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On Asymptotics of Polynomial Eigenfunctions for Exactly-Solvable Differential Operators

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Abstract

In this paper we study the class of differential operators $T = \sum_{j=1}^{k} Q_j D^j$ with polynomial coefficients Q_j and $\deg Q_j \leq j$ with equality for at least one j. We show that if $\deg Q_k < k$ then the root of the nth degree eigenpolynomial p_n of T with the largest absolute value tends to infinity when $n \to \infty$, as opposed to the case when $\deg Q_k = k$. Moreover we present an explicit conjecture and partial results on the growth of the largest root. Based on this conjecture we deduce the algebraic equation satisfied by the asymptotic Cauchy transform of the appropriately scaled eigenpolynomials.

1 Introduction

In this paper we study asymptotic properties of zeros in families of polynomials satisfying certain linear differential equations. Namely, consider a linear differential operator

$$T = \sum_{j=1}^{k} Q_j D^j,$$

where $D = d/dz$ and the Q_j are complex polynomials in a single variable z. We are interested in the case when $\deg Q_j \leq j$ for all j, and in particular $\deg Q_k < k$ for the leading term. Such operators are referred to as *degenerate exactly-solvable operators*, see Definition 1 below. In this paper we study the polynomial eigenfunctions of this operator, that is polynomials satisfying

$$T(p_n) = \lambda_n p_n \tag{1}$$

for some value of the spectral parameter λ_n, where n is a nonnegative integer and $\deg p_n = n$.

The basic motivation for this study comes from two sources: 1) a classical question going back to S. Bochner, and 2) the generalized Bochner problem, which we describe below.

1) In 1929 Bochner asked about the classification of differential equations (1) having an infinite sequence of *orthogonal* polynomial solutions, see [11]. Such a system of polynomials $\{p_n\}_{n=0}^{\infty}$ which are both eigenpolynomials of some finite order differential operator *and* orthogonal with respect to some suitable inner product, are referred to as *Bochner-Krall orthogonal polynomial systems* (BKS), and the corresponding operators are called *Bochner-Krall operators.* It is an open problem to classify all BKS - a complete classification is only known for Bochner-Krall operators of order $k \leq 4$, and the corresponding BKS are various classical systems such as the Jacobi type, the Laguerre type, the Legendre type and the Bessel and Hermite polynomials (see [5]).

Notice that for the operators considered below, the sequence of eigenpolynomials is in general *not* an orthogonal system of polynomials, and can therefore not be studied by means of the extensive theory known for such systems.

2) The problem of a general classisfication of linear differential operators for which the eigenvalue problem (1) has a certain number of eigenfunctions in the form of a finite-order polynomial in some variables, is referred to as *the generalized Bochner problem*, see [15] and [16]. In the former paper a classification of operators possessing infinitely many finite-dimensional subspaces with a basis in polynomials is presented, and in the latter paper a general method has been formulated for generating eigenvalue problems for linear differential operators in one and several variables possessing polynomial solutions.

Definition 1. We call a linear differential operator T of the kth order *exactly-solvable* if it preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$, where $\mathcal{P}_n$ is the linear space of all polynomials of degree less than or equal to n.[1] Or, equivalently, the problem (1) has an infinite sequence of polynomial eigenfunctions *if and only if* the operator T is *exactly-solvable* (see [17]).

The exactly-solvable operators $T = \sum_{j=1}^{k} Q_j D^j$ with $\deg Q_j \leq j$ for all j, split into two major classes: *non-degenerate* and *degenerate*, where in the former case $\deg Q_k = k$, and in the latter case $\deg Q_k < k$ for the leading term. The major difference between these two classes is that in the non-degenerate case the union of all roots of all eigenpolynomials of T is contained in a compact set, contrary to the degenerate case, which we will prove in this paper.

Let us briefly recall our previous results for eigenpolynomials of the non-degenerate exactly-solvable operators. In [1] we proved that asymptotically as

[1] Correspondingly, a linear differential operator of the kth order is called *quasi-exactly-solvable* if it preserves the space $\mathcal{P}_n$ for some fixed n.

$n \to \infty$, the zeros of the nth degree eigenpolynomials p_n of the non-degenerate exactly-solvable operators are distributed according to a certain probability measure which has compact support and which depends *only* on the leading polynomial Q_k. These are our main results from [1]:

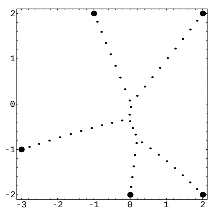
Theorem A. *Let Q_k be a monic polynomial of degree k. Then there exists a unique probability measure μ_{Q_k} with compact support whose Cauchy transform $C(z) = \int \frac{d\mu_{Q_k}(\zeta)}{z-\zeta}$ satisfies $C(z)^k = 1/Q_k(z)$ for almost all $z \in \mathbb{C}$.*

Theorem B. *Let Q_k and μ_{Q_k} be as in Theorem A. Then* supp μ_{Q_k} *is the union of finitely many smooth curve segments, and each of these curves is mapped to a straight line by the locally defined mapping $\Psi(z) = \int Q_k(z)^{-1/k}dz$. Moreover,* supp μ_{Q_k} *contains all the zeros of Q_k, is contained in the convex hull of the zeros of Q_k, is connected and has connected complement.*

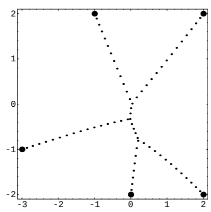
If p_n is a polynomial of degree n we construct the probability measure μ_n by placing the point mass of size $\frac{1}{n}$ at each zero of p_n, and we call μ_n the root measure of p_n. We then have the (main) result:

Theorem C. *Let p_n be the monic degree n eigenpolynomial of a non-degenerate exactly solvable operator T and let μ_n be the root measure of p_n. Then μ_n converges weakly to μ_{Q_k} when $n \to \infty$.*

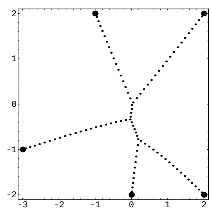
To illustrate, we show the zeros of the polynomial eigenfunctions p_{50}, p_{75} and p_{100} for the non-degenerate exactly-solvable operator $T = Q_5D^5$ where $Q_5 = (z-2+2i)(z+1-2i)(z+3+i)(z+2i)(z-2i-2)$. In the pictures below, the large dots represent the zeros of Q_5 and the small dots represent the zeros of the eigenpolynomials p_{50}, p_{75} and p_{100} respectively:



n=50



n=75



n=100.

As a result of this study, we were then able to prove a special case of a general conjecture describing the leading terms of all Bochner-Krall operators, see [2].

In the present paper we are interested in the class of degenerate exactly-solvable operators, that is operators $T = \sum_{j=1}^{k} Q_jD^j$ where $\deg Q_j \leq j$ for all j with equality for at least one j, and $\deg Q_k < k$. Without loss of generality we assume that the nth degree eigenpolynomial p_n of T is monic. Some well-known classical polynomials, such as the Laguerre polynomials, appear as polynomial solutions to the eigenvalue problem (1) for certain choices on the polynomials coefficients Q_j. Studies on the asymptotic zero behaviour for these polynomials can be found in [4], [7],[9], [13] and [14].

Computer experiments indicate the existence of a limiting measure for the asymptotic zero distribution of the nth degree polynomial eigenfunction p_n of any degenerate exactly-solvable operator after an appropriate scaling. Without such a scaling the roots of p_n tend to infinity when $n \to \infty$, see Theorem 1. Based on calculations involving the Cauchy transform we conjecture how the largest modulus of all roots of p_n grows as $n \to \infty$ for any given degenerate exactly-solvable operator, see Main Conjecture. All experiments performed by the author are consistent with this conjecture (see numerical evidence in Section 4), and we also prove it partially (lower bounds on the largest roots) for some classes of degenerate exactly-solvable operators, see Theorems 3 and 4.

The appropriately scaled eigenpolynomials will then (conjecturally) have nice compactly supported zero distribution in the limit as $n \to \infty$. Under the same assumptions as in Main Conjecture, we then derive the algebraic equation satisfied by the asymptotic Cauchy transform of the scaled eigenpolynomials for any given degenerate exactly-solvable operator (see Main Corollary). From this equation it is possible to obtain detailed information on the asymptotic zero distribution of the scaled eigenpolynomials.

These are our main results:

Theorem 1.[2] *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k, and let r_n be the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T. Then $r_n \to \infty$ as $n \to \infty$.*

Main Conjecture. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k, and denote by j_0 the largest j for which $\deg Q_j = j$. Denote by r_n be the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T. Then*

$$\lim_{n\to\infty} \frac{r_n}{n^d} = c_0,$$

where $c_0 > 0$ is a positive constant and

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j - j_0}{j - \deg Q_j} \right).$$

Based on this Main Conjecture, we now introduce *the scaled eigenpolynomial* $q_n(z) = p_n(n^d z)$, for which the union of all roots are (conjecturally) contained

[2]This theorem is joint work with H. Rullgård.

in a compact set. We then make the following *basic assumption*: assume that $C_n(z) := C_{n,0}(z) = C_{n,1}(z) = \ldots = C_{n,k-1}(z)$ as $n \to \infty$ for the Cauchy transforms[3] of the scaled eigenpolynomial $q_n(z)$ and its derivatives. This means that we assume that the root measures $\mu_n^0, \mu_n^1, \mu_n^2 \ldots, \mu_n^{k-1}$ of $q_n, q_n^{(1)}, q_n^{(2)} \ldots, q_n^{(k-1)}$ respectively, are all equal as $n \to \infty$, and let $C(z) := \lim_{n\to\infty} C_n(z)$ (computer experiments strongly indicate that this assumption is true, see Section 4.3).

Now let $T = \sum_{j=1}^k Q_j D^j = \sum_{j=1}^k \left(\sum_{i=0}^j \alpha_{j,i} z^i\right) D^j$ be a degenerate exactly-solvable operator and denote by j_0 the largest j such that $\deg Q_j = j$. Moreover, with no loss of generality, we make a normalization by assuming that Q_{j_0} is monic, i.e. $\alpha_{j_0,j_0} = 1$. Consider the scaled polynomial $q_n(z) = p_n(n^d z)$, where $p_n(z)$ is the unique and monic nth degree eigenpolynomial of T, and $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$. We then have the following:

Main Corollary. *Assume that $C_n(z) := C_{n,0}(z) = C_{n,1}(z) = \ldots = C_{n,k-1}(z)$ when $n \to \infty$ for the Cauchy transforms of the scaled eigenpolynomial $q_n(z)$ and its derivatives. Then, for almost all complex z in the usual Lebesgue measure on $\mathbb{C}$, the function $C(z) := \lim_{n\to\infty} C_n(z)$ satisfies the following equation:*

$$z^{j_0} C^{j_0}(z) + \sum_{j\in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1.$$

Here A is the set consisting of all j for which the maximum $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained, i.e. $A = \{j : (j-j_0)/(j-\deg Q_j) = d\}$ where d is as above.

In the following theorem we prove a lower bound for the largest modulus of all roots of p_n when $n \to \infty$ for any degenerate exactly-solvable operator:

Theorem 2. *Let $T = \sum_{j=1}^k Q_j D^j = \sum_{j=1}^k \left(\sum_{i=0}^j \alpha_{j,i} z^i\right) D^j$ be a degenerate exactly-solvable operator of order k. Let z_n be the root with the largest modulus, $|z_n| = r_n$, of the unique and monic nth degree eigenpolynomial p_n of T. Then there exists a positive constant $c_0 > 0$ such that*

$$\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0,$$

[3]If q_n is a polynomial of degree n we construct the probability measure μ_n by placing a point mass of size $\frac{1}{n}$ at each zero of q_n. We call μ_n the root measure of q_n. By definition, for any polynomial q_n, the Cauchy transform $C_{n,j}$ of the root measure $\mu_n^{(j)}$ for the jth derivative $q_n^{(j)}$ is defined by

$$C_{n,j}(z) := \frac{q_n^{(j+1)}(z)}{(n-j)q_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z-\zeta},$$

and it is well-known that the measure μ can be reconstructed from C by the formula $\mu = \frac{1}{\pi} \cdot \frac{\partial C}{\partial \bar{z}}$ where $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$.

for any $\gamma < b$ *where*

$$b := \min_{j\in[1,k-1]}^{+} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k} \right),$$

where the notation $\min^{+}$ *means that the minimum is taken only over positive terms* $(k-j+\deg Q_j - \deg Q_k)$.

The following two theorems are partial results supporting Main Conjecture:

Theorem 3. *Let* T *be a degenerate exactly solvable operator of order* k *consisting of precisely two terms:* $T = Q_{j_0}D^{j_0} + Q_k D^k$. *Let* z_n *be the root with the largest modulus of the unique and monic* n*th degree eigenpolynomial* p_n *of* T*, and let* $|z_n| = r_n$. *Then there exists a positive constant* $c > 0$ *such that*

$$\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} \geq c,$$

where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}$.

This result can be generalized, but with certain conditions on the polynomials Q_j for $j > j_0$:

Theorem 4. *Let* T *be a degenerate exactly-solvable operator of order* k. *Denote by* j_0 *the largest* j *such that* $\deg Q_j = j$. *Furthermore, let* $(j - \deg Q_j) \geq (k - \deg Q_k)$ *for every* $j > j_0$. *Let* z_n *be the root with the largest modulus of the unique and monic* n*th degree eigenpolynomial* p_n *of* T*, and let* $|z_n| = r_n$. *Then there exists a positive constant* $c > 0$ *such that*

$$\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} \geq c,$$

where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}$.

Conjecturally, for any degenerate exactly-solvable operator T, the support of the asymptotic zero distribution of the scaled eigenpolynomial q_n is the union of a finite number of analytic curves in the complex plane, which we denote by Ξ_T. We then have the following conjecture:

Conjecture 1.[4] [Interlacing property] *For any family* $\{q_n\}$ *of appropriately*[5] *scaled polynomial eigenfunctions of any degenerate exactly-solvable operator* T*, the zeros of any two consecutive polynomials* q_{n+1} *and* q_n *interlace along* Ξ_T *for all sufficiently large integers* n.[6]

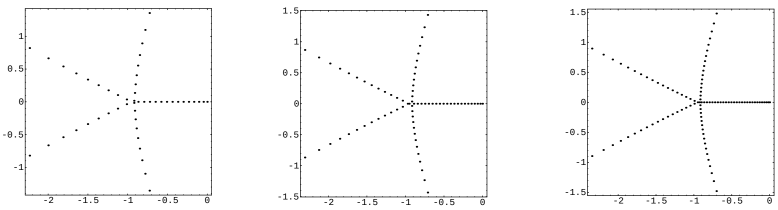
[4] We believe the interlacing property also holds for the non-degenerate exactly-solvable operators, but without such a scaling of the eigenpolynomials.

[5] According to the scaling in Main Conjecture.

[6] The question concerning interlacing was raisedby B. Shapiro. For details see Section 4.4

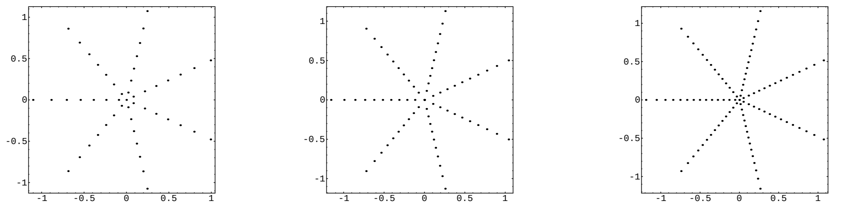
We now present some typical pictures of the zero distribution of the scaled eigenpolynomials for some degenerate exactly-solvable operators. Below, p_n denotes the nth degree monic polynomial eigenfunction of a given operator T, and q_n denotes the corresponding (appropriately) scaled polynomial.

Fig.1: $T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$.



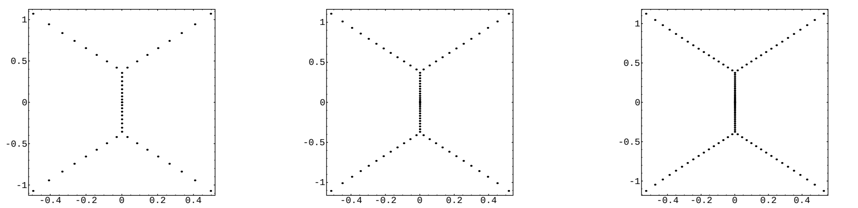
roots of $q_{50}(z) = p_{50}(50z)$ | roots of $q_{75}(z) = p_{75}(75z)$ | roots of $q_{100}(z) = p_{100}(100z)$

Fig.2: $T_2 = z^2D^2 + D^7$.



roots of $q_{50}(z) = p_{50}(50^{5/7}z)$ | roots of $q_{75}(z) = p_{75}(75^{5/7}z)$ | roots of $q_{100}(z) = p_{100}(100^{5/7}z)$

Fig.3: $T_3 = z^3D^3 + z^2D^4 + zD^5$.

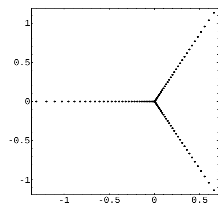


roots of $q_{50}(z) = p_{50}(50^{1/2}z)$ | roots of $q_{75}(z) = p_{75}(75^{1/2}z)$ | roots of $q_{100}(z) = p_{100}(100^{1/2}z)$

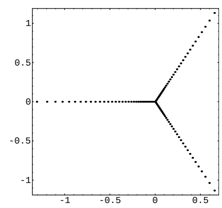
The algebraic equation satisfied by the asymptotic Cauchy transform in Main Corollary indicates that the asymptotic zero distribution of the scaled eigenpolynomials depends only on the term $z^{j_0}D^{j_0}$ and the term(s) $\alpha_{j,\deg Q_j} z^{\deg Q_j} D^j$ of

T for which $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained. To illustrate this fact we present below some pictures of the zero distributions of the scaled eigenpolynomials for some *distinct* operators for which the asymptotic Cauchy transform $C(z)$ of the scaled eigenpolynomials satisfy the *same* equation.

As a first example, consider the operator $T_4 = z^3D^3 + z^2D^5$. Clearly $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right) = (5-3)/(5-2) = 2/3$ and the asymptotic Cauchy transform of the scaled eigenpolynomial $q_n(z) = p_n(n^{2/3}z)$ satisfies the equation $z^3C^3 + z^2C^5 = 1$ in the limit when $n \to \infty$. Now consider the slightly modified operator $\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$ and note that $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is obtained again (only) for $j = 5$ (for $j = 4$ we have $(4-3)/(4-1) = 1/3 < 2/3$ and for $j = 6$ we have $(6-3)/(6-0) = 3/6 = 1/2 < 2/3$). We therefore obtain the same asymptotic equation in $C(z)$ for the scaled eigenpolynomials of $\widetilde{T}_4$ as for the scaled eigenpolynomials of T_4; hence we can consider the added terms z^2D^2, zD^4 and D^6 in $\widetilde{T}_4$ as "irrelevant" for the asymptotic zero distribution. The pictures below clearly illustrate this:

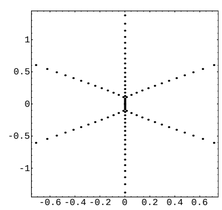


$T_4 = z^3D^3 + z^2D^5$,
roots of $q_{100}(z) = p_{100}(100^{2/3}z)$

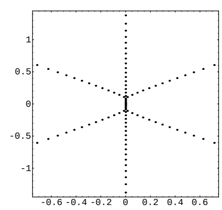


$\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$,
roots of $q_{100}(z) = p_{100}(100^{2/3}z)$

As a second example, consider the operators $T_5 = z^5D^5 + z^4D^6 + z^2D^8$ and $\widetilde{T}_5 = z^2D^2 + z^5D^5 + z^4D^6 + zD^7 + z^2D^8$ whose scaled eigenpolynomials $q_n(z) = p_n(n^{1/2}z)$ both satisfy the Cauchy transform equation $z^5C^5 + z^4C^6 + z^2C^8$ in the limit when $n \to \infty$. In the pictures below one can see that the "irrelevant" terms z^2D^2 and zD^7 of $\widetilde{T}_5$ seem to have no affect on the zero distribution of the scaled eigenpolynomials for sufficiently large n.



$T_5 = z^5D^5 + z^4D^6 + z^2D^8$,
roots of $q_{100}(z) = p_{100}(100^{1/2}z)$



$\widetilde{T}_5 = z^2D^2 + z^5D^5 + z^4D^6 + zD^7 + z^2D^8$,
roots of $q_{100}(z) = p_{100}(100^{1/2}z)$

In the sequel we will settle our Main Conjecture for some special classes of degenerate exactly-solvable operators, and then describe the asymptotic zero distribution of the scaled polynomial eigenfunctions for these operators in detail.

Let us finally mention some possible applications of our results and directions for further reasearch. Operators of the type we consider occur, as was mentioned earlier, in the theory of Bochner-Krall orthogonal systems. A great deal is known about the asymptotic zero distribution of orthogonal polynomials, and by comparing such results with results on the asymptotic zero distribution of eigenpolynomials of degenerate exactly-solvable operators, we believe it will be possible to gain new insight into the nature of BKS.

Acknowledgements. I am sincerely greatful to my PhD advisor Professor B. Shapiro for introducing me to this very fascinating problem and for his constant support during my work. I would also like to thank Professor J-E. Björk and H. Rullgård for stimulating discussions on the topic. My research was supported by Stockholm University.

2 Proofs

We start with the following

Lemma 1. *Let* $T = \sum_{j=1}^{k} Q_j D^j$ *be a degenerate exactly-solvable operator of order* k*. Then, for a sufficiently large integer* n*, there exists a unique constant* λ_n *and a unique monic polynomial* p_n *of degree* n *which satisfy* $T(p_n) = \lambda_n p_n$*. If* $\deg Q_j = j$ *for precisely one value* $j < k$*, then there exists a unique constant* λ_n *and a unique monic polynomial* p_n *of degree* n *which satisfy* $T(p_n) = \lambda_n p_n$ *for* every $n = 1, 2, \ldots$.

Proof of Lemma 1. In [1] we proved that for any exactly-solvable operator T, the eigenvalue problem $T(p_n) = \lambda_n p_n$ can be written as a linear system $MX = Y$, where X is the coefficient vector of the monic nth degree eigenpolynomial p_n with components $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}$, Y is a vector and M is an upper triangular $n \times n$ matrix, both with entries expressible in the coefficients of T. With $T = \sum_{j=1}^{k} Q_j D^j$, $Q_j = \sum_{i=0}^{j} \alpha_{j,i} z^i$, and $p_n(z) = \sum_{i=0}^{n} a_{n,i} z^i$, the eigenvalue λ_n is given by

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!},$$

and the diagonal elements of M are given by

$$M_{i+1,i+1} = \sum_{1 \leq j \leq \min(i,k)} \alpha_{j,j} \frac{i!}{(i-j)!} - \lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \left[\frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right]$$

for $i = 0, 1, \ldots, n-1$. The last equality follows since $i!/(i-j)! = 0$ for $i < j \leq k$ by definition (see Lemma 2 in [1]). In order to prove that p_n is unique we only need to check that the determinant of M is nonzero, which implies that M is invertible and the system $MX = Y$ will have a unique solution. Notice that M is upper triangular, whence its determinant equals the product of its diagonal elements. We now prove that every diagonal element $M_{i+1,i+1}$ is nonzero for all sufficiently large n for all T as above, and for every n if $\deg Q_j = j$ for exactly one j.

From the expression

$$-M_{i+1,i+1} = \sum_{j=1}^{k} \alpha_{j,j} \left[\frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right]$$

it is clear that $M_{i+1,i+1} \neq 0$ for every $i \in [0, n-1]$ and *every* n if $\alpha_{j,j} \neq 0$ for *precisely one* j, that is if $\deg Q_j = j$ for precisely one j - thus we have proved the second part of Lemma 1.

Now assume that $\deg Q_j = j$ for more than one j and denote by j_0 the largest such j (clearly $\alpha_{j_0,j_0} \neq 0$). We then have

$$\begin{aligned} - \quad M_{i+1,i+1} &= \sum_{j=1}^{j_0} \alpha_{j,j} \left[\frac{n!}{(n-j)!} - \frac{i!}{(i-j)!} \right] \\ = \quad \frac{n!}{(n-j_0)!} &\left[\alpha_{j_0,j_0} \left(1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) + \sum_{1 \leq j < j_0} \alpha_{j,j} \frac{(n-j_0)!}{(n-j)!} - \sum_{1 \leq j < j_0} \frac{(n-j_0)! i!}{n!(i-j)!} \right]. \end{aligned}$$

The last two sums on the right-hand side of the equality above tend to zero as $n \to \infty$, since $j_0 > j$ and $i \leq n-1$. Thus for sufficiently large n we have

$$-M_{i+1,i+1} = \frac{n!}{(n-j_0)!} \left[\alpha_{j_0,j_0} \left(1 - \frac{i!/(i-j_0)!}{n!/(n-j_0)!} \right) \right] \neq 0$$

for every $i \in [0, n-1]$, and we have proved the first part of Lemma 1. $\square$

To prove Theorem 1 we need the following lemma. Recall that $\frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z-\zeta} =: C_{n,j}(z)$. Then we have:

Lemma 2. *Let z_n be the root of p_n with the largest modulus, say $|z_n| = r_n$. Then, for any complex number z_0 such that $|z_0| = r_0 \geq r_n$, we have $|C_{n,j}(z_0)| \geq \frac{1}{2r_0}$ for all $j \geq 0$.*

Proof. With ζ being some root of $p_n^{(j)}$ we have $|\zeta| \leq |z_0|$ by Gauss Lucas' Theorem. Thus $\frac{1}{z_0 - \zeta} = \frac{1}{z_0} \cdot \frac{1}{1 - \zeta/z_0} = \frac{1}{z_0} \cdot \frac{1}{1-\theta}$ where $|\theta| = |\zeta/z_0| \leq 1$. With

$w = \frac{1}{1-\theta}$ we obtain

$$|w-1| = \frac{|\theta|}{|1-\theta|} = |\theta||w| \leq |w| \Leftrightarrow |w-1| \leq |w| \Rightarrow Re(w) \geq 1/2.$$

Thus

$$\begin{aligned}
|C_{n,j}(z_0)| &= \left| \int \frac{d\mu_n^{(j)}(\zeta)}{z_0 - \zeta} \right| = \frac{1}{r_0} \left| \int \frac{d\mu_n^{(j)}(\zeta)}{1-\theta} \right| = \frac{1}{r_0} \left| \int w d\mu_n^{(j)}(\zeta) \right| \\
&\geq \frac{1}{r_0} \left| \int Re(w) d\mu_n^{(j)}(\zeta) \right| \geq \frac{1}{2r_0} \int d\mu_n^{(j)}(\zeta) = \frac{1}{2r_0}.
\end{aligned}$$

□

Proof of Theorem 1. Take $T = \sum_{j=1}^{k} Q_j D^j$ and denote by j_0 the largest j such that $\deg Q_j = j$ (clearly $j_0 < k$). From the definition $C_{n,j} = \frac{p_n^{(j+1)}(z)}{(n-j)p_n^{(j)}(z)}$ we get

$$\begin{aligned}
\frac{p_n^{(j)}(z)}{p_n(z)} &= C_{n,0}(z)C_{n,1}(z)\cdots C_{n,j-1}(z) \cdot n(n-1)\cdots(n-j+1) \\
&= \frac{n!}{(n-j)!} \prod_{m=0}^{j-1} C_{n,m}(z).
\end{aligned}$$

With $Q_j(z) = \sum_{i=0}^{\deg Q_j} \alpha_{j,i} z^i$ we have $\lambda_n = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}$, and dividing the eigenvalue equation $T(p_n(z)) = \lambda_n p_n(z)$ by $p_n(z)$ we thus obtain

$$Q_k(z)\frac{p_n^{(k)}}{p_n(z)} + Q_{k-1}(z)\frac{p_n^{(k-1)}}{p_n(z)} + \ldots + Q_1(z)\frac{p_n'}{p_n(z)} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!}$$

$$\Leftrightarrow$$

$$\begin{aligned}
&Q_k(z)\frac{n!}{(n-k)!}\prod_{m=0}^{k-1} C_{n,m}(z) + Q_{k-1}(z)\frac{n!}{(n-k+1)!}\prod_{m=0}^{k-2} C_{n,m}(z) + \ldots \\
&\ldots + Q_1(z)\frac{n!}{(n-1)!}C_{n,0}(z) = \sum_{j=1}^{j_0} \alpha_{j,j}\frac{n!}{(n-j)!}. \qquad (2)
\end{aligned}$$

Now assume that the largest modulus r_n of all roots of p_n (and hence, by Gauss Lucas' Theorem, of any derivative $p_n^{(j)}$) is (strictly) less than some fixed constant $R < \infty$. We can always assume that R is (strictly) larger than the largest absolute value of all roots of Q_k. Now let $\tilde{z}$ be such that $|\tilde{z}| = R$. Then $\frac{1}{2R} \leq |C_{n,j}(\tilde{z})|$ by Lemma 2. Inserting $\tilde{z}$ in equation (2) we obtain:

$$\begin{aligned}
&Q_k(\tilde{z})\frac{n!}{(n-k)!}\prod_{m=0}^{k-1} C_{n,m}(\tilde{z}) + Q_{k-1}(\tilde{z})\frac{n!}{(n-k+1)!}\prod_{m=0}^{k-2} C_{n,m}(\tilde{z}) + \ldots \\
&\ldots + Q_1(\tilde{z})\frac{n!}{(n-1)!}C_{n,0}(\tilde{z}) = \sum_{j=1}^{j_0} \alpha_{j,j}\frac{n!}{(n-j)!}.
\end{aligned}$$

Note that by the choice of $\tilde{z}$ clearly $Q_k(\tilde{z}) \neq 0$ and $p_n(\tilde{z}) \neq 0$. Dividing both sides of this equation by $\frac{n!}{(n-k)!}$ we get

$$\begin{aligned}
& Q_k(\tilde{z}) \prod_{m=0}^{k-1} C_{n,m}(\tilde{z}) \Bigg[1 + \frac{(n-k)!}{(n-k+1)!} \frac{1}{C_{n,k-1}(\tilde{z})} \frac{Q_{k-1}(\tilde{z})}{Q_k(\tilde{z})} + \\
& \frac{(n-k)!}{(n-k+2)!} \frac{1}{C_{n,k-1}(\tilde{z}) C_{n,k-2}(\tilde{z})} \frac{Q_{k-2}(\tilde{z})}{Q_k(\tilde{z})} + \ldots + \frac{(n-k)!}{(n-1)!} \frac{1}{\prod_{m=1}^{k-1} C_{n,m}(\tilde{z})} \frac{Q_1(\tilde{z})}{Q_k(\tilde{z})} \Bigg] \\
= \; & \sum_{j=1}^{j_0} \alpha_{j,j} \frac{(n-k)!}{(n-j)!}. \qquad (3)
\end{aligned}$$

In this equation, the right-hand side tends to zero when $n \to \infty$ since $j_0 < k$. On the other hand, in the left-hand side of (3), the terms in the bracket (except for the constant term 1) all tend to zero when $n \to \infty$, since $\frac{1}{|C_{n,m}(\tilde{z})|} \leq 2R$ and $R < \infty$ by assumption. Thus, for sufficiently large n, we can find a positive constant K_n, with $\lim_{n\to\infty} K_n = 1$, such that the modulus of the left-hand side of equation (3) equals

$$|\mathrm{LHS}| = K_n \cdot |Q_k(\tilde{z})| \prod_{m=0}^{k-1} |C_{n,m}(\tilde{z})| \geq K_n \cdot |Q_k(\tilde{z})| \frac{1}{2^k R^k} = K_0 > 0$$

when $n \to \infty$ for some positive constant $K_0 > 0$, since $R < \infty$. Thus we obtain the contradiction $K_0 \leq 0$ when $n \to \infty$, and therefore the largest modulus r_n of all roots of p_n must tend to infinity when $n \to \infty$. $\square$

In order to prove Theorem 2 we need the following lemma:

Lemma 3. *Let $T = \sum_{j=1}^k Q_j D^j = \sum_{j=1}^k \left(\sum_{i=0}^j \alpha_{j,i} z^i \right) D^j$ be a degenerate exactly-solvable operator of order k. With no loss of generality we assume that Q_k is monic, i.e. $\alpha_{k,\deg Q_k} = 1$. Let z_n be the root with the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T, and let $|z_n| = r_n$. Then the following inequality holds:*

$$1 \leq \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j-\deg Q_k + i}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}.$$

Proof of Lemma 3. From the definition $C_{n,j}(z) = \frac{p_n^{(j+1)}(z)}{(n-j)p^{(j)}(z)}$ we easily derive

$$p^{(j)}(z) = \frac{p_n^{(k)}(z)}{(n-k+1)(n-k+2)\cdots(n-j)\prod_{m=j}^{k-1} C_{n,m}(z)} \quad \forall \quad j < k. \qquad (4)$$

Inserting z_n in our eigenvalue equation $T(p_n(z)) = \lambda_n p_n(z)$ we obtain

$$\sum_{j=1}^{k-1} \left(\sum_{i=0}^{j} \alpha_{j,i} z_n^i \right) p_n^{(j)}(z_n) + \left(\sum_{i=0}^{\deg Q_k} \alpha_{k,i} z_n^i \right) p_n^{(k)}(z_n) = \lambda_n p_n(z_n) = 0.$$

Dividing this equation by $z_n^{\deg Q_k} p_n^{(k)}(z_n)$ we get

$$\sum_{j=1}^{k-1}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{1}{z_n^{\deg Q_k - i}}\right)\frac{p_n^{(j)}(z_n)}{p_n^{(k)}(z_n)} + \sum_{0\le i<\deg Q_k}\alpha_{k,i}\frac{1}{z_n^{\deg Q_k - i}} + 1 = 0,$$

and from this, using (4) and Lemma 2, we obtain the following inequality:

$$\begin{aligned}
1 &= \left|\sum_{j=1}^{k-1}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{1}{z_n^{\deg Q_k - i}}\right)\frac{p_n^{(j)}(z_n)}{p_n^{(k)}(z_n)} + \sum_{0\le i<\deg Q_k}\alpha_{k,i}\frac{1}{z_n^{\deg Q_k - i}}\right| \\
&\le \sum_{j=1}^{k-1}\left|\sum_{i=0}^{j}\alpha_{j,i}\frac{1}{z_n^{\deg Q_k - i}}\right|\frac{|p_n^{(j)}(z_n)|}{|p_n^{(k)}(z_n)|} + \sum_{0\le i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&\le \sum_{j=1}^{k-1}\sum_{i=0}^{j}\frac{|\alpha_{j,i}|}{r_n^{\deg Q_k - i}}\frac{1}{(n-k+1)\cdots(n-j)\prod_{m=j}^{k-1}|C_{n,m}(z_n)|} + \sum_{0\le i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&\le \sum_{j=1}^{k-1}\sum_{i=0}^{j}\frac{|\alpha_{j,i}|}{r_n^{\deg Q_k - i}}\frac{(2r_n)^{k-j}}{(n-k+1)^{k-j}} + \sum_{0\le i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&= \sum_{j=1}^{k-1}\sum_{i=0}^{j}|\alpha_{j,i}|2^{k-j}\frac{r_n^{k-j-\deg Q_k + i}}{(n-k+1)^{k-j}} + \sum_{0\le i<\deg Q_k}\frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}.
\end{aligned}$$

□

Now, using Theorem 1 and Lemma 3, we can prove Theorem 2:

Proof of Theorem 2. Consider the inequality in Lemma 3. Applying Theorem 1 we see that the last sum on the right-hand side of this inequality tends to zero as $n \to \infty$.

Now consider the double sum on the right-hand side of the inequality in Lemma 3. If the exponent $(k - j - \deg Q_k + i)$ of r_n for given i and j is negative or zero, the corresponding term tends to zero when $n \to \infty$ by Theorem 1. Consider the remaining terms in the double sum, namely those for which the exponent $(k - j - \deg Q_k + i)$ of r_n is positive. Assume that $r_n \le c_0(n - k + 1)^\gamma$ where $c_0 > 0$ is a positive constant and $\gamma < \frac{k-j}{k-j+i-\deg Q_k}$ for given $j \in [1, k-1]$ and given $i \in [0, j]$. Then for the corresponding term in the double sum we get

$$\frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} = \left(\frac{r_n}{(n-k+1)^{\frac{k-j}{k-j+i-\deg Q_k}}}\right)^{k-j+i-\deg Q_k} \to 0$$

when $n \to \infty$. Assume that $r_n \le c_0(n - k + 1)^\gamma$ where $c_0 > 0$ is a positive constant and $\gamma < b$, where

$$b = \overset{+}{\min_{\substack{j\in[1,k-1]\\ i\in[0,j]}}} \frac{k-j}{k-j+i-\deg Q_k} = \overset{+}{\min_{j\in[1,k-1]}} \frac{k-j}{k-j+\deg Q_j - \deg Q_k}.$$

The notation $\min^+$ means that we only take the minimum over positive terms $(k - j + \deg Q_j - \deg Q_k)$. (Above we have written the minimum over $i \in [0, j]$, but actually $i \in [0, \deg Q_j]$ for any given j, so since we look for the minimal value we can put $i = \deg Q_j$ in this expression). Then $\gamma < \frac{k-j}{k-j+i-\deg Q_k}$ for *every* $j \in [1, k-1]$ and *every* $i \in [0, j]$; thus every term with positive exponent $(k - j + i - \deg Q_k)$ will tend to zero when $n \to \infty$. Therefore, assuming that $r_n \leq c_0(n - k + 1)^\gamma$ and $\gamma < b$ where b is as above, we get that every term on the right-hand side of the inequality in Lemma 3 tends to zero as $n \to \infty$, and we arrive at the contradiction $1 \leq 0$. From this we conclude that for sufficiently large choices on n there must exist a positive constant $c_0 > 0$ such that $r_n > c_0(n - k + 1)^\gamma$ for all $\gamma < b$, where b is as above, and hence $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0$ for any $\gamma < b$. $\square$

We have conjectured that $\lim_{n\to\infty} \frac{r_n}{n^d} = c_0 > 0$ for the largest modulus r_n of all roots of p_n for all degenerate exactly-solvable operators, where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$ and j_0 is the largest j such that $\deg Q_j = j$. Thus, if the following condition is fulfilled:

$$b := \min_{j\in[1,k-1]}^{+} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k}\right) = \max_{j\in[1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) := d,$$

then there exists a positive constant $c_0 > 0$ such that $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0$ for any $\gamma < d$.

Below we describe two classses of degenerate exactly-solvable operators for which the above condition is satisfied, namely:

Corollary 1. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k such that $\deg Q_j \leq j_0$ for all $j > j_0$, and in particular $\deg Q_k = j_0$, where j_0 is the largest j such that $\deg Q_j = j$. If r_n is the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T, then there exists a positive constant $c_0 > 0$ such that $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0$ for any $\gamma < 1$.*

Proof of Corollary 1. For this class of operators it is conjectured that $\lim_{n\to\infty} \frac{r_n}{n} = c_0 > 0$, since

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-j_0} = 1.$$

The maximum is attained by choosing any $j > j_0$ with $\deg Q_j = j_0$, e.g. $j = k$. Also, for this class of operators we have

$$\begin{aligned} b &:= \min_{j\in[1,k-1]}^{+} \frac{k-j}{k-j+\deg Q_j - \deg Q_k} \\ &= \min_{j\in[1,k-1]}^{+} \frac{k-j}{k-j+\deg Q_j - j_0} = \frac{k-j_0}{k-j_0} = 1, \end{aligned}$$

and the proof is complete by applying Theorem 2. $\square$

Corollary 2. *Let $T = \sum_{j=1}^{k} Q_j D^j$ be a degenerate exactly-solvable operator of order k such that $\deg Q_j = 0$ for all $j > j_0$, where j_0 is the largest j such that $\deg Q_j = j$. Let r_n be the largest modulus of all roots of the unique and monic nth degree eigenpolynomial p_n of T. Then there exists a positive constant $c_0 > 0$ such that $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0$ for any $\gamma < \frac{k-j_0}{k}$.*

Proof of Corollary 2. For this class of operators it is conjectured that $\lim_{n\to\infty} \frac{r_n}{n^{(k-j_0)/k}} = c_0 > 0$, since

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j}\right) = \frac{k-j_0}{k}.$$

Also, for this class of operators we have

$$\begin{aligned} b \quad &:= \quad \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k}\right) \\ &= \quad \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j}\right) = \min_{j\in[1,j_0]} \frac{k-j}{k} = \frac{k-j_0}{k} \end{aligned}$$

where the third equality follows choosing any j such that $\deg Q_j = j$, and the minimum is attained for $j = j_0$ (for $j > j_0$ we get $(k-j)/(k-j+\deg Q_j) = 1 > (k-j_0)/k$), and the proof is complete by applying Theorem 2. $\square$

Remark. For the classes of operators considered in Corollary 1 and Corollary 2 we can actually prove that $\lim_{n\to\infty} \frac{r_n}{n^d} \geq c_0$, where d is as in Main Conjecture, if we assume that we already have the upper bound $\lim_{n\to\infty} \frac{r_n}{n^d} \leq c_1$ for some positive constant c_1, see Section 5.

Proof of Theorem 3. First we note that $\deg Q_{j_0} = j_0$ since there must exist at least one such $j < k$. Let

$$T = Q_{j_0} D^{j_0} + Q_k D^k = \sum_{i=0}^{j_0} \alpha_{j_0,i} z^i D^{j_0} + \sum_{i=0}^{\deg Q_k} \alpha_{k,i} z^i D^k,$$

where $\alpha_{j_0,j_0} \neq 0$, and where we wlog assume that Q_k is monic. From Lemma 3 we get:

$$\begin{aligned} 1 \quad &\leq \quad \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k + k - j_0}}{(n-k+1)^{k-j_0}} + \sum_{0\leq i<\deg Q_k} |\alpha_{k,i}| \frac{1}{r_n^{\deg Q_k - i}} \\ &\leq \quad \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k + k - j_0}}{(n-k+1)^{k-j_0}} + \epsilon, \end{aligned}$$

where we choose n large enough that $\epsilon < 1$ (this is possible since $\epsilon \to 0$ when

$n \to \infty$). Thus for sufficiently large n we have the following inequality:

$$\begin{aligned} c_0 &\le \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{i-\deg Q_k+k-j_0}}{(n-k+1)^{k-j_0}} \\ &\le \sum_{i=0}^{j_0} |\alpha_{j_0,i}| 2^{k-j_0} \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \\ &= K \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}, \end{aligned}$$

where $1-\epsilon = c_0 > 0$ and $K > 0$ since $\alpha_{j_0,j_0} \neq 0$ (the last inequality follows since $i \le j_0$). Thus

$$r_n \ge \frac{c_0}{K}(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}$$

for sufficiently large n, and hence

$$\lim_{n\to\infty} \frac{r_n}{(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}} \ge \frac{c_0}{K} = c > 0.$$

Finally, it is clear that for this two-term operator we have

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k},$$

and we are done. □

Remark. If Q_k is a monomial ($Q_k = z^{\deg Q_k}$), then there exists a positive constant c such that $r_n \ge c(n-k+1)^d$ for *every* n, where $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$ $= \frac{k-j_0}{k-\deg Q_k}$. This is easily seen from the calculations in the proof of Theorem 3 above. Note that the sum $\sum_{0\le i<\deg Q_k} |\alpha_{k,i}| \frac{1}{r_n^{\deg Q_k - i}}$ on the right-hand side of the inequality in Lemma 3 vanishes, and therefore $1 \le K\frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}$ for every n. Also, from the second part of Lemma 1 we know that for this class of operators there exists a unique monic nth degree eigenpolynomial for every n, and the conclusion follows.

Proof of Theorem 4. First, since $j \le k$ and $(j - \deg Q_j) \ge (k - \deg Q_k)$ for every $j > j_0$ for this class of operators, it is clear that

$$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \frac{k-j_0}{k-\deg Q_k}.$$

We assume, with no loss of generality, that Q_k is monic, i.e. $\alpha_{k,\deg Q_k} = 1$. From Lemma 3 we then have the inequality

$$1 \le \sum_{j=1}^{k-1}\sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\le i<\deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}. \tag{5}$$

Clearly the last sum here tends to zero as $n \to \infty$ by Theorem 1. Considering the double sum on the right-hand side of the inequality above it is clear that for every j we have, using $i \leq \deg Q_j$, that

$$\begin{aligned}
&\sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} = \sum_{i=0}^{\deg Q_j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} r_n^{i-\deg Q_j} \\
&= \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \left(2^{k-j} |\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j} 2^{k-j} |\alpha_{j,i}| r_n^{i-\deg Q_j} \right) \\
&= K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}}, \qquad (6)
\end{aligned}$$

where

$$K_j^n = 2^{k-j} |\alpha_{j,\deg Q_j}| + \sum_{i<\deg Q_j} 2^{k-j} |\alpha_{j,i}| r_n^{i-\deg Q_j} > 0,$$

since $\alpha_{j,\deg Q_j} \neq 0$. Also, $K_n^j < \infty$, since $i \in [0, \deg Q_j]$ and thus $(i - \deg Q_j) < 0$ for every j (note that $K_j^n \to 2^{k-j} |\alpha_{j,\deg Q_j}|$ when $n \to \infty$ due to Theorem 1). Thus, with the decomposition
$A = \{j : \deg Q_j = j\}$,
$B = \{j : \deg Q_j < j \quad \textit{and} \quad (k - j + \deg Q_j - \deg Q_k) > 0\}$,
$C = \{j : \deg Q_j < j \quad \textit{and} \quad (k - j + \deg Q_j - \deg Q_k) \leq 0\}$,
and using (6), inequality (5) is equivalent to:

$$\begin{aligned}
1 &\leq \sum_{j=1}^{k-1} \sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\
&= \sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \\
&+ \sum_{j \in C} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}.
\end{aligned}$$

Consider the last two sums on the right hand side of this inequality. They both tend to zero as $n \to \infty$, the last one due to Theorem 1, and the sum over C since we have $(j - \deg Q_j) \geq (k - \deg Q_k) \Leftrightarrow (k - j + \deg Q_j - \deg Q_k) \leq 0$ for every $j \in C$ by assumption, and then applying Theorem 1.
Therefore, in the limit when $n \to \infty$, we get the inequality

$$c_0 \leq \sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \qquad (7)$$

where

$$0 < c_0 = 1 - \sum_{j \in C} K_j^n \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} - \sum_{0 \leq i < \deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}}$$

for sufficiently large n.

Now assume that the set B is empty. This corresponds to an operator with $(j - \deg Q_j) \geq (k - \deg Q_k)$ for every j for which $\deg Q_j < j$. Then the inequality (7) above becomes

$$\begin{aligned} c_0 &\leq \sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} \\ &= \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \left(K_{j_0}^n + \sum_{j \in A \setminus \{j_0\}} K_j^n \frac{1}{(n-k+1)^{j_0-j}} \right) \\ &\leq K_A \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \end{aligned} \tag{8}$$

where $K_A > 0$ is a positive constant (since A is nonempty) which is finite when $n \to \infty$, since $j_0 - j > 0$ for every $j \in A \setminus \{j_0\}$ (recall that j_0 is the largest element in A by definition). Thus for sufficiently large n there exists a positive constant $c = c_0 / K_A > 0$ such that

$$r_n \geq c(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}},$$

and thus

$$\lim_{n \to \infty} \frac{r_n}{(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}} \geq c > 0.$$

Now assume that B is nonempty. Then for sufficiently large n there exists a positive constant $c_0 > 0$ such that (as in the case of empty B) inequality (7) above holds:

$$c_0 \leq \sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}}.$$

For the sum over A we previously concluded in (8) that there exists a positive and finite constant K_A such that

$$\sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} \leq K_A \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}$$

for sufficiently large n, whence we get the following inequality from (7):

$$\begin{aligned} c_0 &\leq \sum_{j \in A} K_j^n \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{j \in B} K_j^n \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} \\ &\leq K_A \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} + \sum_{j \in B} K_j^n \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} \\ &= \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}} \left(K_A + \sum_{j \in B} K_j^n \frac{r_n^{\deg Q_j - j}}{(n-k+1)^{j_0-j}} \right) \\ &\leq K_{AB} \frac{r_n^{k-\deg Q_k}}{(n-k+1)^{k-j_0}}. \end{aligned}$$

Here K_{AB} is a positive and finite constant for sufficiently large n (note that $K_{AB} \to K_A$ when $n \to \infty$, since $(\deg Q_j - j) < 0$ and $j_0 - j > 0$ for every $j \in B$). Thus there exists a positive constant $c = c_0/K_{AB} > 0$ such that

$$r_n \geq c(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}},$$

for sufficiently large choices on n, and thus

$$\lim_{n\to\infty} \frac{r_n}{(n-k+1)^{\frac{k-j_0}{k-\deg Q_k}}} \geq c > 0.$$

□

3 How did we arrive at Main Conjecture?

Our Main Conjecture is based on calculations involving the Cauchy transform and is consistent with all experiments we have performed concerning the zero distribution of eigenpolynomials of degenerate exactly-solvable operators.

With p_n being the unique monic nth degree eigenpolynomial of T, we define the corresponding scaled polynomial $q_n(z) = p_n(n^d z)$, where we need to find the positive number d specific for each operator. For a polynomial q_n of degree n, the Cauchy transform $C_{n,j}$ of the root measure μ_n^j for the jth derivative $q_n^{(j)}$ is given by

$$C_{n,j}(z) := \frac{q_n^{(j+1)}(z)}{(n-j)q_n^{(j)}(z)} = \int \frac{d\mu_n^{(j)}(\zeta)}{z-\zeta}.$$

From this definition we obtain

$$\begin{aligned}
\prod_{i=0}^{j-1} C_{n,i}(z) &= \prod_{i=0}^{j-1} \frac{q_n^{(i+1)}(z)}{(n-j)q_n^{(i)}(z)} \\
&= \frac{q_n^{(1)}(z)}{nq_n(z)} \cdot \frac{q_n^{(2)}(z)}{(n-1)q^{(1)}(z)} \cdot \frac{q_n^{(3)}(z)}{(n-2)q_n^{(2)}(z)} \cdots \\
&\quad \cdots \frac{q_n^{(j-1)}(z)}{(n-j+2)q_n^{(j-2)}(z)} \cdot \frac{q_n^{(j)}(z)}{(n-j+1)q_n^{(j-1)}(z)} \\
&= \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)}.
\end{aligned}$$

Now the basic assumption (see also section 4.3) we make to get our conjecture is the following. Assume that the Cauchy transforms of the scaled polynomial $q_n(z)$ and its derivatives are all equal when $n \to \infty$, i.e. $C_n(z) := C_{n,0}(z) = C_{n,1}(z) = \ldots = C_{n,k-1}(z)$ when $n \to \infty$. This means that we assume that the root measures $\mu_n^0, \mu_n^1, \mu_n^2 \ldots, \mu_n^{k-1}$ of $q_n, q_n^{(1)}, q_n^{(2)} \ldots, q_n^{(k-1)}$ respectively, are all equal as $n \to \infty$. Then

$$C_n^j(z) = \prod_{i=0}^{j-1} C_{n,i}(z) = \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)}, \tag{9}$$

and with $C(z) := \lim_{n\to\infty} C_n(z)$ (we call this function the *asymptotic Cauchy transform* of q_n), we get

$$C^j(z) = \lim_{n\to\infty} C_n^j(z) = \lim_{n\to\infty} \prod_{i=0}^{j-1} C_{n,i}(z) = \frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)}. \tag{10}$$

With $q_n(z) = p_n(n^d z)$ the scaling factor n^d is now appropriately chosen in the sense that we obtain a "nice" equation in the asymptotic Cauchy transform $C(z)$ for the scaled polynomials. Then the asymptotic zero distribution of the scaled polynomials will (conjecturally) be compactly supported.

Let $T = \sum_{j=1}^k \left(\sum_{i=0}^j \alpha_{j,i} z^i\right) D^j$ be a degenerate exactly-solvable operator and denote by j_0 the largest j such that $\deg Q_j = j$. Consider the equation $T(p_n(z)) = \lambda_n p_n(z)$ where

$$\lambda_n = \sum_{j=1}^{k} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} \frac{n!}{(n-j)!} = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1).$$

Clearly this sum ends at j_0 since $\alpha_{j,j} = 0$ for all $j > j_0$ by definition of j_0. We then have

$$T(p_n(z)) = \lambda_n p_n(z)$$

$$\Leftrightarrow$$

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{j} \alpha_{j,i} z^i\right) p_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) p_n(z)$$

Now letting $z \to n^d z$ in this equation we obtain

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{j} \alpha_{j,i} n^{di} z^i\right) p_n^{(j)}(n^d z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) p_n(n^d z),$$

and making the substitution $q_n(z) = p_n(n^d z)$ the equation above will be equivalent to the following:

$$\sum_{j=1}^{k} \left(\sum_{i=0}^{j} \alpha_{j,i} \frac{z^i}{n^{d(j-i)}}\right) q_n^{(j)}(z) = \sum_{j=1}^{j_0} \alpha_{j,j} n(n-1)\cdots(n-j+1) q_n(z).$$

Dividing this equation by $\frac{n!}{(n-j_0)!}q_n(z) = n(n-1)\cdots(n-j_0+1)q_n(z)$ we get

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^{k}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\right)\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} \\
&= \sum_{j=1}^{j_0}\alpha_{j,j}\frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} = \text{RHS} \qquad (11)
\end{aligned}$$

where $\alpha_{j_0,j_0}\neq 0$. Consider the right-hand side (RHS) of equation (11). Since $j\le j_0$, all terms for which $j<j_0$ (if not already zero, which is the case if $\alpha_{j,j}=0$, i.e. if $\deg Q_j<j$) tend to zero when $n\to\infty$, and therefore

$$\text{RHS} = \sum_{j=1}^{j_0}\alpha_{j,j}\frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} \to \alpha_{j_0,j_0}=1 \quad \text{as} \quad n\to\infty.$$

Here we wlog have made a normalization by assuming that Q_{j_0} is monic, i.e. $\alpha_{j_0,j_0}=1$.

Now consider the jth term in the sum on the left-hand side (LHS) of equation (11). Using (9) and (10) we get, for any given j:

$$\begin{aligned}
&\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\cdot\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} = \\
= \;&\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\cdot\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)q_n(z)}\cdot\frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} \\
= \;&\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\cdot C_n^j(z)\cdot\frac{n(n-1)\cdots(n-j+1)}{n(n-1)\cdots(n-j_0+1)} \\
= \;&\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_0-j}}\cdot C_n^j(z)\cdot\frac{n(n-1)\cdots(n-j+1)}{n^j}\frac{n^{j_0}}{n(n-1)\cdots(n-j_0+1)} \\
\to \;&\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_0-j}}C^j(z) \quad \text{when} \quad n\to\infty.
\end{aligned}$$

Thus, for the left-hand side of (11) we have

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^{k}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)}}\right)\frac{q_n^{(j)}(z)}{n(n-1)\cdots(n-j_0+1)q_n(z)} \\
&\to \sum_{j=1}^{k}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_0-j}}\right)C^j(z) \quad \text{when} \quad n\to\infty.
\end{aligned}$$

Adding up we have the following equation satisfied by the asymptotic Cauchy transform C for the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$:

$$\sum_{j=1}^{k}\left(\sum_{i=0}^{j}\alpha_{j,i}\frac{z^i}{n^{d(j-i)+j_0-j}}\right)C^j(z) = 1. \tag{12}$$

In order to make (12) a "nice" equation we need to (in order to avoid infinities in the denominator) impose the following condition on the exponent d of n:

$$d(j-i)+j_0-j \geq 0 \quad \Leftrightarrow \quad d \geq \frac{j-j_0}{j-i}$$

for all $j \in [1,k]$ and all $i \in [0,j]$. Therefore we take $d = \max_{\substack{j\in[1,k]\\ i\in[0,j]}}\left(\frac{j-j_0}{j-i}\right)$, but this maximum is clearly obtained for the maximum value of i for a given j. Since $i \in [0, \deg Q_j]$ for any given j, we may as well put $i = \deg Q_j$. Our condition then becomes[7] $d = \max_{j\in[1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$. But clearly we need only take this maximum over $j \in [j_0+1,k]$, since $j_0 < k$ and therefore there always exists a positive value on d for any operator of the type we consider; thus our condition becomes:

$$d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right).$$

This is how we arrived at the scaling factor n^d. If we put this d into equation (12) and let $n \to \infty$ we obtain an equation satisfied by the asymptotic Cauchy transform of the scaled polynomial $q_n(z) = p_n(n^d z)$ - namely the algebraic equation in Main Corollary.

Arriving at Main Corollary. We insert d in (12), where d is as above (i.e. as in Main Conjecture). We then get the following equation:

$$\sum_{j=1}^{k}\left(\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}}\right)C^j(z) = 1. \tag{13}$$

Denote by $N_{j,i}$ the exponent of n in (13) for given j and i. Thus

$$N_{j,i} = \max_{j\in[j_=+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j.$$

The terms for which this exponent is positive tend to zero as $n \to \infty$.

First we consider j for which $\deg Q_j = j$, and denote, as usual, by j_0 the largest such j. If $j = j_0$, then $i \leq \deg Q_{j_0} = j_0$; thus for $j = j_0$ and $i = j_0$ we

[7]To make sure we do not take this maximum over nonexisting terms we can write $d = \left|\frac{\alpha_{j,\deg Q_j}}{\alpha_{j,\deg Q_j}}\right| \max_{j\in[1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$.

get

$$\begin{aligned}
N_{j_0,j_0} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j_0-j_0)+j_0-j_0=0,
\end{aligned}$$

and for $j=j_0$ and $i<j_0$ we have

$$\begin{aligned}
N_{j_0,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&> \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j_0-j_0)+j_0-j_0=0.
\end{aligned}$$

Thus $N_{j_0,j_0}=0$ and $N_{j_0,i}>0$ for $i<j_0$, and for the term corresponding to $j=j_0$ in (13) we get

$$\sum_{i=0}^{j_0}\alpha_{j_0,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j_0-i)+j_0-j_0}}C^{j_0}(z)\to\alpha_{j_0,j_0}z^{j_0}C^{j_0}(z)=z^{j_0}C^{j_0}(z)$$

in the limit when $n\to\infty$, assuming that Q_{j_0} is monic ($\alpha_{j_0,j_0}=1$).
Now let j be such that $\deg Q_j=j$ and $j<j_0$. Then $i\le\deg Q_j=j$ and

$$\begin{aligned}
N_{j,j} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j=j_0-j>0,
\end{aligned}$$

and for $i<j$ we get

$$\begin{aligned}
N_{j,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&> \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j=j_0-j>0,
\end{aligned}$$

that is $N_{j,i}>0$ for all $j<j_0$ such that $\deg Q_j=j$ and for all $i\le j$. Thus the corresponding terms in (13) tend to zero:

$$\sum_{j\in\{j<j_0:\deg Q_j=j\}}\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{\max_{j\in[j_0+1,j]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}}C^j(z)\to 0$$

when $n\to\infty$ for every $j<j_0$ such that $\deg Q_j=j$.

Now denote by j_m the j for which the maximum $d=\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained. Note that there may be several distinct j for which this maximum is attained![8] Then

$$\begin{aligned}
N_{j_m,degQ_{j_m}} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&= \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j_m-\deg Q_{j_m})+j_0-j_m\\
&= j_m-j_0+j_0-j_m=0,
\end{aligned}$$

and for $i<\deg Q_{j_m}$ we get

$$\begin{aligned}
N_{j_m,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&> \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j_m-\deg Q_{j_m})+j_0-j_m\\
&= j_m-j_0+j_0-j_m=0,
\end{aligned}$$

i.e. $N_{j_m,\deg Q_{j_m}}=0$ and $N_{j_m,i}>0$ for $i<\deg Q_{j_m}$, and for the term corresponding to $j=j_m$ in (13) we get

$$\sum_{i=0}^{\deg Q_{j_m}}\alpha_{j_m,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j_m-i)+j_0-j_m}}C^{j_m}(z)\to\alpha_{j_m,\deg Q_{j_m}}z^{\deg Q_{j_m}}C^{j_m}(z)$$

when $n\to\infty$. In case of several j for which d is attained, we put $A=\{j:(j-j_0)/(j-\deg Q_j)=d:=\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)\}$, and for the corresponding terms in (13) we get

$$\sum_{j\in A}\sum_{i=0}^{\deg Q_j}\alpha_{j,i}\frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}}C^{j}(z)\to\sum_{j\in A}\alpha_{j,\deg Q_j}z^{\deg Q_j}C^{j}(z)$$

when $n\to\infty$. Now consider the remaining terms in (13), namely terms for which $j<j_0$ such that $\deg Q_j<j$, terms for which $j_0<j<j_m$, and terms for which $j_m<j\le k$, (clearly this last case does not exist if $j_m=k$). We start with $j<j_0$ for which $\deg Q_j<j$. Then $i\le\deg Q_j<j$ and

$$\begin{aligned}
N_{j,i} &= \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j\\
&> \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-j)+j_0-j=j_0-j>0,
\end{aligned}$$

[8] Consider, for example, the Laplace type operator (that is with all polynomial coefficients Q_j linear) $T=zD+zD^2+\dots zD^k$. Here $j_0=1$ and the equation satisfied by the asymptotic Cauchy transform of the scaled eigenpolynomial $q_n(z)=p_n(nz)$ is given by $zC(z)+zC^2(z)+\dots zC^k(z)=1$, since the maximum $d=\max_{j\in[2,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)=1$ is attained for *every* $j=2,3,\dots k$.

and the corresponding terms in (13) for which $j < j_0$ such that $\deg Q_j < j$ tend to zero when $n \to \infty$:

$$\sum_{j \in \{j < j_0 : \deg Q_j < j\}} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

when $n \to \infty$.

Now assume that $j_m < k$ and consider $j_m < j \leq k$. Clearly $j_m > j_0$ since the maximum is taken over $j \in [j_0+1, k]$, and therefore $i \leq \deg Q_j < j$ for $j_m < j \leq k$. Also,

$$\max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) = \left(\frac{j_m - j_0}{j_m - \deg Q_{j_m}}\right) > \left(\frac{j-j_0}{j-\deg Q_j}\right),$$

since the maximum is attained for j_m by assumption. Thus we get

$$\begin{aligned} N_{j,i} &= \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j = \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j-i)+j_0-j \\ &> \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \geq \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-\deg Q_j)+j_0-j \\ &= j-j_0+j_0-j = 0, \end{aligned}$$

i.e. $N_{j,i} > 0$ for every $j_m < j \leq k$ and every $i \leq \deg Q_j$. The corresponding terms in (13) therefore tend to zero when $n \to \infty$:.

$$\sum_{j_m < j \leq k} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

as $n \to \infty$.

Finally we consider $j_0 < j < j_m$. Note that this also covers the case $j_{m_1} < j < j_{m_2}$ where the maximum $d = \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained for j_{m_1} and j_{m_2}. Since $i \leq \deg Q_j < j$ we get

$$\begin{aligned} N_{j,i} &= \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j = \left(\frac{j_m-j_0}{j_m-\deg Q_{j_m}}\right)(j-i)+j_0-j \\ &> \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j \geq \left(\frac{j-j_0}{j-\deg Q_j}\right)(j-\deg Q_j)+j_0-j \\ &= j-j_0+j_0-j = 0, \end{aligned}$$

i.e. $N_{j,i} > 0$ for every $j_0 < j < j_m$ and every $i \leq \deg Q_j$. Thus the corresponding terms in (13) tend to zero when $n \to \infty$:

$$\sum_{j_0 < j < j_m} \sum_{i=0}^{\deg Q_j} \alpha_{j,i} \frac{z^i}{n^{\max_{j\in[j=+1,k]0}\left(\frac{j-j_0}{j-\deg Q_j}\right)(j-i)+j_0-j}} C^j(z) \to 0$$

when $n \to \infty$.

Adding up these results we get the following equation from (13) for the asymptotic Cauchy transform $C(z)$ of the scaled eigenpolynomial $q_n(z) = p_n(n^d z)$ where d is as in Main Conjecture:

$$z^{j_0} C^{j_0}(z) + \sum_{j \in A} \alpha_{j,\deg Q_j} z^{\deg Q_j} C^j(z) = 1,$$

where j_0 is the largest j such that $\deg Q_j = j$, and A is the set consisting of all j for which the maximum $d = \max_{j \in [j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$ is attained, i.e. $A = \{j : (j - j_0)/(j - \deg Q_j) = d\}$ where d is as above. □

4 Numerical evidence

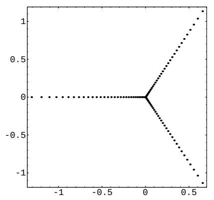
4.1 Evidence for Main Conjecture

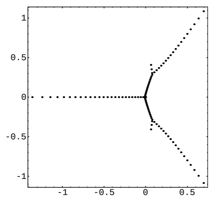
On page 30 we present numerical evidence for Main Conjecture on the asymptotic root growth. We have performed similar computer experiments for a large number of other degenerate exactly-solvable operators, and the results are in all cases consistent with this conjecture.

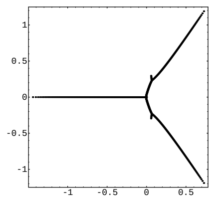
4.2 Comments on Main Corollary

We here present and comment some pictures of the zero distribution for some properly (according to Main Conjecture) scaled eigenpolynomials of some degenerate exactly-solvable operators. In Section 1 we presented pictures of the zero distributions of the scaled eigenpolynomials for some distinct operators for which the Cauchy transform C of the scaled eigenpolynomials satisfy the same equation in the limit as $n \to \infty$. We considered the operator $T_4 = z^3 D^3 + z^2 D^5$ for which $d = 2/3$ and for which the asymptotic Cauchy transform of the scaled eigenpolynomial $q_n(z) = p_n(n^2/3z)$ satisfies the equation $z^3 C^3 + z^2 C^5 = 1$. For the slightly modified operator $\widetilde{T}_4 = z^2 D^2 + z^3 D^3 + z D^4 + z^2 D^5 + D^6$ we noted that d is obtained again (only) for $j = 5$ and we therefore obtain the same equation in C for the scaled eigenpolynomials of $\widetilde{T}_4$ as for the scaled eigenpolynomials of T_4, whence we can consider the added terms $z^2 D^2$, $z D^4$ and D^6 as irrelevant for the zero distribution.

However, instead of D^6, we may add the "more disturbing" term zD^6 to T_4. Consider the operator $\widetilde{\widetilde{T}}_4 = z^2 D^2 + z^3 D^3 + z D^4 + z^2 D^5 + z D^6$ and note that for $j = 6$ we have $(6 - 3)/(6 - 1) = 3/5 = 0.6 < 2/3$ - it is clear that the closer the value $(j - j_0)(j - \deg Q_j)$ of the added term $Q_j D^j$ is to $d = 2/3$, the more disturbing is this term, since, besides the term for which $j = j_0$, it is precisely the terms for which $(j - j_0)/(j - \deg Q_j) = d = 2/3$ that are involved in the asymptotic Cauchy transform equation. Se pictures below.



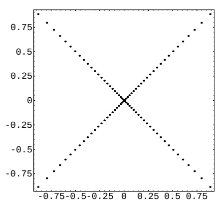


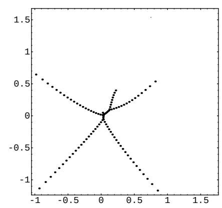


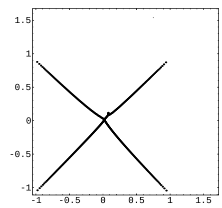
T_4, roots of $q_{100}(z) = p_{100}(100^{2/3}z)$ $\quad$ $\tilde{\tilde{T}}_4$, roots of $q_{100}(z) = p_{100}(100^{2/3}z)$ $\quad$ $\tilde{\tilde{T}}_4$, roots of $q_{600}(z) = p_{600}(600^{2/3}z)$

The term zD^6 should however be irrelevant in the limit when $n \to \infty$ (according to the asymptotic Cauchy transform equation), and experiments indicate that for sufficiently large n the zero distributions for the scaled eigenpolynomials of T_4 and $\tilde{\tilde{T}}_4$ coincide, as they (conjecturally) should.

Note also that it is only the term of highest degree in a given (relevant) Q_j, i.e. $\alpha_{j,\deg Q_j} z^{\deg Q_j}$, that is relevant for the zero distribution of the scaled polynomials in the limit when $n \to \infty$. This is illustrated by the following example, where adding lower degree terms in the (relevant) Q_j clearly does not affect the zero distribution of the scaled eigenpolynomials for large n. Below, $T_6 = z^3D^3 + z^2D^6$, and $\tilde{T}_6 = [(1+13i) + (24i-3)z + 11iz^2 + z^3]D^3 + [(22i - 13) + (-9-14i)z + z^2]D^6$ (note the difference in scaling between the pictures).





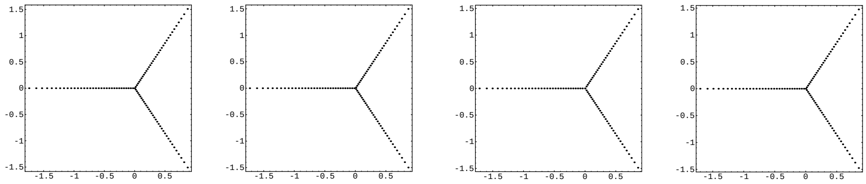


T_6, roots of $q_{100}(z) = p_{100}(100^{3/4}z)$ $\quad$ $\tilde{T}_6$, roots of $q_{100}(z) = p_{100}(100^{3/4}z)$ $\quad$ $\tilde{T}_6$, roots of $q_{500}(z) = p_{500}(500^{3/4}z)$

4.3 On the basic assumption

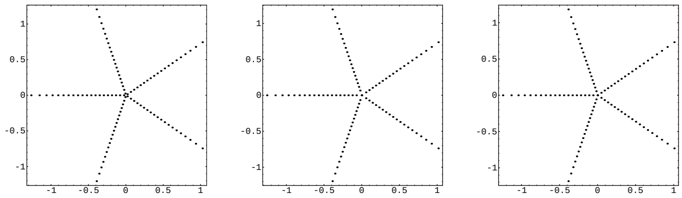
Finally, we show some pictures which support the basic assumption upon which Main Conjecture and Main Corollary are built, namely that the Cauchy transforms for the scaled eigenpolynomial and its derivatves are all equal when $n \to \infty$, i.e. $C_{n,0} = C_{n,1} = \ldots = C_{n,k-1}$ in the limit when $n \to \infty$. This means that we assume that the zero distributions $\mu_n, \mu_n^{(1)}, \ldots, \mu_n^{(k)}$ of the scaled eigenpolynomial and its derivatives $q_n, q_n', \ldots, q_n^{(k)}$ respectively, are all equal when $n \to \infty$. Below, p_n denotes the nth degree monic eigenpolynomial of the given operator, and $q_n = p_n(n^d z)$ denotes the corresponding appropriately scaled polynomial.

Fig. 1: $T_7 = zD + D^3$ and $q_n(z) = p_n(n^{2/3}z)$.

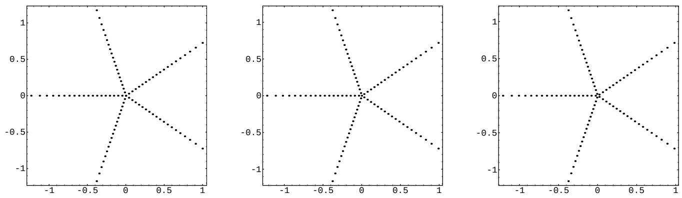


roots of $q_{100}(z)$ roots of $q'_{100}(z)$ roots of $q''_{100}(z)$ roots of $q'''_{100}(z)$

Fig. 2: $T_8 = z^2D^2 + D^5$ and $q_n(z) = p_n(n^{3/5}z)$.

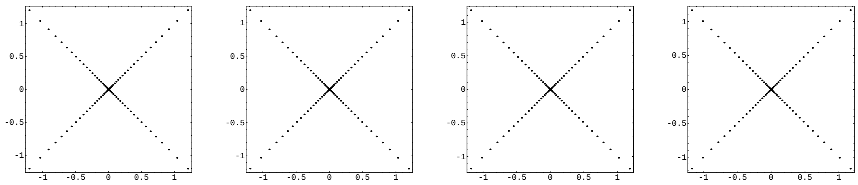


roots of $q_{100}(z)$ roots of $q'_{100}(z)$ roots of $q''_{100}(z)$

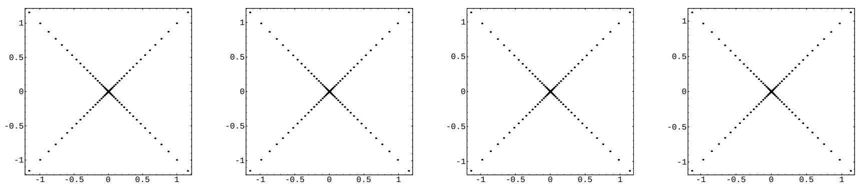


roots of $q'''_{100}(z)$ roots of $q^{(\mathrm{iv})}_{100}(z)$ roots of $q^{(\mathrm{v})}_{100}(z)$

Fig. 3: $T_9 = zD + zD^4 + z^3D^7$ and $q_n(z) = p_n(n^{3/2}z)$.



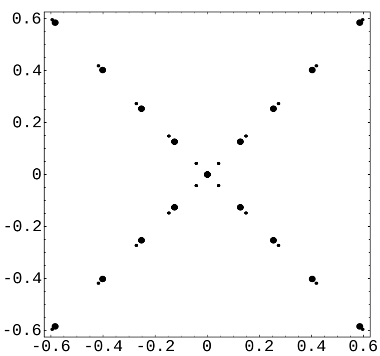
roots of $q_{100}(z)$ roots of $q'_{100}(z)$ roots of $q''_{100}(z)$ roots of $q'''_{100}(z)$



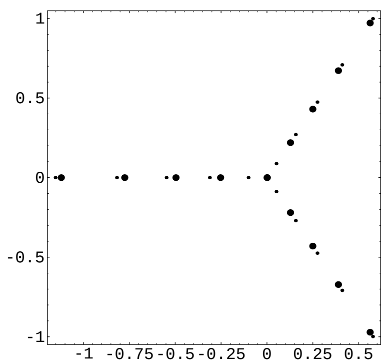
roots of $q^{(\mathrm{iv})}_{100}(z)$ roots of $q^{(\mathrm{v})}_{100}(z)$ roots of $q^{(\mathrm{vi})}_{100}(z)$ roots of $q^{(\mathrm{vii})}_{100}(z)$

4.4 Interlacing property

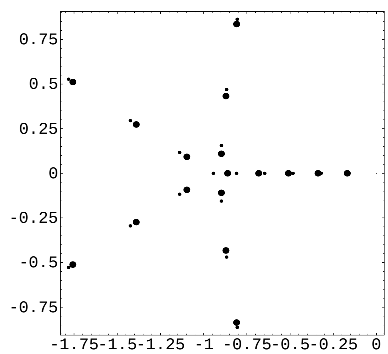
We now state the exact meaning of interlacing on curves in the complex plane. Conjecturally the support of the asymptotic zero distribution of the scaled eigenpolynomial q_n of T is the union of a finite number of analytic curves in the complex plane, which we denote by Ξ_T. When defining the interlacing property some caution is required since the zeros of q_n do not lie exactly on Ξ_T. Thus, identify some sufficiently small neighbourhood $N(\Xi_T)$ of Ξ_T with the normal bundle to Ξ_T by equipping $N(\Xi_T)$ with the projection onto Ξ_T along the fibres which are small curvilinear segments orthogonal to Ξ_T. Then we say that two sets of points in $N(\Xi_T)$ *interlace* if their (orthogonal) projections on Ξ_T interlace in the usual sense. If Ξ_T has singularities one should first remove some sufficiently small neighbourhoods of these singularities and the proceed in the above way on the remaining part of Ξ_T. Conjecture 1 then states that for any sufficiently small neighbourhood $N(\Xi_T)$ of Ξ_T there exists N such that the interlacing property holds for the roots of q_n and q_{n+1} for all $n \geq N$. Below, small dots are roots of q_{n+1} and large dots are roots of q_n for some fixed n.



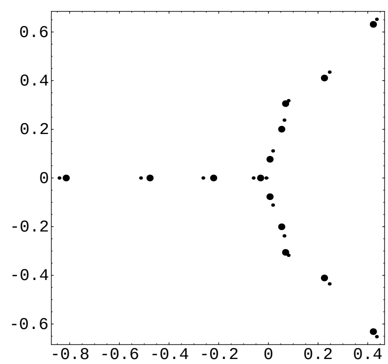
$T = z^2D^2 + z^3D^3 + zD^5$,
roots of q_{25} and q_{24}.



$T = zD + z^2D^2 + D^3$,
roots of q_{20} and q_{19}.



$T = zD + zD^2 + zD^3 + zD^4 + zD^5$,
roots of q_{23} and q_{22}.



$T = z^3D^3 + z^2D^5 + zD^6$,
roots of q_{20} and q_{19}.

Operator	n	r_n experimental	r_n conjectured
$T_1 = zD + zD^2 + zD^3 + zD^4 + zD^5$	50	$2.7 \cdot 50^{0.967595}$	$c_1 \cdot 50^1$
	100	$2.7 \cdot 100^{0.984180}$	$c_1 \cdot 100^1$
	200	$2.7 \cdot 200^{0.992557}$	$c_1 \cdot 200^1$
	250	$2.7 \cdot 250^{0.994272}$	$c_1 \cdot 250^1$
$T_2 = z^2D^2 + D^7$	50	$1.3 \cdot 50^{0.671977}$	$c_2 \cdot 50^{5/7}$
	100	$1.3 \cdot 100^{0.694847}$	$c_2 \cdot 100^{5/7}$
	200	$1.3 \cdot 200^{0.706226}$	$c_2 \cdot 200^{5/7}$
	300	$1.3 \cdot 300^{0.710085}$	$c_2 \cdot 300^{5/7}$
	400	$1.3 \cdot 400^{0.712043}$	$c_2 \cdot 400^{5/7}$
$T_3 = z^3D^3 + z^2D^4 + zD^5$	50	$4/3 \cdot 50^{0.469007}$	$c_3 \cdot 50^{1/2}$
	100	$4/3 \cdot 100^{0.484824}$	$c_3 \cdot 100^{1/2}$
	200	$4/3 \cdot 200^{0.492832}$	$c_3 \cdot 200^{1/2}$
	300	$4/3 \cdot 300^{0.495592}$	$c_3 \cdot 300^{1/2}$
	400	$4/3 \cdot 400^{0.497009}$	$c_3 \cdot 400^{1/2}$
$T_4 = z^3D^3 + z^2D^5$	50	$1.4 \cdot 50^{0.633226}$	$c_4 \cdot 50^{2/3}$
	100	$1.4 \cdot 100^{0.652141}$	$c_4 \cdot 100^{2/3}$
	200	$1.4 \cdot 200^{0.661412}$	$c_4 \cdot 200^{2/3}$
	300	$1.4 \cdot 300^{0.664511}$	$c_4 \cdot 300^{2/3}$
	400	$1.4 \cdot 400^{0.666066}$	$c_4 \cdot 400^{2/3}$
$\widetilde{T}_4 = z^2D^2 + z^3D^3 + zD^4 + z^2D^5 + D^6$	50	$1.4 \cdot 50^{0.632811}$	$\tilde{c}_4 \cdot 50^{2/3}$
	100	$1.4 \cdot 100^{0.651960}$	$\tilde{c}_4 \cdot 100^{2/3}$
	200	$1.4 \cdot 200^{0.661332}$	$\tilde{c}_4 \cdot 200^{2/3}$
	300	$1.4 \cdot 300^{0.664461}$	$\tilde{c}_4 \cdot 300^{2/3}$
	400	$1.4 \cdot 400^{0.666030}$	$\tilde{c}_4 \cdot 400^{2/3}$
$T_5 = z^5D^5 + z^4D^6 + z^2D^8$	50	$1.5 \cdot 50^{0.462995}$	$c_5 \cdot 50^{1/2}$
	100	$1.5 \cdot 100^{0.481684}$	$c_5 \cdot 100^{1/2}$
	200	$1.5 \cdot 200^{0.491066}$	$c_5 \cdot 200^{1/2}$
	300	$1.5 \cdot 300^{0.494304}$	$c_5 \cdot 300^{1/2}$
	400	$1.5 \cdot 400^{0.495971}$	$c_5 \cdot 400^{1/2}$
$\widetilde{T}_5 = z^2D^2 + z^5D^5 + z^4D^6 + zD^7 + z^2D^8$	50	$1.5 \cdot 50^{0.463391}$	$\tilde{c}_5 \cdot 50^{1/2}$
	100	$1.5 \cdot 100^{0.481837}$	$\tilde{c}_5 \cdot 100^{1/2}$
	200	$1.5 \cdot 200^{0.491129}$	$\tilde{c}_5 \cdot 200^{1/2}$
	300	$1.5 \cdot 300^{0.494342}$	$\tilde{c}_5 \cdot 300^{1/2}$
	400	$1.5 \cdot 400^{0.495998}$	$\tilde{c}_5 \cdot 400^{1/2}$
$T_6 = z^3D^3 + z^2D^6$	50	$1.4 \cdot 50^{0.702117}$	$c_6 \cdot 50^{3/4}$
	100	$1.4 \cdot 100^{0.725715}$	$c_6 \cdot 100^{3/4}$
	200	$1.4 \cdot 200^{0.737541}$	$c_6 \cdot 200^{3/4}$
	300	$1.4 \cdot 300^{0.741614}$	$c_6 \cdot 300^{3/4}$
	400	$1.4 \cdot 400^{0.743713}$	$c_6 \cdot 400^{3/4}$
$\widetilde{T}_6 = [(1+13i) + (24i-3)z + 11iz^2 + z^3]D^3$ $+[(22i-13) - (9+14i)z + z^2]D^6$	50	$1.4 \cdot 50^{0.769260}$	$\tilde{c}_6 \cdot 50^{3/4}$
	100	$1.4 \cdot 100^{0.760399}$	$\tilde{c}_6 \cdot 100^{3/4}$
	200	$1.4 \cdot 200^{0.756161}$	$\tilde{c}_6 \cdot 200^{3/4}$
	300	$1.4 \cdot 300^{0.754590}$	$\tilde{c}_6 \cdot 300^{3/4}$
	400	$1.4 \cdot 400^{0.753765}$	$\tilde{c}_6 \cdot 400^{3/4}$

5 Appendix

For the classes of degenerate exactly-solvable operators considered in Corollary 1 and Corollary 2, what we really want is the lower bound $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} \geq c_0 > 0$, since we have conjectured $\lim_{n\to\infty} \frac{r_n}{n^d} = c_0 > 0$, where
$d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$ and j_0 is the largest j such that $\deg Q_j = j$. Recall that in Corollary 1and 2 we obtained the result $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^\gamma} > c_0 > 0$ for any $\gamma < d$.

Here we prove that for a class of operators containing the operators considered in Corollary 1 and 2, the lower bound $r_n \geq c_0(n-k+1)^d$ follows automatically from the inequality in Lemma 3, if we assume that the upper bound $r_n \leq c_1(n-k+1)^d$ holds for large n, where $c_1 > 0$ is a positive contant and $c_0 \leq c_1$.

Theorem 5. *Let T be a degenerate exactly-solvable operator which satisfies the following condition:*

$$b := \overset{+}{\min_{j\in[1,k-1]}} \left(\frac{k-j}{k-j+\deg Q_j - \deg Q_k}\right) = \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right) =: d,$$

where the notation $\min^+$ *means that the minimum is taken only over positive values of* $(k-j+\deg Q_j - \deg Q_k)$. *Assume that* $r_n \leq c_1(n-k+1)^d$ *holds for large* n, *where* $c_1 > 0$ *is a positive constant. Then there exists a positive constant* $c_0 > 0$ *such that* $c_0 \leq c_1$ *and* $r_n \geq c_0(n-k+1)^d$ *for sufficiently large* n, *and thus* $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} = \tilde{c}$, *where* $c_0 \leq \tilde{c} \leq c_1$ *and* $d := \max_{j\in[j_0+1,k]} \left(\frac{j-j_0}{j-\deg Q_j}\right)$.

Proof. From Lemma 3 and using $i \leq \deg Q_j$ for every given j, we have

$$\begin{aligned} 1 &\leq \sum_{j=1}^{k-1}\sum_{i=0}^{j} |\alpha_{j,i}| 2^{k-j} \frac{r_n^{k-j+i-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<\deg Q_k} \frac{|\alpha_{k,i}|}{r_n^{\deg Q_k - i}} \\ &\leq \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j - \deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k - i}} \end{aligned} \tag{14}$$

where $K_j > 0$ is a positive constant. The second sum on the right-hand side of this inequality tends to zero as $n \to \infty$ due to Theorem 1. To prove our theorem we decompose the first sum on the right-hand side of the inequality above into three parts:

- j for which $\frac{k-j}{k-j+\deg Q_j - \deg Q_k} = d$,
  (note that $(k-j+\deg Q_j - \deg Q_k) > 0$ here since $d > 0$),

- j for which $\frac{k-j}{k-j+\deg Q_j-\deg Q_k} > d$,
  (note that $(k-j+\deg Q_j-\deg Q_k) > 0$ here since $d > 0$),

- j for which $(k-j+\deg Q_j-\deg Q_k) \leq 0$,

where $d := \max_{j\in[j_0+1,k]}\left(\frac{j-j_0}{j-\deg Q_j}\right)$. Clearly there are no terms for which $\frac{k-j}{k-j+\deg Q_j-\deg Q_k} < d$ *and* $(k-j+\deg Q_j-\deg Q_k) > 0$, due to the condition $b = d$.

In the first case, for any term for which $(k-j)/(k-j+\deg Q_j-\deg Q_k) = d$, we have

$$\frac{r_n^{k-j+\deg Q_k-\deg Q_k}}{(n-k+1)^{k-j}} = \left(\frac{r_n}{(n-k+1)^d}\right)^{k-j+\deg Q_j-\deg Q_k}.$$

The second part we consider consists of terms for which $(k-j)/(k-j+\deg Q_j-\deg Q_k) > d$, i.e. $d(k-j+\deg Q_j-\deg Q_k) < (k-j)$, and this inequality together with the upper bound $r_n \leq c_1(n-k+1)^d$ gives the following estimation of the corresponding terms in (14):

$$\frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} \leq \frac{c_1(n-k+1)^{d(k-j+\deg Q_j-\deg Q_k)}}{(n-k+1)^{k-j}} \to 0$$

when $n\to\infty$.

The third part we consider consists of the remaining terms, namely terms for which $(k-j+\deg Q_j-\deg Q_k) \leq 0$, since $(k-j) > 0$. But clearly the corresponding terms $r_n^{k-j+\deg Q_j-\deg Q_k}/(n-k+1)^{k-j}$ in (14) tend to zero when $n\to\infty$, using Theorem 1.

Thus, decomposing the first sum on the right-hand side of the last inequality in (14) in this way, we obtain the following inequality :

$$\begin{aligned}
1 &\leq \sum_{j=1}^{k-1} K_j \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k-i}} \\
&\leq \sum_{j\in A} K_j \left(\frac{r_n}{(n-k+1)^d}\right)^{k-j+\deg Q_j-\deg Q_k} \\
&+ \sum_{j\in B} K_j \frac{c_1(n-k+1)^{d(k-j+\deg Q_j-\deg Q_k)}}{(n-k+1)^{k-j}} \\
&+ \sum_{j\in C} K_j \frac{r_n^{k-j+\deg Q_j-\deg Q_k}}{(n-k+1)^{k-j}} + \sum_{0\leq i<i_k} \frac{|\alpha_{k,i}|}{r_n^{i_k-i}}
\end{aligned}$$

where $A = \{j : \frac{k-j}{k-j+\deg Q_j-\deg Q_k} = d\}$, $B = \{j : \frac{k-j}{k-j+\deg Q_j-\deg Q_k} > d\}$ and $C = \{j : (k-j+\deg Q_j-\deg Q_k) \leq 0\}$. The last three sums in on the right-hand side of this inequality tend to zero when $n \to \infty$, the last one due to Theorem 1, the sum over B since $d(k-j+\deg Q_j - \deg Q_k) < (k-j)$, and the sum over C due to Theorem 1.

Thus, when $n \to \infty$, there exists a positive constant $c' > 0$ such that

$$c' \leq \sum_{j \in A} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} \tag{15}$$

where $A = \{j : \frac{k-j}{k-j+\deg Q_j - \deg Q_k} = d\}$ and A is nonempty. If A contains precisely one element, then the sum in the inequality (15) consists of one single term, and we are done; there exists a positive constant c_0 such that $r_n \geq c_0(n-k+1)^d$ for sufficiently large n. But clearly for some operators A will contain more elements.[9] If this is the case, let $m = \min_{j\in A}(k-j+\deg Q_j - \deg Q_k)$ and denote by j_m the corresponding j. Using the upper bound $r_n \leq c_1(n-k+1)^d$ we then get the following inequality from (15):

$$\begin{aligned} c' &\leq \sum_{j \in A} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k} \\ &\leq K_{j_m} \left(\frac{r_n}{(n-k+1)^d} \right)^{m} \\ &+ \sum_{j \in A \setminus \{j_m\}} K_j \left(\frac{r_n}{(n-k+1)^d} \right)^{m} \cdot \left(\frac{r_n}{(n-k+1)^d} \right)^{k-j+\deg Q_j - \deg Q_k - m} \leq \end{aligned}$$

[9]Consider for example the operator $T = zD + D^2 + zD^3 + zD^4$. Then, from Lemma 3, we get the following inequality (here $k = 4$ and $\deg Q_k = 1$):

$$1 \leq \sum_{j=1}^{3} \frac{2^{4-j} r_n^{3-j+\deg Q_j}}{(n-3)^{4-j}} = 8\frac{r_n^3}{(n-3)^3} + 4\frac{r_n}{(n-3)^2} + 2\frac{r_n}{(n-3)}$$

where r_n is the largest modulus of all roots of the unique and monic eigenpolynomial of T. For this operator $d = 1$ and we see that $\frac{4-j}{3-j+\deg Q_j} = d$ for the first ($j = 1$) and the last ($j = 3$) term. Now assuming that $r_n \leq c_1(n-3)$ our inequality becomes

$$\begin{aligned} 1 &\leq 8\frac{r_n^3}{(n-3)^3} + 4\frac{r_n}{(n-3)^2} + 2\frac{r_n}{(n-3)} \\ &\leq 8\frac{r_n}{(n-3)} \cdot \frac{c_1^2(n-3)^2}{(n-3)^2} + 4\frac{c_1(n-3)}{(n-3)^2} + 2\frac{r_n}{(n-3)} \\ &= (8c_1^2+2)\frac{r_n}{(n-3)} + \frac{4c_1}{(n-3)} \end{aligned}$$

where the last term tends to zero as $n \to \infty$. Thus $r_n \geq c_0(n-3)$ for some positive constant c_0 for sufficiently large choices on n.

$$\begin{aligned}
&\leq K_{j_m}\left(\frac{r_n}{(n-k+1)^d}\right)^m + \sum_{j\in A\setminus\{j_m\}} K_j\left(\frac{r_n}{(n-k+1)^d}\right)^m \cdot c_1^{k-j+\deg Q_j - \deg Q_k - m} \\
&= \left(\frac{r_n}{(n-k+1)^d}\right)^m \left(K_{j_m} + \sum_{j\in A\setminus\{j_m\}} K_j \cdot c_1^{k-j+\deg Q_j-\deg Q_k-m}\right) \\
&= \left(\frac{r_n}{(n-k+1)^d}\right)^m K
\end{aligned}$$

where $K > 0$. Thus there exists a positive constant $c_0 = (c'/K)^{1/m} > 0$ such that $r_n \geq c_0(n-k+1)^d$ for sufficiently large choices on n, i.e. $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} \geq c_0$, and thus $\lim_{n\to\infty} \frac{r_n}{(n-k+1)^d} = \tilde{c}$ for some positive contant $\tilde{c}$ such that $c_0 \leq \tilde{c} \leq c_1$. $\square$

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