = |\lambda|^p ||x||$, for every $\lambda \in \mathbb{K}, x \in E$

(iii) $||x+y|| \leq ||x|| + ||y||$, for every $x, y \in E$.

Let now E_i be p_i-normed spaces, $(1 \geq p_i > 0), (1 \leq i \leq m)$.

If the p_i-normed spaces $E_1, \cdots, E_m$ are equal to a p- normed space E, we denote,

$$L(^mE; F) = L(E_1, \cdots E_m; F).$$

the space of continuous maps between the product E^m and F.

An m-linear map $A : E^m \to F$ is called symmetric if

$$A(x_1, \cdots, x_m) = A(x_{\sigma(1)}, \cdots, x_{\sigma(m)}),$$

for any permutation σ of $(1, \cdots, m)$.

Let $L_{as}(^mE; F)$ denote the vector space of all algebraic symmetric m-linear maps of $E^{\,m}$ into F, and let

$$L_s(^mE; F) = L(^mE, F) \cap L_{as}(^mE, F).$$

Then it is easy to check that $L_s(^mE; F)$ is a closed subspace of $L(^mE; F)$ and hence $L_s(^mE; F)$ is a p-Banach space if F is a p-Banach space.

For $m = 0$ we write $L^0(E, F) = L_s(^0E, F) = F$, and for simplicity, when $F = \mathbb{K}$ we write

$$L(^mE) = L(^mE; \mathbb{K}), L_s(^mE) = L_s(^mE, \mathbb{K}).$$

For each m- linear map $A : E^m \to F$ we deøne $A_s : E^m \to F$ by

$$A_s(x_1, \cdots, x_m) = \frac{1}{m!} \sum_{\sigma} A(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) \tag{1}$$

where the summation is over the $m!$ permutation σ of $(1, \cdots, m)$. Hence A_s is a symmetric m-linear map, and it is called the symmetrization of A.

If $A \in L_s(^mE, F)$, then it is clear that $A = A_s$. We note that the map $A \to A_s$ is a continuous map of $L(^mE, F)$ onto $L_s(^mE; F)$ since

$$\|A_s\| \leq \|A\|.$$

1.2. Multilinear Formula. Let $A \in L_a(E_1, \cdots, E_m; F)$, $0 \leq n \leq m$, and $(x_1, \cdots, x_n) \in E_1 \times \cdots \times E_n$.

We deøne $A(x_1, \cdots, x_n)$ by the following formulas :
(1) If $n = m$, $A(x_1, \cdots, x_n) = A(x_1, \cdots, x_m)$.
(2) If $n < m$, $A(x_1, \cdots, x_n) : E_{n+1} \times \cdots \times E_m \to F$
is a mapping deøned by

$$(x_{n+1}, \cdots, x_m) \to A(x_1, \cdots x_n, x_{n+1}, \cdots, x_m).$$

That is, $A(x_1, \cdots, x_n)$ is an $(m-n)$-linear map of $E_{n+1} \times \cdots \times E_m$ into F.When $A \in L_{as}(^mE, F)$ and $x = x_1 = \cdots = x_n$, we write

$$Ax^n = A(x, \cdots, x). \tag{2}$$

We deøne, for convenience $Ax^\circ = a$. This shows that

$$Ax^n \in L_a(^{m-n}E; F) \quad \text{all } n,\ 0 \leq n \leq m.$$

Let $A \in L_a(^mE, F)$, $0 \leq k \leq m$, $x_1, \cdots, x_k \in E$ and $n_1, \cdots, n_k \in N$ with $n_1 + \cdots + n_k = n \leq m$. Then we deøne

$$Ax_1^{n_1} \cdots x_k^{n_k} = A(x_1, \cdots, x_1,\ x_2, \cdots, x_2, \cdots,\ x_k, \cdots, x_k)$$

where x_1 appears n_1 times, x_k appears n_k times, and so on.

Lemma 1.1. (Multinomial Formula) If $A \in L_a(^nE, F)$, then

$$A(x_1 + \cdots + x_k)^n = \sum \frac{n!}{n_1! \cdots n_k!} Ax_1^{n_1} \cdots x_k^{n_k} \tag{3}$$

where the summation is over all k-tuples $(n_1, \cdots, n_k)$ satisfying $n_1 + \cdots + n_k = n$.

Proof. The procedure is obvious for $n = 0$ and $n = 1$. We shall prove the lemma by induction. Assuming the formula is valid for a certain $n>1$, one can readily establish it for $n+1$. Note that if $A \in L_a(^{n+1}E; F)$ then one can write

$$A(x_1 + \cdots + x_k)^{n+1} = A(x_1 + \cdots + x_k)^n (x_1 + \cdots + x_k)^1$$

where in this case $n_1 + ... + n_k = n + 1$ in (3). ■

As a special case of Lemma 2.1, when $k = 2$, we have the following familiar Binomial Formula

$$A(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} Ax^k y^{n-k}. \tag{4}$$

1.3. Polynomials. Let E, F be vector spaces over the same øeld $\mathbb{K}$. A mapping $P : E \to F$ is said to be an m-homogeneous polynomial or a homogeneous polynomial of degree m if there exists an m-linear map $A : E^m \to F$ such that

$$P(x) = Ax^m \tag{5}$$

for all $x \in E$.

For $m = 1$, an m-homogeneous polynomial is simply a linear map of E into F. If $P : E \to F$ is an m-homogeneous polynomial, then $P(rx) = r^m P(x)$ for any $r \in \mathbb{K}$.

We shall denote by $P_a(^mE; F)$ the vector space of all m- homogeneous polynomials of E into F. (The index a in $P_a(^mE; F)$ is to indicate that the m- homogeneous polynomials in this space are not necessarily continuous). For convenience, we agree to write $F = P_a(^\circ E, F)$.

1.4. Polarization Formula. The following theorem provides us with an interesting formula. It relates m- homogeneous polynomials and symmetric m-linear maps.

Theorem 1.2. (Polarization Formula)

Let E and F be p-normed and q-normed spaces, and $A : E^m \to F$ be a symmetric m- linear map. Then

$$A(x_1, \cdots, x_m) = \frac{1}{m!2^m} \sum \epsilon_1, \cdots, \epsilon_m A(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)^m \tag{6}$$

where the summation is taken for all $\epsilon_i \in \{-1, +1\}, i = 1, \cdots, m$.

Proof. By the multinomial formula, we have

$$\begin{aligned} A(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)^m &= \sum_{(n_1, \cdots, n_m)} \frac{m!}{n_1! \cdots n_m!} A(\epsilon_1 x_1)^{n_1} \cdots (\epsilon_n x_m)^{n_m} \\ &= \sum \frac{m!}{n_1! \cdots n_m!} \epsilon_1^{n_1} \cdots \epsilon_m^{n_m} A x_1^{n_1} \cdots x_m^{n_m}. \end{aligned}$$

Multiply both sides by $\sum_{\epsilon_i=\pm 1} \epsilon_1 \cdots \epsilon_m$, we get

$$\sum_{\epsilon_i=\pm 1} \epsilon_1 \cdots \epsilon_m A(\epsilon_1 x_1 + ... + \epsilon_m x_m)^m$$
$$= m! \sum_{\epsilon_i=\pm 1} \frac{1}{n_1!...n_m!} \sum_{\substack{\epsilon_i=\pm 1 \\ 1\leq i\leq m}} \epsilon_1^{n_1+1}...\epsilon_m^{n_m+1} A x_1^{n_1}..x_m^{n_m}.$$

Clearly $\sum \epsilon_1^{n_1+1} \cdots \epsilon_n^{n_m+1} = 0$ if $n_i = 0$ for some $1 \leq i \leq m$. Since $\sum_{\epsilon_i=\pm 1} \epsilon_1^2 \cdots \epsilon_m^2 = 2^m$, we get the desired result;

i.e. $$\frac{1}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1 \\ 1\leq i\leq m}} \epsilon_1 \cdots \epsilon_m A(\epsilon_1 x_1 + \cdots + \epsilon_n x_m)^m = A(x_1, ..., x_m). \blacksquare$$

The following theorem shows the unique correspondence between the classes $P_a(^mE, F)$ and $L_{as}(^mE, F)$.

Theorem 1.3. If $P \in P_a(^mE, F)$, then there exists a unique symmetric m-linear map $A \in L_{as}(^mE, F)$ such that

$$P(x) = Ax^m$$

for all $x \in E$.

Proof. Since $P \in P(^mE; F)$, by deønition, $P(x) = Ax^m$ for some $A \in L_a(^mE, F)$. Then the symmetrization A_s of A satisøes

$$P(x) = A_s x^m.$$

The uniqueness of such a symmetric m-linear map is a consequence of the polarization formula. ■

We have just seen that there is a unique correspondence between the m- homogeneous polynomial P and the symmetric m-linear map A, so to emphasize this we write

$$P = \hat{A}.$$

The following theorem is self-evident.

Theorem 1.4. The mapping

$$A \in L_{as}(^mE, F) \to \hat{A} \in P_a(^mE, F)$$

is a vector space isomorphism.

Example 1.1. Let $E = \mathbb{K}$ and $A : E^m \to F$ be any m-linear map of the form

$$A(\lambda_1, \cdots, \lambda_m) = a\lambda_1 \cdots \lambda_m, \quad a \in F,$$

where $a\lambda_1 \cdots \lambda_m$ denotes $(\lambda_1, \cdots \lambda_m)a$. Therefore, an m-homogeneous polynomial $P : \mathbb{K} \to F$ is of the form

$$P(x) = a\lambda^m, \quad a \in F.$$

In this way, if $F = \mathbb{K}$, we get the classical m-homogeneous polynomial of $\mathbb{K}$ into $\mathbb{K}$. This example motivates and justiøes our deønition of m-homogeneous polynomial.▲

A mapping $P : E \to F$ is called a polynomial if there exists m and $P_k \in P_a({}^kE, F)$, $k = 0, 1, \cdots, m$ such that

$$P = P_0 + P_1 + \cdots + P_m.$$

The addition of this is pointwise, and if $P_m \neq 0$ we say that P is a polynomial of degree m.

The vector space of all polynomials from E to F with respect to pointwise vector operation will be denoted by $P_a(E, F)$.

We have the following result as extension to the classical one for normed spaces, (see [1],Ch.3).

Theorem 1.5. Let $P : E \to F$ be a non-zero polynomial of degree m between locally bounded F-spaces E, F. Then the representation

$$P = P_0 + P_1 + \cdots + P_m \tag{7}$$

is unique. .

Proof. It suŒces to show that ▮

$$P = P_0 + P_1 + \cdots + P_m = 0 \qquad \text{implies} \qquad P_0 = \cdots = P_m = 0.$$

Note that if $P(x) = 0$ for $x \in E$, then we have for any $r \in \mathbb{K}$

$$P(rx) = P_0(x) + rP_1(x) \cdots + r^m P_m(x) = 0.$$

Dividing through by r^m if $r \neq 0$ and making $r \to \infty$, we obtain $P_m = 0$. By induction we then get $P_0 = P_1 = \cdots P_{m-1} = 0$. ■

1.5. Continuous Polynomials. Let us denote by $P(^mE, F)$, the vector space of all continuous m-homogeneous polynomials from a p-normed space E to a q-normed space F with respect to all pointwise vector operations. We write $F = P(^\circ E, F)$.

It is clear that if $P \in P(^mE; F)$, then

$$\begin{aligned} & \sup\left\{\|P(x)\|/\|x\|^{mq/p};\ x \neq 0, \|x\| \leq 1\right\} \\ = & \sup\{\|P(x)\|;\ \|x\| \leq 1\} \\ = & \inf\{M \geq 0;\ \|P(x)\| \leq M\|x\|^{mq/p}\}. \end{aligned}$$

Let $\|P\|$ be the common value of these equalities.. Then $\|P\|$ is a q-norm on $P(^mE, F)$ if F is a q-normed space. This q-norm will be considered throughout the paper.

It is easy to see that

$$\|P(x)\| \leq \|P\|\|x\|^{mq/p} \quad . \tag{8}$$

(see [1]).

As a criterion for the continuity of an m-homogeneous polynomial we present the following theorem between p-normed E and q-normed space F $(1 \geq p, q > 0)$

Theorem 1.6. For $P \in P_a(^mE; F)$, let $A \in L_{as}(^mE; F)$ be such that $P = \hat{A}$. The following statements are equivalent.

(a) $A \in L_s(^mE; F)$.
(b) $P \in P(^mE; F)$.
(c) P is continuous at the origin.
(d) There exists a constant $M > 0$ such that

$$\|P(x)\| \leq M\|x\|^{mq/p}. \tag{9}$$

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$(d) are evident. ▮

We claim that (d) $\Rightarrow$ (a): If $\|P(x)\| \leq M$ for $\|x\| \leq 1$, it follows from the polarization formula that

$$\|A(x_1, \cdots, x_m)\| \leq M/m!$$

for $\|x_i\| \leq \frac{1}{m}$, $i = 1, \cdots, m$, and hence $A \in L_s(^mE; F)$. ■

2. Generalized Martin's Theorem

The following theorem extends Martin's theorem to locally bounded F-spaces.

Theorem 2.1. (Generalized Martin's theorem) For a p-normed space E and a q- normed space F, the mapping

$$A \in L_s(^mE;F) \to \hat{A} \in P(^mE;F)$$

is a vector space isomorphism and a homeomorphism of the ørst onto the second space. Moreover,

$$\|\hat{A}\| \leq \|A\| \leq \frac{m^{mq/p}}{m!} \|\hat{A}\|. \tag{10}$$

Proof. The assertion is trivially true for $m = 0, 1$. So we assume that $m > 1$. It is clear that the mapping is a vector space isomorphism, and it remains to show that the inequality (10) holds.

We have $\|\hat{A}\| \leq \|A\|$ since $\|\hat{A}\| \leq \|A\| \|x\|^{mq/p}$.
We use the polarization formula to get the other inequality

$$\begin{aligned}
\|A(x_1, \cdots, x_m)\| &\leq \frac{1}{m!2^m} \sum \|\hat{A}(\epsilon_1 x_1 + \cdots + \epsilon_n x_m)\| \\
&\leq \frac{1}{m!2^m} \sum \|\hat{A}\|(\|\epsilon_1 x_1 + \cdots + \epsilon_m x_m\|)^{mq/p} \\
&\leq \frac{1}{m!2^m} \sum \|\hat{A}\|(\|x_1\| + \cdots + \|x_m\|)^{mq/p} \\
&\leq \frac{1}{m!} \|\hat{A}\|(\|x_1\| + \cdots + \|x_m\|)^{mq/p}
\end{aligned}$$

since we have 2^m terms in the above summation. Hence if $\|x_i\| \leq 1, \ i = 1, \cdots, m$, we obtain

$$\|A\| \leq \frac{m^{mq/p}}{m!} \|\hat{A}\|. \tag{11}$$

■

Corollary 2.2. If F is a q-Banach space, then $P(^mE;F)$ is a q-Banach space.

2.1. The Generalized Nachbin's Universal Constant $\frac{m^{mq/p}}{m!}$.
The map $A \in L_s(^mE;F) \to \hat{A} \in P(^mE,F)$ considered above is not in general an isometry. We also notice that the coeŒcient $\frac{m^{mq/p}}{m!}$ in the above theorem depends only on m, p and q and not on the p-normed space E and the q-normed space F. Therefore we can consider $m^{mq/p}/m!$ as a universal constant relating the continuous m-homogeneous polynomials and the corresponding symmetric m- linear maps. Of course this constant turns out to be equal to $m^m/m!$ if E and F are p-normed spaces.

The following example shows that we cannot replace the constant $m^{mq/p}/m!$ by a smaller one

Example 2.1. Consider the F-space $E = l^p$ $(1 \geq p > 0)$ of all sequences $x = (x_n)$ of numbers $x_n \in \mathbb{K}$ such that $\|x\| = \sum_1^\infty |x_n|^p < \infty$. For $m > 0$, let $A : E^m \to \mathbb{K}$ be the m-linear map deøned by

$$A(x^1, \cdots, x^m) = x_1^1 x_2^2 \cdots x_m^m;$$

i.e. the product of the diagonal of

$$\begin{aligned} x^1 &= (x_1^1, x_2^1, \cdots, x_m^1, \cdots), \\ x^2 &= (x_1^2, x_2^2, \cdots, x_m^2, \cdots), \\ x^m &= (x_1^m, x_2^m, \cdots, x_m^m, \cdots). \end{aligned}$$

Then A is continuous. The symmetrization A_s of A is now given by

$$A_s(x^1, \cdots, x^m) = \frac{1}{m!} \sum_\sigma x_1^{\sigma(1)} \cdots x_m^{\sigma(m)}$$

where the summation is over all permutations σ of $\{1, 2, \cdots, m\}$.

We claim that $\|A_s\| = \frac{1}{m!}$. In fact

$$\|A_s(x^1, \cdots, x^m)\| \leq \frac{1}{m!} \sum_\sigma (|x_1^{\sigma(1)}| \cdots |x_m^{\sigma(m)}|) \leq \frac{1}{m!} \|x^1\|^{1/p} \cdots \|x^m\|^{1/p}.$$

Hence $\|A_s\| \leq \frac{1}{m!}$. But

$$A_s(e^1, \cdots, e^m) = \frac{1}{m!}.$$

where $e^i = (0, \cdots 0, 1, 0, \cdots)$. Therefore

$$\|A_s\| = 1/m! \quad . \tag{12}$$

Let $\hat{A}_s(x) = A_s(x, \cdots, x)$. Then $\hat{A}_s \in P(^mE; \mathbb{K})$ and

$$\hat{A}_s(x) = x_1 \cdots x_m$$

where $x = (x_1, \cdots, x_m, \cdots)$. Since the geometric mean of positive numbers is always less than or equal to the arithmetic mean, we have

$$\|\hat{A}_s\|^p = |x_1|^p \cdots |x_m|^p \leq \frac{1}{m^m}(|x_1|^p + \cdots + |x_m|^p)^m.$$

Thus $\|\hat{A}_s\| \leq 1/m^{m/p}$. If we take

$$x = (1/m^{1/p}, \cdots, 1/m^{1/p}, 0 \cdots),$$

where $1/m^{1/p}$ appears in the first m terms of x, we obtain

$$\|\hat{A}_s(x)\| = \frac{1}{m^{m/p}}. \tag{13}$$

This shows that $\|\hat{A}_s\| = \frac{1}{m^{m/p}}$,and hence,

$$\|A_s\| = \frac{m^{m/p}}{m!}\|\hat{A}_s\|$$

where $q = 1$ here ▲.

Remark 2.1. If E is a real Hilbert space and F is a Banach space then the mapping $A \in L_s(^mE, F) \to \hat{A} \in P(^mE, F)$ becomes an isometry. If $p = 1$ then the universal constant will equal $\frac{m^m}{m!}$ and the result is due to Nachbin [26].Also for $L^p(\mu), 1 \leq p$, it is due to Sarantopouls [28].

We generalize the polarization formula for the sake of using it in the next section.

2.2. Generalized Polarization Formula. For locally bounded spaces E, F, and $f : E \to F$, let

(14)

$$\varphi_m(f) = \frac{1}{m!2^m}\sum \epsilon_1 \cdots \epsilon_m f(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)\ ,\ x_1, \cdots, x_m \in E$$

where the summation is over all $\epsilon_k = 1$ or -1, $k = 1, \cdots, m$.

If $A_k : E^k \to F$ is a symmetric k-linear map and $P_k = \hat{A}_k$, then

$$\varphi_m(P_k) = \begin{cases} A_m(x_1, \cdots, x_m) & \text{if } k = m \\ 0 \quad \text{if } k < m. \end{cases} \tag{15}$$

We shall call $\varphi_m(f)$ the polarization of f with respect to $x_1, \cdots, x_m$. We have shown the formula for the case $k = m$ in Section 1. If $k < m$, a straightforward computation shows that $\varphi_m(P_k) = 0$ (see the proof of Theorem 1.2)

3. The Space $P(E, F)$

We are now concerned with continuous polynomials from a p-normed space E into a q-normed space F. Let $P(E, F)$ denotes the vector space of all continuous polynomials from E into F.

The following theorem provides some criteria for the continuity of a polynomial P.

Theorem 3.1. Let $P : E \to F$ be a polynomial of degree m from a p-normed space E into a q-normed space F such that

$$P = P_0 + P_1 + \cdots + P_m.$$

Then the following are equivalent :

(a) $P_0, P_1, \cdots, P_m$ are continuous.
(b) P is continuous.
(c) P is continuous at the origin.
(d) P is bounded on the unit ball.

Proof. The implications (a) $\Rightarrow$(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial.

It remains to show (d) $\Rightarrow$ (a). This will be shown by induction. We use the general polarization formula above. We note that

$$\varphi_m(P) = \varphi_m(P_0) + \cdots + \varphi_m(P_m) = \varphi_m(P_m)$$

or

$$\varphi_m(P) = A_m(x_1, \cdots, x_m),$$

where $\hat{A}_m = P_m$. Since P is bounded on the unit, we have $\|P(x)\| \leq M$ on the unit ball $\|x\| < 1$, for some $M > 0$. Now

$$\|A_m(x_1, \cdots, x_m)\| = \|\varphi_m(P)\| \leq \frac{1}{m!}\|P(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)\|.$$

Hence if $\|x_1\|^{1/p} + \cdots + \|x_m\|^{1/p} \leq 1$, then

$$\|A_m(x_1, \cdots, x_m)\| \leq \frac{M}{m!}$$

which shows that A_m is continuous and hence $P_m = \hat{A}_m$ is continuous.

As $P - P_m$ is also bounded on the unit ball, repeating the same argument as above, we can show that P_{m-1} is continuous. Inductively, therefore we conclude that $P_0, P_1, \cdots P_m$ are continuous. ■

4. Banach-Steinhaus Theorem for Polynomials

The following theorem extends Banach-Steinhaus type theorem to homogeneous polynomials between locally bounded F-spaces which are not necessarily locally convex.

Theorem 4.1. Let E be a p-Banach space, F be a q- normed space and (P_n) be a sequence in $P(^mE;F)$. If P is the pointwise limit of the sequence (P_n), then $P \in P(^mE;F)$.

Proof. Let $A_n \in L_s(^mE,F)$ be such that $\hat{A}_n = P_n$ where $P_n \in P(^mE;F)$. The polarization formula, (Section 4) implies that $\lim A_n(x_1,\cdots,x_m)$ exists at each point $(x_1,\cdots,x_m)$ of E^m. Let

$$A = \lim A_n.$$

∎

Then $A \in L_{as}(^mE,F)$.
We claim that $A \in L_s(^mE,F)$. Since

$$L(^mE,F) \simeq L(E,L(^{m-1}E;F)),$$

see [15],if we consider (A_n) as a sequence in $L(E;L(^{m-1}E;F))$, by the Banach-Steinhaus theorem for linear mappings, we obtain $A \in L((E;L(^{m-1}E,F))$.

Hence $A \in L_s(^mE,F)$ as $A_n \in L_s(^mE,F)$ for each n. Now we have

$$\begin{aligned} P(x) &= \lim P_n(x) = \lim \hat{A}_n(x) = \lim A_n(x,\cdots,x) \\ &= A(x,\cdots,x) = \hat{A}(x). \end{aligned}$$

That is, $P = \hat{A}$. This completes the proof. ∎

Corollary 4.2. Let E and F be as in the above theorem and (P_n) be a sequence in $P(E;F)$, whose elements of degree k. That is,

$$P_n = P_{n0} + P_{n1} + \cdots + P_{nk}, \quad P_{nk} \neq 0, \ n = 1,2,.$$

If P_i is the pointwise limit of the sequence (P_{ni}) for each $(1 \leq i \leq k)$, then $P = \sum_{i=1}^{k} P_i \in P(E;F)$, and of degree k.

Proof. By assumption all elements of (P_n) have the same degree k.
P_n is the sum of ønite homogeneous polynomials:

$$P_n = P_{n0} + P_{n1} + \cdots + P_{nk}, \quad P_{nk} \neq 0, \ n = 1,2,..$$

where for each n, P_{nm} are m-homogenous polynomials, $(1 \leq m \leq k)$. Hence by the above theorem each P_{nm} tends to a continuous polynomial

P_m. Consequently, by Theorem 3.1(b), the sums (P_n) will tend to the sum $P \in P(E, F)$ ■.

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