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On Dynamical Behaviour of FitzHugh-Nagumo Systems

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by

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Abstract

In this thesis a class of FitzHugh-Nagumo system is studied. By using the theory of Lyapunov coefficient to analyze Hopf and Bautin bifurcation it is shown that at most two limit cycles can bifurcate from the origin in this case. Further it is shown that there exists choices of parameters such that the maximum number of bifurcating limit cycles is obtained and in this case the inner cycle is unstable while the outer is stable. In the certain case when one of the parameters are assumed to be very small, sufficient conditions are presented ensuring the existence of a unique stable limit cycle. By using a theorem by Lefschetz conditions are given ensuring the existence of at least one stable limit cycle. Also some conditions, apart from the well known Bendixson's criteria, on the parameters giving non-existence of limit cycles are presented. The thesis also contains a complete saddle-node bifurcation analysis based on the Center Manifold Theorem as well as a Bogdanov-Takens bifurcation analysis though not equally complete. Finally the thesis contains sufficient conditions for bounded solutions and a discussion about coupled system of the same class of FitzHugh-Nagumo system.

On dynamical Behavior of FitzHugh-Nagumo Systems
–A summery with some further discussions

In this thesis we deal with a class of polynomial systems

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y) \end{cases} \tag{1}$$

with non-zero parameters $A, B, C, \delta, \varepsilon, \lambda$ and I being an external force which can be a function of t.

The main results

- The main theorem of this thesis is Theorem 5.2 in Paper II. It states that at most two limit cycles can bifurcate form the origin via Hopf bifurcation. Further, it is shown that there exists parameters such that this upper bound is obtained and that in this case the inner cycle is unstable while the outer cycle together with the origin are stable. The proof uses the theory of Lyapunov coefficients and relies heavily on a theorem by Andronov from the 1960's, see [2].
- In the certain case when one of the parameters are assumed to be very small, singular perturbation theory is applied to give sufficient conditions ensuring existence of a unique stable limit cycle.
- Conditions on existence of at least one stable limit cycle are given based on a theorem by Lefschetz. Also some conditions, apart from the well known Bendixson's criteria, on the parameters giving non-existence of limit cycles are presented.
- A complete saddle-node bifurcation analysis based on the Center Manifold Theorem as well as a Bogdanov-Takens bifurcation analysis is presented. In the saddle-node bifurcation, a bifurcation curve is provided.
- It also contains sufficient conditions for bounded solutions and a discussion about coupled system of the same class of FitzHugh-Nagumo system.

Motivation

The FitzHugh-Nagumo system described in (1) is commonly referred to as FitzHugh-Nagumo model because it was based on the system independently used by FitzHugh and Nagumo in the beginning of 1960s "to expose to view part of the inner working mechanism of the Hodgkin-Huxley equations", a popular model in study of neurophysiology since 1952. Later this system has been used in various modelling of biological behavior, ranging from e.g. neurophysiology to modeling of active fibre for cardiac muscle and to implementation of neuron model, see e.g. [3, 7, 5, 8, 15, 20, 24] and references therein.

In order to make our motivation clear, we briefly review the development from Hodgkin-Huxley model to FitzHugh-Nagumo simplified model and current research of our interests. Hodgkin-Huxley model for the action potential of a space clamped squid axon is defined by the four dimensional vector field

$$
(2)\qquad \begin{cases} \dot{v} = I - \left[120m^3h(v+115) + 36n^4(v-12) + 0.3(v+10.599)\right] \\ \dot{m} = (1-m)\Psi\left(\dfrac{v+25}{10}\right) - m\left(4\exp\dfrac{v}{18}\right) \\ \dot{n} = (1-n)0.1\Psi\left(\dfrac{v+10}{10}\right) - n\left(0.125\exp\dfrac{v}{80}\right) \\ \dot{h} = (1-h)0.07\exp\left(\dfrac{v}{20}\right) - \dfrac{h}{1+\exp\frac{v+30}{10}} \\ \Psi(s) = \dfrac{s}{\exp(s)-1} \end{cases}
$$

with variables (v, m, n, h) that represent membrane potential, activation of a sodium current, activation of a potassium current, and inactivation of the sodium current and a parameter I that represents injected current into the space-clamped axon. Although there are improved models the Hodgkin-Huxley model remains the paradigm for conductance-based models of neural system. FitzHugh was the first investigator to apply qualitative phase-plane methods to understanding the Hodgkin-Huxley model. To make headway in gaining analytic insight, FitzHugh first considered the variables that change most rapidly, viewing all others as slowly varying parameters of the system. In this way he derived a reduced two-dimensional system that could be viewed as a phase plane. Note that the voltage convention adopted by FitzHugh $V = v_{\text{out}} - v_{\text{in}}$ is opposite to what subsequently became entrenched in scientific literature.

From the Hodgkin-Huxley equations FitzHugh noticed that the variables V and m change more rapidly than h and n, at least during certain time intervals. By arbitrarily setting h and n to be constant we can isolate a set of two equations which describe a two-dimensional (V, m) phase plane. The elegance of applying phase-plane methods and reduced systems of equations to this rather complicated problem should not be underestimated. Similar ideas are used nowadays in many biological problems, one of the examples is dynamical analysis of cell cycle regulation, e.g. [25].

In 1961 FitzHugh proposed to demonstrate that the Hodgkin-Huxley model belongs to a more general class of systems that exhibit the properties of excitability and oscillations. As a fundamental prototype, the van der Pol oscillator was an example of this class, and FitzHugh therefore used it (after suitable modification). A similar approach was developed independently by Nagumo in 1962. FitzHugh proposed the following equations:

$$
\begin{aligned} \dot{u} &= c[w + u - u^3/3 + I], \\ \dot{w} &= -(u - a + bw)/c. \end{aligned}
$$

In these equations the variable u represents the excitability of the system and could be identified with voltage (membrane potential in the axon); w is a recovery variable, representing combined forces that tend to return the state of the axonal membrane to rest. Finally I is the applied stimulus that leads to excitation (such as input current), or rectangular pulses. In order to obtain suitable behavior, FitzHugh made the following assumptions about the constants a, b and c:

$$
1 - 2b/3 < a < 1, \quad 0 < b < 1, \quad b < c^2.
$$

Today there are variant formulations of this system, used in a variety of biological oscillations. The meaning of the variables and parameters are different from the original FitzHugh-Nagumo's. The formulation we work on is taken from [8] which is widely used in literature.

A short but rather complete description of physiology behind the biological neuron and the corresponding derivation of the Hodgkin-Huxley model and its simplified version, the FitzHugh-Nagumo model, can be found in [20].

The thesis attempts to describing different dynamical behavior exhibited by (1), in terms of the six parameters, as complete as possible, and therefore to gaining analytic insight of this widely used system as much as possible. Although the FitzHugh-Nagumo system is a well-studied object (see e.g. [14, 16, 26, 29, 29, 32]), there are several reasons that motivate the current study.

It is of mathematical interest to know the number of limit cycles for a polynomial system although we do not have ambition to solve this problem completely. A classical solved example is van der Pol equation without external force to which we know that there is only one stable limit cycle. It is also of interest to give a satisfactory picture of how bifurcation takes place and which parameters (of the six) play role in the bifurcation. According to our knowledge, the parameters are essential in biology and different application areas need different parameters. This is one of the reasons we do not *a priori* assume the values of some parameters.

If the above-mentioned problems are considered to be more difficult, we want to mention that there may not exist a definite answer to some seemingly innocent questions, for example, there is still no definite answer, in some parameter settings, to the question that the solution of FitzHugh-Nagumo equations converges to a fixed point or to a limit cycle [16].

Since the FitzHugh-Nagumo model, defined by (1), is used for investigation of a single neuron, in reality we have to study the interconnection and coupling of neurons. In other words, a chain of such systems will be used in more realistic models for instance, [21]. Therefore we believe that a full description of dynamical behavior of the FitzHugh-nagumo system will benefit to understanding more complicated systems based on the FitzHugh-Nagumo system, see "A Case Study on Weakly Coupled FitzHugh-Nagumo Oscillators".

It is most popular in doing dynamical analysis using computer simulations (even with little analysis behind), and research areas such as computational biology is well-established nowadays. Nevertheless some precautions have to be kept in mind. In [16], it was noted that certain bifurcation could take place in a very narrow interval (of the magnitude of 10^{-7} of values of I). It is clear that a rigorous mathematical analysis, if possible, is highly demanded in such a case.

Mathematical analysis behind the results

Now we summarize the approaches to our main results. To make our point clear we shall also discuss some relevant issues during the description. Since it is shown that the solutions are bounded except a bounded set, using Lyapunov function, the questions of existence of Limit cycles, bifurcations and related issues are of interesting.

Existence of limit cycles. The main feature of the FitzHugh-Nagumo model is that it is easier to deal with meanwhile captures the periodic oscillations, the most important phenomenon of the Hodgkin-Huxley. This is the reason for why we are interested in answering question of existence and non-existence of limit cycles and their bifurcation in terms of the parameters. Although many computer simulations indicate the existence of limit cycles, it is often difficult to prove it analytically. Moreover, some cautions must be exercised if a simulation is involved. We shall come back to this point in our conclusion.

The reason why it is difficult to prove the existence of limit cycle is that there is no unique way to deal with the problem. In some situations we can find the solution curve in the phase plane and prove that the curve indeed is closed. Volterra-Lotska population model is such an example. The drawback of this method is that we are not always able to solve the equations.

A common technique to show the existence of limit cycles is applying Poincaré-Bendixson theory and construction of closed curves that bound a limit cycle, e.g. [19]. Based on results in [19] we show in Paper I that for $\lambda + \varepsilon\delta > 0$ the FitzHugh-Nagumo system has at least one stable limit cycle where $A = B = C = 1$. It is worth pointing out that this construction is sometimes very difficult, in particular when limit cycles are close to each other and the scales of variables are small.

By Bendixson's criterion and the criterion for a type of Liénard system, e.g. [30], we are able, in Paper I, to tighten the bound of the parameter regions that ensure non-existence of limit cycles for (1) with $A = B = C = 1$ and without external inputs in comparison with the result in [16].

Although the following topics belong to the same theme we present them in a separate subsection due to their own interest.

Discontinuous periodic solution. Our further study in existing biological relevant literature shows that some of the parameters in the FitzHugh-Nagumo system are very small, for example, ε. This brought our attention to singular perturbation theory, in particular, the so-called discontinuous periodic solution. There is a plenty of excellent Russian literature, see [22, 23, 27]. In our concrete problems we have to reformulate some definitions and results provided in the literature. These are carried out in Section 2 of Paper II. Once the theory is ready for use it is rather simple to find necessary and sufficient conditions to existence of discontinuous periodic solution, See Section 4, Paper II. Furthermore, we prove under some assumptions that for $\varepsilon \ll 1$ the system has a family of limit cycles and these limit cycles are unique for each such ε and the limit of these cycles, as $\varepsilon \to 0$, is the discontinuous periodic solution of the system.

Bifurcations. In this thesis we try to portrait a complete picture of system bifurcations. We study Hopf-Andronov bifurcation, Bautin (or generalized Hopf) bifurcation, saddle-node bifurcation and Bogdanov-Takens (or double-zero) bifurcation. All these have to do with the situation where the linearized system matrix at a fixed point, called $\mathcal{A}$, has at least one eigenvalue with zero real part. The Hopf type of bifurcation occurs when the eigenvalues of $\mathcal{A}$ has a pair of purely imaginary eigenvalues. Saddle node bifurcation appears when one (just one) eigenvalue of $\mathcal{A}$ is zero, and Bogdanov-Takens bifurcation occurs if two eigenvalues are zero.

As a result of Hopf type of bifurcation we are able to show that limit cycles bifurcation from the origin are at most two. The main tool we use is Lyapunov coefficients. To computational purpose we also provide a method to calculate the first and second Lyapunov coefficients which is more efficient in our cases. Note that it is the sign of the Lyapunov coefficients that is important (not the exact expressions).

A complete analysis on saddle node bifurcation is based on the Center Manifold Theorem. Bogdanov bifurcation is also studied, and we prove the existence of smooth bifurcation curve. It is interesting to note that the analysis of Hopf type of bifurcation is rather complete. Here is an example of the bifurcation diagram (from the fixed point which is the origin), where we choose $A = B = C = 1$. Let H be the curve corresponding to the Hopf bifurcation. Along this curve the fixed point, the origin, has eigenvalues $\lambda_{1,2} = \pm i$. The Bautin point (that is the parameters such that the first Lyapunov coefficient becomes zero) separates two branches H_+ and H_- corresponding to a Hopf bifurcation with positive and with negative Lyapunov coefficient respectively (sub-critical and super- critical). A stable limit cycle bifurcates from the origin if we cross H_- from left to right, while an unstable cycle appears if we cross H_+ in the opposite direction. The cycles collide and disappear on the curve T corresponding to a non-degenerate fold bifurcation of the cycles. Along the curve the system has a critical limit cycle. This curve divide the parameter plane into three regions: Region 1: the system has a single stable fixed point and no cycles; Region 2: a unique and stable limit cycle and an unstable fixed point; Region 3: a stable fixed point, and two limit cycles, the outer cycle is stable and the inner cycle (between the outer cycle and the fixed point) is unstable. We place these three regions counterclockwise.

Now let us start our tour around the Bautin point counterclockwise, starting at a point in Region 1, crossing the Hopf bifurcation boundary H_- from Region 1, where the system has a single stable fixed point and no cycle, to Region 2 implies the appearance of a unique and stable limit cycle, which survives when we enter to Region 3. Crossing the Hopf bifurcation boundary H_+ creates an extra unstable cycle inside the first one, while the fixed point regains its stability. Two cycles of opposite stability exists inside Region 2 and disappear at the curve T through a fold bifurcation that leaves a single stable fixed point, thus completing the circle.

Conclusions and further remarks

Finally we will conclude this summary by some remarks on our results and further research investigation.

There are still many problems left unclear. Here is a short list of such questions. How does the external force effect the bifurcation diagram? Is there any unpredictable strange behavior when external force is introduced? How are parameter sets related in different bifurcation e.g. the Hopf bifurcation and the Bogdanov-Takens bifurcation? How do we do a satisfactory analysis in case there are three fixed points?

No doubt a further analysis on the coupled FitzHugh-Nagumo systems is desired. In this case, more complicated bifurcations occur. Since this is a four dimensional system, chaotic behavior may be expected too. This is not an easy issue and computer simulations will be of great help in further mathematical analysis. We believe that the dynamics of this system is very rich.

Now we turn to the question concerning Hudgkin-Huxley model and its simplification by FitzHugh and Nagumo (1). Since the birth of Hodgkin-Huxley equation in neurophysiological modelling, researchers have made considerable effort to try to analyze this system in different way. Extensive efforts have been made to discover chaos in many physical and biological systems including neural systems. Chaotic solutions to the Hodgkin-Huxley equations with periodic forcing [1] and greatly altered parameters [6] have been discovered but not in the original Hodgkin-Huxley model with its original parameters. However, (highly unstable) chaotic solutions were demonstrated numerically by systematic methods in [12], in the Hodgkin-Huxley model with its original parameters, although the existence of chaos was not claimed theoretically. If this numerical evidence is correct, then there is a degree of unpredictability about how the system will respond to stimulation. Here we do not discuss further what other consequences there would be. But one natural question is, does FitzHugh-Nagumo system with constant stimulus give rise chaotic solutions? And do these chaotic solutions (if they are) correspond to the numerical evidence in the original Hodgkin-Huxley model by [12]? What is the implication of these discoveries? Recall that this could happen in a fine scale, which in turn require fine numerical methods. Is there chaos in the brain?

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Paper I

On Existence and Nonexistence of Limit cycles for FitzHugh-Nagumo Class Models*

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Dedicated to Clyde Martin on the occasion of his 60th birthday.

Summary. In this paper we discuss the existence and non-existence of limit cycles of FitzHugh-Nagumo class models. The purpose is to clarify some unclear facts in the literature. We show also that this class of model exhibits double cycle bifurcation in addition to Andronov-Hopf bifurcation.

1 Introduction

In this paper we consider FitzHugh-Nagumo class models common in literature:

$$\begin{cases} \dfrac{du}{dt} = -Cw + Au(B-u)(u-\lambda) + I \\ \dfrac{dw}{dt} = \varepsilon(u - \delta w) \end{cases} \tag{1}$$

where the variable u is the negative of the membrane potential, w is the quantity of refractoriness, and I is the magnitude of stimulating current, the parameter A is to scale the amplitude of the curve, the parameter C affects the coupling strength and ε is added to more easily control the speed of one variable relative to the other. The choice of these parameters together with other three would produce oscillations which are of primary interests in biological contexts. The oscillators produced by system (1) are called *FitzHugh-Nagumo type oscillators* and are common to many biological mechanism at the cellular level. In addition to producing oscillations in the barnacle muscle, the same dynamical structures will appear in mechanistic models of insulin secretion and Ca^{2+} oscillations, see details in [3].

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The system defined by (1) is originated from the simplified Hodgkin-Huxley nerve systems, the famous model of the squid giant axon, see e.g. [2, 3, 8]:

$$\begin{cases} \dfrac{du}{dt} = w - \dfrac{u^3}{3} + u + I \\ \dfrac{dw}{dt} = \rho(a - u - bw) \end{cases} \tag{2}$$

where the parameters ρ, a, b are assumed to satisfy the conditions: $b \in (0, 1)$, $a \in \mathbb{R}$, $\rho > 0$.

The existence and nonexistence of limit cycles for the nerve system (2) were investigated e.g. in [4, 10, 12] in case $I = 0$. It was shown in [9] that the dynamical system (1) with $\varepsilon = C = B = 1$ exhibits a rich structure of bifurcation when $I \neq 0$.

Although the systems (2) and (1) are polynomial systems of same degree, the dynamical behavior can be different, see Section 0.2. Moreover, it is not easy to show under which circumstances there are limit cycles and the question of how many limit cycles the systems have is also left open. Many attempts have been made in simulations in case a current input I is introduced. The purpose of this paper is to give a new criterion under which the system (1) has no limit cycles with three fixed parameters $B = C = \varepsilon = 1$ and with the quantity $I = 0$. Moreover, we shall give a rather complete phase portrait and bifurcation diagram for the system (1) with $A = B = C = 1$, and we demonstrate that this system exhibits the double-cycle bifurcation in addition to the Andronov-Hopf bifurcation.

The paper is organized as follows. In Section 0.2, we point out the differences between the two polynomial systems mentioned above. Then we reformulate (1) to two different types of Liénard equations. Then we analyze the nonexistence limit cycles in Section 0.3. In Section 0.4 we give a proof on existence of limit cycles for the system with a unique unstable fixed point, and we will show that the system undergoes Andronov-Hopf bifurcation (which is well-known), Bautin (or generalized Andronov-Hopf) bifurcation and double cycle bifurcation. And finally we conclude the paper by some further remarks.

2 Preliminaries

In this section we review the theory that is relevant in our study. First consider the system (1) with $B = C = \varepsilon = 1$

$$\begin{cases} \dfrac{du}{dt} = - w + Au(u - \lambda)(1 - u) \\ \dfrac{dw}{dt} = u - \delta w \end{cases} \tag{3}$$

where $0 < \lambda < 1$ and $A, \delta > 0$. The choice of these three parameters is motivated by a desire to get a finer criterion of non-existence of limit cycles than that stated in [9].

A straightforward calculation shows that the system has origin as its only fixed point if and only if

$$(1-\lambda)^2 - \frac{4}{\delta A} < 0 \tag{4}$$

Remark 1. Note that (3) can have either one fixed point as stated above, or two fixed points, origin and $(-(\lambda+1)/2, -(\lambda+1)/(2\delta))$, if $(1-\lambda)^2 - \frac{4}{\delta A} = 0$, or three fixed points, origin and $(u_\pm, w_\pm)$, where

$$u_\pm = -\frac{\lambda}{2} \pm \sqrt{(\lambda-1)^2 - \frac{4}{A\delta}}, \quad w_\pm = \frac{u_\pm}{\delta}$$

if $(1-\lambda)^2 - \frac{4}{\delta A} > 0$. However, (2) has only one fixed point (u_I, w_I) for each $I \in \mathbb{R}$:

$$u_I = \sqrt[3]{(3(I+\frac{a}{b}) + \sqrt{9(I+\frac{a}{b})^2 + 4(\frac{1}{b}-1)^3})/2} + \\ + \sqrt[3]{(3(I+\frac{a}{b}) - \sqrt{9(I+\frac{a}{b})^2 + 4(\frac{1}{b}-1)^3})/2}$$

and

$$w_I = \frac{a - u_I}{b}$$

under the assumptions.

Remark 2. Let (u_e, w_e) be a fixed point, and $h(u) = u(u-\lambda)(1-u)$. It is not hard to show that

(i) if $A\delta h'(u_e) < 1$, then (u_e, w_e) is locally asymptotically stable, for $Ah'(u_e) < \delta$ and a rebeller for $Ah'(u_e) > \delta$;
(ii) if $A\delta h'(u_e) > 1$, then (u_e, w_e) is a saddle point;
(iii) if $A\delta h'(u_e) = 1$, then (u_e, w_e) is unstable for $Ah'(u_e) > \delta$.

Now we make a variable change, to transform the system to a special type of Liénard system, as follows

$$\begin{cases} x = -u, \\ y = w + \delta x. \end{cases}$$

This yields the Liénard system

$$\begin{cases} \dfrac{dx}{dt} = y - F(x) \\ \dfrac{dw}{dt} = -g(x) \end{cases} \tag{5}$$

where

$$\begin{cases} F(x) = x(\delta + A(x+\lambda)(x+1)) \\ g(x) = x(1 + \delta A(x+\lambda)(x+1)) \end{cases} \tag{6}$$

Since the Liénard systems are well-studied planar polynomial systems, our idea is to apply the known results for this system in our particular case. First we note that (4) coincides with the condition

$$xg(x) > 0, \qquad \forall x \neq 0. \tag{7}$$

It is also obvious that

$$F(0) = 0. \tag{8}$$

Clearly, the functions F and g are continuous functions on $\mathbb{R}$ satisfying the Lipschitz condition. Let

$$G(x) = \int_0^x |g(\xi)| d\xi,$$

and

$$M = \min \left\{ \int_0^{\infty} g(x)dx, \int_0^{-\infty} g(x)dx \right\}.$$

It is easy to show that $M = \infty$ and $G(x)$ is strictly increasing. Therefore the inverse of G exists and we denote it by G^{-1}. In the sequel we shall make use of the following theorem for the Liénard systems.

Theorem 1 ([11]). *Suppose that the parameters are chosen such that the origin is the unique fixed point,* $xg(x) > 0$, $\forall x \neq 0$ *and*

$$F(G^{-1}(-w)) \neq F(G^{-1}(w)), \qquad \forall w > 0. \tag{9}$$

Then (5) *has no limit cycles.*

When nonlinear waves in the nerve or muscle fibers or in the heart collide with each other, they mutually annihilate. There are, however, cases where the experiment and theory have shown that the inelasticity of the collisions is not that drastic. [1] considers the collision properties of nonlinear waves in an excitable medium of FitzHugh-Nagumo type, which is paradigmatic to account for quite a variety of biological, biochemical and neurobiological phenomena. It shows that the system

$$\begin{cases} \dfrac{du}{dt} = -w - u(u-1)(u-\lambda) \\ \dfrac{dw}{dt} = \varepsilon(u - \delta w) \end{cases} \tag{10}$$

exhibits bistability for certain parameters, that is one asymptotically stable fixed point and one stable limit cycle coexist. Note that we allow the parameter

λ to be negative. In order to investigate the nonexistence of limit cycles we make a variable change as before to get the type of Liénard system shown (5). Let $x = -u$ and $y = w + \varepsilon\delta x$: Then (10) is transformed into the Liénard system (5) with

$$F(x) = x^3 + (\lambda + 1)x^2 + (\lambda + \varepsilon\delta)x$$
$$g(x) = \varepsilon\delta(x^3 + (\lambda + 1)x^2 + (\lambda + \frac{1}{\delta})x$$

In order to prove the existence of limit cycles, we need another type of Liénard system to be able to apply a theorem by Lefschetz [7]. Now we change the variables u and w to

$$\begin{cases} x =w \\ y =\varepsilon(u - \delta w) \end{cases}$$

the system transforms to

$$\begin{cases} \dfrac{dx}{dt} =y \\ \dfrac{dy}{dt} =- f(x,y)y - \tilde{g}(x) \end{cases} \tag{11}$$

where

$$f(x,y) = \varepsilon(\frac{y^2}{\varepsilon^3} + \frac{3\delta x - (1+\lambda)}{\varepsilon^2}y + \frac{3\delta^2}{\varepsilon}x^2 - \frac{2\delta(1+\lambda)}{\varepsilon}x + \delta + \frac{\lambda}{\varepsilon}) \tag{12}$$
$$\tilde{g}(x) = \varepsilon(\delta^3x^3 - (1+\lambda)\delta^2x^2 + (\lambda\delta + 1)x) \tag{13}$$

Before stating the theorem we introduce the function $\tilde{G}(x)$ defined by

$$\tilde{G}(x) = \int_0^x \tilde{g}(s)ds = \varepsilon(\frac{\delta^3}{4}x^4 - \frac{(1+\lambda)\delta^2}{3}x^3 + \frac{1+\lambda\delta}{2}x^2)$$

Theorem 2 ([7]). *If the following conditions are fulfilled the system* (11) *has at least one stable limit cycle:*

1. *The origin is the only critical point and it is unstable.*
2. *$f(x,y)$ and $\tilde{g}(x)$ are continuous and satisfies a Lipschitz condition $\forall x, y \in \mathbb{R}$.*
3. *$|\tilde{g}(x)| \to \infty$ as $|x| \to \infty$ and $\tilde{g}(x) > 0$ for $x > N$ for some $N > 0$.*
4. *$\frac{\tilde{g}(x)}{\tilde{G}(x)} = O(\frac{1}{|x|})$.*
5. *$\exists M, m, n > 0$ such that $f(x,y) \geq M$ for $|x| \geq n$ and $f(x,y) \geq -m$ for $|x| \leq n$.*

3 Analysis on nonexistence of limit cycles

Now we turn to the analysis of limit cycles of (3).

Proposition 1. *Suppose the origin is the only fixed point and*

$$\left(\lambda - \frac{1}{2}\right)^2 + \frac{3}{4} - \frac{3\delta}{A} < 0. \tag{14}$$

Then the system defined by (3) *has no limit cycles.*

Proof. By Bendixson's criterion we know that the system has no limit cycles if

$$\begin{aligned}&\frac{\partial}{\partial u}(-w - Au(u-1)(u-\lambda)) + \frac{\partial}{\partial w}(u - \delta w)\\ =& -3Au^2 + 2A(\lambda+1)u - \lambda A - \delta\end{aligned}$$

is not equal to zero and does not change sign. This is equivalent to the discriminant of the above polynomial of u is negative, which is in turn (14). □

From now on we assume that the parameters violate (14). To give a finer criterion we turn our interest to the properties of the function F. A straightforward calculation shows that $F(x)$ has three real zeros if and only if

$$A(1-\lambda)^2 - 4\delta > 0 \tag{15}$$

Denote the three zeros of F as

$$\begin{cases}\alpha = 0\\ \beta = -\dfrac{1+\lambda}{2} + \dfrac{\sqrt{A(1-\lambda)^2 - 4\delta}}{2A}\\ \gamma = -\dfrac{1+\lambda}{2} - \dfrac{\sqrt{A(1-\lambda)^2 - 4\delta}}{2A}\end{cases}$$

Clearly $\gamma < 0$. A short calculation shows that $\beta < 0$:

$$-\frac{1+\lambda}{2} + \frac{\sqrt{A(1-\lambda)^2 - 4\delta}}{2A} < -\frac{1+\lambda}{2} + \frac{\sqrt{A^2(1-\lambda)^2}}{2A} = -\lambda < 0$$

To be able to use the Theorem 1, we will show that (9) is satisfied. First we show that the following proposition holds.

Proposition 2. *The condition* (9) *is equivalent to*

$$\big(G(\alpha(\eta)) + G(\beta(\eta)) \neq 0\big) \wedge \big(G(\alpha(\eta)) + G(\gamma(\eta)) \neq 0\big) \quad \forall \eta \in (0, \eta^*], \tag{16}$$

where $\eta^* = F(\overline{x})$, *and* $\overline{x}$ *and* $\underline{x}$ *are the local maximum and local minimum of the function* F, *which are*

$$\underline{x} = -\frac{1}{3}\left((\lambda+1) - \sqrt{(\lambda+1)^2 - 3(\lambda + \frac{\delta}{A})}\right)$$

$$\overline{x} = -\frac{1}{3}\left((\lambda+1) - \sqrt{(\lambda+1)^2 - 3(\lambda + \frac{\delta}{A})}\right)$$

and $\gamma \leq \overline{x} \leq \beta \leq \underline{x} < \alpha$.

Proof. This is an immediate consequence of the fact that $\gamma \le \beta < \alpha = 0$ and the function G is strictly increasing. $\square$

Next we prove

Proposition 3. *If*

$$6 - 3\delta^2 - \delta A(\lambda + (1-\lambda)^2) > 0 \tag{17}$$

then condition (16) *is equivalent to*

$$\begin{aligned}&(6 - 3\delta^2 - \delta A(\lambda + (1-\lambda)^2))(\alpha(\eta) + \beta(\eta))\\ <& - 3\delta\eta + \delta(1+\lambda)(\delta + A\lambda), \quad \forall \eta \in (0, \eta^*].\end{aligned} \tag{18}$$

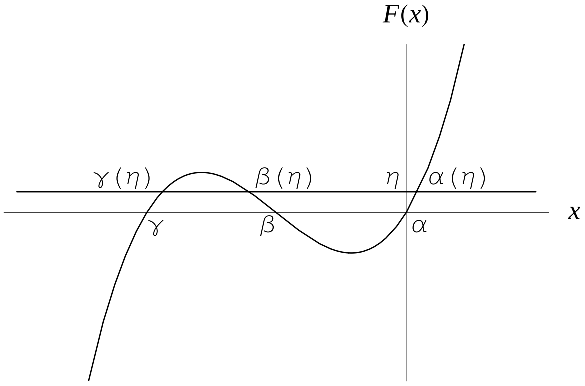

Fig. 1. The zeros of the function F

Proof. From (6)

$$G(x) = \begin{cases} \dfrac{x^2}{12}(3\delta A x^2 + 4\delta A(1+\lambda)x + 6(1+\delta A\lambda), & x \ge 0,\\[2ex] -\dfrac{x^2}{12}(3\delta A x^2 + 4\delta A(1+\lambda)x + 6(1+\delta A\lambda), & x < 0. \end{cases} \tag{19}$$

Using

$$0 = F(\alpha(\eta)) - \eta = A\alpha^3(\eta) + A(1+\lambda)\alpha^2(\eta) + (\delta + \lambda A)\alpha(\eta) - \eta$$

and the similar computation for $\beta(\eta)$ and $\gamma(\eta)$ we get

$$\begin{aligned}
G(\alpha(\eta)) =& \frac{1}{12}((6-3\delta^2-\delta A(\lambda+(1-\lambda)^2))\alpha^2(\eta)+ \\
&+(3\delta\eta-\delta(1+\lambda)(\delta+A\lambda))\alpha(\eta)+\delta(1+\lambda)\eta), \qquad (20)\\
G(\beta(\eta)) =& \frac{1}{12}((6-3\delta^2-\delta A(\lambda+(1-\lambda)^2))\beta^2(\eta)+ \\
&+(3\delta\eta-\delta(1+\lambda)(\delta+A\lambda))\beta(\eta)+\delta(1+\lambda)\eta), \qquad (21)\\
G(\gamma(\eta)) =& \frac{1}{12}((6-3\delta^2-\delta A(\lambda+(1-\lambda)^2))\gamma^2(\eta)+ \\
&+(3\delta\eta-\delta(1+\lambda)(\delta+A\lambda))\gamma(\eta)+\delta(1+\lambda)\eta). \qquad (22)
\end{aligned}$$

Using these equations (16) becomes

$$\begin{aligned}
&(6-3\delta^2-\delta A(\lambda+(1-\lambda)^2))(\alpha(\eta)+\beta(\eta)) \\
&\neq -3\delta\eta+\delta(1+\lambda)(\delta+A\lambda), \qquad \forall\eta\in(0,\eta^*], \qquad (23)\\
&(6-3\delta^2-\delta A(\lambda+(1-\lambda)^2))(\alpha(\eta)+\gamma(\eta)) \\
&\neq -3\delta\eta+\delta(1+\lambda)(\delta+A\lambda), \qquad \forall\eta\in(0,\eta^*], \qquad (24)
\end{aligned}$$

where $\eta^* > 0$. When $\eta \to 0$ the right hand side of (23) and (24) becomes $\delta(1+\lambda)(\delta+A\lambda) > 0$ while the expressions of the left hand side of (23) and (24) is less than zero due to (17). Thus (16) implies that (18) holds. The converse implication is true since $\beta(\eta) > \gamma(\eta)$ for all $\eta \in (0,\eta^*]$. $\square$

In order to analyze inequality (18) it is convenient to introduce a new parameter ξ instead of η. We study the solutions to the equation

$$F(x)-F(\xi)=0, \qquad \text{for } \xi\in[\gamma,\overline{x}). \qquad (25)$$

It is clear that ξ is a root of the equation (25). Denoting the other two roots by $\alpha(\xi),\beta(\xi)$ gives that

$$\begin{cases} \alpha(\xi)+\beta(\xi) = -(1+\lambda+\xi), \\ \alpha(\xi)\beta(\xi) = (1+\xi)(\xi+\lambda)+\dfrac{\delta}{A}. \end{cases}$$

Now (18) becomes

$$\begin{aligned}
H(\xi) :=& -3\delta A\xi^3-3\delta A(1+\lambda)\xi^2-(6(\delta^2-1)+\delta A(1+\lambda)^2)\xi \\
&+(1+\lambda)(6-2\delta^2-\delta A(1-\lambda)^2) > 0, \quad \forall\xi\in(\gamma,\overline{x}]. \qquad (26)
\end{aligned}$$

Since the discriminant of $H'(\xi)$ is greater than zero, H has a local maximum and a local minimum, denoted by $\overline{\xi}$ respectively, $\underline{\xi}$:

$$\begin{cases} \underline{\xi} = -\dfrac{1+\lambda}{3}-\dfrac{\sqrt{6\delta A(1-\delta^2)}}{3\delta A} \\ \overline{\xi} = -\dfrac{1+\lambda}{3}+\dfrac{\sqrt{6\delta A(1-\delta^2)}}{3\delta A} \end{cases}$$

Now we are in the position to prove the following theorem.

Theorem 3. *The system* (5), *and hence the system* (3) *does not have limit cycles if (14) holds or if the following sets of inequalities are satisfied:*

$$(1-\lambda)^2 - \frac{4}{\delta A} < 0 \tag{27}$$
$$3\delta - A((\lambda - \frac{1}{2})^2 + \frac{3}{4}) < 0 \tag{28}$$
$$4\delta - A(1-\lambda)^2 < 0 \tag{29}$$
$$-6 + 3\delta^2 + \delta A((\lambda - \frac{1}{2})^2 + \frac{3}{4}) < 0 \tag{30}$$
$$-H(\gamma) < 0 \tag{31}$$
$$-H'(\gamma) < 0 \tag{32}$$

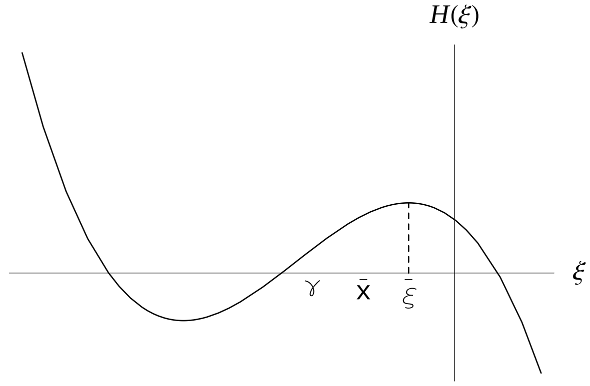

Fig. 2. Illustration of Theorem 3

Note that $\underline{\xi} < \overline{x}$ due to (30).

Proof. Due to (27) theorem (1) can be applied. Condition (28) assures that Proposition 2 is valid and by (30) we can make use of Proposition 3. Thus, it remains to prove that (26) holds. But this is what (31) and (32) assure. □

Remark 3. It is worth pointing out that the parameter set formed by inequalities (27)-(32) is not empty. For example, take $\delta = \frac{1}{9}$ and $\lambda = \frac{2}{3}$ then $4 < A < 66$ satisfies the inequalities.

As a consequence of Theorem 3, Remark 2 and the boundedness of the trajectory ([9]), we have the following stronger statement than Proposition 3 in [9]:

Corollary 1. *Under the condition the conditions* (14) *and* $4\delta > A$ *or the conditions* (27)-(32) *in Theorem 3, the origin is a globally asymptotically stable fixed point.*

As for the nonexistence of limit cycles of the system defined by (10), we first find the fixed points, that are given by the equations

$$\begin{aligned} w =& u(1-u)(u-\lambda) \\ w =& \frac{v}{\delta} \end{aligned}$$

which gives that $u = w = 0$ is the only fixed point if and only if

$$u^2 - (1+\lambda)u + \lambda + \frac{1}{\delta}$$

has no real roots, i.e

$$(1+\lambda)^2 - 4(\lambda + \frac{1}{\delta}) < 0,$$

or equivalently,

$$(\frac{1-\lambda}{2})^2 < \frac{1}{\delta}.$$

Then the characteristic equation of the Jacobian matrix is

$$s^2 + (\varepsilon\delta + \lambda)s + \varepsilon(1 + \lambda\delta) = 0. \tag{33}$$

Thus, the origin is locally asymptotically stable if

$$\mathrm{Re}(-(\varepsilon\delta + \lambda) \pm \sqrt{(\varepsilon\delta + \lambda)^2 - 4\varepsilon(\lambda\delta + 1)} < 0,$$

which is equivalent to, according to Routh test,

$$\varepsilon\delta + \lambda > 0 \quad \text{and} \quad 1 + \lambda\delta > 0, \tag{34}$$

and is unstable if

$$\mathrm{Re}(-(\varepsilon\delta + \lambda) \pm \sqrt{(\varepsilon\delta + \lambda)^2 - 4\varepsilon(\lambda\delta + 1)} > 0$$

Now we state the following theorem without detailed proof, since the analysis is similar.

Theorem 4. *Assume that* (10) *has only one fixed point, that is* $(\frac{\lambda-1}{2})^2 < \frac{1}{\delta}$. *Then system* (10) *has no limit cycles if either*

$$(\lambda - \frac{1}{2})^2 < 3(\delta\varepsilon - \frac{1}{4})$$

holds or one of the following sets of inequalities holds

$$\begin{cases} -(1+\lambda) < 0 \\ -(\lambda + \varepsilon\delta) < 0 \\ 3(\delta\varepsilon - \frac{1}{4}) - (\lambda - \frac{1}{2})^2 < 0 \\ 4\varepsilon\delta - (1-\lambda)^2 < 0 \\ (\lambda+1)^2 + 3(\lambda + \varepsilon\delta) - 6(\lambda + \frac{1}{\delta}) < 0 \\ -\tilde{H}(\tilde{\gamma}) < 0 \\ -\tilde{H}'(\tilde{\gamma}) < 0 \end{cases} \tag{35}$$

or

$$\begin{cases} 1+\lambda<0 \\ -(\lambda+\varepsilon\delta)<0 \\ 3(\delta\varepsilon-\frac{1}{4})-(\lambda-\frac{1}{2})^2<0 \\ 4\varepsilon\delta-(1-\lambda)^2<0 \\ (\lambda+1)^2+3(\lambda+\varepsilon\delta)-6(\lambda+\frac{1}{\delta})<0 \\ \tilde{H}(\underline{\tilde{x}})<0 \\ \tilde{H}(\tilde{\gamma})<0 \\ -\tilde{H}'(\tilde{\gamma})<0 \end{cases} \tag{36}$$

where

$$\begin{aligned} \tilde{H}(\xi) &= -3\xi^3-3(\lambda+1)\xi^2-(6(\frac{1}{\delta}-\varepsilon\delta)-(1+\lambda)^2)\xi \\ &\quad +(\lambda+1)(6(\lambda+\frac{1}{\delta})-2(\lambda+\varepsilon\delta)-(1+\lambda)^2) \\ \underline{\tilde{x}} &= -\frac{\lambda+1}{3}+\frac{1}{3}\sqrt{(\lambda-\frac{1}{2})^2+\frac{3}{4}(1-4\varepsilon\delta)} \\ \Xi &= -\frac{\lambda+1}{3}+\frac{1}{3}\sqrt{6(\frac{1}{\delta}-\varepsilon\delta)} \\ \tilde{\gamma} &= -\frac{\lambda+1}{2}-\frac{1}{2}\sqrt{(1-\lambda)^2-4\varepsilon\delta} \\ \tilde{\mu} &= -\frac{\lambda+1}{2}+\frac{1}{2}\sqrt{(1-\lambda)^2-4\varepsilon\delta} \end{aligned}$$

Remark 4. Note that the fifth inequality in the above sets of inequalities imply that $\Xi<\underline{\tilde{\xi}}$.

Remark 5. It follows that if the above sets of inequalities are satisfied then $\overline{x},\tilde{\overline{x}}>\underline{\xi}$.

Remark 6. Note that neither of the sets of inequalities above is empty, e.g. (35) is satisfied for $\lambda=2$, $\delta=\frac{1}{2}$ and $0<\varepsilon<\frac{1}{2}$ while (35) is satisfied for $\lambda=-4$, $\delta=\frac{1}{8}$ and $0<\varepsilon<50$.

Corollary 2. *Let the conditions in Theorem 4 hold. Then the origin is globally asymptotically stable.*

We only need to show the following proposition.

Proposition 4. *The trajectories of system* (10) *are bounded.*

Proof. Define

$$V(x,y)=\frac{y^2}{2}+\tilde{G}(x)=\frac{y^2}{2}+\frac{\varepsilon}{12}x^2h(x),$$

where $h(x) = 3\delta^3 x^2 - 4(1+\lambda)\delta^2 x + 6(1+\lambda\delta)$. This gives that

$$\dot{V}(x,y) = -y^2 f(x,y)$$

The idea is to show that V is a Lyapunov function outside a bounded set, which implies that the trajectories are bounded.

$$V(x,y) > 0 \quad \forall x, y \neq 0 \iff T = 9(1+\lambda\delta) - 2\delta(1+\lambda)^2 > 0.$$

If $T \leq 0$, then $h(x)$ has two zeros x_1, x_2 and a local minimum point x^* such that $|x^*| < \infty$. Define $\hat{y} = \inf\{y > 0 | y^2 + \frac{\varepsilon \hat{x}^2}{12} h(x^*) > 0\}$. Then $\hat{y} < \infty$. From this it follows that $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 | V(x,y) < 0\}$ is a subset of $\mathcal{A}' = \{(x,y) \in \mathbb{R}^2 | x_1 < x < x_2, |y| < \hat{y}\}$ which is bounded. Turning our attention to $\dot{V}$ we see that

$$\dot{V} < 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(x,y) | y = 0\}$$

This implies

$$S = f(\tilde{x}, \tilde{y}) = \varepsilon\delta - \frac{1}{3}(\lambda - \frac{1}{2})^2 - \frac{1}{4} > 0$$

Let $S < 0$ then f has a local minimum less than zero. But this combined with f being a paraboloid gives the existence of $\breve{x}, \breve{y} < \infty$ such that

$$f(x,y) > 0 \qquad (x,y) \in \mathcal{B}'$$

where $\mathcal{B}' = \{(x,y) \in \mathbb{R}^2 || x| < \breve{x}, |y| < \breve{y}\}$. Thus $\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | \dot{V} > 0\}$ is bounded by $\mathcal{B}'$. Further we have that $\dot{V} = 0$ if $(x,y) \in \mathcal{C}$ where $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 | y = 0 \text{ or } f(x,y) = 0\}$. On this set the solutions to (10) lies on curves defined by $V(x,y) = c$ for constants c. The conclusion is that V is a Lyapunov function on whole of $\mathbb{R}^2$ except for a bounded set and thus the solutions to (10) must be bounded. $\square$

4 Proof of existence of stable limit cycles

In Theorem 4 we assumed $\lambda + \varepsilon\delta > 0$. Now we shall deal with the case $\lambda + \varepsilon\delta < 0$. It turns out that in this case the system has limit cycles. We follow the theorem by Lefschetz Theorem 2 to show the following theorem:

Theorem 5. *System* (10) *has at least one stable limit cycle if* $\varepsilon\delta + \lambda < 0$.

We verify the conditions in Theorem 2. Since $\varepsilon\delta + \lambda < 0$, $(0,0)$ is unstable by (34), and hence the first condition in Theorem 2 is satisfied.

Now both $f(x,y)$ and $\tilde{g}(x)$ are polynomials. So they are continuous and satisfy the Lipschitz condition $\forall x, y \in \mathbb{R}$. Thus condition 2 holds.

It is obvious that $|g(x)| \to \infty$ as $|x| \to \infty$ and that there exists an N such that $g(x) > 0$ for $x > N$ since $\varepsilon, \delta > 0$. From the fact that $\tilde{G}(x)$ is a polynomial of one degree higher than $\tilde{g}(x)$ it follows that $\frac{g(x)}{\tilde{G}(x)} = O(\frac{1}{|x|})$.

Therefore, conditions 3 and 4 are also satisfied. In choosing $M, m, n > 0$ we have a lot of freedom. Taking first derivatives equal to zero yields

$$\tilde{x} = \frac{1+\lambda}{3\delta}, \qquad \tilde{y} = 0.$$

The quadratic form of $f(\tilde{x}, \tilde{y})$ is

$$\begin{aligned} Q(h,k) &= 3\delta^2 h^2 + \frac{1}{\varepsilon^2}k^2 + \frac{3\delta}{\varepsilon}hk \\ &= 3\delta^2(h + \frac{1}{2\varepsilon\delta}k)^2 + \frac{1}{4\varepsilon^2}k^2 > 0 \qquad h, k \neq 0. \end{aligned}$$

Thus $f(\tilde{x}, \tilde{y}) = \varepsilon\delta - \frac{1}{3}(\lambda - \frac{1}{2})^2 - \frac{1}{4}$ is a global minimal point of the paraboloid f. Thus, if $f(\tilde{x}, \tilde{y}) > 0$ we can choose $M = f(\tilde{x}, \tilde{y})$ and m, n arbitrary. On the other hand, if the converse is true there exist $M, n > 0$ such that $f(x, y) > M$ for $|x| > n$ and $f(x, y) > -m = f(\tilde{x}, \tilde{y})$ for $|x| < n$ since the graph of f is a paraboloid. Now the proof of Theorem 5 is complete.

5 Double cycle bifurcation

Let us assume that the origin is the only stable fixed point of the system defined by (10), which is the main topic of this section. Now we vary the parameter λ we see that an Andronov-Hopf bifurcation occurs when $\lambda + \varepsilon\delta = 0$. Then the fixed point becomes unstable and bifurcates to at least one stable limit cycle as shown in Theorem 5. However, the analysis in Section 0.3 showed that there is a gap in parameter space where the origin is the only globally asymptotically stable fixed point (nonexistence of limit cycles, see Theorem 4) and where limit cycles could possibly exist.

In the following we give a numerical example that shows that limit cycles occur in pair when we allow $\lambda < 0$ we see that there is a stable fixed point and a stable limit cycle, e.g. take $\varepsilon = 0.015$, $\delta = 3.5$ and $\lambda = -0.045$, and furthermore, there are at least two limit cycles. These parameters λ, δ and ε lie outside the parameter set formed by (35).

The bifurcation diagram of such a system is very interesting. It is possible to find a combination of the parameters where the first Lyapunov coefficient (see e.g. [6]) becomes zero, thus a Bautin (or generalized Andronov-Hopf bifurcation) occurs, in fact on the boundary of the parameter set formed by (35) and (35).. Furthermore, double-cycle bifurcation occurs when bistability exists. A completely theoretical analysis is given in a forthcoming paper, where both the first and second Lyapunov coefficients are investigated.

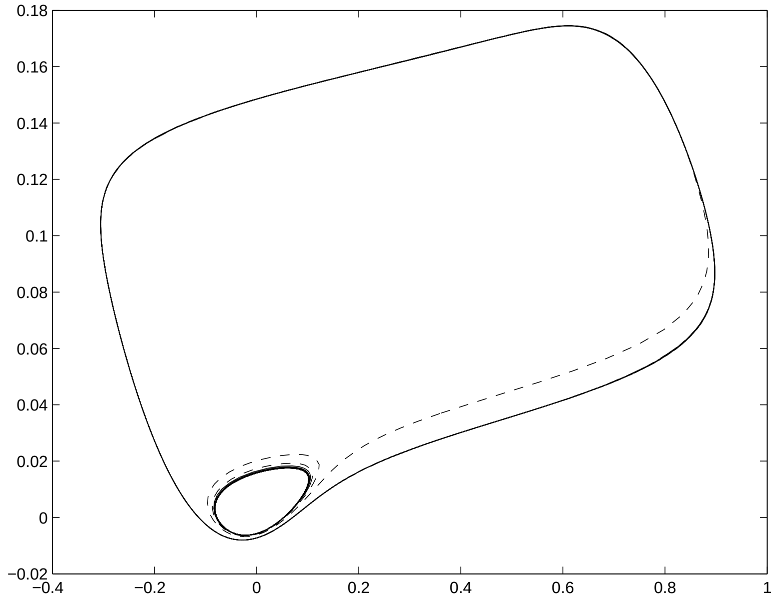

Fig. 3. Phase portrait of (10). There is bistability between the limit cycle and the asymptotically stable fixed point and two limit cycles occur. The inner cycle is unstable and the outer cycle is stable.

6 Conclusion

In this paper we gave some finer criteria for nonexistence of limit cycles for FitzHugh-Nagumo type of models, and hence a affirmative answer to the origin to be a globally asymptotically stable fixed point. Further, we proved the existence of stable limit cycles for the system (10). We also gave a numerical example that showed occurrence of double-cycle bifurcation in case bistability exists.

In our opinion it is a very hard problem to give the exact number of limit cycles of the FitzHugh-Nagumo system, since the analysis on how many limit cycles the FitzHugh-Nagumo class model has is in fact a problem that belongs to the second part of Hilbert's 16th problem.

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Paper II

Applied Singular Perturbation Theory and Bifurcation of FitzHugh-Nagumo Systems

Abstract

The main result of this paper is Theorem 5.2 stating that no more than two limit cycles can bifurcate from the origin via Hopf bifurcation and that there exists choices of parameters such that this upper bound is obtained. We also show that for these parameters the inner cycle is unstable while the outer cycle together with the origin are stable. The proof uses Lyapunov coefficients and relies on a theorem by Andronov see [1]. By applied singular perturbation theory we also give sufficient condition for existence of a unique stable limit cycle under a certain assumption. Finally we present a saddle-node and Bogdanov-Takens bifurcation analysis.

1 Introduction

This article is concerned with the analysis of the FitzHugh-Nagumo class model

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y), \end{cases} \tag{1}$$

with non-zero parameters $A, B, C, \delta, \varepsilon, \lambda$.

Due to the fact that this system is frequently used, e.g. biologic modelling in brain research and to some extend cardiac movements, it is of great importance to get a good understanding of it. This is reflected in the number of articles written in the area see for example [2, 3, 5, 6, 11, 13, 15, 17, 18]. When modelling using this system of equations with an added stochastic disturbance, it is desirable to know if the observed behavior is due to the disturbance or is already built into the deterministic system.

This paper is organized as follows: The second section contains a mixture of known classical results and some worked up versions of these, covering the areas of singular perturbation theory and Lyapunov coefficients. For our purpose we study the boundedness of solutions in Section 3. In the fourth section conditions on some of the parameters are presented ensuring existence of a unique limit cycle as a result of singular perturbation theory. The main results in Section 5 is an upper bound of the maximal number of limit cycles that can bifurcate from the origin via Hopf bifurcation and determination of the sign of the second Lyapunov coefficient. In Section 6 respectively, Section 7 we study saddle-node and Bogdanov-Takens bifurcation, which should be viewed as a preparation for further investigation of coupled system of the above FitzHugh-Nagumo class type. After the conclusions in section 8 follows an appendix with some plots of numerically solved system which verifies the theoretic results of section 4 and section 5.

2 Preliminaries

This section contains both classical results and some worked up results on singular perturbation theory and Lyapunov coefficients.

2.1 Singular perturbation analysis

This section is concerned with periodic solutions of planar systems of the form

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$. In the case when $\varepsilon \gg 1$ the rescaling of time $t = \varepsilon\tau$ transforms (2) into

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \theta \dfrac{dy}{dt} = g(x,y), \end{cases} \tag{3}$$

where $\theta = \frac{1}{\varepsilon} \ll 1$ and thus there is no need for analyzing this case separately. Before going into a detailed discussion and definition of *discontinuous periodic solutions*, DPS, and the possible existence of limit cycles of system (2) let us start with a sketch of the behavior of its solutions. Assume that we start at a point (x_0, y_0) such that $f(x_0, y_0) \neq 0$. Since ε is very small, the trajectories $(x(t), y(t))$, of (2) will move with almost constant velocity in the y-direction in comparison to the velocity in the x-direction. This type of motion will continue until a point $(x(t_1), y(t_1))$ is reached such that $f(x(t_1), y(t_2))$ is of the same order as ε. Thus for $t \in [0, t_1]$ the following approximation of (2) makes sense.

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(x,y_0) \\ \dfrac{dy}{dt} = 0. \end{cases} \tag{4}$$

The fixed points of the approximated system are given by the set $F_{\tilde{y}} = \{(x, \tilde{y}) \in \mathbb{R}^2 | f(x, \tilde{y}) = 0\}$ and the stability is determined by the sign of $\frac{\partial f}{\partial x}$ along S where S is given by

$$S = \cup_{\tilde{y} \in \mathbb{R}} F_{\tilde{y}}.$$

Thus the trajectories of (4) will approach points of S such that $\frac{\partial f(x,y)}{\partial x} < 0$. Close to such points the velocities of $x(t)$ and $y(t)$ will be of the same order and thus the position of the equilibrium of (4) will start to move, i.e the fixed points will change continuously between different branches of $F_{\tilde{y}}$. Thus, the trajectories will lie in a small neighborhood of S until a point $(x(t_2), y(t_2))$ is reached where $\frac{\partial f(x(t_2), y(t_2))}{\partial x} = 0$ and $\frac{\partial^2 f(x(t_2), y(t_2))}{\partial x^2} \neq 0$. For $t \in [t_1, t_2]$ the following approximation, also referred to as the reduced system, of (2) is valid,

$$\begin{cases} f(x,y) = 0 \\ \frac{dy}{dt} = g(x,y). \end{cases} \tag{5}$$

At the so-called *break point* $(x(t_2), y(t_2))$, the stability is disturbed and the trajectories of (2) are again approximated by (4) and everything is repeated until the trajectories either reach a fixed point of (2) or there are no break points and the trajectories are trapped in a small neighborhood of the set S. The motion of system (4) is called *fast flow* while the motion of the system (5) is referred to as the *slow flow*. After this discussion it should come as no surprise that under certain conditions, system (2) has a family of periodic solutions,L_ε, parametrized by ε. In order to state the major theorem of this section we first need to define the concept of *discontinuous periodic solutions*, DPS. We start by introducing the following sets already mentioned above:

- $S = \{(p, q) \in \mathbb{R}^2 | f(p, q) = 0\}$,
- $K = \{(p, q) \in S | \frac{\partial f(p,q)}{\partial x} = 0\}$,
- $L = \{(p, q) \in S | \frac{\partial f(p,q)}{\partial x} < 0\}$,

For the sake of exposition we make the following remarks about solutions to the approximated systems (4) and (5) respectively.

Remark 2.1. *Given a point $(x_0, y_0) \in \mathbb{R}^2 - S$, there exist a unique solution $x_{FF}(t)$ to the approximation (4) satisfying $(x_{FF}(0), y_0) = (x_0, y_0)$ and*

$$\lim_{t\to\infty} (x_{FF}(t), y_0) \in L.$$

Remark 2.2. *By assumption, the Implicit Function Theorem can be applied to the equation $f(x, y) = 0$ in the above approximation. This gives the existence of a function*

$$h : \mathbb{R} \supset U \to V \subset \mathbb{R}$$

such that $x = h(y)$. Thus equation (5) may be written as

$$\frac{dy}{dt} = g(h(y), y). \tag{6}$$

For $(x_0, y_0) \in L$ there exists a unique solution, $y_{SF}(t)$, to equation (6) for $t \in [0, T_{(x_0,y_0)})$ where $T_{(x_0,y_0)}$ satisfies the condition

$$\lim_{t\to T_{(x_0,y_0)}} (h(y_{SF}(t)), y_{SF}(t)) \in K$$

and $(h(y_{SF}(0)), y_{SF}(0)) = (x_0, y_0)$.

With all these preparations done, the definition of DPS is rather straight forward although rather lengthy.

Definition 2.1. *Construct a sequence $\{p^k_{(x,y)}\}$ of points in $\mathbb{R}^2$ in the following way.*

(i) Let $p^0_{(x,y)} = (x, y) \in \mathbb{R}^2 - S$.

(ii) Let $p^1_{(x,y)} = (x_1, y) \in L$, where

$$x_1 = \lim_{t\to\infty} x_{FF}(t)$$

and $x_{FF}(t)$ is the solution to the approximated system (4) with initial condition $x_{FF}(0) = x$, (see Remark 2.1). Define

$$FF_1 = \{(x_{FF}(t), y) | t \in [0, \infty)\}.$$

(iii) *Let* $p^2_{(x,y)} = (h(y_2), y_2) \in K$*, where*

$$y_2 = \lim_{t \to T} y_{SF}(t)$$

and $y_{SF}(t)$ *is the solution to the approximated system* (5) *with initial condition* $y_{SF}(0) = y$*, (see Remark 2.2). Define*

$$SF_1 = \{(h(y_{SF}(t)), y_{SF}(t)) | t \in [0, T)\}.$$

(iv) *Let* $p^3_{(x,y)} = (x_3, y_2) \in L$*, where*

$$x_3 = \lim_{t \to \infty} x_{FF}(t)$$

and $x_{FF}(t)$ *is the solution to the approximated system* (5) *which has the property that*

$$\lim_{t \to -\infty} x_{FF}(t) = h(y_2)$$

(see Remark 2.1). Define

$$FF_2 = \{(x_{FF}(t), y_2) | t \in \mathbb{R}\} \cup \{h(y_2), y_2\}.$$

The sequence $\{p^k_{(x,y)}\}$ *is now continued in exactly the way. System* (2) *is said to have a DPS,* Γ_0*, if there exist* $m, n \in \mathbb{N}^+$ *such that* $m < n$ *and*

$$p^m_{(x,y)} = p^n_{(x,y)}$$

for some $(x, y) \in \mathbb{R}^2$*. Further more, this DPS is given by*

$$\Gamma_0 = \cup_{k=m}^{n} (FF_k \cup SF_k).$$

Now, finally all preparations have been made in order to state the following theorem by Mishchenko and Rosov, Theorem 14 in [4].

Theorem 2.1. *Let*

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases} \tag{7}$$

be a dynamical system on the plane and assume that the following conditions are satisfied:

(i) *All second derivatives of* f *and* g *are continuous at each point in the plane.*

(ii) *At all points of* S *it holds true that*

$$f_x^2(x, y) + f_y^2(x, y) > 0. \tag{8}$$

(iii) *For all points in* K *it holds that*

$$\frac{\partial^2 f(p, q)}{\partial x^2} \neq 0. \tag{9}$$

(iv) *There are no fixed points in the set* $K \cup L$*.*

(v) *System* (7) *has a DPS, denoted* L_0*.*

Then for each sufficiently small ε *system* (7) *has a unique stable limit cycle,* L_ε*, and further more*

$$\lim_{\varepsilon \to 0} L_\varepsilon = L_0. \tag{10}$$

2.2 Lyapunov coefficients

In this section we give a theoretical background to the theory of Lyapunov coefficients that will be used in section 5 for analysis of our specific class of FitzHugh-Nagumo systems. As a starting point we assume that a dynamical system is given in the form

$$\begin{cases} \dfrac{dx}{dt} = f(x,y,\mu) \\ \dfrac{dy}{dt} = g(x,y,\mu), \end{cases} \tag{11}$$

where f and g are real valued analytic functions on $U \times V \times W \subset \mathbb{R}^3$ and that $(0,0,\mu_0)$ is a fixed point of (11). Associated to the system and this point is the system matrix, $\mathcal{A}$, defined by

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where all derivatives are calculated at the fixed point.

Remark 2.3. *The eigenvalues of $\mathcal{A}$ solves the equation*

$$s^2 - \sigma(\mu_0)s + \Delta(\mu_0) = 0,$$

where $\sigma(\mu_0) = trace(\mathcal{A})$ and $\Delta(\mu_0) = det(\mathcal{A})$.

It is a well known fact that the origin is structurally stable iff $\Re(s_i(\mu_0)) \neq 0$, the real part of $s_i(\mu_0)$, where $s_i(\mu_0)$ are the eigenvalues of $\mathcal{A}$ for $i = 1,2$. To put it in another way, for all values of μ close to μ_0 the solutions of the perturbed systems restricted to a small neighborhood of the origin are topologically equivalent if $\Re(s_i(\mu_0)) \neq 0$.

Definition 2.2. *Given a real analytic dynamical system*

$$\mathcal{B} : \begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

we say that another system

$$\tilde{\mathcal{B}} : \begin{cases} \dfrac{dx}{dt} = \tilde{f}(x,y) \\ \dfrac{dy}{dt} = \tilde{g}(x,y) \end{cases}$$

is δ close up to order k to system $\mathcal{B}$ in $G \subset \mathbb{R}^2$ if

$$D_i < \delta, \quad i = 1,2 \tag{12}$$

where

$$D_1 = \max_{(x,y)\in G} |f^{r+s}_{x^r y^s} - \tilde{f}^{r+s}_{x^r y^s}|, \quad r+s = 0,1,\cdots,k \tag{13}$$

and

$$D_2 = \max_{(x,y)\in G} |g^{r+s}_{x^r y^s} - \tilde{g}^{r+s}_{x^r y^s}|, \quad r+s = 0,1,\cdots,k \tag{14}$$

Thus bifurcation can only occur when at least one of the eigenvalues have zero real part. We are going to study the case where $s_i(\mu_0) = \pm\imath\Delta(\mu_0)$, where $\imath^2 = -1$, with $\Delta(\mu_0) > 0$. To this end we assume that the eigenvalues of $\mathcal{A}$ are given by $s(\mu) = \alpha(\mu) \pm \imath\beta(\mu)$ and that

$$\begin{cases} \sigma(\mu_0) = 0 \\ \Delta(\mu_0) > 0 \end{cases} \tag{15}$$

for some parameter value $\mu = \mu_0$. By Remark 2.3 and assumption (15) $\beta(\mu) > 0$ for all μ in a small neighborhood of μ_0 since $\beta(\mu) = \sqrt{\sigma^2(\mu) - 4\Delta(\mu)}$ and the determinant and trace functions are continuous. With these assumptions, the system (11) can be put in the canonical form

$$\begin{cases} \dfrac{d\xi}{dt} = \alpha(\mu)\xi - \beta(\mu)\eta + f(\xi, \eta, \mu) \\ \dfrac{d\eta}{dt} = \beta(\mu)\xi + \alpha(\mu)\eta + g(\xi, \eta, \mu) \end{cases} \tag{16}$$

by the real non-singular transformation

$$\begin{cases} \xi = (a - \alpha(\mu))y + cx \\ \eta = \beta(\mu)y. \end{cases} \tag{17}$$

By introducing a complex variable $z = \xi + \imath\eta$ system (16) is transformed into a single complex differential equation

$$\dot{z} = (\alpha(\mu) + \imath\beta(\mu))z + f(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\imath}) + \imath g(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\imath}) := s(\mu)z + H(z, \bar{z}, \mu) \tag{18}$$

where H must be analytic since both f and g are. The following two lemmas show that in a neighborhood of the origin (18) can be put in a canonical form by a smooth change of coordinates.

Lemma 2.1. *Let*

$$\dot{z} = s(\mu)z + c_1(\mu)z^2\bar{z} + \dots c_k(\mu)z^{k+1}\bar{z}^k + h(z, \bar{z}, \mu), \tag{19}$$

where

$$h(z, \bar{z}, \mu) = \sum_{r+s \geq 2(k+1)} h_{rs}(\mu)\frac{z^r\bar{z}^s}{r!s!}. \tag{20}$$

Then there exists a smooth change of variables

$$z = w + \psi(w, \bar{w}, \mu) \tag{21}$$

such that

$$\dot{w} = s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + o(|w|^{2(k+1)+1}), \tag{22}$$

where

$$\psi(w, \bar{w}, \mu) = \sum_{r+s=2(k+1)} \psi_{rs}(\mu)\frac{w^r\bar{w}^s}{r!s!}. \tag{23}$$

Proof. We assume that there is a change of coordinates satisfying (21) and (22). Using (19) and (21) we obtain

$$\begin{aligned}
\dot{z} &= s(\mu)(w+\psi(w,\bar{w},\mu)) + c_1(\mu)(w+\psi(w,\bar{w},\mu))^2(\bar{w}+\bar{\psi}(w,\bar{w},\mu)) + \cdots + \\
&\quad + c_k(\mu)(w+\psi(w,\bar{w},\mu))^{k+1}(\bar{w}+\bar{\psi}(w,\bar{w},\mu))^k \\
&\quad + \sum_{r+s\geq 2(k+1)} h_{rs}(\mu)\frac{(w+\psi(w,\bar{w},\mu))^r(\bar{w}+\bar{\psi}(w,\bar{w},\mu))^s}{r!s!} \\
&= s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + \\
&\quad + \sum_{r+s=2(k+1)} (s(\mu)\psi_{rs}(\mu) + h_{rs}(\mu))\frac{w^r\bar{w}^s}{r!s!} + o(|w|^{2(k+1)+1}). \qquad (24)
\end{aligned}$$

On the other hand taking derivatives of (21) and using (22) yields

$$\begin{aligned}
\dot{z} &= \dot{w} + \psi_w(\mu)(w,\bar{w})\dot{w} + \psi_{\bar{w}}(\mu)(w,\bar{w})\dot{\bar{w}} \\
&= s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + \\
&\quad + \sum_{r+s=2(k+1)} \psi_{rs}(\mu)(rs(\mu) + s\bar{s}(\mu))\frac{w^r\bar{w}^s}{r!s!}. \qquad (25)
\end{aligned}$$

Comparing (24) with (25) gives following expression for $\psi_{rs}(\mu)$

$$\psi_{rs}(\mu) = \frac{h_{rs}(\mu)}{(r-1)s(\mu) + s\bar{s}(\mu)}. \qquad (26)$$

Since

$$(r-1)s(\mu_0) + s s(\bar{\mu_0}) = \imath\beta(\mu_0)(r-1-s) = \imath\beta(\mu_0)(2(k+1)-(2s+1)) \neq 0 \quad (27)$$

the change of variables is smooth for $|\mu-\mu_0|$ small. Equation (22) now follows from (26) and (23). $\square$

Lemma 2.2. *Let*

$$\dot{z} = s(\mu)z + c_1 z^2\bar{z} + \cdots + c_k z^{k+1}\bar{z}^k + \tilde{h}(z,\bar{z}), \qquad (28)$$

where

$$\tilde{h}(z,\bar{z}) = \sum_{r+s\geq 2(k+1)+1} \tilde{h}_{rs}\frac{z^r\bar{z}^s}{r!s!}. \qquad (29)$$

Then there exists a smooth change of variables

$$z = w + \phi(w,\bar{w}) \qquad (30)$$

such that

$$\dot{w} = s(\mu)w + c_1 w^2\bar{w} + \cdots + c_k w^{k+1}\bar{w}^k + c_{k+1}w^{k+2}\bar{w}^{k+1} + o(|w|^{2(k+2)}), \quad (31)$$

where

$$\phi(w,\bar{w}) = \sum_{r+s=2(k+1)+1} \phi_{rs}\frac{w^r\bar{w}^s}{r!s!}. \qquad (32)$$

Proof. Using the same method as in the preceding lemma gives

$$\phi_{rs} = \frac{\tilde{h}_{rs}}{(r-1)s(\mu) + s\bar{s}(\mu)}. \tag{33}$$

The big different here is that the denominator is zero for $r = k+2$ and $s = k+1$. In order to get a smooth change of variables we put $\phi_{k+2,k+1} = 0$. This results in

$$c_{k+1} = \frac{\tilde{h}_{k+2,k+1}}{(k+2)!(k+1)!}. \tag{34}$$

Note that by using Lemma 2.1 and Lemma 2.2 alternately all Lyapunov coefficients can be calculated via equation 34. □

When determining the number of limit cycles of system (16) in a small neighborhood of the origin the following lemma is useful since it shows that the number remains the same if the higher order terms are dropped.

Lemma 2.3. *The system*

$$\dot{w} = s(\mu)w + c_1(\alpha)w^2\bar{w} + \cdots + c_k(\alpha)w^{k+1}w^k + O(|w|^{2(k+1)}) \tag{35}$$

has the same number of limit cycles as

$$\dot{w} = s(\mu)w + c_1(\alpha)w^2\bar{w} + \cdots + c_k(\alpha)w^{k+1}w^k \tag{36}$$

in a small neighborhood of the origin.

Proof. Introduce the notation $c_r(\alpha) = a_r(\alpha) + \imath b_r(\alpha)$ and convert (35) into a system of differential equations in polar coordinates

$$\begin{cases} \dot{r} = \alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta) \\ \dot{\theta} = \beta + b_1 r^2 + \cdots b_k r^{2k} + \Psi(r,\theta), \end{cases} \tag{37}$$

where $\Phi(r,\theta) = O(|r|^{2(k+1)})$ and $\Psi(r,\theta) = O(|r|^{2k+1})$. Eliminating the time dependence we get

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta)}{\beta + b_1 r^2 + \cdots b_k r^{2k} + \Psi(r,\theta)} = \\ &= \frac{\alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta)}{\beta} \times \\ &\quad \frac{1}{1 + \frac{r^2}{\beta}(b_1 + \cdots b_k r^{2(k-1)} + \frac{\Psi(r,\theta)}{r^2}}. \end{aligned} \tag{38}$$

Since we are interested in the behavior in a small neighborhood of the origin r is assumed very small. Further more we have by assumption that $\beta \neq 0$. Thus we can use the formula for geometric series on the last term in (38).

$$
\begin{aligned}
&\frac{1}{1+\frac{r^2}{\beta}(b_1+\cdots b_k r^{2(k-1)}+\frac{\Psi(r,\theta)}{r^2}} = \\
&= 1+\frac{r^2}{\beta}\sum_{\substack{n_1,\cdots,n_{k+1}\\ n_1+\cdots+n_{k+1}=2}} \binom{2}{n_1,\cdots,n_{k+1}} b_1^{n_1}(b_2r^2)^{n_2}\cdots(b_kr^{2(k-1)})^{n_k}\left(\frac{\beta\Psi(r,\theta)}{r^2}\right)^{n_{k+1}} + \\
&\quad +\cdots+ \\
&\quad +\left(\frac{r^2}{\beta}\right)^k \sum_{\substack{n_1,\cdots,n_{k+1}\\ n_1+\cdots+n_{k+1}=k}} +\binom{k}{n_1,\cdots,n_{k+1}} b_1^{n_1}(b_2r^2)^{n_2}\cdots(b_kr^{2(k-1)})^{n_k}\left(\frac{\beta\Psi(r,\theta)}{r^2}\right)^{n_{k+1}} \\
&\quad + O(r^{2(k+1)}).
\end{aligned} \tag{39}
$$

Inserting the above expression in (38) gives

$$
\frac{dr}{d\theta} = \frac{1}{\beta}\left(\alpha r + a_1 r^3 + \cdots + a_k r^{2k+1} + d_2 r^2 + \cdots + d_{2k} r^{2k} + R(r,\theta)\right), \tag{40}
$$

where the coefficients d_s neither depend on Ψ nor Φ and $R(r,\theta) = O(r^{2(k+1)})$. Let $r(\theta; r_0)$ be the solution to (40) with initial value r_0. Since the origin is assumed to be a fixed point $r(\theta; 0) = 0$. Thus the Taylor expansion of $r(\theta; r_0)$ around the origin is given by

$$
r(\theta, r_0) = u_1(\theta) r_0 + \cdots + u_{2k+1}(\theta) r_0^{2k+1} + O(|r_0|^{2(k+1)}). \tag{41}
$$

Differentiating (41) and using (40) shows that the differential equations for $u_s(\theta)$ is independent of the term $R(r,\theta)$. The number of limit cycles can now be determined as the number of positive real fixed points, r_0, of $r(2\pi, r_0)$. Since the functions $u_s(\theta)$ are independent of $R(r,\theta)$ the number of small positive real fixed points will be the same when the term $O(|r_0|^{2(k+1)})$ is dropped in equation (35) $\square$

Remark 2.4. *From* (37) *we see that the trajectories will move in a counter-clockwise direction since* $\beta(\mu) > 0$ *for all* μ *close to* μ_0 *by assumption and* r *is assumed to be small.*

The following corollary, saying that the behavior of the system is determined by the the functions $c_r(\mu)$ at $\mu = \mu_0$ and further that it is only necessary to compute the first nonzero coefficient, is useful in applications.

Corollary 2.1. *In order to study the bifurcations of a general system*

$$
\begin{cases}
\dfrac{dx}{dt} = f(x,y,\mu) \\
\dfrac{dy}{dt} = g(x,y,\mu)
\end{cases} \tag{42}
$$

in a neighborhood of the bifurcation value $\mu = \mu_0$ *it is sufficient to calculate the first nonzero coefficient* $c_k(\mu_0)$.

Proof. Since all changes of variables transforming (11) into

$$\begin{cases} \dot{r} = \alpha(\mu)r + a_1(\mu)r^3 + \cdots a_k(\mu)r^{2k+1} + \Phi(r,\theta,\mu) \\ \dot{\theta} = \beta(\mu) + b_1(\mu)r^2 + \cdots b_k(\mu)r^{2k} + \Psi(r,\theta,\mu) \end{cases} \tag{43}$$

are smooth the functions $c_r(\mu) = a_r(\mu) + \imath b_r(\mu)$ are also smooth. Thus for μ close to μ_0 the leading terms in the expressions for $\dot{r}$ and $\dot{\theta}$ are given by $a_k(\mu_0)$ and $b_k(\mu_0)$ respectively where $c_k(\mu)$ is the first non-vanishing coefficient at $\mu = \mu_0$. □

Definition 2.3. *Let $c_m(\mu_0) = a_m(\mu_0) + \imath b_m(\mu_0)$. Then the real number $a_m(\mu_0)$ is called the m-th Lyapunov coefficient often denoted by L_m.*

Definition 2.4. *The origin is said to be a focus of multiplicity m of system* (16) *if L_m is the first non-vanishing Lyapunov coefficient.*

Let us now turn to the question about how many limit cycles that can bifurcate from a focus of a certain multiplicity. This question is answered by the following two theorems that can be found in [1].

Theorem 2.2. *If O $(0,0)$ is a multiple focus of multiplicity k $(k \geq 1)$ of a dynamic system $(\mathcal{A})$ of class $N \geq 2+1$ or of analytical class, then*

1. *there exist $\varepsilon_0 > 0$ and $\delta_0 >$ such that any system $(\tilde{\mathcal{A}})$ δ_0-close to rank $2k+1$ to system $(\mathcal{A})$ has at most k closed paths in $U_{\varepsilon_0}(O)$;*

2. *for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, there exists a system $(\tilde{\mathcal{A}})$ of class N or (respectively) of analytical class which is δ-close to rank $2k+1$ to $(\mathcal{A})$ and has k closed paths in $U_\varepsilon(O)$.*

Theorem 2.3. *Let O $(0,0)$ be multiple focus of multiplicity k of a dynamic system $(\mathcal{A})$ of class $N \geq 2k+1$ or of analytical class, and let ε_0 and δ_0 be positive numbers defined by the first part of Theorem 2.2 and such that any system $(\tilde{\mathcal{A}})$ δ_0-close to $(\mathcal{A})$ has a single equilibrium state in $U_{\varepsilon_0}(O)$ which is a focus. Then*

1. *for any ε and δ, $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$, and for any s, $1 \leq s \leq k$, there exists a system $(\mathcal{B})$ of class N (or respectively,analytical) which is δ-close to rank $2k+1$ to system $(\mathcal{A})$ and has in $U_\varepsilon(O)$ precisely s closed paths.*

2. *if system $(\mathcal{B})$ is δ_0-close to rank $2k+1$ to system $(\mathcal{A})$ and has k limit cycles in U_{ε_0}, all these cycles, and likewise the focus of system $(\mathcal{B})$ lying in U_{ε_0}, are structurally stable (simple) i.e. they are either stable or unstable.*

Applying the above theorems to (16) gives the following bifurcation table

We have seen that the Lyapunov coefficients plays a crucial role for the behavior of solutions of a system with a pair of purely imaginary eigenvalues. The following formulas show how to calculate them explicitly.

$$\begin{aligned} L_1 = & \frac{\pi}{4\sqrt{\Delta}}[3(a_{30} + b_{03} + a_{12} + b_{21}] + \\ & -\frac{\pi}{4\Delta}[2(a_{20}b_{20} - a_{02}b_{02}) - a_{11}(a_{02} + a_{20}) + b_{11}(b_{02} + b_{20})] \end{aligned} \tag{44}$$

	$\delta < \frac{-AB\lambda}{\varepsilon}$	$\delta = \frac{-AB\lambda}{\varepsilon}$	$\delta > \frac{-AB\lambda}{\varepsilon}$
$L_1 > 0,\ \sigma'(\mu_0) > 0$	Origin stable Unstable L.C	Origin unstable No L.C.	Origin unstable No L.C.
$L_1 > 0,\ \sigma'(\mu_0) < 0$	Origin unstable No L.C.	Origin unstable No L.C.	Origin stable Unstable L.C.
$L_1 < 0,\ \sigma'(\mu_0) > 0$	Origin stable No L.C.	Origin stable No L.C.	Origin unstable Stable L.C.
$L_1 < 0,\ \sigma'(\mu_0) < 0$	Origin unstable Stable L.C.	Origin stable No L.C.	Origin stable No L.C.

Table 1: Hopf bifurcation table for system (11)

$$\begin{aligned}
L_2 = -\frac{\pi}{24}&[a_{02}b_{20}(5a_{02}b_{11} + 10a_{02}a_{20} + 4{b_{11}}^3 + 11a_{20}b_{11} + 6{a_{20}}^2 - \\
&- 5a_{11}b_{20} - 10b_{20}b_{02} - 4{a_{11}}^2 - 11a_{11}b_{02} - 6{b_{02}}^2) + a_{20}b_{02}(6{b_{02}}^2 - \\
&- 5a_{11}b_{02} + 10b_{02}b_{20} - 2{a_{11}}^2 - 5a_{11}b_{20} + 5a_{20}b_{11} - 6{a_{20}}^2 - 10a_{20}a_{02} + \\
&+ 2{b_{11}}^2 + 5a_{02}b_{11}) + a_{02}b_{02}(5{b_{11}}^2 - {a_{11}}^2 - 6a_{11}b_{02}) - a_{20}b_{20}(5{a_{11}}^2 - \\
&- {b_{11}}^2 - 6a_{20}b_{11}) + {a_{11}}^3(a_{20} + a_{02}) - {b_{11}}^3(b_{02} + b_{20}) - 5{b_{20}}^2(a_{12} + 3b_{03}) + \\
&+ {b_{02}}^2(3b_{21} - 6a_{12} - 5a_{30}) + {a_{11}}^2(a_{12} + a_{30}) + b_{20}b_{02}(5b_{21} - 5a_{12} - \\
&- 9b_{03} + 5a_{30}) - b_{20}a_{11}(4a_{12} + 9b_{03} + 5a_{30}) + b_{02}a_{11}(3b_{21} - a_{12} + \\
&+ 4a_{30}) - 5{a_{02}}^2(b_{21} + 3a_{30}) + {a_{20}}^2(3a_{12} - 6b_{21} - 5b_{03}) + \\
&+ {b_{11}}^2(b_{21} + b_{03}) + a_{20}a_{02}(5a_{12} - 5b_{21} - 9a_{30} + 5b_{03}) - \\
&- a_{02}b_{11}(4b_{21} + 9a_{30} + 5b_{03}) + a_{20}b_{11}(3a_{12} - b_{21} + 4b_{03}) + \\
&+ 4b_{20}b_{11}(2b_{30} + b_{12}) + b_{02}b_{11}(7b_{20} - a_{21} + 5b_{12} + a_{03}) + \\
&+ 2a_{11}b_{11}(a_{03} + b_{30}) + 2a_{20}b_{20}(8b_{30} - 5a_{21} - b_{12}) + 2a_{20}b_{02}(4b_{30} - \\
&- 5a_{21} - 5b_{12} + 4a_{03}) + a_{20}a_{11}(b_{30} + 5a_{21} - b_{12} + 7a_{03}) - \\
&- 2a_{02}b_{20}(a_{21} + b_{12}) + 2a_{02}b_{02}(8a_{03} - 5b_{12} - a_{21}) + 4a_{02}a_{11}(2a_{03} + \\
&+ a_{21}) + b_{11}(5b_{04} - b_{22} + 2a_{13} - 3b_{40}) + a_{02}(2b_{22} + 20b_{04} + 5a_{13} + \\
&+ 3a_{31}) + a_{20}(4b_{22} + 22b_{04} + 7a_{13} - 6b_{40} + 9a_{31}) - \\
&- b_{20}(2a_{22} + 20a_{40} + 5b_{31} + 3b_{13}) - a_{11}(5a_{40} - a_{22} + 2b_{31} - 3a_{04}) + \\
&+ 3a_{21}(2a_{30} + b_{03} + a_{12}) - 3b_{12}(2b_{03} + a_{30} + b_{21}) + 3a_{03}(a_{12} + \\
&+ 3b_{03}) - 3b_{30}(b_{21} + 3a_{30}) - b_{02}(4a_{22} + 22a_{40} + 7b_{31} - 6a_{04} + 9b_{13}) + \\
&+ 3b_{41} + 3b_{23} + 15b_{05} + 15a_{50} + 3a_{32} + 3a_{14}]. \qquad (45)
\end{aligned}$$

In both formulas it is assumed that the system is in the form

$$\begin{cases} \dfrac{dx}{dt} = y + \tilde{f}(x, y, \mu_0) \\ \dfrac{dy}{dt} = -x + \tilde{g}(x, y, \mu_0) \end{cases}$$

If not, the transformation

$$\begin{cases} y_1 = x_1 \\ y_2 = -\dfrac{ax_1 + bx_2}{\sqrt{\Delta}} \end{cases}$$

does the trick. The coefficients a_{rs} and b_{rs} are the Taylor coefficients of f and g after the above change of coordinates, and are given by

$$\begin{cases} a_{rs} = \displaystyle\sum_{k=s}^{r+s} \binom{k}{s} \frac{(-1)^k a^{(k-s)} \Delta^{\frac{s}{2}}}{(r+s-k)!k!b^k} f_{r+s-k,k} \\ b_{rs} = \displaystyle\sum_{k=s}^{r+s} \binom{k}{s} \frac{(-1)^{k+1} a^{(k-s)} \Delta^{\frac{s-1}{2}}}{(r+s-k)!k!b^k} (af_{r+s-k,k} + bg_{r+s-k,k}). \end{cases}$$

Remark 2.5. *The exact expressions of the Lyapunov coefficients can be different by different transformation of coordinates. However the signs are invariant.*

3 Boundedness of solutions

In this section we are going to present necessary conditions under which our system

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y) \end{cases} \tag{46}$$

has bounded solutions. This analysis depends on the construction of a Lyapunov function. We will restrict to the case $I = 0$. By making the change of variables

$$\begin{cases} u = y \\ v = \varepsilon(x - \delta y), \end{cases} \tag{47}$$

the system can be written in the Lie'nard form

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -vf(u,v) - g(u), \end{cases} \tag{48}$$

where

$$\begin{cases} f(u,v) = \varepsilon A(\dfrac{v^2}{\varepsilon^3} + \dfrac{3\delta u - (B+\lambda)}{\varepsilon^2} v + \dfrac{3\delta^2}{\varepsilon} u^2 - \dfrac{2\delta(B+\lambda)}{\varepsilon} u + \dfrac{\varepsilon\delta + AB\lambda}{A\varepsilon} \\ \quad g(u) = -A\varepsilon u(\delta^3 u^2 - \delta^2(B+\lambda)u + \dfrac{C + AB\delta\lambda}{A}. \end{cases} \tag{49}$$

Define

$$V(u,v) = \frac{v^2}{2} + G(u), \tag{50}$$

where

$$G(u) = \int_0^u g(s)ds. \tag{51}$$

Then, with some restrictions on the parameters, V is a Lyapunov function on the whole $\mathbb{R}^2$ except for a bounded set by the following lemma.

Lemma 3.1. *Let*

(i) $A > 0$

(ii) $\varepsilon > 0$

(iii) $\delta > 0$

Then $V(u,v) = \frac{v^2}{2} + G(u)$ *is a Lyapunov function on* $\mathbb{R}^2$ *except for a bounded set.*

Proof. Using the given assumptions, the proof of corollary 1 in [19] goes through. □

Corollary 3.1. *Let the assumptions of Lemma 3.1 hold. Then the solutions to system* (46) *are bounded.*

4 Analysis of a class of FitzHugh-Nagumo systems with respect to singular perturbation theory

In this section we are going to find conditions on the parameters such that Theorem 2.1 is applicable. In order to transform our system to the special form of the previous section we make the time scaling $\tau = \varepsilon t$ and assume that $0 < \varepsilon \ll 1$. Doing so gives us the system

$$\begin{cases} \varepsilon \dfrac{dx}{d\tau} = -Cy + Ax(B-x)(x-\lambda) := f(x,y) \\ \dfrac{dy}{d\tau} = x - \delta y := g(x,y). \end{cases} \tag{52}$$

In the sequel the symbol τ will be replaced by t. The verification of the first two conditions of Theorem 2.1 is easy since

(i) The functions f and g are polynomials and therefore have continuous second derivatives.

(ii)
$$f_x^2(x,y) + f_y^2(x,y) = A^2(x^2 + 4(\lambda + B)^2) > 0 \tag{53}$$

Thus in order to apply Theorem 2.1 it remains to find conditions on the parameters such that the remaining conditions are satisfied. For this particular system the sets S, K and L are given by

$$\begin{cases} S = \{(x,y) \in \mathbb{R}^2 | y = -\dfrac{A}{C}x(x-B)(x-\lambda) := F(x)\} \\ K = \{(x_1, F(x_1)), (x_2, F(x_2))\} \\ L = (L_{A^-} \cup L_{A^+}), \end{cases} \tag{54}$$

where x_i are the real roots of $F'(x) = 0$ for $i = 1, 2$ and

$$L_{A^-} = \{(x,y) \in \mathbb{R}^2 | y = F(x), x_1 < x < x_2, A < 0\}$$

and

$$L_{A^+} = \{(x,y) \in \mathbb{R}^2 | y = F(x), x < x_1 \vee x > x_2, A > 0\}.$$

The approximation (5) of system (52) can be written as

$$\frac{dx}{dt} = \frac{x - \delta F(x)}{F'(x)} \tag{55}$$

for $(x,y) \in L$ and where $F(x)$ is defined in the expression of S. The following lemma gives necessary conditions for system (52) to have a DPS.

Lemma 4.1. *If system* (52) *has a DPS then*

(i) $(B-\lambda)^2 + B\lambda > 0$

(ii) $A > 0$

(iii) $\frac{dx}{dt} > 0, \quad x < x_1$

(iv) $\frac{dx}{dt} < 0, \quad x > x_2$

Proof. Let Γ_0 be a DPS of (52). Since the minimum number of sections a DPS can be composed of is four, two slow and two fast, we conclude that system (52) must have at least two breakpoints, i.e. $|K| \geq 2$. By definition $|K|$ is the cardinality of K. But from the definition of K we know that $|K| \leq 2$ and thus we have shown that $|K| = 2$. This however is equivalent to the first condition. From the fact that $|K| = 2$ we know that Γ_0 is composed of exactly two sections of slow flow and two sections of fast flow. From the definition of DPS we know that these are disjoint sets and that the sections of slow flow are subsets of L. If $A < 0$ then L consists of only one connected component. This contradicts that the sections of slow flow are disjoint and separated by at least one breakpoint. Thus by assumption that all parameters are nonzero we deduce that $A > 0$. Since $F(x_1) < F(x_2)$ for $A > 0$ and Γ_0 is the DPS the remaining two statements hold true. $\square$

That the above conditions are not sufficient can be seen from the fact that there might be a fixed point of (52) on L making a DPS impossible. A more exact answer is given by the following Lemma.

Lemma 4.2. *Let α_i be the roots of $x - \delta F(x)$, x_i be roots of $F'(x)$ and let p_j satisfy $F(x_i) = F(p_i)$. Then system* (52) *has a DPS iff*

(i) $\alpha_i \notin [p_2, x_1]$

(ii) $\alpha_i \notin [x_2, p_1]$

(iii) $(B - \lambda)^2 + B\lambda > 0$

(iv) $A > 0$

(v) $\frac{dx}{dt} > 0, \quad x < x_1$

(vi) $\frac{dx}{dt} < 0, \quad x > x_2$

Proof. By Lemma 4.1 conditions $(iii) - (vi)$ are necessary for existence of a DPS. With the extra conditions $(i) - (ii)$ a DPS is given by

$$\Gamma_0 = \cup_{k=1}^{2}(SF_k \cup FF_k)$$

where

$$SF_1 = \{(x, y) \in \mathbb{R}^2 | y = F(x), p_2 \leq x < x_1\} \cap L$$

$$FF_1 = \{(x, y) \in \mathbb{R}^2 | x_1 \leq x < x_2, y = F(x_1)\}$$

$$SF_2 = \{(x, y) \in \mathbb{R}^2 | y = F(x), x_2 < x \leq p_1\} \cap L$$

$$FF_2 = \{(x, y) \in \mathbb{R}^2 | x_1 < x \leq p_2, y = F(x_2)\}$$

The SF_i are the sections of slow flow and the SF_i are the sections of fast flow for $i = 1, 2$. $\square$

4.1 A unique fixed point

When the origin is a unique fixed point of (52), i.e

$$(B+\lambda)^2 - \frac{4C}{A\delta} < 0 \tag{56}$$

the calculations are quite straightforward. From Lemma 4.2 we know that the sign of $\dot{x}$ is crucial for the existence of DPS and from (55) it is given by

$$\mathrm{sign}(\frac{dx}{dt}) = \mathrm{sign}(x - \delta F(x))\mathrm{sign}(F'(x)). \tag{57}$$

An analysis of the separate parts of the above expression shows that

$$\mathrm{sign}(x - \delta F(x)) = \begin{cases} -\mathrm{sign}(\frac{\delta A}{C}) & \text{for } x < 0 \\ \mathrm{sign}(\frac{\delta A}{C}) & \text{for } x > 0, \end{cases} \tag{58}$$

and

$$\mathrm{sign}(F'(x)) = \begin{cases} -\mathrm{sign}(\frac{A}{C}) & \text{for } x < x_1, x > x_2 \\ \mathrm{sign}(\frac{A}{C}) & \text{for } x_1 < x < x_2. \end{cases} \tag{59}$$

Combining (59) and (58) we end up with the following expression for $\mathrm{sign}(\frac{dx}{dt})$:

$$\mathrm{sign}(\frac{dx}{dt}) = \begin{cases} \mathrm{sign}(\delta) & \text{for } p_2 < x < x_1 \\ -\mathrm{sign}(\delta) & \text{for } x_2 < x < p_1. \end{cases} \tag{60}$$

By application of Lemma 4.2 we have thus shown the following Theorem:

Theorem 4.1. *Let the origin be a unique fixed point of system* (52)*. Then it has a DPS iff the following conditions are fulfilled*

(i) *Condition* $(iii) - (iv)$ *of Lemma 4.2.*

(ii) $x_1 < 0 < x_2$

(iii) $\delta > 0$.

4.2 Three fixed points

This section is devoted to the problem of determining conditions under which (52) has DPS while at the same time it has three distinct fixed points. This assumption requires an extended analysis of $\mathrm{sign}(x_0 - \delta F(x_0))$. Let $I_k = (a_k, b_k)$ where

$$a_k = \begin{cases} -\infty & k = 1 \\ \alpha_{k-1} & k = 2, 3, 4 \end{cases} \tag{61}$$

and

$$b_k = \begin{cases} \alpha_k, & k = 1, 2, 3 \\ \infty, & k = 4. \end{cases} \tag{62}$$

Then

$$\mathrm{sign}(x - \delta F(x)) = (-1)^k \mathrm{sign}(\frac{\delta A}{C}), \quad x \in I_k. \tag{63}$$

Lemma 4.3. *Define p_1 and p_2 by $f(x_i) = f(p_i)$ for $i = 1, 2$ respectively. Assume further that all elements in $S = \{x_1, x_2, \alpha_1, \alpha_2, \alpha_3\}$ are different where α_i are the zeros of $x - \delta F(x)$. A necessary condition for (52) to have a DPS is that the elements of S divide the interval $[p_2, p_1]$ into an even number of subintervals.*

Proof. First note that $\dot{x}$ only changes sign at points of S. Since they are different by assumption, corresponding to each of these elements there is a change in sign for exactly one of $(x - \delta F(x))$ and $F'(x)$. By Lemma 4.2 it is necessary that $\dot{x}$ have opposite sign in the intervals $[p_2, x_1]$ and $[x_2, p_1]$. Thus $\dot{x}$ must shift sign an odd number of times. But this is equivalent to S dividing $[p_2, p_1]$ into an even number of subintervals. □

Using Lemma 4.3 yields four possibilities for the existence of a DPS emerging from the cases of three and five elements of S lying in the interval $[p_2, p_1]$. These are

C_1: $x_1 < \alpha_1 < x_2$ and $\alpha_2, \alpha_3 > p_1$;

C_2: $x_1 < \alpha_2 < x_2$ and $\alpha_1 < p_2, \quad \alpha_3 > p_1$;

C_3: $x_1 < \alpha_3 < x_2$ and $\alpha_1, \alpha_2 < p_2$;

C_4: $x_1 < \alpha_1 < \alpha_2 < \alpha_3 < x_2$.

In these cases $\text{sign}(\dot{x})$ can be described as follows

$$\text{sign}(\frac{dx}{dt}) = \begin{cases} (-1)^{i(C_k)-1}\text{sign}(\delta) & \text{for } x \in [p_2, x_1] \\ (-1)^{i(C_k)}\text{sign}(\delta) & \text{for } x \in [x_2, p_1]. \end{cases} \tag{64}$$

where $i(C_k) := k$. Thus we see that δ has to be greater than zero in the cases with an even number and less than zero for the odd numbered cases in order for a DPS to exist. Therefore we have shown the following theorem.

Theorem 4.2. *Assume that system (52) has three fixed points. Then it has a DPS iff the following conditions are fulfilled*

(i) Condition (iii) – (iv) of Lemma 4.2.

(ii) One of the cases C_k holds

(iii) $\delta < 0$ for k even and $\delta > 0$ for k odd

4.3 Unstable periodic solutions

Letting $\tau = -t$ in (52) leads to

$$\begin{cases} \varepsilon \dfrac{dx}{d\tau} = Cy - Ax(B - x)(x - \lambda) := \tilde{f}(x, y) = -f(x, y) \\ \dfrac{dy}{dt} = -x + \delta y := \tilde{g}(x, y) = -g(x, y). \end{cases} \tag{65}$$

Proceeding as earlier we get the approximation (5) of system (65) to be

$$\frac{dx}{d\tau} = -\frac{x - \delta F(x)}{F'(x)} = -\frac{dx}{dt}, \tag{66}$$

for $(x, y) \in L$. In this case

$$\frac{\partial \tilde{f}}{\partial x} = -\frac{\partial f}{\partial x} \Rightarrow \mathrm{sign}(\frac{\partial \tilde{f}}{\partial x}) = \begin{cases} \mathrm{sign}(A), & x < x_1, x > x_2 \\ -\mathrm{sign}(A), & x_1 < x < x_2. \end{cases} \tag{67}$$

Using that

$$\mathrm{sign}(\frac{dx}{d\tau}) = -\mathrm{sign}(\frac{dx}{dt})$$

equation (60) yiels that

$$\mathrm{sign}(\frac{dx}{d\tau}) = \begin{cases} -\mathrm{sign}(\delta) & \text{for } p_2 < x < x_1 \\ \mathrm{sign}(\delta) & \text{for } x_2 < x < p_1 \end{cases} \tag{68}$$

if the origin are the unique fixed point. Further more, applying (64) to this system gives that

$$\mathrm{sign}(\frac{dx}{d\tau}) = \begin{cases} (-1)^{i(c_k)}\mathrm{sign}(\delta), & x \in [p_2, x_1] \\ (-1)^{i(c_k)+1}\mathrm{sign}(\delta), & x \in [x_2, p_1], \end{cases} \tag{69}$$

if there are three fixed points. In analogue with Theorems 4.1 and 4.2 the following conclusions hold true:

Theorem 4.3. *Let the origin be a unique fixed point of system* (65)*. Then it has a DPS iff the following conditions hold true*

(i) $(B - \lambda)^2 + B\lambda > 0$

(ii) $A < 0$

(iii) $\delta < 0$

Theorem 4.4. *Assume that system* (65) *has three fixed points. Then it has a DPS iff the following conditions hold true*

(i) $(B - \lambda)^2 + B\lambda > 0$

(ii) $A < 0$

(iii) One of the cases C_k *holds*

(iv) $\delta > 0$ *for* k *even and* $\delta < 0$ *for* k *odd*

Remark 4.1. *Since A has to be greater than zero for positive time and less than zero for negative time, stable and unstable periodic solutions to* (52) *emerging from DPS cannot coexist.*

4.4 Analysis for large values of ε

What about the case $\varepsilon \gg 1$? A scaling of time, $t = \varepsilon\tau$, transforms (52) to

$$\begin{cases} \dfrac{dx}{d\tau} = -Cy + Ax(B - x)(x - \lambda) := f(x, y) \\ \theta\dfrac{dy}{dt} = x - \delta y := g(x, y) \end{cases} \tag{70}$$

where $\theta = \frac{1}{\varepsilon} \ll 1$. For this system the sets S, L and K are given by

$$\begin{cases} S = \{(x,y) \in \mathbb{R}^2 | y = \dfrac{x}{\delta} := \tilde{F}(x)\} \\ K = \emptyset \\ L = \{(x,y) \in S | \delta > 0\}. \end{cases} \tag{71}$$

Remark 4.2. *Since K is empty there can not exist any DPS.*

4.5 A unique limit cycle

After the investigation of occurrences of DPS for system (52) we are now in a position to apply Theorem 2.1. Thus if we put the pieces together we arrive at the following theorems for one and three fixed points respectively.

Theorem 4.5. *Let the assumptions of Theorem 4.1 hold. Then for ε sufficiently small, system* (52) *has a family of limit cycles L_ε. Further more, these limit cycles are unique for every such ε and*

$$\lim_{\varepsilon \to 0} L_\varepsilon = L_0 \tag{72}$$

where L_0 is the DPS of the system.

Theorem 4.6. *Assume that system* (52) *has three fixed points and that*

(i) *Condition $(iii) - (iv)$ of Lemma 4.2.*

(ii) $x_1 < \alpha_1 < \alpha_2 < \alpha_3 < x_2$

(iii) $\delta < 0$

Then for ε sufficiently small, system (52) *has a family of limit cycles L_ε. Further these limit cycles are unique for every such ε and*

$$\lim_{\varepsilon \to 0} L_\varepsilon = L_0 \tag{73}$$

where L_0 is the DPS of the system.

Both theorems are proved by verification of the conditions in Theorem 2.1. Notice that the possible cases C_1, C_2 and C_3 for DPS do not satisfy the fourth condition in Theorem 2.1. Thus they have to be pulled out. Finally, observe also that the conditions on A and δ in the above theorems are very natural since they ensures boundedness of the solutions by Corollary 3.1.

5 Hopf and Bautin bifurcation

In the previous section we deduced conditions ensuring a unique limit cycle. Although this is a very nice result it has the drawback that ε was assumed to be very small. Natural questions to ask are if there exist limit cycles not occurring as a result of singular perturbation and how many they are and what about their stability. A standard method frequently used to answer the first question is the Hopf Bifurcation Theorem. In order to find some answers to the remaining ones it is very useful to use the theory of Lyapunov coefficients introduced in the preliminaries. It should be remarked that all analysis in this section is focused on the behavior in a small neighborhood of the origin.

5.1 $I = 0$

The class of FitzHugh-Nagumo systems that we are interested in are, as stated before, of the form

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x-B)(x-\lambda) \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y). \end{cases} \tag{74}$$

The system matrix for this system is given by

$$\mathcal{A} = \begin{pmatrix} -AB\lambda & -C \\ \varepsilon & -\delta\varepsilon. \end{pmatrix}$$

First of all one verifies that $\sigma = -AB\lambda - \varepsilon\delta$ and $\Delta = AB\delta\varepsilon\lambda + \varepsilon C$. Since we are interested in the case when (74) has a pair of purely imaginary eigenvalues, the following restriction on the parameters must hold

$$\begin{cases} \delta = \delta^* = -\dfrac{AB\lambda}{\varepsilon} \\ \Delta = -A^2B^2\lambda^2 + C\varepsilon > 0. \end{cases} \tag{75}$$

An application of Hopf Bifurcation Theorem shows that our system has a periodic solution for all sets of nonzero parameters $(A, B, C, \delta, \varepsilon, \lambda)$ satisfying (75) and $\varepsilon \neq 0$. For a more thorough investigation we use the theory of Lyapunov coefficients introduced in the preliminaries. According to formula (44) the first Lyapunov coefficient is given by

$$L_1(\delta*) = \frac{-A\pi}{4\Delta^{\frac{3}{2}}}[2A^2B\lambda^3 + A^2B^2\lambda^2 + 2A^2B^3\lambda + 3C\varepsilon] = \frac{-A\pi}{4\Delta^{\frac{3}{2}}}p_1(\lambda). \tag{76}$$

An easy computation shows that $\frac{d}{d\lambda}(p_1(\lambda) - 3\varepsilon)$ has only one zero since $-11A^4B^4 < 0$ for all values of A, B, C, ε. Thus, for every ε, there exists a unique solution $\bar{\lambda} = \bar{\lambda}(\varepsilon)$ to the equation $L_1 = 0$ for every ε. From (76) the sign of L_1 is given by

$$\text{sign}(L_1) = -\text{sign}(A)\text{sign}(B)\text{sign}(\lambda - \bar{\lambda}) \tag{77}$$

and thus there are eight possibilities:

I $A > 0$, $B > 0$ and $\lambda < \bar{\lambda}$;

	$\delta < \frac{-AB\lambda}{\varepsilon}$	$\delta = \frac{-AB\lambda}{\varepsilon}$	$\delta > \frac{-AB\lambda}{\varepsilon}$
Cases I-IV, $\varepsilon < 0$	Origin stable Unstable L.C	Origin unstable No L.C.	Origin unstable No L.C.
Cases I-IV, $\varepsilon > 0$	Origin unstable No L.C.	Origin unstable No L.C.	Origin stable Unstable L.C.
Cases V-VIII, $\varepsilon < 0$	Origin stable No L.C.	Origin stable No L.C.	Origin unstable Stable L.C.
Cases V-VIII, $\varepsilon > 0$	Origin unstable Stable L.C.	Origin stable No L.C.	Origin stable No L.C.

Table 2: Hopf bifurcation table for system (74)

II $A > 0$, $B < 0$ and $\lambda > \bar{\lambda}$;

III $A < 0$, $B > 0$ and $\lambda > \bar{\lambda}$;

IV $A < 0$, $B < 0$ and $\lambda < \bar{\lambda}$;

V $A < 0$, $B < 0$ and $\lambda > \bar{\lambda}$;

VI $A < 0$, $B > 0$ and $\lambda < \bar{\lambda}$;

VII $A > 0$, $B < 0$ and $\lambda < \bar{\lambda}$;

VIII $A > 0$, $B > 0$ and $\lambda > \bar{\lambda}$.

Application of the Table 1 on our system gives the following bifurcation table At the parameter value $\lambda = \bar{\lambda}$, the first Lyapunov coefficient can not be used to draw any conclusions. In this case the second coefficient, L_2, has to be calculated. Using (45) the following expression for L_2 is obtained

$$L_2 = -\frac{\pi A^3 B\bar{\lambda}}{24\Delta^{\frac{5}{2}}}[16A^2B\bar{\lambda}^3 + 23A^2B^2\bar{\lambda}^2 + 16A^2B^3\bar{\lambda} + 9C\varepsilon] = -\frac{\pi A^3 B\bar{\lambda}}{24\Delta^{\frac{5}{2}}}p_2(\bar{\lambda}). \tag{78}$$

Theorem 5.1. *If $A, B, C, \varepsilon, \lambda \neq 0$, $\delta = \frac{-AB\lambda}{\varepsilon}$ ($\sigma = 0$) and $-A^2B^2\lambda^2 + C\varepsilon > 0$ ($\Delta > 0$) the origin of system (74) is a multiple focus of order not larger than two.*

Proof. The conditions on σ and Δ are equivalent to saying that the origin is a multiple focus. From (76) and (78) and the assumption of nonzero parameters it follows that

$$\begin{cases} L_1 = 0 \\ L_2 = 0 \end{cases} \Leftrightarrow \begin{cases} p_1 = p_1(A, B, C, \varepsilon, \lambda) = A^2B\lambda(2\lambda^2 + B\lambda + 2B^2) + 3C\varepsilon = 0 \\ p_2 = p_2(A, B, C, \varepsilon, \lambda) = A^2B\lambda(16\lambda^2 + 23B\lambda + 16B^2) + 9C\varepsilon = 0 \end{cases} \tag{79}$$

From this we conclude that $L_1 = L_2 = 0 \Rightarrow \lambda = -B$ by elimination of the term $C\varepsilon$. Replacement of λ by $-B$ in the expression for p_1 gives that $A^2B^4 = C\varepsilon$. But this contradicts the assumption that $\Delta > 0$. Thus we conclude that L_1 and L_2 can not be zero simultaneously and by definition the origin can not be a multiple focus of order greater than two since $L_1 = 0 \Rightarrow L_2 \neq 0$. $\square$

Corollary 5.1. *Let the assumption of Theorem 5.1 hold. If $L_1 = 0$ then $L_2 < 0$ for all parameters fulfilling the requirements.*

Proof. Let $A = B = 1$ and $C\varepsilon = \frac{1}{3}$. Further, choose λ such that $L_1 = 0$. This is equivalent to solve the equation

$$p_1 = 2\lambda^3 + \lambda^2 + 2\lambda + 1 = 0. \tag{80}$$

The solutions to (80) are given by $\lambda = \pm\imath$ and $\lambda = \frac{-1}{2}$. Thus $\lambda = \frac{-1}{2}$ is the only real value of λ making $L_1 = 0$. It is also easy to verify that this value together with $A = B = 1$ and $C\varepsilon = \frac{1}{3}$ satisfy the assumptions ($\Delta = \frac{-1}{4} + \frac{1}{3} > 0$). Plugging in this choice of parameters in (78) we see that $L_2 < 0$. But since $L_2 : \mathbb{R}^5 \to \mathbb{R}$ is continuous and $L_2 \neq 0$, by theorem 5.1 it must hold that $L_2 < 0$ for all choices of parameters fulfilling the assumptions. □

Theorem 5.2. *Let the assumptions of Theorem 5.1 hold and assume that $L_1 = 0$. Then*

(i) There exist parameters such that two limit cycles bifurcate from the origin and this is the maximal number of bifurcating limit cycles.

(ii) A necessary condition for this is that the origin is stable.

(iii) The inner cycle will be unstable while the outer stable.

Proof. *(i)* Apply Theorem 5.1 and Theorem 2.2.

(ii) – *(iii)* By Corollary 5.1 the outer cycle has to be stable from the outside and by Theorem 2.3 part 2 it is thus stable also from the inside. By part 1 of this Theorem two is the maximal number of limit cycles and thus the inner cycle has to be unstable from the outside. Applying Theorem 2.3 part 2 once again shows that the inner cycle is unstable and thus the origin has to be stable.

□

Remark 5.1. *When $L_1 = 0$, the bifurcation is also called Bautin bifurcation or generalized Hopf bifurcation. The points in the parameter space where Bautin bifurcation takes place are said to be Bautin points.*

5.2 $I \neq 0$

Now we turn to study the full system

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x - B)(x - \lambda) + I \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y). \end{cases} \tag{81}$$

The equilibrium points of (81) are given by the equations

$$\begin{cases} x = \delta y \\ y = -\dfrac{A}{C}x(x - B)(x - \lambda) + \dfrac{I}{C}, \end{cases} \tag{82}$$

which is equivalent to

$$k(y) := y^3 - \frac{\lambda + B}{delta} y^2 + \frac{AB\lambda\delta + C}{A\delta^3} y = \frac{I}{A\delta^3}. \tag{83}$$

Restricting to the case of a unique fix point is equivalent to saying that $k'(y)$ does not have any real zeros, i.e. the parameters must satisfy the condition

$$\frac{A\delta(\lambda - B)^2 + AB\lambda\delta - 3C}{A\delta} < 0. \tag{84}$$

Now, let (x_0, y_0) denote the unique fixed point and make the change of variables

$$\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0. \end{cases} \tag{85}$$

In thees coordinates the system takes the form

$$\begin{cases} \dot{\tilde{x}} = -C\tilde{y} - A\phi(\tilde{x}) + C_1 \\ \dot{\tilde{y}} = \varepsilon(\tilde{x} - \delta\tilde{y}) + C_2, \end{cases} \tag{86}$$

where

$$\begin{cases} \phi(\tilde{x}) = -A(\tilde{x}^3 + \tilde{x}^2(3x_0 - (\lambda + B)) + \tilde{x}(3x_0^2 - 2x_0(B + \lambda) + B\lambda) \\ \quad C_1 = I - Cy_0 - Ax_0(x_0 - B)(x_0 - \lambda) \\ \quad C_2 = \varepsilon(x_0 - \delta y_0). \end{cases} \tag{87}$$

Remark 5.2. *By definition of x_0 and y_0 we conclude that $C_1 = C_2 = 0$.*

Depending on the value of x_0 two different cases appear. First assume that

$$x_0 \in (\alpha - \beta, \alpha + \beta), \tag{88}$$

where

$$\begin{cases} \alpha = \dfrac{\lambda + B}{3} \\ \beta = \dfrac{-2\sqrt{(\lambda - B)^2 - 2B\lambda}}{3}. \end{cases} \tag{89}$$

In this case there exists γ_1 and γ_2 such that

$$\phi(\tilde{x}) = -A\tilde{x}(\tilde{x} - \gamma_1)(\tilde{x} - \gamma_2) \tag{90}$$

and which are given by

$$\begin{cases} \gamma_1 + \gamma_2 = 3x_0 - (\lambda + B) \\ \quad \gamma_1\gamma_2 = (x_0 - B)(x_0 - \lambda) + x_0(2x_0 - (\lambda + B). \end{cases} \tag{91}$$

Thus (81) can be written as

$$\begin{cases} \dot{\tilde{x}} = -C\tilde{y} - A\tilde{x}(\tilde{x} - \gamma_1)(\tilde{x} - \gamma_2) \\ \dot{\tilde{y}} = \varepsilon(\tilde{x} - \delta\tilde{y}) \end{cases} \tag{92}$$

and therefore the analysis of this system is exactly the same as for the case $I = 0$ with the exception that γ_1 and γ_2 no longer are independent parameter but are related via the equations (91). When x_0 does not satisfies (88) all calculation has to be made all over again. This will not be done here except for the first Lyapunov coefficient given by

$$\begin{aligned} L_1(x_0) = & \frac{A^3\pi}{4\Delta^{\frac{3}{2}}}(-27x_0^4 + 36(\lambda + B)x_0^3 - 18(\lambda + B)^2x_0^2 + 4(\lambda + B)^3x_0 + \\ & - (2B\lambda^3 + B^2\lambda^2 + 2B^3\lambda) - \frac{3C\varepsilon}{A^2}). \end{aligned} \tag{93}$$

6 Saddle-node bifurcation

This section is concerned with the case of exact one eigenvalue of the system matrix at a fixed point being zero. In this case the fixed point is called a saddle-node point. We are especially interested in finding a parameter μ and a fixed point $(x_0(\mu_0), y_0(\mu_0))$, for $\mu = \mu_0$, such that there exists a smooth curve of equilibriums in $\mathbb{R}^2 \times \mathbb{R}$ which passes through $(x_0(\mu_0), y_0(\mu_0), \mu_0)$ and is tangent to the plane $\mathbb{R}^2 \times \{\mu_0\}$. Further, it should hold that a new fixed point is born when μ crosses μ_0 from one direction and that the fixed point disappear when μ crosses μ_0 in the other direction. In this case we say that $(x_0(\mu_0), y_0(\mu_0))$ undergoes a saddle-node bifurcation.

6.1 $I = 0$

In this case the equilibrium points of our system are given by $(0, 0)$ and

$$\begin{cases} x_0 = \dfrac{B+\lambda}{2} \pm \dfrac{\sqrt{(B+\lambda)^2 - \frac{4C}{A\delta}}}{2} \\ y_0 = \dfrac{x}{\delta}. \end{cases} \tag{94}$$

Regardless of which of the parameters A, B, C, δ or λ we choose as a bifurcation parameter there can not exist a $\mu_0 \in \mathbb{R} - \{0\}$ such that

$$\lim_{\mu \to \mu_0} (x_0(\mu), y_0(\mu)) = (0, 0). \tag{95}$$

Thus there can not be a saddle-node bifurcation at the origin. Assume now that system (74) has two fixed points, i.e. that

$$(B+\lambda)^2 - \frac{4C}{A\delta} = 0. \tag{96}$$

Then

$$(x_0, y_0) = \left(\frac{B+\lambda}{2}, \frac{B+\lambda}{2\delta} \right) \tag{97}$$

and by making the change of variables

$$\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0, \end{cases} \tag{98}$$

the system matrix becomes

$$\mathcal{A} = \begin{pmatrix} -A(-\frac{(B+\lambda)^2}{4} + B\lambda) & -C \\ \varepsilon & \varepsilon\delta \end{pmatrix}.$$

Exactly one of the eigenvalues are zero iff

$$\begin{cases} \Delta = 0 \\ \sigma \neq 0. \end{cases} \tag{99}$$

Using equation (96) gives that

$$\Delta = \frac{A\varepsilon\delta}{2}(B^2 + \lambda^2) \neq 0 \tag{100}$$

since we are assuming that all parameters are nonzero. Thus there can not be any saddle-node bifurcation from the fixed point (x_0, y_0).

6.2 $I \neq 0$

Let I be the bifurcation parameter of the full system

$$\begin{cases} \dfrac{dx}{dt} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{dy}{dt} = \varepsilon(x - \delta y). \end{cases} \tag{101}$$

For $I = 0$ the eigenvalues of the system matrix at the origin are given by

$$s_{1,2} = \frac{\sigma}{2} \pm \frac{\sqrt{\sigma^2 - 4\Delta}}{2}$$

and thus it is necessary that

$$\begin{cases} \sigma = -(AB\lambda + \varepsilon\delta) \neq 0 \\ \Delta = AB\delta\varepsilon + C\varepsilon = 0 \end{cases} \tag{102}$$

for the origin to be a saddle-noddle point. In this case the eigenvalues are given by 0 and σ with corresponding eigenvectors given by

$$\begin{cases} e_0 = \begin{pmatrix} 1 & -\frac{AB\lambda}{C} \end{pmatrix}^T \\ e_\sigma = \begin{pmatrix} 1 & \frac{\delta\varepsilon}{C} \end{pmatrix}^T. \end{cases} \tag{103}$$

Introduce the change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e_0 & e_\sigma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \frac{C}{AB\lambda + \delta\varepsilon} \begin{pmatrix} \frac{\delta\varepsilon}{C} & -1 \\ \frac{AB\lambda}{C} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{104}$$

After some calculations one obtain the system

$$\begin{cases} \dfrac{du}{dt} = \dfrac{A\delta\varepsilon}{AB\lambda + \delta\varepsilon}[(B+\lambda)(u+v)^2 - (u+v)^3 + \dfrac{I}{A}] \\ \dfrac{dv}{dt} = -(AB\lambda + \delta\varepsilon)v + \dfrac{A^2B\lambda}{AB\lambda + \delta\varepsilon}[(B+\lambda)(u+v)^2 - (u+v)^3 + \dfrac{I}{A}] \end{cases} \tag{105}$$

in the new variables. If we introduce the equation $\dot{I} = 0$ we get the new system

$$\begin{cases} \dfrac{du}{dt} = \dfrac{A\delta\varepsilon}{AB\lambda + \delta\varepsilon}[(B+\lambda)(u+v)^2 - (u+v)^3 + \dfrac{I}{A}] \\ \dfrac{dI}{dt} = 0 \\ \dfrac{dv}{dt} = -(AB\lambda + \delta\varepsilon)v + \dfrac{A^2B\lambda}{AB\lambda + \delta\varepsilon}[(B+\lambda)(u+v)^2 - (u+v)^3 + \dfrac{I}{A}] \end{cases} \tag{106}$$

living in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. By assumption this system has two zero eigenvalues and one nonzero. Thus by the Center Manifold theorem there exist a locally defined smooth 2-dimensional invariant manifold $\Sigma \subset \mathbb{R}^3$ such that the tangent plane to Σ is spanned by the coordinate axis of u and I. Thus Σ may be expressed as a graph

$$v = h(u, I) = au^2 + buI + cI^2 + O(3), \tag{107}$$

where $O(3)$ means terms of order $u^3, u^2 I, uI^2$ and I^3. Further, by theorem 5.2 in [14] system (106) is topological equivalent to

$$\begin{cases} \dfrac{du}{dt} = \dfrac{A\delta\varepsilon}{AB\lambda + \delta\varepsilon}[(B+\lambda)(u+v)^2 - (u+v)^3 + \dfrac{I}{A}] \\ \dfrac{dI}{dt} = 0 \\ \dfrac{dv}{dt} = -(AB\lambda + \delta\varepsilon)v. \end{cases} \tag{108}$$

Due to this fact we only need to study the subsystem of u and I. If we assume that $B \neq -\lambda$ the following steps in section 3.3 in [14] are smooth

Step2 Introduction of the new parameter

$$\tilde{\mu} = \frac{\delta\varepsilon}{AB\lambda + \delta\varepsilon} I. \tag{109}$$

Step3 The time-scaling

$$t = \left| \frac{AB\lambda + \delta\varepsilon}{A\delta\varepsilon(B+\lambda)} \right| \tau \tag{110}$$

and introduction of the new parameter

$$\mu = \tilde{\mu} \left| \frac{AB\lambda + \delta\varepsilon}{A\delta\varepsilon(B+\lambda)} \right| \tau. \tag{111}$$

With these transformations and the use of (107) the differential equation for u takes the form

$$\frac{du}{dt} = \mu + su^2 + O(3), \tag{112}$$

where

$$s = \operatorname{sign}\left(\frac{AB\lambda + \delta\varepsilon}{A\delta\varepsilon(B+\lambda)} \right). \tag{113}$$

By Lemma 3.1 in [14] the term $O(3)$ may be dropped. Thus we end up with

$$\frac{du}{dt} = \mu + su^2. \tag{114}$$

The bifurcation diagrams of (114) are shown in Figures 6.2 and 6.2.

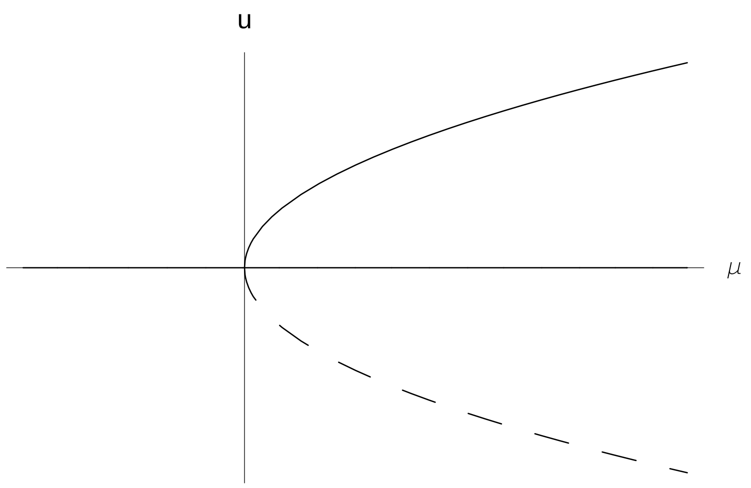

Figure 1: Bifurcation diagram of (114) for $s = -1$. The dashed curve consists of unstable fixed points while the solid are stable.

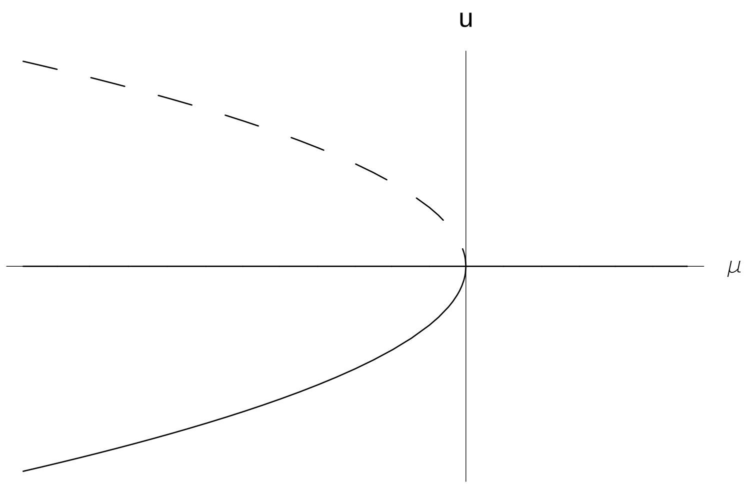

Figure 2: Bifurcation diagram of (114) for $s = 1$. The dashed curve consists of unstable fixed points while the solid are stable.

7 Bogdanov-Takens bifurcation

In the previous section we discussed the case when our system had exactly one zero eigenvalue. Now we are going to study the case when both eigenvalues of the system matrix at the origin are zero. This is called the Bogdanov-Takens bifurcation. Since the eigenvalues at the origin are given by

$$s_{1,2} = \frac{\sigma}{2} \pm \frac{\sqrt{\sigma^2 - 4\Delta}}{2} \tag{115}$$

the condition is that

$$\begin{cases} \sigma = -(AB\lambda + \varepsilon\delta) = 0 \Leftrightarrow \delta_0 = -\dfrac{AB\lambda_0}{\varepsilon} \\ \Delta = AB\lambda\varepsilon\delta + \varepsilon C = 0 \Leftrightarrow \lambda_0^2 = \dfrac{\varepsilon C}{A^2B^2}. \end{cases} \tag{116}$$

We are now looking for vectors v_0, v_1, w_0 and w_1 such that

$$\begin{cases} \mathcal{A}v_0 = 0 \\ \mathcal{A}v_1 = v_0 \\ \mathcal{A}^T w_1 = 0 \\ \mathcal{A}^T w_0 = w_1 \\ v_0^T w_0 = 1 \\ v_1^T w_1 = 1 \end{cases} \tag{117}$$

with $\mathcal{A}$ being the system matrix at the origin for $\delta = \delta_0$ and $\lambda = \lambda_0$. After some calculations we see that the following set of vectors fulfill this requirements

$$\begin{cases} v_0 = \begin{pmatrix} -\frac{AB\lambda_0}{\varepsilon} & 1 \end{pmatrix}^T \\ v_1 = \begin{pmatrix} \frac{1}{\varepsilon} & 0 \end{pmatrix}^T \\ w_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \\ w_1 = \begin{pmatrix} \varepsilon & AB\lambda_0 \end{pmatrix}^T. \end{cases} \tag{118}$$

After making the change of variables

$$\begin{cases} u = \begin{pmatrix} x & y \end{pmatrix} w_0 \\ v = \begin{pmatrix} x & y \end{pmatrix} w_1 \end{cases} \tag{119}$$

and calculating the Taylor coefficients a_{20}, b_{20} and b_{11} for $\dot{u}$ and $\dot{v}$, Theorem 8.4 in [14] can be applied yielding that our system is topological equivalent to the system

$$\begin{cases} \dfrac{d\eta_1}{dt} = \eta_2 \\ \dfrac{d\eta_2}{dt} = \beta_1 + \beta 2\eta_1 + \eta_1^2 + k\eta_1\eta_2 \end{cases} \tag{120}$$

for $k = \mathrm{sign}[b_{20}(a_{20} + b_{11})]$ and β_1 and β_2 being parameters. In our case

$$k = \mathrm{sign}[-\frac{4A^3BC\lambda_0(B+\lambda_0)^2}{\varepsilon}]. \tag{121}$$

Explicit calculations of the parameters β_1 and β_2 is left for future study.

8 Conclusion

The main result of this paper is that we have shown that at most two limit cycles can bifurcate from the origin via Hopf and Boutin bifurcations. In connection to this we have also shown that if the first Lyapunov coefficient is zero then the second one is always less than zero. A consequence of this is that the origin has to be stable if two limit cycles are to bifurcate from it. Further, if there is such a bifurcation the outer limit cycle is stable while the inner one is unstable. In the particular case of one special parameter being very small we provide sufficient conditions for the existence of a unique stable limit cycle. A complete saddle-node bifurcation picture as well as a Bodganov-Taken bifurcation analysis together with sufficient conditions for boundedness of solutions are also given.

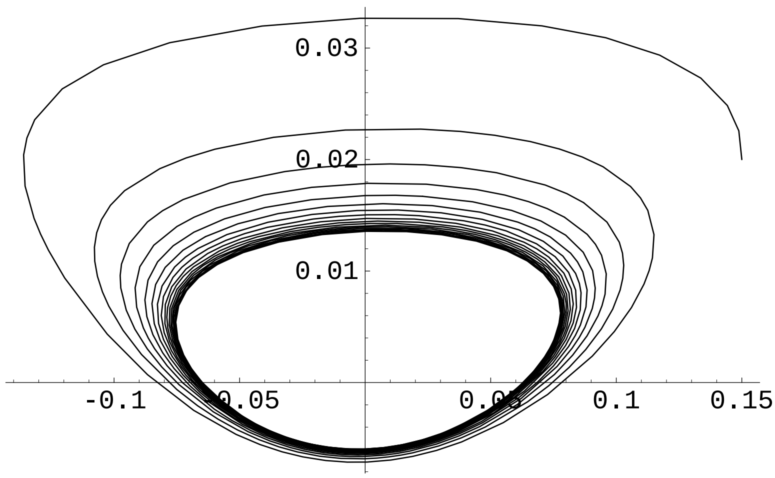

Figure 3: The parameter values are; $A = B = C = 1$, $\delta = 0.5$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is a unique unstable fixed point, $L_1 < 0$ and $\delta < \delta^*$. Further more, $x_0 = 0.15$ and $y_0 = 0.02$.

Appendix

Remark 8.1. *Plot 6 and 7 reflects the local propery of the Lyapunov coefficients and the conclusions presented in Table 2.2. The existence of the stable limit cycle indicated in the picture can be verified by Theorem* 5 *in [19].*

Remark 8.2. *In order to indicate the existence of the claimed unstable limit cycle for the parameters as in plots 8 and 9, the time is reversed in these plots.*

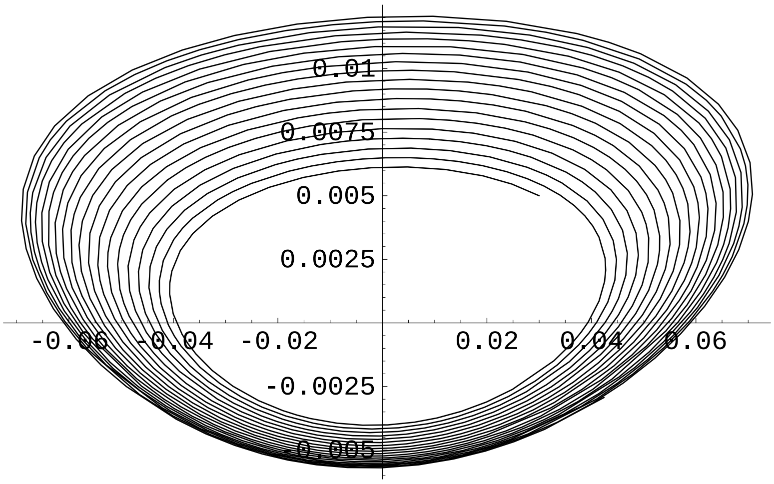

Figure 4: The parameter values are; $A = B = C = 1$, $\delta = 0.5$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is an unique unstable fixed point, $L_1 < 0$ and $\delta < \delta^*$. Further more, $x_0 = 0.03$ and $y_0 = 0.005$.

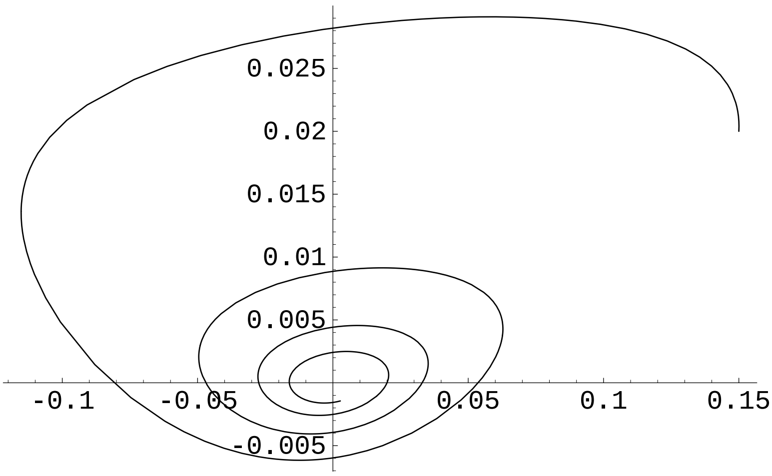

Figure 5: The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is a unique stable fixed point, $L_1 < 0$ and $\delta > \delta^*$. Further more, $x_0 = 0.15$ and $y_0 = 0.02$.

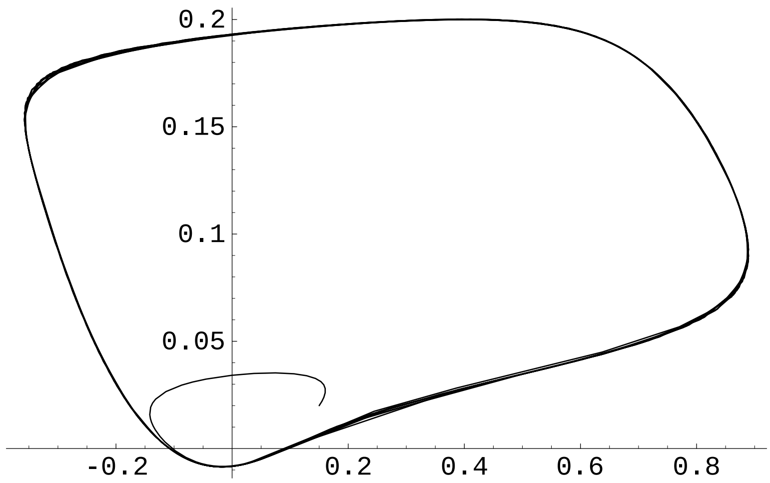

Figure 6: The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique unstable fixed point, $L_1 > 0$ and $\delta < \delta^*$. Further more, $x_0 = 0.15$ and $y_0 = 0.02$.

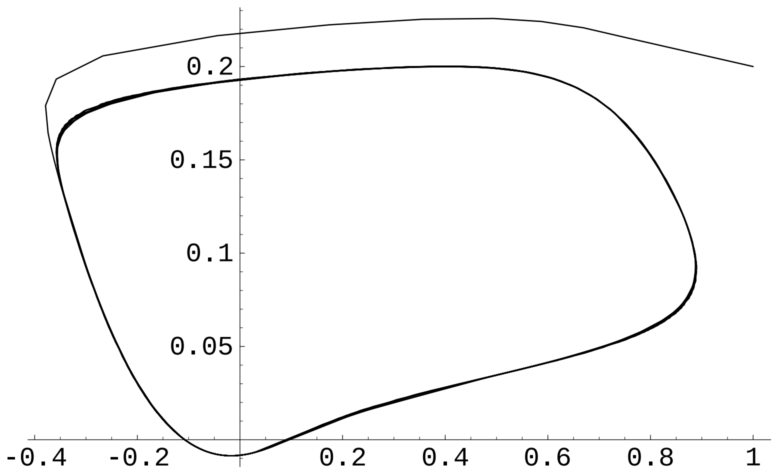

Figure 7: The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique unstable fixed point, $L_1 > 0$ and $\delta < \delta^*$. Further more, $x_0 = 1$ and $y_0 = 0.2$.

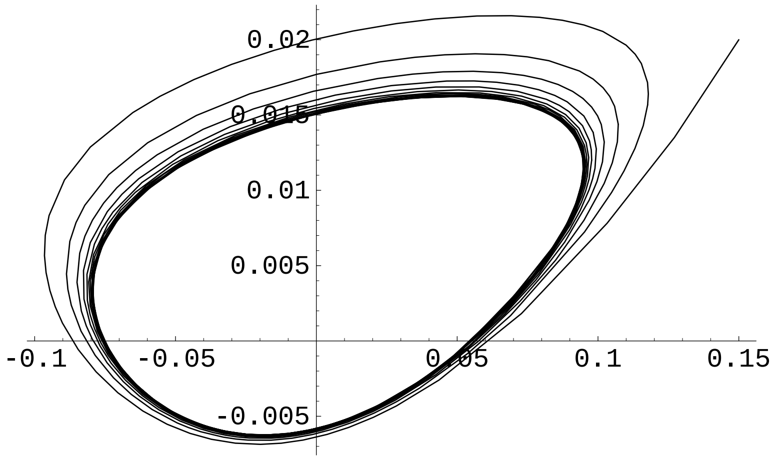

Figure 8: The parameter values are; $A = B = C = 1$, $\delta = 3$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique stable fixed point, $L_1 > 0$ and $\delta > \delta^*$. Further more, $x_0 = 0.15$ and $y_0 = 0.02$ and the time is reversed.

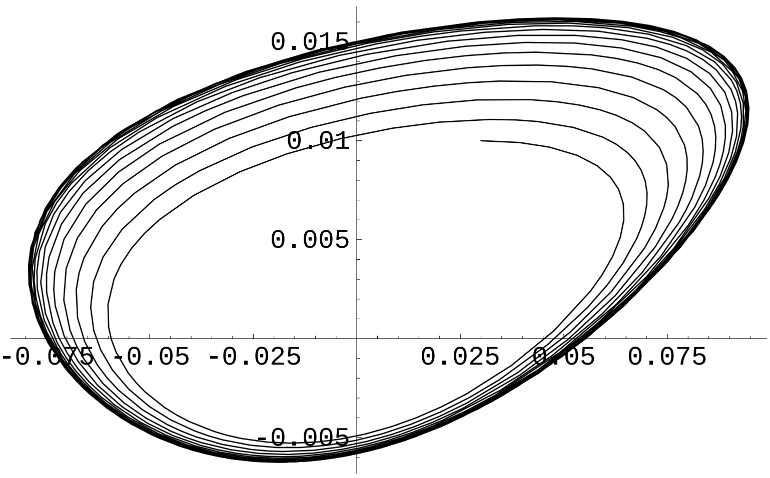

Figure 9: The parameter values are; $A = B = C = 1$, $\delta = 3$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique stable fixed point, $L_1 > 0$ and $\delta > \delta^*$. Further more, $x_0 = 0.03$ and $y_0 = 0.01$ and the time is reversed.

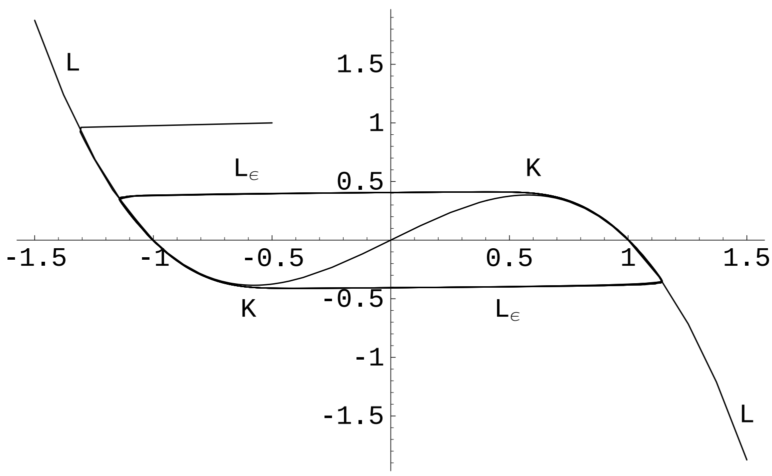

Figure 10: The parameter values are; $A = B = C = \delta = 1$, $\varepsilon = 0.015$ and $\lambda = -1$. Thus the origin is a unique unstable fixed point and the conditions in Theorem 4.5 are satisfied.

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On dynamical Behavior of FitzHugh-Nagumo Systems –A Case Study on Weakly Coupled FitzHugh-Nagumo Oscillators

Consider the following weakly coupled FitzHugh-Nagumo systems,

$$\begin{cases} \dfrac{dx_1}{dt} = -C_1 y_1 - A_1 x_1 (x_1 - B_1)(x_1 - \lambda_1) + \gamma_{12}(x_1 - x_2) \\ \dfrac{dy_1}{dt} = \varepsilon_1 (x_1 - \delta_1 y_1) \\ \dfrac{dx_2}{dt} = -C_2 y_2 - A_2 x_2 (x_2 - B_2)(x_2 - \lambda_2) + \gamma_{21}(x_2 - x_1) \\ \dfrac{dy_2}{dt} = \varepsilon_2 (x_2 - \delta_2 y_2) \end{cases}$$

where the parameters are nonzero and the coupling constant γ's are very small.

This model in original FitzHugh version is designed to emulate two neurons linked with electrical coupling, i.e. coupling via the flow of ions through the gap junctions between neurons. From a biological point of view, if we consider the two neurons to be in a similar region of the brain, it is likely that the parameter values will be similar but not identical. Thus the focus should be on the case where the parameters are such that both neurons will be capable of exhibiting the same qualitative behavior. Nevertheless, to give a basis from which to start, we consider the behavior when two neurons with identical parameters are coupled together, and further we assume that the coupling is symmetric, that is the γ's are the same, i.e. we drop the index in the parameters

$$\begin{cases} \dfrac{dx_1}{dt} = -C y_1 - A x_1 (x_1 - B)(x_1 - \lambda) + \gamma(x_1 - x_2) \\ \dfrac{dy_1}{dt} = \varepsilon (x_1 - \delta y_1) \\ \dfrac{dx_2}{dt} = -C y_2 - A x_2 (x_2 - B)(x_2 - \lambda) + \gamma(x_2 - x_1) \\ \dfrac{dy_2}{dt} = \varepsilon (x_2 - \delta y_2) \end{cases}$$

It is clear that the restriction of identical neurons and the symmetric coupling implies the property that the system is invariant under transformation $(x_1, y_1, x_2, y_2) \to (x_2, y_2, x_1, y_1)$. In turn the set $\{(x_1, y_, x_2, y_2) | x_1 = x_2, y_1 = y_2\}$ is invariant, i.e. the flow initiated at a point in this set remains in the set. Physically, the invariance of this subset means that if the two neurons start with identical conditions then their subsequent behavior will also be identical.

8.1 Possible bifurcations

As a standard analysis, we can start by determining the fixed points of the equations, and compute the system matrix $\mathcal{A}$ at the fixed points. If none of the eigenvalues of $\mathcal{A}$ are on the imaginary axis then the local stability of the fixed points is rather simple. However, the global stability is harder. If eigenvalues lie on the imaginary axis, then many types of bifurcations can occur since the system is four dimensional.

There can be one, two, three and four zero eigenvalues, there can be one or two zero eigenvalues and a pair of purely imaginary eigenvalues, there can be one pair of purely imaginary eigenvalues, and there can be two different pairs of eigenvalues, or two double eigenvalues. For each eigenvalue the length of Jordan form also plays important role. Each case is a possible bifurcation.

From the analysis in Paper II we anticipate the saddle-node bifurcation from the origin as well as Hopf, Bautin and Bogdanov-Takens bifurcations. Moreover, we expect pitchfork, fold-Hopf, Hopf-Hopf bifurcations. Furthermore, we will not be surprised if the $\mathbb{Z}_2$ symmetric Hopf bifurcation takes place since the system is symmetric. Maybe our main point is that the system exhibits more complicated dynamics than limit cycles. This was shown in a study of coupled van der Pol oscillators, [1]. In this short note we give some examples.

Numerical examples and discussions

Let now $A = B = C = 1$. We have simulated the system for different parameters. Further we fix $\varepsilon = 0.015, \lambda = -0.04$, and $\delta = 3.56667$. Here are the simulations of three sets of parameters by varying the coupling constants γ. The time length of each simulation is 150000. Moreover, to have a systematic search we have a linear transformation of variables: $(x_1, y_1, x_2, y_2) \leftrightarrow (X_1, Y_1, X_2, Y_2)$. Note that this does not effect the characters of dynamics.

1. $\gamma = 0.077$: The simulation seems to show that there is a stable limit cycle.
2. $\gamma = 0.078$: It looks like that it undergoes a periodic doubling.
3. $\gamma = 0.086$: What can we say about this?

To get affirmative answers on these simulations it is required further analysis. This is left for future investigation.

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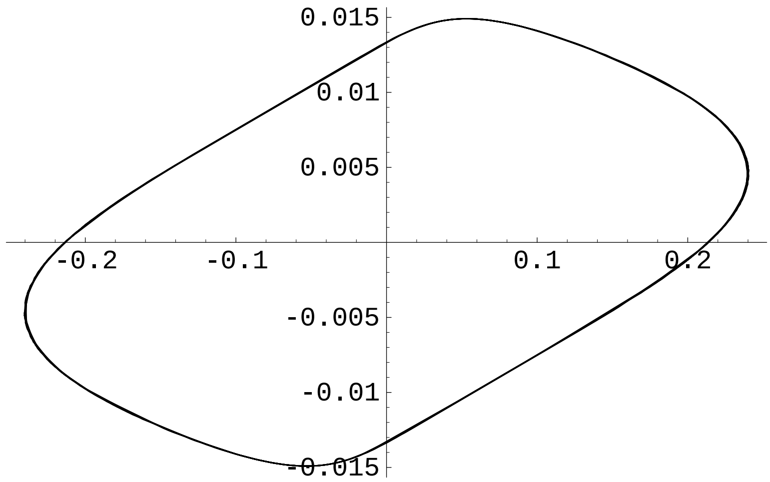

Figure 11: $\gamma = 0.0077$: the phase plane X_1, Y_1.

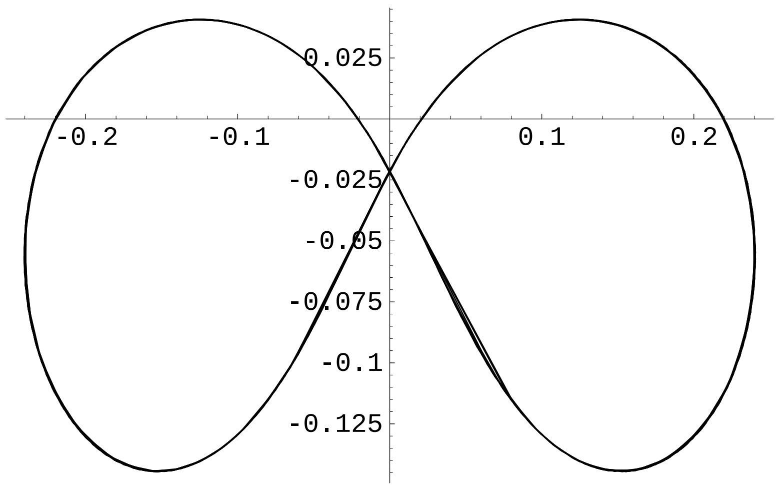

Figure 12: $\gamma = 0.0077$: the phase plane X_1, X_2.

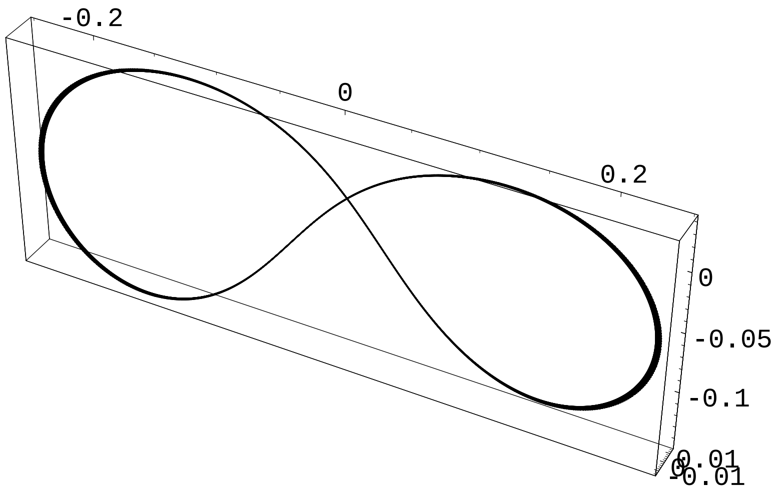

Figure 13: $\gamma = 0.0077$: the three dimensional $X_1, Y_1, X2$.

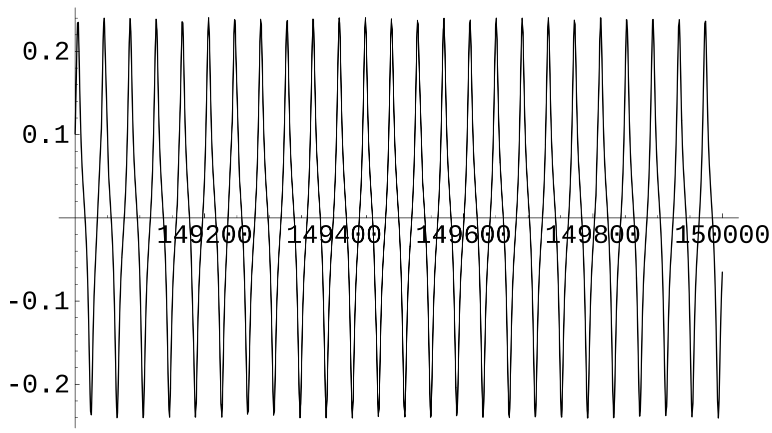

Figure 14: $\gamma = 0.0077$: X_1 vs t.

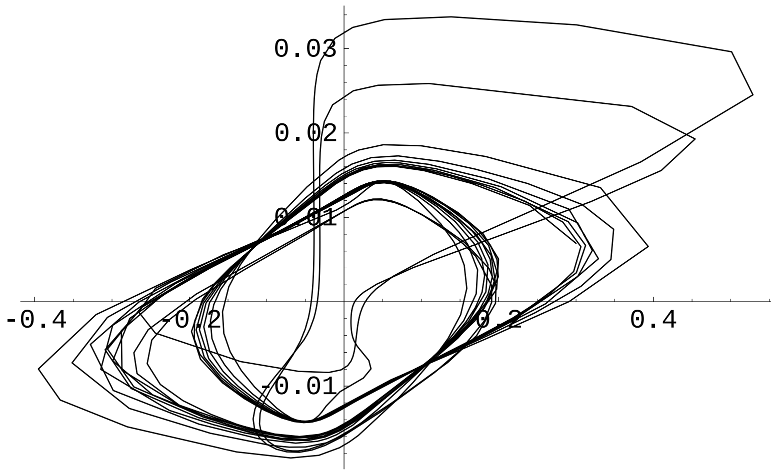

Figure 15: $\gamma = 0.0078$: the phase plane X_1, Y_1.

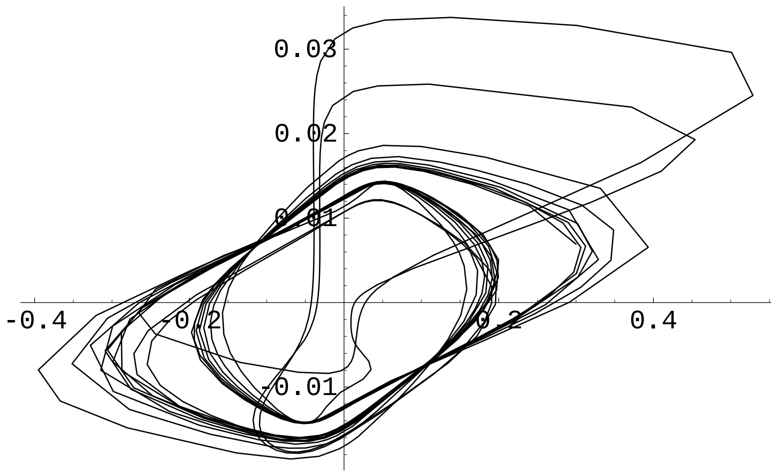

Figure 16: $\gamma = 0.0078$: the phase plane X_1, X_2.

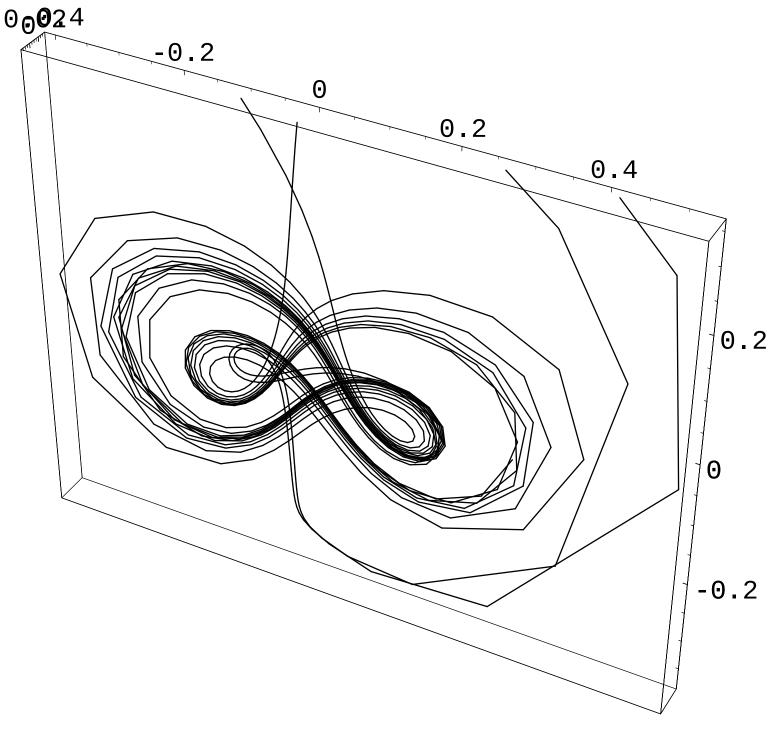

Figure 17: $\gamma = 0.0078$: the three dimension X_1, Y_1, X_2.

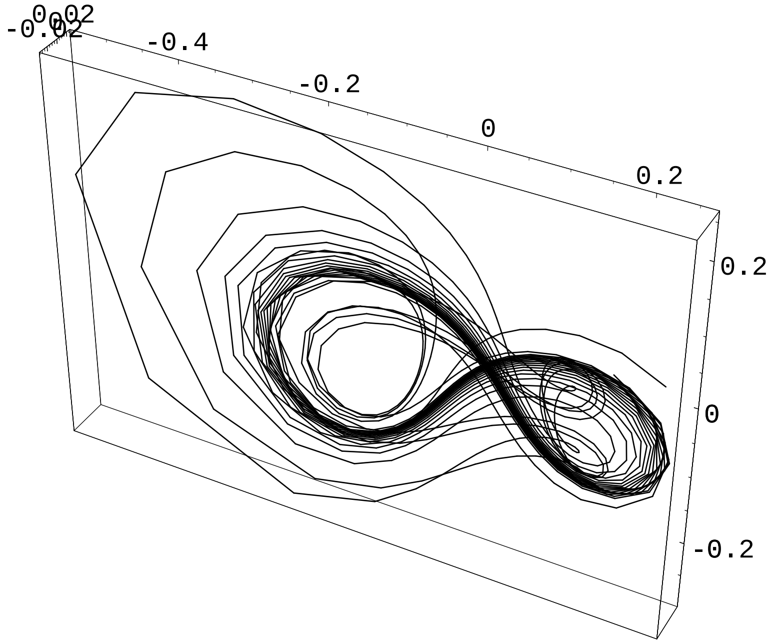

Figure 18: $\gamma = 0.0086$: the three dimension X_1, Y_1, X_2.