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Weyl's limit circles for Hain-Lüst operator whenever quasiregularity conditions are not satisfied

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On the essential spectrum of a class of singular matrix differential operators. II Weyl's limit circles for Hain-Lüst operator whenever quasiregularity conditions are not satisfied

Pavel Kurasov, Igor Lelyavin, and Serguei Naboko

Abstract. The essential spectrum of the singular matrix differential operator of mixed order determined by the following operator matrix

$$\begin{pmatrix} -\dfrac{d}{dx}\rho(x)\dfrac{d}{dx} + q(x) & \dfrac{d}{dx}\dfrac{\beta(x)}{x} \\ \\ -\dfrac{\beta(x)}{x}\dfrac{d}{dx} & \dfrac{m(x)}{x^2} \end{pmatrix}.$$

is studied. Investigation of the essential spectrum of the corresponding self-adjoint operator is continued but now without assuming that the quasiregularity conditions are satisfied. New conditions that guarantee that the operator is semibounded from below are derived. It is proven that the essential spectrum of any self-adjoint operator associated with the matrix differential operator is given by range $\left(\frac{m\rho-\beta^2}{\rho x^2}\right)$ in the case where the quasiregularity conditions are not satisfied.

1. Introduction

Matrix differential operators of mixed order attracted a lot of attention recently due to their interesting and unexpected spectral properties. Investigating problem related to magnetohydrodynamics it has been discovered that such operators may have so-called singularity essential spectrum - the essential spectrum connected entirely with the singular point of the operator [**4, 10, 11, 12, 14, 19, 27**]. Mathematically rigorous studies of matrix differential operators of mixed order have been carried out in [**2, 3, 7, 8, 9, 13, 18, 21, 22, 23, 29**]. In order to investigate the phenomenon of singularity essential spectrum it was suggested in [**17**](see this article for a detailed description of recent developments in this area) to study the following matrix ordinary differential operator

$$L = \begin{pmatrix} -\frac{d}{dx}\rho(x)\frac{d}{dx} + q(x) & \frac{d}{dx}\frac{\beta(x)}{x} \\ -\frac{\beta(x)}{x}\frac{d}{dx} & \frac{m(x)}{x^2} \end{pmatrix} \tag{1}$$

in the Hilbert space $\mathcal{H} = L_2[0,1] \oplus L_2[0,1]$. This operator is singular if the function β is not equal to zero or the function m does not have second order zero at the origin. It is natural to use the following assumptions on the coefficients

$$\rho, q, \beta, m \in C^2[0,1], \tag{2}$$

and that the density function ρ is positive definite

$$\rho(x) \geq \rho_0 > 0. \tag{3}$$

Singular matrix differential operators with coefficients of mixed order appear in different problems related to applications in physics and engineering, in particular in magnetohydrodynamics. It appeared that these operators have interesting structure of the spectrum and therefore attracted attention of specialists in spectral theory. The operators appearing in realistic problems are rather complicated and their spectral analysis leads to tedious calculations which make it difficult to study the interplay between the matrix coefficients. It appears to us that the operator (1) is the simplest matrix differential operator possessing the following spectral property: its essential spectrum can not be obtained as a limit as $\epsilon \to 0^+$ of the essential spectrum of the same differential operator restricted to the interval $[\epsilon, 1]$. Note that the differential order of the coefficients and the orders of the singularities cancel in the formal determinant of the operator L: the differential order of the product of the diagonal coefficients is $2+0$ and of the antidiagonal is $1+1$. Similar for the orders of the singularities $0+2 = 1+1$. This property allows the unusual interplay between the matrix coefficients.

It has been proven that the essential spectrum of the corresponding operator $\mathbf{L}$ is bounded if and only if the following quasiregularity conditions are satisfied

$$\left\{ \begin{array}{rcl} \rho m - \beta^2|_{x=0} & = & 0, \\ \frac{d}{dx}\left(\rho m - \beta^2\right)|_{x=0} & = & 0. \end{array} \right. \tag{4}$$

It appears that under these conditions the essential spectrum consists of two parts: regularity and singularity spectra. The first part of the essential spectrum is determined by the behavior of the coefficients in the whole interval $x \in [0,1]$ and is given by

$$\mathcal{R}\left(\frac{\rho m - \beta^2}{\rho x^2}\right). \tag{5}$$

This spectrum can be obtained by considering the sequence of regular matrix differential operators on the intervals $(\epsilon, 1]$ as $\epsilon \to 0$. The second part of the essential spectrum is called the singularity spectrum is determined by the limits of the coefficients at the origin, i.e. exclusively by the singularity. This spectrum cannot be obtained as a limit described above. The singularity spectrum is equal to

$$\left[\frac{l_0}{4 + \frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right] \tag{6}$$

where $l_0 = \lim_{x\to 0}\left(\frac{\rho m - \beta^2}{m x^2}\right)$. A similar operator has been studied later in [**18**] under quasiregularity conditions as well.

In the current article we study the case where the essential spectrum of the matrix differential operator is not necessarily bounded, but the operator is just semibounded from below. This assumption is natural in numerous physical applications.

The differential expression L does not determine the self-adjoint operator in $\mathcal{H}$ uniquely and therefore on the first step it is natural to associate with L a certain minimal operator $\mathbf{L}_{\min}$. Since the end point $x = 1$ is regular for the matrix differential operator we decided to define $\mathbf{L}_{\min}$ on the set of functions from $C_0^\infty(0,1] \oplus C_0^\infty(0,1]$ satisfying certain symmetric boundary condition at the end point $x = 1$. Consider the transformed derivative

$$w_U(x) = -\rho(x)u_1'(x) + \frac{\beta(x)}{x}u_2(x). \tag{7}$$

Then any symmetric boundary condition at the regular point $x = 1$ can be written as [**29, 17**]

$$w_U(1) = h_1 u_1(1), \quad h_1 \in \mathbb{R} \cup \{\infty\}. \tag{8}$$

So the minimal operator $\mathbf{L}$ is defined by (1) on the domain

$$\mathrm{Dom}\,(\mathbf{L}_{\min}) = \left\{U \in C_0^\infty(0,1] \oplus C_0^\infty(0,1],\ w_U(1) = h_1 u_1(1)\right\}. \tag{9}$$

Note that the domain includes infinitely many times differentiable functions vanishing in a neighborhood of the origin. In what follows we are going to keep the same notation for the closure of the operator $\mathbf{L}_{\min}$ in $\mathcal{H}$.

In the current article we concentrate our attention to the case where the quasiregularity conditions are not satisfied. It is proven in the following section that the differential operator is semibounded from below if and only if conditions (10-12) are satisfied. These condition include quasiregularity conditions as a special case. Then the Friedrichs extension of the minimal operator is described. Finally it is proven that the essential spectrum of any self-adjoint operator associated with (1) is given just by (5) in the case where conditions (10-12) are satisfied but the quasiregularity conditions not. Thus the following striking fact is proven: the singularity essential spectrum (6) for the matrix differential operator is present only if the quasiregularity conditions are satisfied, provided that the the operator is singular and semibounded. Under the same assumptions it is proven that the singularity spectrum appears if and only if the Hain-Lüst operator is in the limit point case at the singular point following Weyl's classification.

2. Semiboundedness

It has been proven in [**17**] that the essential spectrum of any self-adjoint operator associated with the differential expression (1) is bounded if and only if the quasiregularity conditions (4) are satisfied.

PROPOSITION 1. *([**17**], Lemma 3.1) Under the assumptions (2) and (3) on the coefficients $\rho, \beta, m,$ and q the quasiregularity conditions are fulfilled if and only if the essential spectrum of at least one (and hence any) self-adjoint extension of $\mathbf{L}_{\min}$ is bounded.*

Therefore if the quasiregularity conditions are not satisfied, every self-adjoint operators associated with (1) has an unbounded essential spectrum. It is natural to examine the questions under which conditions this essential spectrum is semibounded from below. It is more or less clear that the deficiency indices of the minimal operator are finite (This fact will be proven mathematically rigorously later.) Therefore just the same conditions on the coefficients guarantee that both the minimal operator and any its self-adjoint extension are semibounded from below.

LEMMA 1. *The operator $\mathbf{L}_{\min}$ is semibounded from below if and only if one of the following three conditions is satisfied*

$$\rho m - \beta^2\big|_{x=0} > 0, \tag{10}$$

$$\rho m - \beta^2 \,|_{x=0} = 0, \quad \frac{d}{dx}\left(\rho(x)m(x) - \beta^2(x)\right)\Big|_{x=0} > 0, \tag{11}$$

$$\rho m - \beta^2\Big|_{x=0} = 0, \quad \frac{d}{dx}\left(\rho(x)m(x) - \beta^2(x)\right)\Big|_{x=0} = 0.^1 \tag{12}$$

PROOF. We are going to prove sufficiency and necessity of these conditions separately.

Sufficiency It is enough to show that the quadratic form associated with the minimal operator $\mathbf{L}_{\min}$ is semibounded from below under conditions (10-12), i.e. that the following inequality holds

$$\langle \mathbf{L}_{\min} U, U\rangle \geq C \parallel U \parallel^2_{\mathcal{H}} \tag{13}$$

with a certain real constant C.

Let $U \in C_0^\infty(0,1] \oplus C_0^\infty(0,1]$ and satisfy the boundary condition (8) at $x = 1$, then the quadratic form can be calculated using integration by parts

(14)

$$\begin{aligned}\langle \mathbf{L}U, U\rangle &= \langle \rho u_1', u_1'\rangle + w_U(0)\overline{u_1(0)} + \langle q u_1, u_2\rangle - 2\Re\langle \tfrac{\beta}{x} u_2, u_1'\rangle + \langle \tfrac{m}{x^2} u_2, u_2\rangle \\ &= \parallel \tfrac{1}{\sqrt{\rho}} w_U \parallel^2 + h_1 |u_1(1)|^2 + \left\langle \tfrac{\rho m - \beta^2}{\rho x^2} u_2, u_2 \right\rangle + \langle q u_1, u_1\rangle.\end{aligned}$$

Note that we used the fact that the support of U does not contain the origin and the function U satisfies symmetric boundary condition (8) at $x = 1$. The second term in (14) vanishes in the special case $h_1 = \infty$ (Dirichlet boundary condition at $x = 1$). Let us show that under conditions (10-12) the quadratic form is semibounded from below.

We are going to prove first that the sum of the first two terms is bounded from below with respect to $\parallel U \parallel^2$. This proof is trivial if $h_1 > 0$ (both terms are nonnegative) and $h_1 = \infty$ (the second term is absent and the first term is non-negative).

Consider the case $h_1 < 0$. Let us prove that $|u_1(1)|^2$ is infinitesimally bounded with respect to $\parallel u_1' \parallel_{L_2(1/2,1)}$ and $\parallel U \parallel_{L_2(1/2,1)}$ (inequality (15) below). We consider the following obvious estimate

$$|u_1(1)|^2 \leq 2\int_x^1 |u_1(t)u_1'(t)|dt + |u_1(x)|^2$$

[1]Note that this condition (12) just coincides with the quasiregularity condition, which guarantees boundedness (from above and from below) of the essential spectrum.

and integrate it over the interval $[1/2, 1]$

$$\begin{aligned}\frac{1}{2}|u_1(1)|^2 &\leq 2\int_{1/2}^1\int_{1/2}^1 |u_1(t)u_1'(t)|dtdx + \int_{1/2}^1 |u_1(x)|^2dx \\ &\leq \frac{1}{2}\left(\epsilon \parallel u_1' \parallel^2_{L_2(1/2,1)} + \frac{4}{\epsilon} \parallel u_1 \parallel^2_{L_2(1/2,1)}\right) + \parallel u_1 \parallel^2_{L_2(1/2,1)}.\end{aligned}$$

This inequality implies that

$$|u_1(1)|^2 \leq \epsilon \parallel u_1' \parallel^2_{L_2(1/2,1)} + \left(\frac{4}{\epsilon}+2\right) \parallel u_1 \parallel^2_{L_2(1/2,1)}. \tag{15}$$

On the other hand the first term in (14) (which is clearly positive) can be estimated from below using triangle inequality and the fact that the following function is uniformly bounded $\frac{\beta(x)}{\sqrt{\rho(x)x}} \leq C_1$ for $x \in [1/2, 1]$

$$\begin{aligned}\parallel \frac{1}{\sqrt{\rho}} w_U(x) \parallel^2_{L_2(0,1)} &\geq \parallel \frac{1}{\sqrt{\rho}} w_U(x) \parallel^2_{L_2(1/2,1)} \\ &\geq \parallel \sqrt{\rho} u_1' \parallel^2_{L_2(1/2,1)} - \parallel \frac{\beta}{\sqrt{\rho x}} u_2 \parallel^2 \\ &\geq \rho_0 \parallel u_1' \parallel^2_{L_2(1/2,1)} - C_1 \parallel U \parallel^2.\end{aligned} \tag{16}$$

In the case $h_1 < 0$ choosing $\epsilon = \frac{\rho_0}{|h_1|}$ we conclude that the sum of the first two terms in (14) is semibounded from below with respect to the norm in $\mathcal{H}$.

The third term $\left\langle \frac{\rho m - \beta^2}{x^2\rho} u_2, u_2 \right\rangle$ in (14) is uniformly bounded if the quasiregularity condition (12) is satisfied. If one of the conditions (10) or (11) is satisfied, then the function $\frac{\rho m-\beta^2}{x^2\rho}$ is positive in a certain neighborhood of the origin, say $[0, r]$, so that the scalar product can be decomposed into the sum of two integrals, one positive and one uniformly bounded in the norm of $\mathcal{H}$:

$$\left\langle \frac{\rho m - \beta^2}{x^2\rho} u_2, u_2 \right\rangle = \int_0^r \frac{\rho m - \beta^2}{x^2\rho}|u_2|^2dt + \int_r^1 \frac{\rho m - \beta^2}{x^2\rho}|u_2|^2dt.$$

The last term $\langle qu_1, u_1\rangle$ is bounded as well, since the function q is uniformly bounded.

We have proven that under conditions (10), (11), or (12) the quadratic form of the minimal operator is semibounded from below.

Necessity Suppose that (10-12) are not satisfied, i.e. parameters of the operator satisfy one of the following two conditions:

$$\rho m - \beta^2\big|_{x=0} < 0, \tag{17}$$

$$\rho m - \beta^2\big|_{x=0} = 0, \; \frac{d}{dx}\left(\rho m - \beta^2\right)\big|_{x=0} < 0. \tag{18}$$

If one of these conditions is satisfied, then the function $\rho m - \beta^2$ is negative in a certain interval $(0, \epsilon) \subset (0,1)$. It may be equal to zero at the origin, but the order of the zero cannot be higher than 1. It follows, that there exists a certain positive number $C_2 > 0$ such that the following inequality holds

$$\rho m - \beta^2 < -C_2 \rho x \Rightarrow \frac{\rho m - \beta^2}{x^2 \rho} < -\frac{C_2}{x}, \tag{19}$$

for all $x \in (0, \epsilon)$. Using this fact we are going to construct a sequence of functions $U^k \in \mathrm{Dom}\,(\mathbf{L}_{\min}) \subset \mathcal{H}$ with the following properties:
1. $\| U^k \|$ is uniformly bounded;
2. $\langle \mathbf{L}_{\min} U^k, U^k \rangle$ tends to $-\infty$ as $k \to \infty$.

The last term in (14) is uniformly bounded, since q is a bounded function, and therefore does not affect the divergence of $\langle \mathbf{L} U^k, U^k \rangle$ to $-\infty$. In addition the sequence we are going to construct will have the following property

$$w_{U^k} \equiv 0 \Leftrightarrow (u_1^k)' = \frac{\beta(x)}{x\rho(x)} u_2^k. \tag{20}$$

The second component of U^k can be chosen equal to

$$u_2^k = \begin{cases} \sin(\ln x - \ln \epsilon), & x \in (\epsilon e^{-2\pi k}, \epsilon), \\ 0, & \text{otherwise}. \end{cases}$$

Then in order to satisfy (20) we choose the first component equal to

$$u_1^k(x) = \int_0^x \frac{\beta(t)}{t\rho(t)} u_2^k(t) dt.$$

It is clear that the functions u_2^k are uniformly bounded and therefore $\| u_2^k \|$ are uniformly bounded as well. Due to oscillation properties of u_2^k the functions u_1^k are uniformly bounded as well. This implies that both $|u_1^k(1)|^2$ and $\| u_1^k \|$ are uniformly bounded. We conclude that the sequence $\| U^k \|$ is uniformly bounded. The corresponding quadratic form $\langle LU^k, U^k \rangle$ given by (14) tends to $-\infty$, since the first term in (14) vanishes, the second and fourth terms are uniformly bounded and the third term due to (19) can be estimated as

$$\begin{aligned} \left\langle \frac{\rho m - \beta^2}{\rho x^2} u_2, u_2 \right\rangle &\leq -C_2 \left\langle \frac{1}{x} u_2^k, u_2^k \right\rangle \\ &= -C_2 \int_{\epsilon e^{-2\pi k}}^{\epsilon} \frac{\sin^2(\ln x - \ln \epsilon)}{x} dx \searrow 0, \end{aligned}$$

and therefore tends to $-\infty$. The sequence constructed satisfies conditions 1 and 2, but it does not belong to the domain of $\mathbf{L}$, since the functions are not infinitely many times differentiable at the points $x = \epsilon e^{-2\pi k}, \epsilon$. To get infinitely differentiable functions one can smooth U^k out without changing drastically the norm and the value of the quadratic form. □

Thus we have proven that the matrix differential operator L is semibounded from below if only if conditions (10-12) are satisfied. The assumption that the operator is semibounded is standard in studied of different physical problems.

3. Deficiency indices of the minimal operator

The following theorem has already been proven in [**17**]. We decided to reformulate it in order to adjust it to the notations used in the current article.[2]

PROPOSITION 2. *(Following Theorem 4.1 from* [**17**]*) The operator* $\mathbf{L}_{\min}$ *is a symmetric operator in the Hilbert space* $\mathcal{H}$ *with finite equal deficiency indices.*
1) If the operator matrix L *is singular quasiregular (i.e.* $m(0) \neq 0$ *and quasiregularity conditions are satisfied), then the deficiency indices of* $\mathbf{L}_{\min}$ *are trivial and the operator* $\mathbf{L}_{\min}$ *is self-adjoint.*
2) If the operator matrix is regular or is not quasiregular then the deficiency indices of $\mathbf{L}_{\min}$ *are equal to* $(1,1)$*. The self-adjoint extensions of* $\mathbf{L}_{\min}$ *are described by boundary conditions using the following alternatives covering all possibilities:*
a) If $\rho(0)m(0) - \beta^2(0) \neq 0$ *or* $\beta(0) = 0$*, then the first component* u_1 *of any vector from the domain of the adjoint operator* $\mathbf{L}^*_{\min}$ *is continuous in the closed interval* $[0,1]$*. All self-adjoint extensions of the operator* $\mathbf{L}_{\min}$ *are described by the* **standard** *boundary condition at* $x = 0$ [3]

$$\omega_U(0) = h_0 u_1(0), h_0 \in \mathbf{R} \cup \{\infty\}. \tag{21}$$

b) If $\rho(0)m(0) - \beta^2(0) = 0$, $\frac{d}{dx}\left(\rho m - \beta^2\right)(0) \neq 0$*, and* $\beta(0) \neq 0$*, then the first component* u_1 *of any vector from the domain of the adjoint*

[2] Notation $\mathbf{L}_{\min}$ was used in [**17**] to denote the symmetric operator determined by the differential expression L on the domain $C_0^\infty(0,1) \oplus C_0^\infty(0,1)$ consisting of functions with compact support separated from the point $x = 1$. In the current article the domain of $\mathbf{L}_{\min}$ contains functions with support not necessarily separated from $x = 1$, but satisfying the standard boundary condition (7) at this endpoint.

[3] In the case $h_0 = \infty$, the corresponding boundary condition should be written as $u_1(0) = 0$ or $c_U = 0$

operator $\mathbf{L}_{\min}^*$ *admits the asymptotic representation*

$$u_1(x) = k w_U(0) \ln x + c_U + o(1), \text{ as } x \to 0, \tag{22}$$

where $k = -\frac{\beta^2(0)}{\rho(0)} \frac{1}{\frac{d}{dx}(\rho m - \beta^2)|_{x=0}}$ *and* c_U *is an arbitrary constant depending on* U. *Then all self-adjoint extensions of the operator* $\mathbf{L}_{\min}$ *are described by the* **nonstandard** *boundary condition* [3]

$$\omega_U(0) = h_0 c_U, \ h_0 \in \mathbf{R} \cup \{\infty\}. \tag{23}$$

Information concerning the deficiency indices of $\mathbf{L}_{\min}$ *and self-adjoint local boundary conditions is collected in the following table*

<table>
<tr><td rowspan="2"></td><td rowspan="2">$\rho(0)m(0) - \beta^2(0) \neq 0$</td><td colspan="2">$\rho(0)m(0) - \beta^2(0) = 0$</td></tr>
<tr><td>$\frac{d}{dx}(\rho m - \beta^2)|_{x=0} \neq 0$</td><td>$\frac{d}{dx}(\rho m - \beta^2)|_{x=0} = 0$</td></tr>
<tr><td>$\beta(0) = 0$</td><td>*indices (1,1)*
standard b.c. (21)</td><td>*indices (1,1)*
standard b.c. (21)</td><td>*indices (1,1)*
standard b.c. (21)</td></tr>
<tr><td>$\beta(0) \neq 0$</td><td>*indices (1,1)*
standard b.c. (21)</td><td>*indices (1,1)*
nonstandard b.c. (23)</td><td>*indices (0,0)*
self-adjoint</td></tr>
</table>

This proposition implies in particular that deficiency indices of $\mathbf{L}_{\min}$ are always finite and equal. Therefore there always exists a family of self-adjoint operators associated with the differential expression L. Every operator from such family is an extension of $\mathbf{L}_{\min}$ and the essential spectrum does not depend on the particular extension chosen.

4. Friedrichs extension

We have seen that the differential expression L does not necessarily determine a unique self-adjoint operator in $\mathcal{H}$. In the case $\mathbf{L}_{\min}$ is semibounded it is natural to associate with L the Friedrichs extension of $\mathbf{L}_{\min}$. This extension is studied in the current section. Note that this question has already been studied in a more general context in [**16**], but we provide a detailed analysis for the operator under investigation.

THEOREM 1. *The Friedrichs extension of the symmetric operator* $\mathbf{L}_{\min}$ *is described by boundary conditions at* $x = 0$ *depending on the their type (and properties of the coefficients of course) as follows:*

A) If the operator $\mathbf{L}_{\min}$ *is self-adjoint, then no boundary condition at the origin is needed and the Friedrichs extension just coincides with* $\mathbf{L}_{\min}$. *This case occurs if the coefficients of the operator matrix satisfy the following conditions*

$$\rho m - \beta^2\big|_{x=0} = 0 \quad \frac{d}{dx}\left(\rho m - \beta^2\right)\big|_{x=0} = 0 \ \text{ and } \ \beta(0) \neq 0. \tag{24}$$

B) If the extensions of $\mathbf{L}_{\min}$ are described by the standard boundary condition at the origin (see (21)), then the Friedrichs extension corresponds to the condition

$$u_1(0) = 0. \tag{25}$$

This case occurs if the coefficients of the operator matrix satisfy one of the following two conditions

$$\rho m - \beta^2\big|_{x=0} > 0, \tag{26}$$

or

$$\rho m - \beta^2\big|_{x=0} = 0, \ \ \frac{d}{dx}\left(\rho m - \beta^2\right)\big|_{x=0} > 0 \ \ \text{and} \ \ \beta(0) = 0. \tag{27}$$

C) If the extensions of $\mathbf{L}_{\min}$ are described by the non-standard boundary condition at the origin (see (23)), then the Friedrichs extension corresponds to the condition

$$w_U(0) = 0. \tag{28}$$

This case occurs if the coefficients of the operator matrix satisfy the following conditions

$$\rho m - \beta^2\big|_{x=0} = 0, \frac{d}{dx}\left(\rho m - \beta^2\right)\big|_{x=0} > 0 \ \ \text{and} \ \ \beta(0) \neq 0. \tag{29}$$

PROOF. The statement formulated in part A is trivial and is included for the sake of completeness only. We are going to consider the two remaining cases separately, but the same idea will be used. It will be proven that every function from the domain of the Friedrichs extension necessarily satisfies one of the boundary conditions ((25) or (28) depending on their type) describing self-adjoint extensions of $\mathbf{L}_{\min}$. This will be enough to determine the boundary conditions describing the Friedrichs extension, since the operator $\mathbf{L}_{\min}$ is closed and has deficiency indices $(1,1)$. Really every function satisfying the boundary conditions corresponding to two different self-adjoint extensions necessarily belongs to the domain of the original symmetric operator $\mathbf{L}_{\min}$. Therefore to establish the boundary conditions describing the Friedrichs extension it is enough to prove that the functions from the domains of these extensions satisfy (25) and (28) respectively.

B) To construct the Friedrichs extension one has to consider the closure of the domain $\mathrm{Dom}\,(\mathbf{L}_{\min})$ with respect to the following quadratic form which is positive for all sufficiently large values of A

(30)

$$[U,U] \equiv \langle (\mathbf{L}_{\min} + A)U, U\rangle = \langle \mathbf{L}_{\min} U, U\rangle + A \parallel U \parallel^2$$

$$= \parallel \frac{1}{\sqrt{\rho}} w_U \parallel^2 + \langle \frac{\rho m - \beta^2}{x^2 \rho} u_2, u_2\rangle + \langle q u_1, u_1\rangle + h_1 |u_1(1)|^2 + A \parallel U \parallel^2 .$$

The terms $\langle q u_1, u_1\rangle$ and $h_1|u_1(1)|^2$ can be estimated through the other terms. Indeed the estimate for $\langle q u_1, u_1\rangle$ is trivial

$$|\langle q u_1, u_1\rangle| \leq \max |q(x)| \parallel u_1 \parallel^2 \leq \max |q(x)| \parallel U \parallel^2 . \tag{31}$$

To get the estimate for $h_1|u_1(1)|^2$ consider the triangle inequality

$$\begin{aligned} \parallel \tfrac{1}{\sqrt{\rho}} w_U \parallel^2 &\geq \parallel \frac{1}{\sqrt{\rho}} w_U \parallel^2_{L_2(1/2,1)} \\ &\geq \frac{\sqrt{\rho_0}}{2} \parallel u_1' \parallel^2_{L_2(1/2,1)} - \text{const} \parallel \frac{1}{x} u_2 \parallel^2_{L_2(1/2,1)} . \end{aligned}$$

Then using (15) we get

$$\begin{aligned} |u_1(1)|^2 &\leq \epsilon \parallel u_1' \parallel^2_{L_2(1/2,1)} + \left(\frac{4}{\epsilon} + 2\right) \parallel u_1 \parallel^2_{L_2(1/2,1)} \\ &\leq \epsilon \frac{2}{\sqrt{\rho_0}} \parallel \frac{1}{\sqrt{\rho}} w_U \parallel^2_{L_2(1/2,1)} + \text{const}(\epsilon) \parallel U \parallel^2_{L_2(1/2,1)} \qquad (32) \\ &\leq \epsilon \frac{2}{\sqrt{\rho_0}} \parallel \frac{1}{\sqrt{\rho}} w_U \parallel^2 + \text{const}(\epsilon) \parallel U \parallel^2 . \end{aligned}$$

It follows that the quadratic form $[U,U]$ given by (30) for sufficiently large values of A is equivalent to the following quadratic form

$$Q(U,U) = \parallel \frac{1}{\sqrt{\rho}} w_U \parallel^2 + B \langle \frac{\rho m - \beta^2}{x^2 \rho} u_2, u_2\rangle + A \parallel U \parallel^2, \tag{33}$$

where A and B are certain positive real numbers. (The parameter A appeared here may differ slightly from the one used in (30).)

Let us study the two possible cases (26) and (27) separately:

Case 1 Let condition (26) be satisfied: $\rho m - \beta^2|_{x=0} > 0$. On a certain interval $(0,\epsilon)$, $\epsilon > 0$ the function $\frac{\rho m - \beta^2}{\rho}$ is strictly positive. Consider any sequence $U^k \in \text{Dom}\,(\mathbf{L}_{\min})$, $k = 1, 2, \ldots$ having support on $(0,\epsilon)$ and converging in the norm given by $Q(U,U)$. Since the form Q can be estimated from below as follows

$$Q(U,U) \geq C_3 \parallel \frac{1}{x} u_2 \parallel^2_{L_2(0,\epsilon)}, \quad C_3 > 0$$

and therefore for any sequence U^k converging with respect to $Q(U,U)$ the sequence $\frac{1}{x}u_2^k$ is a Cauchy sequence in $L_2(0,\epsilon)$. On the other hand the same quadratic form can also be estimated as

$$Q(U,U) \geq \| \frac{1}{\sqrt{\rho}} w_U \|^2_{L_2(0,\epsilon)} = \| -\sqrt{\rho}u_1' + \frac{\beta}{\sqrt{\rho}}\frac{1}{x}u_2 \|^2_{L_2(0,\epsilon)} . \tag{34}$$

Therefore $-\sqrt{\rho}(u_1^k)'$ is a Cauchy sequence in $L_2(0,\epsilon)$. Taking into account that $Q(U,U) \geq C_4 \| u_1 \|^2_{L_2(0,\epsilon)}$, $C_4 > 0$, we conclude that u_1^k is a Cauchy sequence with respect to the norm of $W_2^1(0,\epsilon)$. Every function u_1^k is equal to zero at the origin and therefore the limit function u_1 satisfies the Dirichlet boundary condition (25).

The assumption that the support os U^k belongs to $(0,\epsilon)$ is not very restrictive. Let U^k be any sequence from Dom $(\mathbf{L}_{\min})$ converging in the norm $Q(U,U)$. Consider in addition any cut-off function $\psi \in C_0^\infty(0,1]$, identically equal to 1 on the interval $[\epsilon,1]$. Then the sequence ψU^k converges to a function from Dom $(\mathbf{L}_{\min})$ and therefore the sequence $(1-\psi)U^k$ is a Cauchy sequence with respect to $Q(U,U)$ having support on the interval $(0,\epsilon)$. We have proven that every such sequence converges to a function satisfying Dirichlet condition at the origin. It follows that the limit of U^k satisfies the same condition. Thus the Friedrichs extension of the operator $\mathbf{L}_{\min}$ is described by (25) in this case.

Case 2 Let condition (27) be satisfied: $\rho m - \beta^2|_{x=0} = 0$, $\frac{d}{dx}(\rho m - \beta^2)|_{x=0} > 0$ and $\beta(0) = 0$. Again there exists $\epsilon > 0$ such that the function $\rho m - \beta^2 > 0$ for $x \in (0,\epsilon)$. Consider an arbitrary sequence $U^k \in$ Dom $(\mathbf{L}_{\min})$, $k = 1,2,...$ converging in the norm given by $Q(U,U)$. We assume again that supports of all functions U^k belong to $(0,\epsilon)$. Condition (27) imply that for a certain $\epsilon > 0$ the following estimate holds

$$Q(U,U) \geq C_5 \| \frac{1}{\sqrt{x}} u_2 \|^2_{L_2(0,\epsilon)}, \; C_5 > 0, \tag{35}$$

and therefore $\frac{1}{\sqrt{x}}u_2^k$ is a Cauchy sequence in $L_2(0,\epsilon)$. The estimate (34) can be modified as

$$Q(U,U) \geq \| -\sqrt{\rho}u_1' + \frac{\beta}{\sqrt{\rho}}\frac{1}{\sqrt{x}} \times \frac{1}{\sqrt{x}}u_2 \|^2_{L_2(0,\epsilon)} .$$

Since $\beta \in C^1(0,1)$ and $\beta(0) = 0$, the function $\frac{\beta}{\sqrt{\rho}}\frac{1}{\sqrt{x}}$ is bounded and therefore $-\sqrt{\rho}(u_1^k)'$ is a Cauchy sequence in $L_2(0,\epsilon)$. Taking into account that u_1^k is a Cauchy sequence we conclude that u_1^k converges with respect to the norm of $W_2^1(0,\epsilon)$ and therefore satisfy Dirichlet boundary condition (25) at the origin. The same reasoning as in Case 1 may be applied to modify the proof for sequences U^k not necessarily having support in $(0, \epsilon)$. Thus the Friedrichs extension of the operator $\mathbf{L}_{\min}$ is also described by (25).

C) Under condition (29) the form $[U, U]$ is again equivalent to the form $Q(U, U)$. As in Case 2, there exists *esilon* > 0, such that $\rho m - \beta^2 > 0$ for $x \in (0, \epsilon)$. Consider any sequence $U^k \in \mathrm{Dom}\,(\mathbf{L}_{\min})$, $k = 1, 2, \ldots$ converging in the norm given by $Q(U, U)$. The estimate (35) holds and it follows that $\frac{1}{\sqrt{x}} u_2^k$ is a Cauchy sequence in $L_2(0, \epsilon)$. On the other hand $\frac{1}{\sqrt{\rho}} w_{U^k}$ is also a Cauchy sequence in $L_2(0, \epsilon)$, as well, since

$$\sqrt{x} \frac{1}{\sqrt{\rho}} w_{U^k} = -\sqrt{\rho}\sqrt{x} u_1^{k\prime} + \frac{\beta}{\sqrt{\rho}} \frac{1}{\sqrt{x}} u_2.$$

It follows that $\sqrt{x} u_1^{k\prime}$ is a Cauchy sequence in $L_2(0, \epsilon)$. It follows that the functions belonging to the domain of the Friedrichs extension in particular satisfy

$$\sqrt{x} u_1' \in L_2(0, \epsilon). \tag{36}$$

Let us remind that in the case under investigation every function from the domain of the adjoint operator $\mathbf{L}_{\min}^*$ as well as its Friedriches extension possesses the asymptotic representation (22). Every function possessing this representation satisfies (36) if and only if $w_U(0) = 0$, i.e. only if the function satisfies the non-standard boundary condition (28). It follows that the Friedrichs extension is the extension described by the boundary condition (28).

□

5. The essential spectrum: the quasi-regularity conditions are not fulfilled

This is the main section of the article and it is devoted to the calculation of the essential spectrum of any self-adjoint operator associated with the differential expression L. This question has been solved in the case where the quasiregularity conditions are satisfied (see [**17**]). These condition guarantee that the operator is bounded. Therefore in this section we concentrate our attention to the case where the quasiregularity conditions (4) are not satisfied, but conditions (10-12) are fulfilled.

THEOREM 2. *Let $\mathbf{L}_{min}$ be semi-bounded from below. Suppose that the quasi-regularity conditions (4) are not satisfied. Then the essential spectrum of any self-adjoint extension $\mathbf{L}$ of the operator $\mathbf{L}_{\min}$ is given by*

$$\sigma_{\mathrm{ess}}(\mathbf{L}) = \overline{\mathcal{R}\{\frac{m - \frac{\beta^2}{\rho}}{x^2}\}}. \tag{37}$$

PROOF. Let us make the change of variables

$$x \to y \quad \begin{cases} x = e^{-y}, \\ dx = -e^{-y} dy = -x dy, \end{cases} \tag{38}$$

which transforms the interval $[0,1]$ into the semiaxis $[0,\infty)$. The points 0 and 1 are mapped into the points ∞ and 0 respectively. This change of variables determines the following unitary correspondence between the Hilbert spaces $L_2(0,1)$ and $L_2(0,\infty)$:

$$\begin{array}{lclcl} \Phi & : & \psi(x) & \mapsto & \widetilde{\psi}(y) = \psi(e^{-y}) e^{-\frac{y}{2}}; \\ \Phi^{-1} & : & \widetilde{\psi}(y) & \mapsto & \psi(x) = \dfrac{1}{\sqrt{x}} \widetilde{\psi}(-\ln x). \end{array} \tag{39}$$

The differential operator L is transformed into the following differential operator K acting on two-component functions on $[0,\infty)$:
(40)

$$K = \begin{pmatrix} -\dfrac{d}{dy}\dfrac{\rho}{x^2}\dfrac{d}{dy} + (q(x) + \dfrac{\rho'_x}{2x} - \dfrac{3\rho}{4x^2}) & -\dfrac{d}{dy}\dfrac{\beta}{x^2} + \dfrac{\beta}{2x^2} \\ \dfrac{\beta}{x^2}\dfrac{d}{dy} + \dfrac{\beta}{2x^2} & \dfrac{m}{x^2} \end{pmatrix} \equiv \begin{pmatrix} A & C^* \\ C & D \end{pmatrix}.$$

In what follows we are going to use both variables x and y simultaneously hoping that this will not lead to misunderstanding.

Let us consider the minimal (symmetric) operator $\mathbf{K}_{\min}$ being the closure of the differential operator K considered on the domain of functions from $C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)$ (arbitrarily many times differentiable functions with compact support on $[0,\infty)$ not necessarily separated from the origin) satisfying the standard boundary condition at the origin, which can be recalculated from (8)

$$\widetilde{\omega}_U(0) = \widetilde{h}_1 \widetilde{u}_1(0), \qquad \text{where} \qquad \widetilde{h}_1 = h_1 - \frac{\rho(1)}{2} \in \mathbb{R} \cup \{\infty\}. \tag{41}$$

The analysis of the operator $\mathbf{K}_{\min}$ is equivalent to the analysis of the operator $\mathbf{L}_{\min}$ carried out in the preceding sections, since these two operators are connected by the unitary transformation (39). Hence the deficiency indices of the operator $\mathbf{K}_{\min}$ are $(0,0)$ or $(1,1)$ depending on the properties of the coefficients as $y \to \infty$. It is not hard to reformulate these conditions but we are not going to do that, since our aim is to calculate the essential spectrum, which does not depend on the particular extension of the minimal operator - all extensions have just the same essential spectrum, since the deficiency indices are finite.

Consider the resolvent equation

$$(\mathbf{K}_{\min} - \mu)^{-1} F = U$$

for sufficiently small negative values of $\mu \ll -1$. For smooth F and U

$$F \in \mathcal{R}(\mathbf{K}_{\min}|_{\mathrm{Dom}\,(\mathbf{K}_{\min}) \cap C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)});$$

$$U \in C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)$$

the equation can be written as

$$f_1 = (A-\mu)u_1 + C^* u_2, \qquad f_2 = Cu_1 + (D-\mu)u_2.$$

Using the fact that the operator $D - \mu = \frac{m}{x^2} - \mu$ is invertible for sufficiently small negative $\mu \ll -1$ ($m|_{x=0} > 0$ or $m'|_{x=0} > 0$ if $m|_{x=0} = 0$) component u_2 can be excluded from the system by first resolving the second equation

$$u_2 = (D-\mu)^{-1} f_2 - (D-\mu)^{-1} C u_1$$

and then substituting this expression into the first equation

$$f_1 = ((A-\mu) - C^*(D-\mu)^{-1}C)u_1 + C^*(D-\mu)^{-1} f_2. \tag{42}$$

Hence in order to calculate u_1 one needs to invert the so-called Hain-Lüst operator

$$T(\mu) = (A - \mu I) - C^*(D - \mu I)^{-1} C. \tag{43}$$

Let us consider the minimal operator $T_{\min}$ corresponding to this differential expression and defined on the functions from the domain $C_0^\infty(0,\infty)$. This operator can be written in the following form:

(44)

$$\begin{aligned} T_{\min}(\mu) &= -\frac{d}{dy}\Big(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}\Big)\frac{d}{dy} \\ &\quad + \Big\{q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m-\mu x^2)} - x\frac{d}{dx}\Big(\frac{\beta^2}{2x^2(m-\mu x^2)}\Big) - \mu\Big\} \\ &= -\frac{d}{dy} V_\mu \frac{d}{dy} + W_\mu, \end{aligned}$$

where we use the following notations:

$$V_\mu = \frac{\widehat{v}_\mu}{x^2}, \quad W_\mu = \frac{1}{4}\frac{\widehat{v}_\mu}{x^2} + \frac{x}{2}\frac{d}{dx}\Big(\frac{\widehat{v}_\mu}{x^2}\Big) + q(x) - \mu, \tag{45}$$

$$\widehat{v}_\mu = \frac{\rho m - \beta^2 - \rho\mu x^2}{m - \mu x^2}. \tag{46}$$

In what follows it will be convenient to separate the three only possible sets of parameters which guarantee that the operator $\mathbf{L}_{\min}$ is semibounded from below but the quasiregularity conditions are not satisfied

- Case A

$$(\rho m - \beta^2)|_{x=0} > 0. \tag{47}$$

(This condition implies in particular that $m(0) > 0$.)

- Case B

$$(\rho m - \beta^2)|_{x=0} = 0, (\rho m - \beta^2)'_x|_{x=0} > 0 \text{ and } m(0) = 0. \tag{48}$$

(This condition implies in particular that $m'_x(0) > 0$.)

- Case C

$$(\rho m - \beta^2)|_{x=0} = 0, (\rho m - \beta^2)'|_{x=0} > 0 \text{ and } m(0) > 0. \tag{49}$$

In what follows we are going to refer to these cases as Case A, B or C.

LEMMA 2. *Under conditions of Theorem 2 the action of the operator $T_{\min}$ can be written using one of the following two representations*

$$T_{min}(\mu) = e^y(-\frac{d}{dy}v_\mu \frac{d}{dy} + w_\mu)e^y, \tag{50}$$

with

$$w_\mu = \frac{1}{4}\widehat{v}_\mu - \frac{1}{2}x(\widehat{v}_\mu)'_x + (q-\mu)x^2, \qquad v_\mu = \widehat{v}_\mu, \tag{51}$$

and

$$T_{min}(\mu) = e^{\frac{y}{2}}(-\frac{d}{dy}v_\mu \frac{d}{dy} + w_\mu)e^{\frac{y}{2}}, \tag{52}$$

$$w_\mu = (q-\mu)x, \qquad v_\mu = \frac{\widehat{v}_\mu}{x}. \tag{53}$$

Comment The first representation (50) will be used in Cases A and B. In the Case C the function $\hat{v}_\mu$ is vanishing at zero and therefore it is natural to use the function $v_\mu = \hat{v}_\mu/x$ (instead of $v_\mu = \hat{v}_\mu$). This leads to the second representation (52). Therefore in what follows we are going to use the definition (51) for the function v_μ in the Cases A and B, and definition (53) - in the Case C.

PROOF. We are going to prove representations (50) and (52) separately starting from the first one. Consider formula (44) for the Hain-Lüst operator

$$T_{min}(\mu) = -\frac{d}{dy}\frac{v_\mu}{x^2}\frac{d}{dy} + W_\mu = -\frac{d}{dy}e^y v_\mu e^y \frac{d}{dy} + W_\mu$$

Using the following commutation relation for the operator of multiplication by a certain differentiable function $\varphi(y)$ and the operator of the first differentiation

$$\frac{d}{dy}\varphi = \varphi\frac{d}{dy} + \varphi'_y, \tag{54}$$

the expression for $T_{\min}$ can be transformed as follows

$$\begin{aligned} &T_{min}(\mu) \\ &= -e^y\frac{d}{dy}v_\mu e^y\frac{d}{dy} - e^y v_\mu e^y\frac{d}{dy} + W_\mu \\ &= -e^y\frac{d}{dy}v_\mu\frac{d}{dy}e^y - e^y v_\mu\frac{d}{dy}e^y + e^y\frac{d}{dy}v_\mu e^y + e^y v_\mu e^y + W_\mu \\ &= e^y(-\frac{d}{dy}v_\mu\frac{d}{dy} + (v_\mu)'_y + v_\mu + x(\frac{1}{4}\frac{v_\mu}{x^2} + \frac{x}{2}\frac{d}{dx}(\frac{v_\mu}{x^2}) + q(x) - \mu)x)e^y. \end{aligned}$$

Taking into account that $\frac{d}{dy} = -x\frac{d}{dx}$ and $\frac{d}{dx}(\frac{v_\mu}{x^2}) = \frac{(v_\mu)'_x}{x^2} - \frac{2v_\mu}{x^3}$ we get the desired representation

$$T_{min}(\mu) = e^y(-\frac{d}{dy}v_\mu\frac{d}{dy} + \frac{1}{4}v_\mu - \frac{1}{2}x(v_\mu)'_x + (q-\mu)x^2)e^y.$$

To get representation (52) we use similar calculations to obtain first

$$T_{min}(\mu) = e^{\frac{y}{2}}(-\frac{d}{dy}v_\mu\frac{d}{dy} + \frac{1}{2}(v_\mu)'_y + \frac{1}{4}v_\mu + xW_\mu)e^{\frac{y}{2}},$$

and then

$$T_{min}(\mu) = e^{\frac{y}{2}}(-\frac{d}{dy}v_\mu\frac{d}{dy} + (q-\mu)x)e^{\frac{y}{2}}.$$

□

The following Lemma proves that the function v_μ is always positive definite for negative μ with sufficiently large absolute value.

LEMMA 3. *Let conditions of Theorem 2 be satisfied. Then the function v_μ is positive definite for sufficiently small $\mu \ll -1$, i.e. there exist $c > 0$ and $\mu_0 \in \mathbb{R}$, such that*

$$\mu \le \mu_0 \Rightarrow v_\mu(x) \ge c. \tag{55}$$

PROOF. The function v_μ is given by different formulas (51) and (53) in the cases A, B and C. Therefore let us separate the proof into three parts corresponding to these three situations.

Case A Let condition A be satisfied. Then the function v_μ is given by

$$v_\mu = \frac{\rho m - \beta^2 - \rho\mu x^2}{m - \mu x^2}. \tag{56}$$

Choose negative μ satisfying the following two inequalities:

$$\mu < \frac{m(x)}{x^2} \quad \text{and} \quad \mu < \frac{1}{\rho_0}\frac{\rho(x)m(x) - \beta^2(x)}{x^2}. \tag{57}$$

This is possible, since the functions $m(x)$ and $\rho(x)m(x) - \beta^2(x)$ are continuous in $[0,1]$ and attain positive values at the origin. This implies in particular that the functions are positive in a certain interval $[0,\epsilon]$, $\epsilon > 0$. Therefore having in mind negative values of the spectral parameter μ, it can be chosen satisfying the inequalities (57) in the interval $x \in [\epsilon, 1]$, where the quotients are continuous functions and therefore are bounded from below.[4]

Under these conditions both the numerator and the denominator of the function v_μ are continuous positive definite functions. Thus the function v_μ is positive definite as well.

Case B Let condition B be satisfied. The function v_μ is again given by formula (56). Choose μ satisfying the following two inequalities

$$\mu < \frac{m'(x)}{2x} \quad \text{and} \quad \mu < \frac{1}{\rho_0}\frac{(\rho(x)m(x) - \beta^2(x))'}{2x}, \tag{58}$$

which is possible, since the functions $m'(x)$ and $(\rho(x)m(x) - \beta^2(x))'$ are continuous and attain positive values at the origin.

Under these conditions the function v_μ is given by a quotient of two functions which are positive definite for any positive x. Moreover the limit of v_μ as $x \to 0$ is positive

$$\lim_{x\to 0} v_\mu(x) = \frac{(\rho m - \beta^2)'_x|_{x=0}}{m'_x|_{x=0}} > 0.$$

Therefore the function v_μ can be considered as a continuous function on the compact interval $[0,1]$ and therefore attains its minimum, which is clearly positive, since the function has positive limits at the end points of the interval $[0,1]$ and is positive everywhere inside the open interval $(0,1)$. Therefore this function is positive definite.

Case C Let condition C be satisfied. The function v_μ is now given by formula (53)

$$v_\mu = \frac{\rho m - \beta^2 - \rho\mu x^2}{x(m - \mu x^2)}.$$

[4] The same reasoning will be used in Cases B and C below.

Choose μ satisfying the following two inequalities

$$\mu < \frac{m(x)}{x^2} \quad \text{and} \quad \mu < \frac{1}{\rho_0}\frac{(\rho(x)m(x) - \beta^2(x))'}{2x} \tag{59}$$

and apply the same arguments as in Cases A and B.

The function v_μ is then again given by a quotient of two functions which are positive for any $x \in (0, 1]$. Moreover the limit of v_μ as $x \to 0$ is positive

$$\lim_{x\to 0} v_\mu(x) = \frac{(\rho m - \beta^2)'_x|_{x=0}}{m(0)} > 0.$$

The same argument as in the Case B leads to the conclusion that the function v_μ is positive definite. □

LEMMA 4. *Let conditions of Theorem 2 be satisfied. Then in the Cases A and B the function w_μ is positive definite for sufficiently small $\mu \ll -1$, i.e. the following inequality is satisfied*

$$w_\mu(x) \geq c > 0 \tag{60}$$

with a certain positive constant c. In the Case C the function w_μ/x is positive definite, i.e. the following inequality is satisfied

$$w_\mu(x) \geq cx, \tag{61}$$

where c is a certain positive constant.

PROOF. Let us consider the cases A, B and C separately.

Cases A and B The function w_μ is given by

$$w_\mu = \frac{1}{4}\hat{v}_\mu - \frac{1}{2}x(\hat{v}_\mu)'_x + (q - \mu)x^2.$$

The function $h(x) \equiv \frac{1}{4}\hat{v}_\mu - \frac{1}{2}x(\hat{v}_\mu)'_x$ is a continuous function on $[0, 1]$ attaining positive value at the origin $h(0) = \frac{1}{4}v_\mu(0) > 0$. Therefore there exists μ_1 such that $h(x) - \mu_1 x^2$ is positive definite, i.e. there exists $c > 0$ such that $h(x) - \mu x^2 > c$. It follows that for $\mu < \mu_1 - \parallel q \parallel_\infty$ the function w_μ is positive definite, i.e. satisfies (60).

Case B The function w_μ is now given by

$$w_\mu = (q - \mu)x.$$

Choosing $\mu < - \parallel q \parallel_\infty$ we guarantee that $w_\mu > cx$, where c is a certain positive constant.

□

LEMMA 5. *Let conditions of Theorem 2 be satisfied. Then the operator $T_{\min}(\mu)$ is positive definite for all sufficiently small $\mu \ll -1$ uniformly with respect to μ, i.e. the following estimate is valid*

$$T_{\min} \geq c > 0. \tag{62}$$

PROOF. Cases A and B Consider the quadratic form of the operator $T_{\min}$. Let $u \in C_0^\infty$ then

$$\begin{aligned} \langle u, T_{\min} u\rangle &= \langle \frac{d}{dy} e^y u, v_\mu \frac{d}{dy} e^y u\rangle + \langle e^y u, w_\mu e^y u\rangle \\ &\geq \min w_\mu(x) \parallel e^y u \parallel^2 \geq c \parallel u \parallel^2, \end{aligned}$$

where the constant c taken from the estimate (60) from Lemma 4. The estimate is valid uniformly with respect to μ, provided it is sufficiently small.

Case C The quadratic form may be estimated as

$$\begin{aligned} \langle u, T_{\min} u\rangle &= \langle \frac{d}{dy} e^{y/2} u, v_\mu \frac{d}{dy} e^{y/2} u\rangle + \langle e^{y/2} u, w_\mu e^{y/2} u\rangle \\ &\geq \inf \frac{w_\mu(x)}{x} \parallel u \parallel^2 \geq c \parallel u \parallel^2, \end{aligned}$$

where c is the positive constant from (61). □

This Lemma implies in particular that the operator $T_{\min}$ is boundedly invertible. Let us denote by $T(\mu)$ the Friedrichs extension of the minimal operator $T_{\min}$ - an extension having the same lower bound as the minimal operator. This extension will be called Hain-Lüst operator in what follows.

Consider the following resolvent operator

(63)

$$\begin{aligned} &M(\mu) \equiv (K_{min} - \mu)^{-1} \\ &= \begin{pmatrix} T^{-1}(\mu) & -T^{-1}(\mu)[C^*(D-\mu I)^{-1}] \\ -[(D-\mu I)^{-1}C]T^{-1}(\mu) & (D-\mu I)^{-1} + [(D-\mu I)^{-1}C]T^{-1}(\mu)[C^*(D-\mu I)^{-1}] \end{pmatrix} \\ &=: \begin{pmatrix} G_{11}(\mu) & G_{12}(\mu) \\ G_{21}(\mu) & G_{22}(\mu) \end{pmatrix}. \end{aligned}$$

In what follows we are going to show that the operators $G_{11}(\mu), G_{12}(\mu)$, and $G_{21}(\mu)$ are compact for sufficiently small negative values of the spectral parameter μ, which will imply that the essential spectrum of $M(\mu)$ is determined exclusively by $G_{22}(\mu)$.[5]

[5]This is in contrast to the case where the quasiregularity conditions (4) are satisfied. In that case the essential spectrum is determined by the operator $G_{11}(\mu)$ as well [**17**].

LEMMA 6. *Let conditions of Theorem 2 hold true. Then the operator* $G_{11}(\mu) \equiv T_{\min}^{-1}(\mu)$ *is compact for sufficiently small values of* $\mu \ll -1$.

PROOF. To proof this lemma we are going to show that the operator $T_{\min}^{-1}(\mu)$ maps every bounded set to a compact set. In other words, we are going to show that every set bounded in the graph norm of $T_{\min}(\mu)$ is compact in $L_2[0,\infty)$ for sufficiently small negative values of μ. Let $u, \| u \| \leq 1$ is mapped by $T_{\min}^{-1}(\mu)$ to v

$$v = T_{\min}^{-1}(\mu)u \Rightarrow T_{\min}(\mu)v = u$$
$$\Rightarrow \langle v, T_{\min}(\mu)v\rangle = \langle u, v\rangle.$$

Then Lemma 5 implies that every set bounded in the graph norm is bounded in the norm associated with the quadratic form of the operator

$$c \| v \|^2 \leq |\langle v, T_{\min} v\rangle| \leq \| v \| \| u \|$$
$$\Rightarrow \langle v, T_{\min} v\rangle \leq C.$$

We conclude that to proof this lemma it is enough to show, that every bounded with respect to the quadratic form $\langle v, T_{\min} v\rangle$ subset of $\mathrm{Dom}\,(T_{\min})$ is compact in the Hilbert space $L_2[0,\infty)$. In other words it is enough to show that the set of functions

$$\left\{ u \in C_0^\infty[0,\infty) | u(0) = 0, \left\langle e^{\alpha y} \left(-\frac{d}{dy} v_\mu \frac{d}{dy} + w_\mu \right) e^{\alpha y} u, u \right\rangle \leq 1 \right\}$$

is compact, where $\alpha = 1$ in the Cases A and B, and $\alpha = 1/2$ in the Case C. Taking into account estimates on the functions v_μ and w_μ (Lemmas 3 and 4) we conclude that for sufficiently small μ to prove the compactness of $T_{\min}(\mu)$ it is enough to show the compactness of the following sets:
Cases A and B

$$S_{AB} \equiv \left\{ u \in C_0^\infty[0,\infty) | u(0) = 0, \| \frac{d}{dy} \left(e^y u \right) \|^2 + \| e^y u \|^2 \leq 1 \right\} \tag{64}$$

and
Case C

$$S_C \equiv \left\{ u \in C_0^\infty[0,\infty) | u(0) = 0, \| \frac{d}{dy} \left(e^{y/2} u \right) \|^2 + \| u \|^2 \leq 1 \right\} \tag{65}$$

Let us construct a compact ϵ-net for these sets. Consider a set of cut-off functions χ_N with the following properties

$$\chi_N \in C_0^\infty[0,\infty), \;\; \chi_N(y) = \begin{cases} 1, & y \leq N, \\ 0, & y \geq N+1. \end{cases}$$

Then the tails $(1-\chi_N)u$ of functions from the sets $S_{AB,C}$ can be estimated as follows:
Cases A and B.

$$\| e^y u \| \leq 1 \Rightarrow e^N \| (1-\chi_N)u \| \leq 1 \rightarrow \| (1-\chi_N)u \| \leq e^{-N}.$$

Case C.

$$e^{y/2}u(y) = \int_0^y (e^{\tilde{y}/2}u(\tilde{y}))' d\tilde{y}$$

$$\Rightarrow |e^{y/2}u(y)| \leq \int_0^y |(e^{\tilde{y}/2}u(\tilde{y}))'| d\tilde{y} \leq \sqrt{\int_0^y 1 d\tilde{y} \int_0^y |(e^{\tilde{y}/2}u(\tilde{y}))'|^2 d\tilde{y}} \leq \sqrt{y}$$

$$\Rightarrow |u(y)| \leq \sqrt{y} e^{-y/2} \Rightarrow \int_N^\infty |u(y)|^2 dy \leq (N+1)e^{-N}$$

$$\Rightarrow \| (1-\chi_N)u \| \leq \sqrt{N+1} e^{-N/2}.$$

These estimates show that taking sufficiently large N the sets $\chi_N S_{AB,C} \equiv \{\chi_N u | u \in S_{AB,C}\}$ the sets approximate the sets $S_{AB,C}$ with arbitrary precision ϵ. The set $S_{AB,C}$ are bounded in the metrics of $W_2^1[0, N+1]$. Since the embedding of $W_2^1[0, N+1]$ into $L_2[0, N+1]$ is compact for any finite N (Rellich Theorem), these sets form a compact ϵ-net for the set S. Hence the operator $T_{\min}(\mu)$ is compact. □

LEMMA 7. *Under the conditions of Theorem 2 the operators* $\frac{d}{dy} T_{\min}^{-1}(\mu)$ *and* $T_{\min}^{-1}(\mu)\frac{d}{dy}$ *are compact for sufficiently small negative values of* μ.

PROOF. The two operators under consideration

$$\frac{d}{dy} T_{\min}^{-1}(\mu) \quad \text{and} \quad T_{\min}^{-1}(\mu)\frac{d}{dy}$$

are formally mutually adjoint. Therefore it is enough to proof the compactness of only one of them, say $\frac{d}{dy} T_{\min}^{-1}(\mu)$. This operator is compact if and only if it maps every bounded set, say $B_1 \equiv \{u; \| u \| \leq 1\}$ onto a compact set. For arbitrary u consider the function $v = T_{\min}^{-1}(\mu)u$. Then the operator $\frac{d}{dy} T_{\min}^{-1}(\mu)$ is compact if and only if the set

$$S \equiv \{v : \| T_{\min}(\mu)v \| \leq 1\}$$

is compact in the norm of $W_2^1[0,\infty)$. During the proof of the previous lemma we have already shown that the sets $\chi_N S$ form an ϵ-net for the set S in the norm of $L_2[0,\infty)$. Let us show that these sets form an ϵ-net even in the norm of $W_2^1[0,\infty)$. It remains to show that for sufficiently large N the first derivatives can be estimated uniformly. Let $\alpha = 1$ in the Cases A and B and $\alpha = 1/2$ in the Case C, then we have

$$\| \frac{d}{dy}[(1-\chi_N)v] \|$$

$$= \| \frac{d}{dy}\left[(1-\chi_N)e^{-\alpha y}e^{\alpha y}v\right] \|$$

$$\leq \| (1-\chi_N)'v \| + \| (1-\chi_N)(-\alpha)v \| + \| (1-\chi_N)e^{-\alpha y}\frac{d}{dy}(e^{\alpha y}v) \|$$

$$\leq (c+\alpha) \| v \|_{L_2[N,\infty)} + e^{-\alpha N} \| \frac{d}{dy}(e^{\alpha y}v) \|,$$

where $c = \max|\chi_N'|$. We have already proven that the first term tends to zero as $N \to \infty$. The sets $\chi_N S$ are compact since the estimate in the Cases A and B

$$\langle T_{\min}v, v\rangle \geq C \| \frac{d}{dy}(e^y u) \|^2 + \| e^y u \|^2$$

implies that $T_{\min}^{-1/2}(\mu)$ is a bounded operator from $L_2[0,N]$ onto $W_2^1[0,N]$. It follows that the operator $T_{\min}^{-1}(\mu)$ is a bounded operator from $L_2[0,N]$ onto $W_2^2[0,N]$. Similar estimate holds in the Case C. Therefore the operator $\frac{d}{dy}T_{\min}^{-1}(\mu)$ is a bounded operator from $L_2[0,N]$ onto $W_2^1[0,N]$ and it is a compact operator as an operator in $L_2[0,N]$, since the embedding of $W_2^1[0,N]$ in $L_2[0,N]$ is compact. $\square$

The last two lemmas imply that the operators $G_{12}(\mu)$ and $G_{21}(\mu)$ are compact. Really the operator $G_{12}(\mu)$ can be written as

$$G_{12}(\mu) = T_{\min}^{-1}(\mu)\frac{d}{dy}\frac{\beta}{m-\mu x^2} - T_{\min}^{-1}(\mu)\frac{\beta}{2(m-\mu x^2)}.$$

In the Cases A and C the C^1-function $m(x)$ has positive value at the origin. In the Case B the derivative $m'(0)$ is positive. Hence the function $m - \mu x^2$ is positive definite for sufficiently small values of μ. Therefore the operator $B_1 = \frac{\beta}{m-\mu x^2}$ is bounded and thus the operator

$$G_{12}(\mu) = T_{\min}^{-1}(\mu)\frac{d}{dy}B - \frac{1}{2}T_{\min}^{-1}(\mu)B$$

is compact. The operator

$$G_{21}(\mu) = -B\frac{d}{dy}T_{\min}^{-1}(\mu) - \frac{1}{2}BT_{\min}^{-1}(\mu)$$

is compact as well.

Thus the essential spectrum of the matrix operator $M(\mu)$ coincides with the essential spectrum of the operator $G_{22}(\mu)$ up to the spectral point $\mu = 0$, which may be ignored during the calculation of the essential spectrum of the operator K.

Consider the Cases A and B first. Then the operator $G_{22}(\mu)$ has the form

$$\begin{aligned} G_{22}(\mu) &= \frac{1}{D-\mu I} + \frac{1}{D-\mu I} C T^{-1}(\mu) C^* \frac{1}{D-\mu I} \\ &= \frac{x^2}{m-\mu x^2} + \frac{\beta}{m-\mu x^2}\left(\frac{d}{dy}+\frac{1}{2}\right) T^{-1}(\mu)\left(-\frac{d}{dy}+\frac{1}{2}\right)\frac{\beta}{m-\mu x^2} \\ &\doteq \frac{x^2}{m-\mu x^2} - \frac{\beta}{m-\mu x^2}\frac{d}{dy}T^{-1}(\mu)\frac{1}{2}\frac{\beta}{m-\mu x^2} \\ &\doteq \frac{x^2}{m-\mu x^2} + \frac{\beta e^{-y}}{m-\mu x^2}\frac{d}{dy}\frac{1}{\frac{d}{dy}v_\mu\frac{d}{dy} - w_\mu}\frac{d}{dy}\frac{\beta e^{-y}}{m-\mu x^2}, \end{aligned}$$

where we used representation (50) and commutation relation (54) to get the last expression.

In what follows we are going to use the calculus of pseudodifferential operators and therefore Fourier transform on $\mathbb{R}$. Let us consider a new operator $M(\mu)$ defined in the space $L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ by the same matrix differential expression (63), where all functions involved are extended as even functions to the whole real axis. The essential spectrum of the new operator coincides up to multiplicity with the essential spectrum of the original operator [**24**]. Using the operator $p = \frac{1}{i}\frac{d}{dy}$ the operator $G_{22}(\mu)$ can now be written as follows

$$\hat{G}_{22} \doteq \frac{x^2}{m-\mu x^2} + \frac{\beta e^{-y}}{m-\mu x^2} p \frac{1}{p v_\mu p + w_\mu} p \frac{\beta e^{-y}}{m-\mu x^2}.$$

Let us use the following fact proven in [**17**]

PROPOSITION 3. *(Lemma 8 from* [**17**]*) Let the real valued function $f(y)$ be positive bounded and separated from zero*

$$0 < c \le f(y) \le C$$

for some $c, C \in R_+$. Let the function $g(y)$ be bounded and the operator

$$L \equiv p f(y) p + g(y)$$

be self-adjoint and invertible in $L_2(\mathbb{R})$. Suppose that the operator

$$pL^{-1}p$$

be bounded. Then for any bounded function $h(y)$ such that $lim_{y\to\infty}h(y) = 0$ the following equality holds in Calkin algebra

$$pL^{-1}ph \doteq \frac{h}{f}.$$

Applying this proposition to the operator $\hat{G}_{22}$ we get

$$\begin{aligned}
\hat{G}_{22} &\doteq \frac{x^2}{m-\mu x^2} + \frac{\beta x}{m-\mu x^2}\frac{1}{v_\mu}\frac{x\beta}{m-\mu x^2} \\
&= \frac{x^2}{m-\mu x^2} + \frac{\beta x}{m-\mu x^2}\frac{m-\mu x^2}{\rho m - \beta^2\rho\mu x^2}\frac{x\beta}{m-\mu x^2} \\
&= \frac{\rho x^2}{\rho m - \beta^2 - \rho\mu x^2} \\
&= \frac{1}{\frac{m}{x^2} - \frac{\beta^2}{\rho x^2} - \mu}.
\end{aligned} \tag{66}$$

Using Weyl's theorem for compact perturbations [**15**] and Dunford spectral mapping theorem [**5**] we conclude that the essential spectrum of any self-adjoint extension of the operator $K_{\min}$ and therefore any self-adjoint extension of $L_{\min}$ can be calculated as

$$\sigma_{\text{ess}}(L) = Range\{\frac{m - \frac{\beta^2}{\rho}}{x^2}\}. \tag{67}$$

In the Case C similar calculations can be carried out. The operator $G_{22}(\mu)$ given by

$$G_{22}(\mu) = \tfrac{x^2}{m-\mu x^2}$$

$$+\tfrac{1}{m-\mu x^2}\left(\beta\tfrac{d}{dy} + \beta/2\right)e^{-y/2}\left(-\tfrac{d}{dy}v_\mu\tfrac{d}{dy} + w_\mu\right)^{-1}e^{-y/2}\left(-\tfrac{d}{dy}\beta + \beta/2\right)\tfrac{1}{m-\mu x^2}$$

may be continued as a pseudodifferential operator to the whole axis

$$\begin{aligned}
\hat{G}_{22}(\mu) &\doteq \frac{x^2}{m-\mu x^2} + \frac{1}{m-\mu x^2}\beta p e^{-y/2}\left(p v_\mu p + w_\mu\right)^{-1} e^{-y/2} p \beta \frac{1}{m-\mu x^2} \\
&\doteq \frac{x^2}{m-\mu x^2} + \frac{1}{m-\mu x^2}\beta e^{-y/2} p \left(p v_\mu p + w_\mu\right)^{-1} p e^{-y/2} \beta \frac{1}{m-\mu x^2} \\
&\doteq \frac{x^2}{m-\mu x^2} + \frac{1}{m-\mu x^2}\beta \sqrt{x} \frac{1}{v_\mu} \sqrt{x} \beta \frac{1}{m-\mu x^2} \\
&= \tfrac{x^2}{m-\mu x^2} + \tfrac{1}{m-\mu x^2}\beta^2 \tfrac{x^2(m-\mu x^2)}{\rho m - \beta^2 - \rho\mu x^2} \tfrac{1}{m-\mu x^2} \\
&= \frac{1}{\frac{m}{x^2} - \frac{\beta^2}{\rho x^2} - \mu},
\end{aligned}$$

where we again used commutation relations (54) and representations (53) and (45) for the function v_μ. This accomplishes the proof.[6] □

It is clear that the set $\mathcal{R}\left(\frac{\rho m - \beta^2}{\rho x^2}\right)$ is unbounded from above in all three Cases A,B and C. On the other hand if the quasiregularity conditions are satisfied, then the essential spectrum is bounded from above as well as from below, but in addition to the described branch of essential spectrum a new branch of essential spectrum is present. This new branch of the essential spectrum has been called singularity spectrum, since it cannot be obtained as a limit of the essential spectra of differential operators given by restrictions of the differential operators L to intervals $(\epsilon, 1]$. This singularity spectrum is absent if the quasiregularity conditions are not fulfilled. In the following section we are going to discuss the relations between the singularity spectrum and properties of Hain-Lüst operator.

6. Weyl circles for Hain-Lüst operator and quasiregularity conditions

In this section we are going to investigate the relations between the quasiregularity conditions and the properties of the Hain-Lüst operator,

[6]Note that in the considered case the entries G_{11}, G_{12} an G_{21} are compact operators. This case differs drastically from the case where the quasiregularity conditions are satisfied [**17**]. In the latter case a certain more elaborated technique had to be used (Lemma on the essential spectrum of the triple sum of operators in Banach space, [**17**], see also [**25, 26**]).

which is a second order differential operator on the interval $x \in [0,1]$. This operator is given by the following differential expression

$$T(\mu) = -\frac{d}{dx}\left(\frac{\rho m - \beta^2}{m - \mu x^2} - \frac{\rho\mu}{m - \mu x^2}x^2\right)\frac{d}{dx} + q(x) - \mu. \tag{68}$$

In order to avoid inessential difficulties we are going to study this operator for sufficiently small values of the parameter μ, i.e. we assume that $\mu \le \mu_0 \ll -1$. The minimal operator $T_{\min}(\mu)$ determined by (68) is defined on $C^\infty(0,1)$ - the set of smooth functions with compact support separated from the end points of the interval $x \in [0,1]$. The maximal operator - adjoint to $T_{\min}(\mu)$ in $L_2[0,1]$ is defined by the same differential expression (68) on the domain $\{f \in L_2[0,1] : T(\mu)f \in L_2[0,1]\}$. The operator $T(\mu)$ is formally symmetric and it can be made self-adjoint by introducing proper boundary conditions at the end points. The endpoint $x = 1$ is always regular and we assume that certain symmetric condition is imposed at this point. The point $x = 0$ is singular and in order to investigate it Weyl's limit point-limit circle theory [**6, 28**] will be used. These studies will tell us whether it is necessary to introduce additional boundary condition at the origin in order to make $T(\mu)$ self-adjoint (provided μ is negative and sufficiently small). It will be more convenient to use the y-representation (38). In this representation the singulat point $x = 0$ corresponds to $y = +\infty$.

THEOREM 3. *Let standard assumptions (2) and (3) on the coefficients of the operator L given by (1) be satisfied. Suppose that the operator $\mathbf{L}$ is semibounded. Then the Hain-Lüst operator $T(\mu)$ is in the limit point case at $x = 0$ ($y = \infty$) for sufficiently small values of μ, i.e. for $\mu \ll -1$, if and only if the quasiregularity conditions (4) are satisfied, but the operator is not regular. In other words, the Hain-Lüst operator is in the limit circle case at $x = 0$ ($y = \infty$) for sufficiently small values of $\mu \ll -1$ if and only if either the quasiregularity conditions (4) are not satisfied or the operator L is regular.*

PROOF. We are going to consider five different cases covering all possible values of the coefficients. The Cases A,B and C coincide with the ones introduced first during the proof of Theorem 2. The Case D covers all coefficients satisfying quasiregularity conditions, but which are not regular. The last case (Case E) is added for the sake of completeness and corresponds to the regular operator L.

Cases A and B

Suppose that conditions (47) or (48) are satisfied. Then in accordance with Lemma 2 the operator $T(\mu)$ can be written in the form

(50)

$$T(\mu) = e^y \left(-\frac{d}{dy} v_\mu \frac{d}{dy} + w_\mu \right) e^y$$

with the functions v_μ and w_μ given by (51). These functions satisfy the following inequalities due to Lemmas 3 and 4:

$$v_\mu(x) \geq c > 0, \quad w_\mu(x) \geq c > 0,$$

for sufficiently small values of μ. Let us study the asymptotics as $y \to \infty$ of the solutions to the equation

$$T(\mu) f = if, \tag{69}$$

which is a second order differential equation. Straightforward calculations transform this equation into the standard form

(70)

$$\frac{d^2}{dy^2} f + \left(2 + (\frac{d}{dy} v_\mu)/v_\mu \right) \frac{d}{dy} f + \left(\frac{1}{2} (\frac{d}{dy} v_\mu)/v_\mu + \frac{3}{4} + \frac{\mu - q + i}{e^{2y} v_\mu} \right) f = 0.$$

The functions $(\frac{d}{dy} v_\mu)/v_\mu$ and $\frac{\mu - q + i}{e^{2y} v_\mu}$ are exponentially small as $y \to \infty$ and therefore it is natural to expect that the asymptotics of the solutions is just the same as for the equation

$$\frac{d^2}{dy^2} f + 2 \frac{d}{dy} f + \frac{3}{4} f = 0 \Rightarrow f(y) = C_1 e^{-\frac{1}{2} y} + C_2 e^{-\frac{3}{2} y}.$$

Let us discuss how to prove this fact for general second order differential equation [**20**]

$$\frac{d^2}{dy^2} f + (a_1 + g_1(y)) \frac{d}{dy} f + (a_2 + g_2(y)) f = 0, \tag{71}$$

where a_1, a_2 are certain real constants and $g_{1,2}$ are real valued functions tending to zero exponentially fast as $y \to \infty$. One may get rid of the first derivative by introducing the new function h as follows

$$f(y) = e^{-\frac{1}{2}(a_1 y + \int^y g_1(\tilde{y}) d\tilde{y})} h(y). \tag{72}$$

The equation (71) transforms as

$$\frac{d^2}{dy^2} h + \left(-\frac{1}{4} (a_1 + g_1)^2 - \frac{1}{2} g'_{1y} + a_2 + g_2 \right) h = 0. \tag{73}$$

It will be more convenient to introduce the following notations

$$\begin{aligned} c &= \sqrt{\frac{1}{4} a_1^2 - a_2}; \\ g &= \frac{1}{2} a_1 g_1 + \frac{1}{4} g_1^2 + \frac{1}{2} g'_{1y} - g_2. \end{aligned} \tag{74}$$

Using these notations equation (73) can be written as follows

$$\frac{d^2}{dy^2}h - c^2 h - g(y)h = 0. \tag{75}$$

We are interested to prove that the solutions to this equation have the same asymptotics as solution to the free equation

$$\frac{d^2}{dy^2}h - c^2 h = 0 \Rightarrow h = C_1 e^{cx} + C_2 e^{-cx}. \tag{76}$$

Suppose that the potential g satisfies the following estimate

$$|g(y)| \leq a(y) e^{-2cy}, \tag{77}$$

where a is a certain L_1 - function. Then the growing solution to equation (75) satisfies the following Volterra type equation

$$h(y) = e^{cy} + \int_y^\infty \frac{1}{2c}\left(e^{c(y-\tilde{y})} - e^{-c(y-\tilde{y})}\right) g(\tilde{y}) h(\tilde{y}) d\tilde{y}. \tag{78}$$

This integral equation has a solution having the asymptotics $h_1(y) \sim e^{cy}$ if the potential q satisfies the estimate (77). In this case the solution may be obtained by the method of successive approximations. Then there is another linear independent solution with the asymptotics $h_2(y) \sim e^{-cy}$. This follows easily from the fact that h_1 and h_2 are solutions to one and the same second order differential equation and therefore their Wronskian is constant.

Let us return back to the studies of equation (70). Comparison with equation (71) gives us

$$\begin{array}{rclrcl} a_1 & = & 2, & g_1 & = & \left(\frac{d}{dy} v_\mu\right)/v_\mu, \\ a_2 & = & \frac{3}{4}, & g_2 & = & \frac{1}{2}\left(\frac{d}{dy} v_\mu\right)/v_\mu + \frac{\mu - q + i}{v_\mu} e^{-2y}, \end{array}$$

$$\Rightarrow \left\{ \begin{array}{rcl} c & = & \sqrt{\frac{1}{4}4 - \frac{3}{4}} = \frac{1}{2} \\ g & = & \frac{1}{2}\left(\frac{d}{dy} v_\mu\right)/v_\mu + \frac{1}{4}\left(\left(\frac{d}{dy} v_\mu\right)/v_\mu\right)^2 \\ & & +\frac{1}{2}\frac{d}{dy}\left(\left(\frac{d}{dy} v_\mu\right)/v_\mu\right) - \frac{\mu - q + i}{v_\mu} e^{-2y}. \end{array} \right.$$

The potential g can be simplified as

$$g = \left(-\frac{1}{4}\left(\frac{v'_{\mu x}}{v_\mu}\right)^2 + \frac{1}{2}\frac{v''_{\mu xx}}{v_\mu} - \frac{\mu - q + i}{v_\mu}\right) e^{-2y}.$$

Since the function v_μ never vanish for sufficiently small $\mu \ll -1$ and all functions are two times differentiable, the expression in brackets is uniformly bounded. It follows that g satisfies the necessary estimate

$$|g(y)| \leq \text{const } e^{-2y}.$$

Hence solutions to (75) have asymptotics

$$h_1 \sim e^{\frac{1}{2}y} \text{ and } h_2 \sim e^{-\frac{1}{2}y}.$$

The corresponding solutions to (69) have the following behavior

$$f_1 \sim e^{-\frac{1}{2}y} \text{ and } f_2 \sim e^{-\frac{3}{2}y}. \tag{79}$$

Both solutions are square integrable in the neighborhood of $y = +\infty$, i.e. the case of Weyl's limit circle occurs.

Case C

This case can be investigated using similar method. The Hain-Lüst operator can be written in the form (52)

$$T(\mu) = e^{y/2}\left(-\frac{d}{dy}v_\mu\frac{d}{dy} + w_\mu\right)e^{y/2}$$

with the coefficients satisfying (53)

$$v_\mu \geq c > 0, \quad w_\mu \geq cx, c > 0$$

for sufficiently small values of μ. Then the equation (69) can be transformed into

$$\frac{d^2}{dy^2}f + \left(1 + (\frac{d}{dy}v_\mu)/v_\mu\right)\frac{d}{dy}f + \left(\frac{1}{4} + \frac{1}{2}(\frac{d}{dy}v_\mu)/v_\mu + \frac{\mu - q + i}{e^y v_\mu}\right)f = 0.$$

Comparison with equation (71) gives

$$\begin{array}{rclrcl} a_1 & = & 1, & g_1 & = & \left(\frac{d}{dy}v_\mu\right)/v_\mu, \\ a_2 & = & \frac{1}{4}, & g_2 & = & \frac{1}{2}\left(\frac{d}{dy}v_\mu\right)/v_\mu + \frac{\mu - q + i}{v_\mu}e^{-y}, \end{array}$$

$$\Rightarrow \left\{ \begin{array}{rcl} c & = & \sqrt{\frac{1}{4}4 - \frac{3}{4}} = \frac{1}{2} \\ g & = & \frac{1}{4}\left(\left(\frac{d}{dy}v_\mu\right)/v_\mu\right)^2 + \frac{1}{2}\frac{d}{dy}\left(\left(\frac{d}{dy}v_\mu\right)/v_\mu\right) - \frac{\mu - q + i}{v_\mu}e^{-y}. \end{array}\right.$$

The potential g can be simplified as

$$g = \left(-\frac{1}{4}e^{-y}\left(\frac{v'_{\mu x}}{v_\mu}\right)^2 + \frac{1}{2}\frac{v'_{\mu x}}{v_\mu} + \frac{1}{2}e^{-y}\frac{v''_{\mu xx}}{v_\mu} - \frac{\mu - q + i}{v_\mu}\right)e^{-y}.$$

The expression in brackets is again uniformly bounded. It follows that g satisfies the necessary estimate

$$|g(y)| \leq \text{const } e^{-y}.$$

Hence solutions to (75) have asymptotics

$$h \sim C_1 y + C_2.$$

The corresponding solutions to (69) have the following behavior

$$f \sim (C_1 y + C_2) e^{-\frac{1}{2}y} \tag{80}$$

and both solutions are square integrable in the neighborhood of $y = +\infty$, i.e. the operator $T(\mu)$ is in the limit circle case.

Case D under construction

Assume that the coefficients satisfy quasiregularity conditions (4) but the operator is not regular. This implies in particular that $m|_{x=0} \neq 0$. Otherwise the quasiregularity conditions would imply that $\beta|_{x=0} = 0$ and therefore m has second order zero at the origin.

The coefficients V_μ and W_μ of the Hain-Lüst operator (44)

$$T(\mu) = -\frac{d}{dy} V_\mu \frac{d}{dy} + W_\mu$$

are uniformly bounded and positive definite functions for sufficiently small values of $\mu \ll -1$. It follows that the quadratic form of $T(\mu)$ is equivalent to the quadratic form of the second derivative operator $-\frac{d^2}{dy^2} + 1$. The later operator is in the limit point case at $y = +\infty$. Hence the Hain-Lüst operator is in the limit point case as well.

Case E

Suppose that the operator L is regular, i.e. the functions β and m have first order, respectively second order zeroes at the origin. In this case $\hat{v}_\mu$ is uniformly bounded and positive definite for sufficiently small values of μ, i.e. $\mu \ll -1$. Therefore this case is similar to Cases A and B just considered. The asymptotics of the solution is given by the same formula (79) and all solutions are square integrable. □

This theorem implies that the singularity spectrum for the operator L appears if and only if the Hain-Lüst operator is in the limit point case at the singular point. We believe that this observation is crucial for the existence of the singularity spectrum even in a more general settings. It is planned to continue studies of this phenomena for more sophisticated singular matrix differential operator.

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