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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR DISCRETE-TIME STRICTLY POSITIVE REAL MATRICES

WEIGUO GAO AND YISHAO ZHOU

Abstract. The purpose of this note is to establish some relations between Kalman-Yakubovich-Popov (KYP) lemma and different versions of a strictly positive real rational matrix with minimal realization for discrete-time systems. This note also deals with the KYP lemma for realizations which are not minimal but asymptotically stable and observable.

Keywords: Strictly positive real, Kalman-Yakubovich-Popov lemma, minimality

1. Introduction

The purpose of this note is to clarify some equivalent conditions of the so-called strictly positive real matrices for discrete-time systems. Both frequency-domain and time-domain conditions will be discussed. (Strictly) positive real matrices have been very important concepts in stability analysis and many engineering applications, see e.g. [2, 3, 6, 4, 7, 9, 10, 13]. For convenience we shall call $\mathbb{C}_+$ (strictly) positive real matrices for continuous-time systems and $\mathbb{D}$ (strictly) positive real matrices for discrete-time systems, where $\mathbb{C}_+ := \{z|\mathrm{Re}(z) > 0\}$, and $\mathbb{D} := \{z||z| > 1\}$. Also we shall use the abbreviations PR and SPR for positive real and strictly positive real, respectively. In the literature the following definition is often used (e.g. [1, 3]).

Definition 1.1. An $m \times m$ matrix $H(z)$ of real rational functions is said to be $\mathbb{C}_+$-SPR if

(i) all elements are analytic in the closed right-half plane $\overline{\mathbb{C}}_+$;
(ii) $H(j\omega) + H^T(-j\omega) > 0$ for all real values of ω.

The matrix of rational functions $H(z)$ is said to be $\mathbb{D}$-SPR if

(i)$_d$ all elements are analytic in $\overline{\mathbb{D}}$;
(ii)$_d$ $H(e^{j\omega}) + H^T(e^{-j\omega}) > 0$ for $0 \le \omega \le 2\pi$.

Sometimes it is required that the inequality in condition (ii) hold for all $\omega \in \mathbb{R} \cup \{\infty\}$. We call this condition for (ii$'$). For the scalar case it was pointed out in [12] that (i) and (ii) are only necessary, while (i) and (ii$'$) are only sufficient for the Lur'e equations [10, 9, 13] to be satisfied. So another definition for $\mathbb{C}_+$-SPR was introduced in [12] with motivation from network theory.

Definition 1.2. A rational matrix $H(z)$ is $\mathbb{C}_+$-SPR if $H(z-\mu)$ is $\mathbb{C}_+$-PR[1] for some $\mu > 0$.

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[1] A rational matrix $H(z)$ is $\mathbb{C}_+$-PR if

(a) all elements are analytic in $\overline{\mathbb{C}}_+$; and poles of any element of $H(z)$ on the $j\omega$-axis are distinct, and the associated residue matrices of $H(z)$ are ≥ 0;

A natural question arises here. What is the counterpart of this defintion for discrete-time systems? If there is such a definition, is it equivalent to Definition 1.1? We shall answer these questions in this note.

Frequency domain necessary and sufficient conditions for $\mathbb{C}_+$-SPR were investigated in [8, 14], and a series of time domain and frequency domain conditions for $\mathbb{C}_+$-SPR were discussed in [15]. It is often true that theory for continuous-time systems will be valid for discrete-time systems by some technical modifications. However, it was observed in [5] that the $\mathbb{C}_+$-SPR matrices and the $\mathbb{D}$-SPR matrices defined in Definition 1.1 are not completely parallel, at least in the scalar case. We shall in this note show how Definition 1.2 is defined for discrete-time systems and point out the differences between discrete and continuous time, and derive some necessary and sufficient conditions for $\mathbb{D}$-SPR matrices.

Throughout this note, we denote the symmetric positive (semi-)definite matrix A by $A > 0$ ($A \geq 0$), and the congruence of the matrices A and B by $A \sim B$. Moreover we make a general assumption: $H(z) + H^T(z^{-1})$ (in discrete time) has rank m almost everywhere in the complex plane and $H(z)$ is a non-zero rational matrix.

Let $H(z) = D + C(zI - A)^{-1}B$ with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ be a minimal realization, and A be Hurwitz. From the PR lemma [2] it follows that $H(z)$ is $\mathbb{C}_+$-PR matrices , if and only if there exist matrices $P \geq 0, P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ that satisfy the the Lur'e equations continuous-time Lur'e equations

$$A^T P + PA = -Q^T Q,\ B^T P - C = W^T Q,\ W^T W = D + D^T.$$

In continuous time several different conditions have been used to define SPR as discussed in the previous section. All these are related to (ii) in Definition 1.1. The following conditions were considered in [15].

(1) There exist matrices $P > 0,\ L > 0,\ P, L \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ that satisfy

$$A^T P + PA = -Q^T Q - L,\ B^T P - C = W^T Q,\ W^T W = D + D^T. \tag{1.1}$$

(2) For all real ω,

$$H(j\omega) + H^T(-j\omega) > 0; \tag{1.2}$$

(3) For all real ω,

$$H(j\omega) + H^T(-j\omega) > 0, \quad \text{and} \quad \lim_{\omega\to\infty} \omega^2(H(j\omega) + H^T(-j\omega)) \geq 0; \tag{1.3}$$

(4) For some $\eta > 0$

$$H(j\omega) + H^T(-j\omega) \geq \eta I \tag{1.4}$$

It was proved [15] that (1.3) is equivalent to

$$H(j\omega - \mu) + H^T(-j\omega - \mu) \geq 0, \quad \text{for some } \mu > 0, \tag{1.5}$$

which, together with (i), implies that $H(z - \mu)$ is PR for some $\mu > 0$, and therefore, it is equivalent to Definition 1.2. Moreover, It was shown in [15] that (3) $\Rightarrow$ (2),

(b) $H(j\omega) + H^T(-j\omega) \geq 0$ for all real ω which are not poles of any element of $H(j\omega)$.

Similarly, $H(z)$ is $\mathbb{D}$-PR if

(a)$_\mathrm{d}$ all elements are analytic in $\overline{\mathbb{D}}$; and poles of any element of $H(z)$ on the unit circle are distinct, and the associated residue matrices of $H(z)$ are ≥ 0;

(b)$_\mathrm{d}$ $H(e^{j\omega}) + H^T(e^{-j\omega}) \geq 0$ for $0 \leq \omega \leq 2\pi$ which are not poles of any element of $H(e^{j\omega})$.

they imply (1) and (4) respectively. Furthermore, if $D + D^T > 0$ then (4) $\Rightarrow$ (3) and if $D = 0$ then (4) $\Rightarrow$ (2). The minimality condition is relaxed in [4] for the equivalence between $\mathbb{C}_+$-SPR by Definition 1.2 and Lur'e equations. A natural question arises now. Are there any similar results for $\mathbb{D}$-SPR matrices? We shall answer this question in next section. To our best knowledge, this has not been discussed in literature.

2. Main results

For the sake of exposition, we first assume that the realization (A, B, C, D) of the rational matrix $H(z)$ is minimal. Let A be a Schur matrix. We shall prove the following theorem.

Theorem 2.1. *Given $H(z) = D + C(zI - A)^{-1}B$ with all elements analytic in $\overline{\mathbb{D}}$. Consider the following conditions*

1) *There exist matrices $P > 0$, $L > 0$, $P, L \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ that satisfy the Lur'e equations*

$$A^T PA - P = -Q^T Q - L,\ C - B^T PA = W^T Q,\ W^T W = D + D^T - B^T PB. \quad (2.1)$$

1′) *Same as* 1) *except L is related to P by*

$$L = \mu(2 - \mu)P$$

for some $0 < \mu < 1$

2) *For all $0 \le \omega \le 2\pi$*

$$H(e^{j\omega}) + H^T(e^{-j\omega}) > 0. \quad (2.2)$$

3) *There exists $\eta > 0$ such that for all $0 \le \omega \le 2\pi$*

$$H(e^{j\omega}) + H^T(e^{-j\omega}) \ge \eta I. \quad (2.3)$$

4) *The discrete-time Lur'e equations (2.1) with $L = 0$ and Q replaced $Q/(1-\mu)$ are satisfied by the realization $(A/(1-\mu), B, C/(1-\mu), D)$ replacing corresponding to $H((1-\mu)z)$ for some $0 < \mu < 1$.*

5) *For all $0 \le \omega \le 2\pi$, there exists $0 < \mu < 1$ such that*

$$H\left((1-\mu)e^{j\omega}\right) + H^T\left((1-\mu)e^{-j\omega}\right) \ge 0. \quad (2.4)$$

Then 1′) *and* 2)–5) *are equivalent, and they imply* 1). *Moreover,* 1) *implies the rest if W is nonsingular.*

Remark 2.2. Conditions 1), 2) and 3) in the unit circle case correspond to (1), (2) and (4) in the half-plane case, respectively. Condition 5) is the counterpart of (1.5). From the above theorem we see that (1.3) does not have the discrete-time counterpart, and the result in the unit circle case is stronger than that in the half-plane case. The proofs of 1′) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) $\Leftrightarrow$ 2) are similar to those in [15, 14]. For completeness we give the whole proof.

Proof. The proof of equivalence 1′) $\Leftrightarrow$ 4) is straightforward algebraic manipulation. The implication 3) to 2) is trivial.
2) $\Rightarrow$ 3): Assume condition 3) is false. Then there would exist $\{u_n, \|u_n\| = 1\}$ and $\{\omega_n, 0 \le \omega_n \le 2\pi\}$ such that

$$0 \le \langle (H(e^{j\omega_n}) + H^T(e^{-j\omega_n}))u_n, u_n\rangle \le \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} \langle (H(e^{j\omega_n}) + H^T(e^{-j\omega_n}))u_n, u_n\rangle = 0.$$

As $\{\omega_n\}$ is bounded, it contains convergent subsequences $\{\omega_{n_k}\}$. Let its limit be ω_0. For $\{u_{n_k}\}$, there is a convergent subsequence denoted again by $\{u_{n_k}\}$. Then, because $H(z)$ is analytic outside the unit circle,

$$\lim_{k\to\infty} \langle (H(e^{j\omega_0})+H^T(e^{-j\omega_0}))u_{n_k}, u_{n_k}\rangle = \lim_{k\to\infty} \langle (H(e^{j\omega_{n_k}})+H^T(e^{-j\omega_{n_k}}))u_{n_k}, u_{n_k}\rangle = 0,$$

which conflicts the condition $H(e^{j\omega_0}) + H^T(e^{-j\omega_0}) > 0$.
5) $\Rightarrow$ 2): Since $\Phi(e^{j\omega}) := H((1-\mu)e^{j\omega}) + H^T((1-\mu)e^{-j\omega}) \geq 0$ for all $\omega \in [0, 2\pi]$, and $\Phi(e^{j\omega}) = \Phi^T(e^{-j\omega})$ for all $\omega \in [0, 2\pi]$, By Youla's factorization theorem, [3], we have that

$$H((1-\mu)z) + H^T(((1-\mu)z)^{-1}) = W^T(((1-\mu)z)^{-1})W((1-\mu)z)$$

where $W((1-\mu)z)$ is analytic in $|z| \geq 1$ and has rank m in $|z| > 1$. Now letting $(1-\mu)z = e^{j\omega}$ yields

$$\det(H(e^{j\omega}) + H^T(e^{-j\omega})) = \det W^T(e^{-j\omega}) \det W(e^{j\omega}) \neq 0.$$

for all $\omega \in [0, 2\pi]$. Since $H(e^{j\omega}) + H^T(e^{-j\omega}) \geq 0$ for all $\omega \in [0, 2\pi]$, we have $H(e^{j\omega}) + H^T(e^{-j\omega}) > 0$, for all $\omega \in [0, 2\pi]$.
2) $\Rightarrow$ 1): By Theorem 12.6.5 in [7] there exists a symmetric matrix $Y \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} -Y + A^TYA & C^T + A^TYB \\ C + B^TYA & D + D^T + B^TYB \end{bmatrix} > 0.$$

Since the realization is minimal we can show that there is a negative definite solution Y to the above matrix inequality. Choose now $L = -Y + A^TYA - (C^T + A^TYB)(D + D^T + B^TYB)^{-1}(C + B^TYA)$. By above inequality $L > 0$. Now a straightforward calculation yields

$$\begin{bmatrix} -Y + A^TYA - L & C^T + A^TYB \\ C + B^TYA & D + D^T + B^TYB \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & D + D^T + B^TYB \end{bmatrix} \geq 0.$$

Therefore the matrix in the left hand side can be factorized as $\begin{bmatrix} Q^T \\ W^T \end{bmatrix} [\,Q\ W\,]$, where $Q \in \mathbb{R}^{m\times n}$, and $W \in \mathbb{R}^{m\times m}$, which implies the Lur'e equations

$$-Y + A^TYA - L = Q^TQ, C + B^TYA = W^TQ, D + D^T + B^TYB = W^TW.$$

Thus we have shown that there is $P = -Y > 0$ and W and Q that satisfy (2.1).
1) $\Rightarrow$ 2): If W is nonsingular, then $W^TW = D + D^T - B^TPB > 0$. Hence, the backward deduction in the proof of 2) $\Rightarrow$ 1) is valid, using the standard linear algebra argument.
4) $\Rightarrow$ 5): Same as 1) $\Rightarrow$ 2) where $L = \mu(2-\mu)P$.
5) $\Rightarrow$ 4): The standard positive real to $H((1-\mu)z)$, [3].
2) $\Rightarrow$ 5): By matrix inversion formula, we have

$$\begin{aligned} &H((1-\mu)e^{j\omega}) + H^T((1-\mu)e^{-j\omega}) \\ =&H(e^{j\omega}) + H^T(e^{-j\omega}) + \mu\left[Ce^{j\omega}((e^{j\omega}I - A)^{-1}((1-\mu)e^{j\omega}I - A)^{-1}B\right. \\ &\left.+B^Te^{-j\omega}(e^{-j\omega}I - A^T)^{-1}((1-\mu)e^{-j\omega}I - A^T)^{-1}C^T\right]. \end{aligned}$$

Hence, for any $v \in \mathbb{C}^m$

$$v^*H((1-\mu)e^{j\omega})v \geq v^*H(e^{j\omega})v - \mu\|C\|\,\|B\|\|(e^{j\omega}I - A)^{-1}\|\,\|((1-\mu)e^{j\omega}I - A)^{-1}\|\,\|v\|^2.$$

Notice that for sufficiently small $\mu > 0$ the matrix $A/(1-\mu)$ is still Schur. Thus,

$$\|(e^{j\omega}I - A)^{-1}\| \le \frac{1}{|1-\|A\||}, \quad \|((1-\mu)e^{j\omega}I - A)^{-1}\| \le \frac{1}{|1-\mu-\|A\|\,|}$$

Hence, by 3) (which is equivalent to 2)), there is an $\eta > 0$ such that

$$\begin{aligned} v^*H((1-\mu)e^{j\omega})v &\ge \eta\|v\|^2 - \frac{\mu\|C\|\,\|B\|\,\|v\|^2}{|1-\|A\|\,|\,|1-\mu-\|A\|\,|} \\ &= \|v\|^2\left(\eta - \frac{\mu\|C\|\,\|B\|}{|1-\|A\|\,|\,|1-\mu\|A\|\,|}\right). \end{aligned}$$

For sufficiently small μ the right hand side of the above inequality is nonnegative. This completes the proof. □

Note that the minimality condition was only used to assure the existence of positive definite solution to the Lur'e equations. Clearly, above theorem shows the differences between the discrete and the continuous cases. In addition, for SISO systems it is easy to see that

i) The $\mathbb{D}$-PR implies that $d > 0$;
ii) The $\mathbb{C}_+$-PR implies that $d \ge 0$, and if $d = 0$, we must have $c^Tb \neq 0$.

Corollary 2.3. *Given an $m \times m$ rational matrix $H(z) = D + C(zI - A)^{-1}B$ such that $H(z) + H^T(z^{-1})$ has rank m almost everywhere in $\mathbb{C}$, then*

(i) $H(z)$ is $\mathbb{D}$-SPR if and only if all elements of $H(z)$ are analytic in the $\overline{\mathbb{D}}$ and there exists $\eta > 0$ such that $H(e^{j\omega}) + H^T(e^{-j\omega}) \ge \eta I$, for all $0 \le \omega < 2\pi$;
(ii) $H(z)$ is $\mathbb{D}$-SPR if and only if $H((1-\mu)z)$ is $\mathbb{D}$-PR, $\forall \mu \in (0, \mu^*)$ for some $0 < \mu^* < 1$;
(iii) $H(z)$ with minimal realization (A, B, C, D) is $\mathbb{D}$-SPR if and only if there exist a positive definite symmetric matrix P, matrices Q and W, and a positive constant $\mu < 1$ such that

$$\begin{aligned} A^TPA - P &= -Q^TQ - \mu(2-\mu)P \\ C - B^TPA &= W^TQ \\ W^TW &= D + D^T - B^TPB. \end{aligned} \tag{2.5}$$

3. KYP lemma without minimality

In this section we show that the item (iii) in the corollary is also valid for non-minimal realizations of the rational matrix $H(z)$. First we prove the discrete-time counterpart of a theorem by [14]. The proofs of both theorems are similar to those of the continuous-counterparts. However, they are more technical due to the nature of discrete-time systems.

Theorem 3.1. *Let $H(z) = C(zI - A)^{-1}B + D$ be an $m \times m$ real rational matrix such that $H(z) + H^T(z^{-1})$ has rank m almost everywhere in $\mathbb{C}$, where A is a Schur matrix, (A, B) is controllable. Then, given $\mu > 0$, and an arbitrary positive definite matrix $L \in \mathbb{R}^{n\times n}$, $H(z)$ is SPR for sufficient small μ if and only if there exist a positive definite symmetric matrix P, matrices $Q \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such* *that*

$$\begin{aligned} A^TPA - P &= -Q^TQ - \mu L \\ C - B^TPA &= W^TQ \\ W^TW &= D + D^T - B^TPB. \end{aligned} \tag{3.1}$$

Proof. Sufficiency: Note that $H((1-\mu)z) = C((1-\mu)z - A)^{-1}B + D$ is analytic in $|z| \geq 1$ $\forall \mu \in (0, \mu^*)$ for some $\mu^* > 0$. As before, we have that $H((1-\mu)z) = H(z) + \mu G(z)$, where $G(z) = zC(zI - A)^{-1}((1-\mu)z - A)^{-1}B$. Using (3.1), a straightforward calculation yields

$$\begin{aligned} &H((1-\mu)e^{j\omega}) + H^T((1-\mu)e^{-j\omega}) \\ =&H(e^{j\omega}) + H(e^{-j\omega}) + \mu(G(e^{j\omega}) + G^T(e^{-j\omega})) \\ =&W^TW + B^TPB + W^TQ(e^{j\omega} - A)^{-1}B + B^T(e^{-j\omega} - A^T)^{-1}Q^TW \\ &+ B^TPA(e^{j\omega}I - A)^{-1}B + B^T(e^{-j\omega} - A^T)^{-1}A^TPB + \mu(G(e^{j\omega}) + G^T(e^{-j\omega})) \\ =&(W^T + Q^T(e^{-j\omega} - A^T)^{-1}B^T)(W + B(e^{j\omega} - A)^{-1}Q) \\ &+ \mu B^T(e^{-j\omega} - A^T)^{-1}L(e^{j\omega} - A)^{-1}B + \mu(G(e^{j\omega}) + G^T(e^{-j\omega})). \end{aligned}$$

Now for some $v \in \mathbb{R}^{m\times m}$, such that $v^*B^T(e^{-j\omega} - A^T)^{-1}L(e^{j\omega} - A)^{-1}Bv = 0$, then $Bv = 0$. This implies that $v^*(G^T(e^{-j\omega}) + G(e^{j\omega}))v = 0$. Therefore, for some $0 < \mu_0 < \mu^*$ and any $\mu \in (0, \mu_0)$, $\mu B^T(e^{-j\omega} - A^T)^{-1}L(e^{j\omega} - A)^{-1}B + \mu(G(e^{j\omega}) + G^T(e^{-j\omega})) \geq 0$, for all $\omega \in [0, 2\pi]$. So $H(z)$ is SPR by Corollary 2.3.
Necessity: Assume that $H(z)$ is SPR. By Theorem 2.1, $H(e^{j\omega}) + H^T(e^{-j\omega}) > 0$, $\forall \omega \in [0, 2\pi]$. Then, for all $\mu \in (0, \mu^*)$ for some $\mu^* > 0$, and any $L \in \mathbb{R}^{n\times n}$,

$$H(e^{j\omega}) + H^T(e^{-j\omega}) - \mu B^T(e^{-j\omega} - A^T)^{-1}L(e^{j\omega} - A)^{-1}B \geq 0, \forall \omega \in [0, 2\pi].$$

Note that this implies that the Popov function

$$\Sigma(z) = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix} \begin{bmatrix} -\mu L & B \\ B^T & D + D^T \end{bmatrix} \begin{bmatrix} (z^{-1}I - A^T)^{-1}B^T \\ I \end{bmatrix} \geq 0$$

on the unit circle for all $\mu \in (0, \mu^*)$ for some $\mu^* > 0$. By Youla's theorem and Theorem 16.5.2 in [7], there are matrices $W \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{m\times n}$ such that

$$\begin{aligned} &H(e^{j\omega}) + H^T(e^{-j\omega}) - \mu B^T(e^{-j\omega} - A^T)^{-1}L(e^{j\omega} - A)^{-1}B \\ =&(W^T + B^T(e^{-j\omega} - A^T)^{-1}Q^T)(W + Q(e^{j\omega} - A)^{-1}B) \end{aligned} \tag{3.2}$$

For this Q, we have a unique positive definite solution P from the Lyapunov equation

$$A^TPA - P = -Q^TQ - \mu L$$

since the right-hand side is strictly negative definite, showing that the first equation in (3.1) holds. Setting $z = e^{j\omega}$ in (3.2). Then the Laurent series of both sides of (3.2) converges in the neighborhood of the unit circle, $N_{|z|=1}$, because A has all eigenvalues inside the unit circle. Comparing the coefficients in the Laurent series of (3.2), we get the following equalities

$$D + D^T = W^TW + B^T\sum_{i=1}^{\infty}(A^T)^{i-1}(Q^T + \mu L)A^{i-1}\,B$$

$$CA^{i-1}B = W^TQA^{i-1}B + B^T\sum_{i=1}^{\infty}(A^T)^{i-1}(Q^T + \mu L)A^iB, \quad i = 1, 2, ...$$

The series $\sum_{i=1}^{\infty}(A^T)^{i-1}(Q^T+\mu L)A^{i-1}$ converges since A is Schur. Moreover, it converges to P by the Lyapunov equation above. Hence, we obtain the equalities

$$\begin{aligned} D+D^T &= W^TW+B^TPB \\ CA^{i-1}B &= W^TQA^{i-1}B+B^TPA^iB, \quad i=1,2,... \end{aligned}$$

proving that the second equation in (3.1) is valid. Finally combining the controllability of (A,B) and the second equation in the above equalities, we obtain $C=W^TQ+B^TPA$, which is the last equation in (3.1). $\square$

Now we turn to the KYP lemma without restriction of minimal realization.

Theorem 3.2. *Let $H(z)=C(zI-A)^{-1}B+D$ be an $m\times m$ transfer matrix such that $H(z)+H^T(z^{-1})$ has rank m almost everywhere in the complex plane, where A is a Schur and nonsingular matrix, and (C,A) is observable. Assume that if there are multiple eigenvalues, then all of them are controllable modes or all of them are uncontrollable modes. Then, $H(z)$ is SPR if and only if there exist a positive definite symmetric matrix P, matrices Q and W, and a positive constant $\mu<1$ such that*

$$\begin{aligned} A^TPA-P &= -Q^TQ-\mu(2-\mu)P \\ B^TPA-C &= W^TQ \\ W^TW &= D+D^T-B^TPB. \end{aligned} \tag{3.3}$$

Proof. Sufficiency: For $\epsilon<\mu$, the first equation in (3.1) implies that

$$\frac{A^T}{1-\epsilon}P\frac{A}{1-\epsilon}-P=-\frac{Q^TQ+(\mu-\epsilon)(2-\mu-\epsilon)P}{(1-\epsilon)^2}<0.$$

Therefore, $A/(1-\epsilon)$ is Schur, hence, $H((1-\epsilon)z)$ is analytic in $|z|\geq 1$. Using (3.3)

$$\begin{aligned} &H((1-\epsilon)e^{j\omega})+H^T((1-\epsilon)e^{-j\omega}) \\ =&W^TW+B^TPB+W^TQ_\epsilon(e^{j\omega}I-A_\epsilon)^{-1}B+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}Q_\epsilon^T \\ &+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}(P-A_\epsilon^TPA_\epsilon)(e^{j\omega}I-A_\epsilon)^{-1}B \\ =&W^TW+B^TPB+W^TQ_\epsilon(e^{j\omega}I-A_\epsilon)^{-1}B+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}Q_\epsilon^T \\ &+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}\left(\frac{\epsilon(2-\epsilon)}{(1-\epsilon)^2}P+Q_\epsilon^TQ_\epsilon+\frac{\mu(2-\mu)}{(1-\epsilon)^2}P\right)(e^{j\omega}I-A_\epsilon)^{-1}B \\ =&\left(W^T+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}Q_\epsilon^T\right)\left(W+Q_\epsilon(e^{j\omega}I-A_\epsilon)^{-1}B\right) \\ &+B^T(e^{-j\omega}I-A_\epsilon^T)^{-1}\left(\frac{(\mu-\epsilon)(2-\mu-\epsilon)}{(1-\epsilon)^2}P\right)(e^{j\omega}I-A_\epsilon)^{-1}B\geq 0, \end{aligned}$$

where $A_\epsilon=A/(1-\epsilon), C_\epsilon=C/(1-\epsilon), Q_\epsilon=Q/(1-\epsilon)$. Therefore, $H((1-\mu)z)$ is PR, that is $H(z)$ is SPR, by Theorem 2.1.

Necessity: Assume that $H(z)$ is SPR, by Theorem 2.1 $H((1-\mu)z)$ is PR. Let $H(z)=C(zI-A)^{-1}B+D$ be a non-minimal realization with A Schur and (C,A) observable. Since (A,B) is not controllable, there is a nonsingular matrix T such that $TAT^{-1}=\begin{bmatrix}A_c & A_{12}\\ 0 & A_{nc}\end{bmatrix}$, $TB=\begin{bmatrix}B_c\\ 0\end{bmatrix}$, $CT^{-1}=\begin{bmatrix}C_c & C_{nc}\end{bmatrix}$, where the induces c and nc stand for the blocks related to controllable respectively uncontrollable states. According to the assumption, the eigenvalues of A_c are not the eigenvalues of A_{nc}. Then $H(z)=C_c(zI-A_c)^{-1}B_c+D$ is a minimal realization. Consequently we have a non-minimal realization (A_μ, B, C_μ, D) of $H((1-\mu)z)$, and a minimal realization $(A_{c,\mu}, B_c, C_{c,\mu}, D)$ of $H((1-\mu)z)$, where $A_\mu = A/(1-\mu), C_\mu = C/(1-\mu)$, and $A_{c,\mu} = A_c/(1-\mu), C_{c,\mu} = C_c/(1-\mu)$. Let $U(z) = H((1-\mu)z)$. By Youla's theorem of spectral factorization, there is an $m \times m$ stable spectral factor $V(z)$ such that

$$U(z) + U^T(z^{-1}) = V^T(z^{-1})V(z) \tag{3.4}$$

since $U(z)$ is PR and $H(z) + H^T(z^{-1})$ has rank m almost everywhere in $\mathbb{C}$. By the PR lemma, there are matrices $W \in \mathbb{R}^{m\times m}$ and $Q_{c,\mu}$ (with appropriate dimension) such that $(A_{c,\mu}, B_c, Q_{c,\mu}, W)$ is a minimal realization of $V(z)$. Then, (A_μ, B, Q_μ, W) is a non-minimal realization with $Q_\mu = \begin{bmatrix} Q_{c,\mu} & Q_{nc,\mu} \end{bmatrix} T$ such that $(Q_{nc,\mu}, A_{nc,\mu})$ is observable. It is straightforward to show that

$$\left(\begin{bmatrix} A_\mu & \\ & A_\mu^{-T} \end{bmatrix}, \begin{bmatrix} B \\ -A_\mu^{-T} C_\mu^T \end{bmatrix}, \begin{bmatrix} C_\mu & B^T A_\mu^{-T} \end{bmatrix}, D + D^T - B^T A_\mu^{-T} C_\mu^T \right)$$

and

$$\left(\begin{bmatrix} A_\mu & \\ -A_\mu^{-T} Q_\mu^T Q_\mu & A_\mu^{-T} \end{bmatrix}, \begin{bmatrix} B \\ -A_\mu^{-T} Q_\mu^T W \end{bmatrix}, \begin{bmatrix} Q_\mu^T W - B^T A_\mu^{-T} Q_\mu^T Q_\mu & B^T A_\mu^{-T} \end{bmatrix}, \right.$$
$$\left. W^T W - B^T A_\mu^{-T} Q_\mu^T W \right)$$

are non-minimal realizations of $U(z) + U^T(z^{-1})$ and $V^T(z^{-1})V(z)$, respectively, under the condition that A is nonsingular. Now (Q_μ, A_μ) is observable, (because $(Q_{c,\mu}, A_{c,\mu})$ and $(Q_{nc,\mu}, A_{nc,\mu})$ are observable, and the controllable and uncontrollable modes are different), and A_μ is Schur, there exists a unique positive definite solution P to the Lyapunov equation

$$A_\mu^T P A_\mu - P = -Q_\mu^T Q_\mu \tag{3.5}$$

or equivalently,

$$A^T P A - P = -Q^T Q - \mu(2-\mu)P$$

where $Q = (1-\mu)Q_\mu$, that is the first equation in (3.3).

Using the matrix $S = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$ to change the coordinates for the non-minimal realization of $V^T(z^{-1})V(z)$ we obtain the following equivalent realization

$$\left(\begin{bmatrix} A_\mu & \\ & A_\mu^{-T} \end{bmatrix}, \begin{bmatrix} B \\ -PB - A_\mu^{-T} Q_\mu^T W \end{bmatrix}, \begin{bmatrix} Q_\mu^T W - B^T A_\mu^{-T} Q_\mu^T Q_\mu + B^T A_\mu^{-T} P & B^T A_\mu^{-T} \end{bmatrix}, \right.$$
$$\left. W^T W - B^T A_\mu^{-T} Q_\mu^T W \right)$$

By (3.4), we can show that the stable (respectively unstable) parts of the realizations of $U(z) + U^T(z^{-1})$ and $V^T(z^{-1})V(z)$ are identical. Hence, considering only the stable part yields

$$D + D^T - B^T A_\mu^{-T} A_\mu^T = W^T W - B^T A_\mu^{-T} Q_\mu^T W \tag{3.6}$$

$$C_\mu = Q_\mu^T W - B^T A_\mu^{-T} Q_\mu^T Q_\mu + B^T A_\mu^{-T} P \tag{3.7}$$

Substituting (3.5) in (3.6) and (3.7) yields

$$D + D^T - B^T P B = W^T W$$
$$C_\mu = Q_\mu^T W - B^T P A_\mu.$$

The first equation is the last equation in (3.3), and the second one is the second equation in (3.3) by substituting $Q = (1-\mu)Q_\mu$. Thus the theorem is proved. □

4. Conclusion

We have proved a series of equivalent conditions for strictly positive real matrices for discrete-time systems, and we have also shown that the so-called Kalman-Yakubovich-Popov lemma is still valid by replacing the minimality condition for realizations of a positive real matrix by the observability of an asymptotically stable system.

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