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ON SOME LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS FOR DESCRIPTOR SYSTEMS

G. KURINA

Abstract. The work deals with linear-quadratic optimal control problems with constant coefficients when the state equation is unresolved with respect to the derivative. In the first two problems, the performance index is the sum of the integral of a quadratic form with respect to the control and quadratic forms with respect to the differences between the output variable values in the fixed points and given values. Two cases are researched, namely, when the initial value for the part of the state variable is given and when additional constraints for boundary points of the state variable are absent. For the third problem, output variable values in the fixed points are given. The sufficient and necessary control optimality conditions have been established for considered problems. The solvability of these optimal control problems has been proved. The examples are given which show that the necessary control optimality conditions are not valid under general assumptions. It has been pointed out that the problems on smoothing and interpolating splines are the particular cases of the studied optimal control problems.

Key words: linear-quadratic optimal control problems, descriptor systems, necessary and sufficient control optimality conditions, solvability of optimal control problems, interpolating and smoothing splines.

1. Introduction

For the last twenty years, many papers devoted to the study of optimal control problems with state equation unresolved with respect to the derivative have been published (see, for example, the reviews [1,2]). The most part of these publications deals with systems with constant coefficients. The systems, unresolved with respect to the derivative, are frequently named in the scientific literature as descriptor systems. Linear-quadratic optimal control problems for descriptor systems with variable coefficients have been studied in [3,4] under the most general conditions.

In the present work, linear-quadratic optimal control problems with constant coefficients are studied when the state equation is

$$\frac{d(Ex(t))}{dt} = Ax(t) + Bu(t), \tag{1.1}$$

the output variable is

$$y(t) = Cx(t), \tag{1.2}$$

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and the performance index has the form

$$J(u,y) = \frac{1}{2}\sum_{j=0}^{N+1}\langle y(t_j)-y_j, F_j(y(t_j)-y_j)\rangle + \frac{1}{2}\int_0^T \langle u(t), Ru(t)\rangle dt, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in appropriate spaces, $t \in [0,T]$, $t_0 = 0$, $t_{N+1} = T$, $0 < t_1 < \cdots < t_N < T$; t_j, $j = 1,...,N+1$, are fixed; $x(t) \in X$, $y(t) \in Y$, $u(t) \in U$; X, Y, U are real finite-dimensional Euclidean spaces; $E, A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y), F_j \in \mathcal{L}(Y)$, $R \in \mathcal{L}(U)$; the operators $F_j, j = 0,...,N+1$, and R are symmetric, F_j are positive semidefinite, and R is positive definite, the elements $y_j \in Y$, $j = 0,...,N+1$, are given.

In section 2, the value $Ex(0)$ is given and $F_0 = 0$. In section 3, the value $Ex(0)$ is free. In section 4, $y(t_j)$ are given, $j = 1,...,N$. The sufficient and necessary control optimality conditions have been established for considered problems. The solvability of these control problems has been proved. The examples, showing that the necessary control optimality conditions are not valid under general assumptions, are given in the ends of sections 2,3.

In section 5, it is pointed out that the problems on smoothing and interpolating splines are the particular cases of the optimal control problems studied in sections 3,4.

The illustrative examples are presented in sections 2 - 5.

The obtained results are unknown even for the case $E = I$.

It should be noted that the considered approach to solving problem (1.1)-(1.3) from section 3, when $E = I$, is different from the method of solving a similar problem from [5]. In contrast to [5], we do not assume that $u(t)$, y_j, $j = 1,...,N+1$, are scalar, the matrices $diag(F_1,...,F_{N+1})$, C^*F_0C are positive definite and $y_0 = 0$.

The paper [6] deals with the necessary control optimality condition for nonlinear optimal control problems with the state equation resolved with respect to the derivative, consisting in the closening of the object with fixed points in the indicated order when resources are limited, the integral part in the performance index is absent in this case. We note that the relations from [6] for determining the adjoint variable in the maximum principle for linear quadratic problems are different from the relations in this work.

The necessary control optimality condition for one optimal control problem with the constraint for the state variable in an intermediate point was considered in [7].

We will assume that admissible controls $u(\cdot)$ are piecewise continuous functions on $[0,T]$ ensuring the solvability of a state equation with given conditions for the state variable, trajectories $x(\cdot)$ of a state equation are piecewise continuous functions satisfying the state equation almost everywhere such that $Ex(\cdot)$ are continuous. For the definiteness we can suppose that control functions and further adjoint variables are continuous to the right in the points of the discontinuity. When $t = 0$ and $t = T$ we assume the continuity to the right and to the left, respectively.

Further, the asterisk with the notation of an operator denotes the adjoint operator for this operator.We denote by P and Q the orthogonal projectors of the space X onto $\ker E$ and $\ker E^*$ respectively corresponding to the orthogonal decompositions

$$X = \ker E \oplus \mathrm{Im}\, E^* = \ker E^* \oplus \mathrm{Im}\, E.$$

By E_+ we will denote the inverse operator for the operator $(I-Q)E(I-P)$: $\mathrm{Im}E^* \to \mathrm{Im}E$.

2. Problem with fixed $Ex(0)$

2.1. Problem statement

Let us consider the problem of minimizing the functional

$$J(u,y)=\frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j)-y_j,F_j(y(t_j)-y_j)\rangle+\frac{1}{2}\int_0^T\langle u(t),Ru(t)\rangle dt \tag{2.1}$$

with respect to trajectories of the system

$$\frac{d(Ex(t))}{dt}=Ax(t)+Bu(t), \tag{2.2}$$

$$Ex(0)=x^0, \tag{2.3}$$

$$y(t)=Cx(t), \tag{2.4}$$

where the element $x^0\in \mathrm{Im}\,E$ is given.

2.2. Sufficient control optimality conditions

Theorem 2.1 (sufficient control optimality condition). *The control $u_*(\cdot)$, given by the formula*

$$u_*(t)=R^{-1}B^*\psi(t), \tag{2.5}$$

where $\psi(\cdot)$ is a solution of the problem

$$E^*\frac{d\psi(t)}{dt}=-A^*\psi(t),\ t\neq t_j, \tag{2.6}$$

$$E^*(\psi(t_j-0)-\psi(t_j+0))=-C^*F_j(y_*(t_j)-y_j),\quad j=1,...,N, \tag{2.7}$$

$$E^*\psi(T)=-C^*F_{N+1}(y_*(T)-y_{N+1}), \tag{2.8}$$

$y_*(t)=Cx_*(t)$, $x_*(\cdot)$ is a trajectory of system (2.2), (2.3) corresponding to the control $u_*(\cdot)$, is an optimal control for problem (2.1) - (2.4).

Proof. Let $u(\cdot)$ be an arbitrary admissible control and $x(\cdot)$ be a corresponding trajectory, $y(t)=Cx(t)$. Taking into account relation (2.1), we have

$$J(u,y)-J(u_*,y_*)=\frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j)-y_*(t_j),F_j(y(t_j)-y_*(t_j))\rangle$$

$$+\frac{1}{2}\int_0^T\langle u(t)-u_*(t),R(u(t)-u_*(t))\rangle dt+\Delta, \tag{2.9}$$

where

$$\Delta=\sum_{j=1}^{N+1}\langle F_j(y_*(t_j)-y_j),y(t_j)-y_*(t_j)\rangle+\int_0^T\langle Ru_*(t),u(t)-u_*(t)\rangle dt.$$

Now we transform the expression for Δ using equalities (2.2) - (2.8) and the formula of integration by parts

$$\Delta = -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \sum_{j=1}^{N}\langle -E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle$$
$$+ \int_0^T \langle B^*\psi(t), u(t) - u_*(t)\rangle dt$$
$$= -\langle E^*\psi(T), x(T) - x_*(T)\rangle - \sum_{j=1}^{N}\langle E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle$$
$$- \int_0^T \langle \psi(t), Ax(t) - Ax_*(t)\rangle dt + \int_0^T \langle \psi(t), \frac{d(Ex(t))}{dt} - \frac{d(Ex_*(t))}{dt}\rangle dt$$
$$= -\sum_{j=1}^{N}\langle E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle$$
$$+ \sum_{j=1}^{N}\langle \psi(t_j - 0) - \psi(t_j + 0), E(x(t_j) - x_*(t_j))\rangle = 0.$$

As F_j, R are non-negative definite, then the optimality $u_*(\cdot)$ follows from (2.9).

□

2.3. Necessary control optimality conditions

Now we give the conditions ensuring the unique solvability of system (2.6) - (2.8) for the adjoint variable $\psi(\cdot)$ when $y_*(\cdot)$ is fixed.

Furthermore, we suppose that the next conditions I, II are satisfied.

I. $Im\, C^*F_j \subseteq Im\, E^*, \quad j = 1, ..., N+1.$

II. The operator $QAP : \ker E \to \ker E^*$ *has the inverse operator* $(QAP)^{-1} : \ker E^* \to \ker E$.

If $E = I$ then condition I is fulfilled. Condition II means that differential algebraic equation (2.1) has the tractability index 1. (The corresponding definition see, for example, in [4].)

Lemma 2.1 *Under conditions I,II, system (2.6)-(2.8) with fixed* $y_*(\cdot)$ *is uniquely solvable.*

Proof. Taking into account conditions I, II, we obtain from (2.6)-(2.8) the equalities

$$Q\psi(t) = -(PA^*Q)^{-1}PA^*(I-Q)\psi(t),$$
$$\frac{d(I-Q)\psi(t)}{dt} = -E_+^*(I-P)A^*(I - (PA^*Q)^{-1}PA^*)(I-Q)\psi(t),$$
$$(I-Q)\psi(T) = -E_+^*(I-P)C^*F_{N+1}(y_*(T) - y_{N+1}),$$
$$(I-Q)\psi(t_j - 0) - (I-Q)\psi(t_j + 0) = -E_+^*(I-P)C^*F_j(y_*(t_j) - y_j),\ j = 1, ..., N.$$

Considering the last system sequentially on the segments $[t_N, T], [t_{N-1}, t_N], \ldots, [0, t_1]$, when $y_*(t)$ is fixed, it is not difficult to see that this system has the unique solution. □

Now we show that the following theorem is valid.

Theorem 2.2 (necessary control optimality condition). *Under conditions I, II, an optimal control* $u_*(\cdot)$ *for problem (2.1)-(2.4) is given by formula (2.5), where* $\psi(\cdot)$ *is a solution of system (2.6)-(2.8),* $y_*(t) = Cx_*(t)$, $x_*(\cdot)$ *is a corresponding optimal trajectory.*

Proof. Let $u(\cdot)$ be an arbitrary admissible control and $x(\cdot)$ be a corresponding trajectory, $y(t) = Cx(t)$. We will find the expression for the difference $J(u, y) - J(u_*, y_*)$. Using relations (2.9), (2.4), (2.8), (2.7), (2.3), (2.6), (2.2), we have the following sequence of the equalities

$$
\begin{aligned}
&J(u,y) - J(u_*,y_*)\\
&= - \langle E^*\psi(T), x(T) - x_*(T)\rangle - \sum_{j=1}^{N}\langle E^*(\psi(t_j-0) - \psi(t_j+0)), x(t_j) - x_*(t_j)\rangle\\
&\quad + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt + \frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j) - y_*(t_j), F_j(y(t_j) - y_*(t_j))\rangle\\
&\quad + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt
\end{aligned}
$$

$$
\begin{aligned}
&= - \int_{t_N}^{T} \frac{d}{dt}\langle E^*\psi(t), x(t) - x_*(t)\rangle dt - \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\frac{d}{dt}\langle E^*\psi(t), x(t) - x_*(t)\rangle dt\\
&\quad - \int_0^{t_1}\frac{d}{dt}\langle E^*\psi(t), x(t) - x_*(t)\rangle dt - \sum_{j=1}^{N}\langle E^*\psi(t_j+0) - E^*\psi(t_j-0), x(t_j) - x_*(t_j)\rangle\\
&\quad - \sum_{j=1}^{N}\langle E^*(\psi(t_j-0) - \psi(t_j+0)), x(t_j) - x_*(t_j)\rangle + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt\\
&\quad + \frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j) - y_*(t_j), F_j(y(t_j) - y_*(t_j))\rangle + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt
\end{aligned}
$$

$$
\begin{aligned}
&= - \int_0^T \langle -A^*\psi(t), x(t) - x_*(t)\rangle dt - \int_0^T \langle \psi(t), Ax(t) + Bu(t) - Ax_*(t) - Bu_*(t)\rangle dt\\
&\quad + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt + \frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j) - y_*(t_j), F_j(y(t_j) - y_*(t_j))\rangle\\
&\quad + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^T \langle Ru_*(t) - B^*\psi(t), u(t) - u_*(t)\rangle dt + \frac{1}{2}\sum_{j=1}^{N+1}\langle y(t_j) - y_*(t_j), F_j(y(t_j) - y_*(t_j))\rangle\\
&\quad + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt.
\end{aligned}
$$

Using standard arguments, we obtain from here that $Ru_*(t) - B^*\psi(t) = 0$. The theorem is proved. □

2.4. Solvability of problem following from control optimality condition

Let us introduce the notation $S = BR^{-1}B^*$.

Theorem 2.3 *If conditions I, II are satisfied, then the system*

$$
\frac{d(Ex(t))}{dt} = Ax(t) + S\psi(t), \tag{2.10}
$$

$$Ex(0) = x^0, \tag{2.3}$$
$$E^* \frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \tag{2.6}$$
$$E^*\psi(T) = -C^*F_{N+1}(Cx(T) - y_{N+1}), \tag{2.11}$$
$$E^*(\psi(t_j - 0) - \psi(t_j + 0)) = -C^*F_j(Cx(t_j) - y_j), \quad j = 1, ..., N, \tag{2.12}$$

has the unique solution.

Proof. At first, we establish the uniqueness of the solution, i.e. we prove the absence of a non-trivial solution for system (2.10), (2.3), (2.6), (2.11), (2.12), when $x^0 = 0, \quad y_j = 0, \quad j = 1, ..., N + 1$.

So, we assume now that $x^0 = 0, \quad y_j = 0, \quad j = 1, ..., N + 1$. We multiply equation (2.10) scalarly by $\psi(t)$ and we multiply equation (2.6) scalarly by $x(t)$. After adding the obtained results we have

$$\frac{d}{dt}\langle Ex(t), \psi(t)\rangle = \langle S\psi(t), \psi(t)\rangle, \quad t \neq t_j.$$

Integrating this equality consecutively with respect to the segments $[t_j, t_{j+1}]$, $j = 0, ..., N$, and adding the obtained results, we have

$$\sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \frac{d}{dt}\langle Ex(t), \psi(t)\rangle dt = \int_0^T \langle S\psi(t), \psi(t)\rangle dt.$$

From here, using equalities (2.11), (2.12), we obtain

$$\sum_{j=0}^{N}\langle Ex(t), \psi(t)\rangle\Big|_{t_j+0}^{t_{j+1}-0} = \sum_{j=0}^{N}(\langle x(t_{j+1}), E^*\psi(t_{j+1} - 0)\rangle - \langle x(t_j), E^*\psi(t_j + 0)\rangle)$$
$$= \sum_{j=0}^{N}\langle x(t_{j+1}), E^*\psi(t_{j+1} - 0)\rangle - \sum_{j=0}^{N}\langle x(t_j), E^*\psi(t_j + 0)\rangle$$
$$= \sum_{j=1}^{N+1}\langle x(t_j), E^*\psi(t_j - 0)\rangle - \sum_{j=0}^{N}\langle x(t_j), E^*\psi(t_j + 0)\rangle$$
$$= \langle x(T), E^*\psi(T)\rangle + \sum_{j=1}^{N}\langle x(t_j), E^*(\psi(t_j - 0) - \psi(t_j + 0))\rangle$$
$$= -\sum_{j=1}^{N+1}\langle x(t_j), C^*F_jCx(t_j)\rangle = \int_0^T \langle S\psi(t), \psi(t)\rangle dt.$$

As F_j, S are non-negative definite, from the last relation the next equality follows

$$S\psi(t) \equiv 0.$$

Then we obtain from (2.10), (2.3) the system

$$\frac{d(Ex(t))}{dt} = Ax(t), \quad Ex(0) = 0,$$

which has, in view of condition II, the unique solution

$$x(t) \equiv 0.$$

Now, from (2.6), (2.11), (2.12), we obtain the system

$$E^* \frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j,$$

$$E^*\psi(T) = 0, \quad E^*(\psi(t_j - 0) - \psi(t_j + 0)) = 0, \quad j = 1, ..., N,$$

which has, in view of condition II, the unique solution

$$\psi(t) \equiv 0.$$

The uniqueness of the system (2.10), (2.3), (2.6), (2.11), (2.12) solution is proved.

In view of condition II, system (2.10), (2.3), (2.6), (2.11), (2.12) is reduced to the linear system resolved with respect to the derivative. Namely, we obtain from this system the relations

$$\frac{d(I-P)x}{dt} = E_+(I-Q)APx + E_+(I-Q)A(I-P)x$$
$$+E_+(I-Q)SQ\psi + E_+(I-Q)S(I-Q)\psi, \tag{2.13}$$
$$0 = QAPx + QA(I-P)x + QSQ\psi + QS(I-Q)\psi, \tag{2.14}$$
$$(I-P)x(0) = E_+(I-Q)x^0, \tag{2.15}$$
$$(I-P)E^*\frac{d(I-Q)\psi}{dt} = -(I-P)A^*Q\psi - (I-P)A^*(I-Q)\psi, \quad t \neq t_j, \tag{2.16}$$
$$0 = PA^*Q\psi + PA^*(I-Q)\psi, \tag{2.17}$$
$$(I-P)E^*(I-Q)\psi(T) = -(I-P)C^*F_{N+1}(C(I-P)x(T) - y_{N+1}), \tag{2.18}$$
$$\begin{aligned}&(I-P)E^*((I-Q)\psi(t_j-0) - (I-Q)\psi(t_j+0))\\ &= -(I-P)C^*F_j(C(I-P)x(t_j) - y_j), \quad j = 1, ..., N.\end{aligned} \tag{2.19}$$

Here we have used the relations $F_jCP = 0, \quad j = 1, ..., N+1$, following from condition I. For brevity, the argument t is omitted in (2.13), (2.14), (2.16), (2.17). In view of condition II, we can express $Q\psi$ by $(I-Q)\psi$ from (2.17) and Px by $(I-P)x, Q\psi, (I-Q)\psi$ from (2.14). Substituting the expressions for $Q\psi, Px$ into (2.13), (2.16) we obtain from (2.13)-(2.19) for $v = (I-P)x, w = (I-P)E^*(I-Q)\psi$ in the space $\mathrm{Im}A^* \times \mathrm{Im}A^*$ the system of the form

$$\frac{dv}{dt} = \tilde{A}v + \tilde{S}w,$$
$$\frac{dw}{dt} = -\tilde{A}^*w, \quad t \neq t_j,$$
$$v(0) = v^0, \quad w(T) = -\tilde{F}v(T) - \tilde{y}_{N+1},$$
$$w(t_j - 0) - w(t_j + 0) = -\tilde{F}_jv(t_j) - \tilde{y}_j, \quad j = 1, ..., N.$$

The last system has the unique solution as the corresponding homogeneous system has the trivial solution only (We have shown it earlier). The theorem is proved. □

2.5. Necessary control optimality condition is not valid in general case

In this section we will show by the example that the necessary control optimality condition from Theorem 2.2 is not valid for general descriptor systems. We consider the initial value problem

$$\begin{aligned}\frac{dx_1}{dt} &= x_3 + u, \quad x_1(0) = x_1^0,\\ \frac{dx_2}{dt} &= x_2 - x_3 + u, \quad x_2(0) = 0,\\ 0 &= -x_2\end{aligned} \tag{2.20}$$

for $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3, \quad t \in [0,1], \quad x_1^0 \in \mathbb{R}$.

The descriptor system (2.20) is regular with index 2. For every piecewise continuous function $u(\cdot)$ we have an unique solution

$$\begin{gathered} x_1(t) = x_1^0 + 2\int_0^t u(s)ds, \\ x_2(t) = 0, \\ x_3(t) = u(t), \quad t \in [0,1]. \end{gathered} \tag{2.21}$$

It should be noted that the example, showing that the sufficient optimality condition is not necessary, has been given in [8] for the linear-quadratic problem with a standard performance index. In contrast to [8], we consider the minimized performance index of the form

$$J(u,x) = J(u) = \frac{1}{2}x_1^2(\frac{1}{2}) + \frac{1}{2}x_1^2(1) + \frac{1}{2}\int_0^1 u^2(t)dt \tag{2.22}$$

which has the special property to be also a cost function for the controlled explicit ordinary differential equation

$$\frac{dx_1}{dt} = 2u, \quad x_1(0) = x_1^0. \tag{2.23}$$

Consider $\mu(\cdot) \in \mathrm{PC}([0,1],\mathbb{R})$ to be the adjoint variable for the problem (2.22), (2.23).We write the problem for the adjoint variable from Theorem 2.1

$$\begin{gathered} \frac{d\mu(t)}{dt} = 0, \quad t \neq \frac{1}{2}, \\ \mu(1) = -x_{1*}(1), \\ \mu\left(\frac{1}{2} - 0\right) - \mu\left(\frac{1}{2} + 0\right) = -x_{1*}\left(\frac{1}{2}\right). \end{gathered} \tag{2.24}$$

Here we have $N = 1, T = 1, E = 1, A = 0, B = 2, C = 1, F_1 = F_2 = 1, R = 1, t_1 = \frac{1}{2}, \quad y_1 = y_2 = 0$, $x_{1*}(t)$ is a trajectory corresponding to the control

$$u_*(t) = 2\mu(t).$$

From system (2.24) we obtain the relations

$$\begin{gathered} \mu(t) = \begin{cases} c_1, & 0 \le t < \frac{1}{2}, \\ c_2, & \frac{1}{2} \le t \le 1, \end{cases} \\ c_2 = -x_{1*}(1), \\ c_1 - c_2 = -x_{1*}\left(\frac{1}{2}\right), \end{gathered} \tag{2.25}$$

where c_1 and c_2 are constants.

Then the trajectory $x_{1*}(\cdot)$ is a continuous solution of the system

$$\begin{gathered} \frac{dx_{1*}}{dt} = \begin{cases} 4c_1, & 0 \le t < \frac{1}{2}, \\ 4c_2, & \frac{1}{2} < t \le 1, \end{cases} \\ x_{1*}(0) = x_1^0. \end{gathered}$$

As the function x_{1*} must be continuous, from the last system we have

$$x_{1*}(t) = \begin{cases} x_1^0 + 4c_1 t, & 0 \le t \le \frac{1}{2}, \\ x_1^0 + 2(c_1 - c_2) + 4c_2 t, & \frac{1}{2} \le t \le 1. \end{cases}$$

From here, taking into account (2.25), we obtain the system with respect to c_1, c_2

$$\begin{aligned} c_2 &= -x_1^0 - 2(c_1 - c_2) - 4c_2, \\ c_1 - c_2 &= -x_1^0 - 2c_1. \end{aligned}$$

Hence

$$c_1 = -\frac{4}{11}x_1^0, \quad c_2 = -\frac{1}{11}x_1^0.$$

So, the control problem (2.22), (2.23) has the solution

$$u_*(t) = \begin{cases} -\dfrac{8}{11}x_1^0, & 0 \le t < \dfrac{1}{2}, \\ \\ -\dfrac{2}{11}x_1^0, & \dfrac{1}{2} \le t \le 1, \end{cases}$$

$$x_{1*}(t) = \begin{cases} (1 - \dfrac{16}{11}t)x_1^0, & 0 \le t \le \dfrac{1}{2}, \\ \\ \dfrac{1}{11}(5 - 4t)x_1^0, & \dfrac{1}{2} \le t \le 1, \end{cases} \tag{2.26}$$

$$J(u_*) = \frac{2}{11}(x_1^0)^2.$$

It is obvious that the control function u_* is also an optimal control for the optimization problem (2.20), (2.22). The corresponding optimal trajectory can be found from (2.21).

Now we write the problem (2.6)-(2.8) for the adjoint variable $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))^T \in \mathbb{R}^3$ corresponding to the problem (2.20), (2.22)

$$\begin{aligned} \frac{d\psi_1}{dt} &= 0, \\ \frac{d\psi_2}{dt} &= -\psi_2 + \psi_3, \\ 0 &= -\psi_1 + \psi_2, \quad t \neq \frac{1}{2}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \psi_1\left(\frac{1}{2} - 0\right) - \psi_1\left(\frac{1}{2} + 0\right) &= -x_{1*}\left(\frac{1}{2}\right), \\ \psi_2\left(\frac{1}{2} - 0\right) - \psi_2\left(\frac{1}{2} + 0\right) &= 0, \end{aligned}$$

$$\psi_1(1) = -x_{1*}(1), \tag{2.28}$$

$$\psi_2(1) = 0. \tag{2.29}$$

Here $N = 1, \quad T = 1, \quad E = \mathrm{diag}(1, 1, 0), \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix},$

$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad C = I, \quad F_1 = F_2 = \mathrm{diag}(1, 0, 0), \quad t_1 = \dfrac{1}{2}, \quad y_1 = y_2 = (0, 0, 0)^T.$

It is not difficult to see that the equality (2.27), when $t = 1$, contradicts in the general case to the relations (2.28), (2.29), (2.26), as if $x_1^0 \neq 0$, then $\psi_1(1) = -\dfrac{1}{11}x_1^0 \neq 0$.

Thus the optimal control problem (2.20), (2.22) has a solution but for $x_1^0 \neq 0$ the problem for the adjoint variable from Theorem 2.1 is not solvable, i.e. in this example for $x_1^0 \neq 0$ the necessary control optimality condition from Theorem 2.2 is not valid.

It should be noted that for the problem (2.20), (2.22) the condition II from Theorem 2.2 (necessary control optimality condition) is not valid. Indeed , we have for this problem $\mathrm{Ker}E = \mathrm{Ker}E^* = \left\{ \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, a \in \mathbb{R} \right\}$, $P = Q = \mathrm{diag}(0,0,1)$. Therefore, $QAP = 0$.

Remark 2.1. Some results from this section, when $E = I$, are announced in [9].

3. Problem with free $Ex(0)$

3.1. Problem statement

Let us consider the problem of minimizing the functional

$$J(u,y) = \frac{1}{2}\sum_{j=0}^{N+1}\langle y(t_j) - y_j, F_j(y(t_j) - y_j)\rangle + \frac{1}{2}\int_0^T \langle u(t), Ru(t)\rangle dt \tag{3.1}$$

with respect to trajectories of the system

$$\frac{d(Ex(t))}{dt} = Ax(t) + Bu(t), \tag{3.2}$$

$$y(t) = Cx(t). \tag{3.3}$$

3.2. Sufficient control optimality conditions

Theorem 3.1 (sufficient control optimality condition). *The control $u_*(\cdot)$, given by the formula*

$$u_*(t) = R^{-1}B^*\psi(t), \tag{3.4}$$

where $\psi(\cdot)$ is a solution of the problem

$$E^*\frac{d\psi(t)}{dt} = -A^*\psi(t),\ t \neq t_j, \tag{3.5}$$

$$E^*\psi(0) = C^*F_0(y_*(0) - y_0), \tag{3.6}$$

$$E^*(\psi(t_j - 0) - \psi(t_j + 0)) = -C^*F_j(y_*(t_j) - y_j), \quad j = 1, ..., N, \tag{3.7}$$

$$E^*\psi(T) = -C^*F_{N+1}(y_*(T) - y_{N+1}), \tag{3.8}$$

$y_*(t) = Cx_*(t)$, $x_*(\cdot)$ is a trajectory of system (3.2) corresponding to the control $u_*(\cdot)$, is an optimal control for problem (3.1) - (3.3).

Proof. Let $u(\cdot)$ be an arbitrary admissible control and $x(\cdot)$ be a corresponding trajectory, $y(t) = Cx(t)$. Taking into account relation (3.1), we have

$$\begin{aligned} J(u,y) - J(u_*,y_*) = &\frac{1}{2}\sum_{j=0}^{N+1}\langle y(t_j) - y_*(t_j), F_j(y(t_j) - y_*(t_j))\rangle \\ &+\frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt + \Delta, \end{aligned} \tag{3.9}$$

where

$$\Delta = \sum_{j=0}^{N+1} \langle F_j(y_*(t_j) - y_j), y(t_j) - y_*(t_j)\rangle + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt.$$

Now we transform the expression for Δ using equalities (3.2) - (3.8) and the formula of integration by parts

$$\begin{aligned}
\Delta =& \langle E^*\psi(0), x(0) - x_*(0)\rangle + \sum_{j=1}^{N}\langle -E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle \\
&- \langle E^*\psi(T), x(T) - x_*(T)\rangle + \int_0^T \langle B^*\psi(t), u(t) - u_*(t)\rangle dt \\
=& \langle E^*\psi(0), x(0) - x_*(0)\rangle - \sum_{j=1}^{N}\langle E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle \\
&- \langle E^*\psi(T), x(T) - x_*(T)\rangle -_0^T \langle \psi(t), Ax(t) - Ax_*(t)\rangle dt \\
&+ \int_0^T \langle \psi(t), \frac{d(Ex(t))}{dt} - \frac{d(Ex_*(t))}{dt}\rangle dt \\
=& \langle E^*\psi(0), x(0) - x_*(0)\rangle - \sum_{j=1}^{N}\langle E^*(\psi(t_j - 0) - \psi(t_j + 0)), x(t_j) - x_*(t_j)\rangle \\
&- \langle E^*\psi(T), x(T) - x_*(T)\rangle - \int_0^T \langle \psi(t), Ax(t) - Ax_*(t)\rangle dt \\
&+ \sum_{j=0}^{N}\langle \psi(t), Ex(t) - Ex_*(t)\rangle\Big|_{t_j+0}^{t_{j+1}-0} - \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \langle \frac{d\psi(t)}{dt}, Ex(t) - Ex_*(t)\rangle dt \\
=& -\int_0^T \langle \psi(t), A(x(t) - x_*(t))\rangle dt + \int_0^T \langle A^*\psi(t), x(t) - x_*(t)\rangle dt = 0.
\end{aligned}$$

As F_j, R are non-negative definite, then the optimality $u_*(\cdot)$ follows from (3.9). □

3.3. Necessary control optimality conditions

Theorem 3.2 (necessary control optimality condition). *Let $u_*(\cdot)$ be an optimal control for problem (3.1)-(3.3), condition II is satisfied, system (3.5)-(3.8), where $y_*(t) = Cx_*(t)$, $x_*(\cdot)$ is an optimal trajectory, has a solution $\psi(\cdot)$. Then $u_*(\cdot)$ is defined by formula (3.4).*

Proof. Let $u(\cdot)$ be an arbitrary control and $x(\cdot)$ be a corresponding trajectory, $y(t) = Cx(t)$. It should be noted that if condition II is valid then any piecewise continuons on [0,T] function with values in U is an admissible control.

Taking into account equations (3.5), (3.2), we obtain the equalities

$$\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}}\frac{d}{dt}\langle\psi(t),Ex(t)\rangle dt=\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}}(\langle\frac{d\psi(t)}{dt},Ex(t)\rangle+\langle\psi(t),\frac{dEx(t)}{dt}\rangle)dt$$
$$=\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}}\langle\psi(t),Bu(t)\rangle dt=\int_0^T\langle\psi(t),Bu(t)\rangle dt$$
$$=\sum_{j=0}^{N}(\langle\psi(t_{j+1}-0),Ex(t_{j+1})\rangle-\langle\psi(t_j+0),Ex(t_j)\rangle)$$
$$=\sum_{j=1}^{N+1}\langle E^*\psi(t_j-0),x(t_j)\rangle-\sum_{j=0}^{N}\langle E^*\psi(t_j+0),x(t_j)\rangle$$
$$=\langle E^*\psi(T),x(T)\rangle+\sum_{j=1}^{N}\langle E^*(\psi(t_j-0)-\psi(t_j+0)),x(t_j)\rangle-\langle E^*\psi(0),x(0)\rangle.$$

So we have the identity

$$-\int_0^T\langle B^*\psi(t),u(t)\rangle dt+\langle E^*\psi(T),x(T)\rangle+\sum_{j=1}^{N}\langle E^*(\psi(t_j-0)-\psi(t_j+0)),x(t_j)\rangle$$
$$-\langle E^*\psi(0),x(0)\rangle=0. \tag{3.10}$$

If we use this identity for the pair $(u_*(\cdot),x_*(\cdot))$ and subtract the obtained identity from (3.10) we will have the following identity

$$-\int_0^T\langle B^*\psi(t),u(t)-u_*(t)\rangle dt+\langle E^*\psi(T),x(T)-x_*(T)\rangle$$

$$+\sum_{j=1}^{N}\langle E^*(\psi(t_j-0)-\psi(t_j+0)),x(t_j)-x_*(t_j)\rangle-\langle E^*\psi(0),x(0)-x_*(0)\rangle=0.$$

In view of the last identity and (3.3), (3.6) - (3.8), we obtain from (3.9) the following relation

$$J(u,y)-J(u_*,y_*)=\int_0^T\langle Ru_*(t)-B^*\psi(t),u(t)-u_*(t)\rangle dt$$

$$+\frac{1}{2}\sum_{j=0}^{N+1}\langle y(t_j)-y_*(t_j),F_j(y(t_j)-y_*(t_j))\rangle+\frac{1}{2}\int_0^T\langle u(t)-u_*(t),R(u(t)-u_*(t))\rangle dt.$$

As $u_*(\cdot)$ is an optimal control then the last expression must be non-negative for every control $u(\cdot)$ and corresponding values $x(t_j), j=0,...,N+1$. We take $Ex(0)=Ex_*(0)$. Then for sufficiently small values $u(t)-u_*(t)$ we must have

$$\int_0^T\langle Ru_*(t)-B^*\psi(t),u(t)-u_*(t)\rangle dt\geq 0.$$

From here the relation (3.4) follows.

The theorem is proved. □

3.4. Solvability of problem following from control optimality condition

Let us use the previous notation $S = BR^{-1}B^*$ and assume that the following conditions I', III are fulfilled in addition to condition II.

I'. $Im\, C^*F_j \subseteq Im\, E^*, \quad j = 0, ..., N+1.$

III. The operator $F_0C(I-P) : Im\, E^* \to Y$ *is injective.*

Theorem 3.3 *If conditions I', II, III are satisfied, then the system*

$$\frac{d(Ex(t))}{dt} = Ax(t) + S\psi(t), \tag{3.11}$$

$$E^*\frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \tag{3.12}$$

$$E^*\psi(0) = C^*F_0(Cx(0) - y_0), \tag{3.13}$$

$$E^*(\psi(t_j - 0) - \psi(t_j + 0)) = -C^*F_j(Cx(t_j) - y_j), \quad j = 1, ..., N, \tag{3.14}$$

$$E^*\psi(T) = -C^*F_{N+1}(Cx(T) - y_{N+1}) \tag{3.15}$$

has an unique solution.

Proof. We use the scheme of the Theorem 2.3 proof. At first, we establish the uniqueness of the solution, i.e. we prove the absence of a non-trivial solution for the last system when $y_j = 0, \quad j = 0, ..., N+1$.

So, we assume now that $y_j = 0, \quad j = 0, ..., N+1$. We multiply equation (3.11) scalarly by $\psi(t)$ and we multiply equation (3.12) scalarly by $x(t)$. After adding the obtained results we have

$$\frac{d}{dt}\langle Ex(t), \psi(t)\rangle = \langle S\psi(t), \psi(t)\rangle, \quad t \neq t_j.$$

Integrating this equality consecutively with respect to the segments $[t_j, t_{j+1}], \quad j = 0, ..., N$, and adding the obtained results, we have

$$\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \frac{d}{dt}\langle Ex(t), \psi(t)\rangle dt = \int_0^T \langle S\psi(t), \psi(t)\rangle dt.$$

From here, using equalities (3.13) - (3.15), we obtain

$$\sum_{j=0}^{N}\langle Ex(t), \psi(t)\rangle\Big|_{t_j+0}^{t_{j+1}-0} = \sum_{j=1}^{N+1}(\langle x(t_j), E^*\psi(t_j - 0)\rangle - \sum_{j=0}^{N}\langle x(t_j), E^*\psi(t_j + 0)\rangle$$

$$= \langle x(T), E^*\psi(T)\rangle + \sum_{j=1}^{N}\langle x(t_j), E^*(\psi(t_j - 0) - \psi(t_j + 0))\rangle - \langle x(0), E^*\psi(0)\rangle$$

$$= -\sum_{j=0}^{N+1}\langle x(t_j), C^*F_jCx(t_j)\rangle = \int_0^T \langle S\psi(t), \psi(t)\rangle dt.$$

As F_j, S are non-negative definite, from the last relation the next equalities follow

$$S\psi(t) \equiv 0, \quad F_jCx(t_j) = 0, \quad j = 0, ..., N+1.$$

Then we have for $x(t)$ the problem

$$\frac{d(Ex(t))}{dt} = Ax(t), \quad F_jCx(t_j) = 0, \quad j = 0, ..., N+1.$$

In view of condition I', the following equality is valid $F_jCP = 0$. Then, using the condition III, we obtain $(I-P)x(0) = 0$, hence, taking into account condition II, we have the unique solution

$$x(t) \equiv 0.$$

Now, from (3.12) - (3.15), we obtain the system

$$E^* \frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \quad j = 1, ..., N,$$

$$E^*\psi(0) = 0, \quad E^*(\psi(t_j - 0) - \psi(t_j + 0)) = 0, \quad E^*\psi(T) = 0,$$

which has, in view of condition II, the unique solution

$$\psi(t) \equiv 0.$$

The uniqueness of the system (3.11) - (3.15) solution is proved.

In view of conditions I', II, system (3.11) - (3.15) is reduced to the linear system resolved with respect to the derivative. Namely, we obtain from this system the relations

$$\frac{d(I-P)x}{dt} = E_+(I-Q)APx + E_+(I-Q)A(I-P)x$$

$$+E_+(I-Q)SQ\psi + E_+(I-Q)S(I-Q)\psi,$$

$$0 = QAPx + QA(I-P)x + QSQ\psi + QS(I-Q)\psi,$$

$$(I-P)E^*\frac{d(I-Q)\psi}{dt} = -(I-P)A^*Q\psi - (I-P)A^*(I-Q)\psi, \quad t \neq t_j,$$

$$0 = PA^*Q\psi + PA^*(I-Q)\psi,$$

$$(I-P)E^*(I-Q)\psi(0) = (I-P)C^*F_0(C(I-P)x(0) - y_0),$$

$$(I-P)E^*((I-Q)\psi(t_j-0) - (I-Q)\psi(t_j+0)) = -(I-P)C^*F_j(C(I-P)x(t_j) - y_j),$$

$$j = 1, ..., N,$$

$$(I-P)E^*(I-Q)\psi(T) = -(I-P)C^*F_{N+1}(C(I-P)x(T) - y_{N+1}).$$

Here we have used the relations $F_jCP = 0, \quad j = 0, ..., N+1$, following from condition I'. For brevity, the argument t is omitted.

In view of condition II, we can express $Q\psi$ by $(I-Q)\psi$ from the fourth equation of the last system and Px by $(I-P)x, (I-Q)\psi$ using the second equation of the last system. Substituting the expressions for $Q\psi, Px$ into the first and the third equations of the last system we obtain for $v = (I-P)x, \quad w = (I-P)E^*(I-Q)\psi$ in the space $\mathrm{Im}A^* \times \mathrm{Im}A^*$ the system of the form

$$\frac{dv}{dt} = \tilde{A}v + \tilde{S}w,$$

$$\frac{dw}{dt} = -\tilde{A}^*w, \quad t \neq t_j,$$

$$w(0) = \tilde{F}_0 v(0) - \tilde{y}_0,$$

$$w(t_j - 0) - w(t_j + 0) = -\tilde{F}_j v(t_j) - \tilde{y}_j, \quad j = 1, ..., N,$$

$$w(T) = -\tilde{F}_{N+1}v(T) - \tilde{y}_{N+1}.$$

The last system has the unique solution as the corresponding homogeneous system has the trivial solution only (We have shown it earlier). The theorem is proved. $\square$

3.5. Illustrative examples

At first, we consider two examples with state equation resolved with respect to the derivative. The first simplest example has the obvious solution, which we will find using Theorem 3.1.

Example 3.1. Let the state equation be

$$\frac{dx}{dt} = 2u,$$

and the performance index be

$$J = \frac{1}{2}(x^2(0) + x^2(\frac{1}{2}) + x^2(1)) + \frac{1}{2}\int_0^1 u^2(t)dt.$$

The system (3.5) - (3.8) in this case has the form

$$\frac{d\psi}{dt} = 0, \quad t \neq \frac{1}{2},$$
$$\psi(0) = x_*(0),$$
$$\psi\left(\frac{1}{2} - 0\right) - \psi\left(\frac{1}{2} + 0\right) = -x_*\left(\frac{1}{2}\right),$$
$$\psi(1) = -x_*(1).$$

Here we have $N = 1$, $T = 1$, $E = 1, A = 0, B = 2, C = 1, F_0 = F_1 = F_2 = 1$, $R = 1$, $t_1 = \frac{1}{2}$, $y_0 = y_1 = y_2 = 0$, $x_*(\cdot)$ is a trajectory corresponding to the control

$$u_*(\cdot) = 2\psi(\cdot).$$

From the system for adjoint variable we obtain the relations

$$\psi(t) = \begin{cases} c_1, & 0 \leq t < \frac{1}{2}, \\ c_2, & \frac{1}{2} \leq t \leq 1, \end{cases}$$
$$c_1 = x_*(0),$$
$$c_1 - c_2 = -x_*\left(\frac{1}{2}\right),$$
$$c_2 = -x_*(1),$$

where c_1 and c_2 are constants.

Then the trajectory x_* is a continuous solution of the system

$$\frac{dx_*}{dt} = \begin{cases} 4c_1, & 0 \leq t < \frac{1}{2}, \\ 4c_2, & \frac{1}{2} < t \leq 1. \end{cases}$$

As the function x_* must be continuous then from the last system we have

$$x_*(t) = \begin{cases} c_1 + 4c_1 t, & 0 \leq t \leq \frac{1}{2}, \\ 3c_1 - 2c_2 + 4c_2 t, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

From here, in view of the relations for c_1, c_2, it follows

$$c_1 - c_2 = -c_1 - 2c_1,$$
$$c_2 = -3c_1 + 2c_2 - 4c_2.$$

Hence $c_1 = c_2 = 0$, and $x_*(t) \equiv 0, u_*(t) \equiv 0, I(u_*) = 0$. We have obtained the result, which is evident, taking into account the form of the performance index and the state equation in this example.

Example 3.2. The state equation is

$$\frac{dx}{dt} = 2u,$$

the performance index is

$$J = \frac{1}{2}((x(0)-1)^2 + x^2(\frac{1}{2}) + x^2(1)) + \frac{1}{2}\int_0^1 u^2(t)dt.$$

System (3.5) - (3.8) in this case has the form

$$\frac{d\psi}{dt} = 0, \quad t \neq \frac{1}{2},$$

$$\psi(0) = x_*(0) - 1,$$

$$\psi\left(\frac{1}{2} - 0\right) - \psi\left(\frac{1}{2} + 0\right) = -x_*\left(\frac{1}{2}\right),$$

$$\psi(1) = -x_*(1).$$

Here we have $N = 1$, $T = 1$, $E = 1, A = 0, B = 2, C = 1, F_0 = F_1 = F_2 = 1$, $R = 1$, $t_1 = \frac{1}{2}$, $y_0 = 1, y_1 = y_2 = 0$, $x_*(\cdot)$ is a trajectory corresponding to the control

$$u_*(\cdot) = 2\psi(\cdot).$$

From the system for adjoint variable we obtain the relations

$$\psi(t) = \begin{cases} c_1, & 0 \le t < \frac{1}{2}, \\ c_2, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$c_1 = x_*(0) - 1,$$

$$c_1 - c_2 = -x_*\left(\frac{1}{2}\right),$$

$$c_2 = -x_*(1),$$

where c_1, c_2 are constants.

Then the trajectory $x_*(\cdot)$ is a continuous solution of the system

$$\frac{dx_*}{dt} = \begin{cases} 4c_1, & 0 \le t < \frac{1}{2}, \\ 4c_2, & \frac{1}{2} < t \le 1. \end{cases}$$

As the function $x_*(\cdot)$ must be continuous then from the last system we have

$$x_*(t) = \begin{cases} c_3 + 4c_1 t, & 0 \le t \le \frac{1}{2}, \\ c_3 + 2(c_1 - c_2) + 4c_2 t, & \frac{1}{2} \le t \le 1. \end{cases}$$

From here, in view of the relations for c_1, c_2, it follows

$$\begin{aligned} &c_1 = c_3 - 1, \\ &c_1 - c_2 = -c_3 - 2c_1, \\ &c_2 = -c_3 - 2(c_1 - c_2) - 4c_2. \end{aligned}$$

Hence $c_1 = -\dfrac{4}{15}, c_2 = -\dfrac{1}{15}, c_3 = \dfrac{11}{15}$, and

$$x_*(t) = \begin{cases} \dfrac{11}{15} - \dfrac{16}{15}t, & 0 \leq t \leq \frac{1}{2}, \\ \dfrac{1}{3} - \dfrac{4}{15}t, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$u_*(t) = \begin{cases} -\dfrac{8}{15}, & 0 \leq t < \frac{1}{2}, \\ -\dfrac{2}{15}, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$I(u_*) = \frac{2}{15}.$$

Example 3.3. Now we consider the problem of minimizing the functional

$$J = \frac{1}{2}((x_1(0) - 1)^2 + x_1^2(\frac{1}{2}) + x_1^2(1)) + \frac{1}{2}\int_0^1 u^2(t)dt$$

on trajectories of the regular descriptor system with index 2

$$\frac{dx_1}{dt} = x_3 + u,$$
$$\frac{dx_2}{dt} = x_2 - x_3 + u,$$
$$0 = -x_2$$

for $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3, t \in [0, 1]$.

It is not difficult to see that the solving this problem is reduced to the solving the problem from example 3.2.

Using the solution from example 3.2, we obtain the solution of the considered problem

$$x_{1*}(t) = \begin{cases} \dfrac{11}{15} - \dfrac{16}{15}t, & 0 \leq t \leq \frac{1}{2}, \\ \dfrac{1}{3} - \dfrac{4}{15}t, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$x_{2*}(t) = 0,$$

$$x_{3*}(t) = u_*(t) = \begin{cases} -\dfrac{8}{15}, & 0 \leq t < \frac{1}{2}, \\ -\dfrac{2}{15}, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$I(u_*) = \frac{2}{15}.$$

Let us write the problem (3.5)-(3.8) for the adjoint variable $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))^T$ from Theorem 3.1 corresponding to the considered problem. Taking into account the expression for $x_{1*}(t)$, we have

$$\frac{d\psi_1}{dt} = 0,$$
$$\frac{d\psi_2}{dt} = -\psi_2 + \psi_3,$$

$$0 = -\psi_1 + \psi_2, \quad t \neq \frac{1}{2}, \tag{3.16}$$

$$\psi_1(0) = x_{1*}(0) - 1 = -\frac{4}{15},$$

$$\psi_2(0) = 0,$$

$$\psi_1\left(\frac{1}{2} - 0\right) - \psi_1\left(\frac{1}{2} + 0\right) = -x_{1*}\left(\frac{1}{2}\right) = -\frac{3}{15},$$

$$\psi_2\left(\frac{1}{2} - 0\right) - \psi_2\left(\frac{1}{2} + 0\right) = 0,$$

$$\psi_1(1) = -x_{1*}(1) = -\frac{1}{15}, \tag{3.17}$$

$$\psi_2(1) = 0. \tag{3.18}$$

Here $N = 1$, $T = 1$, $E = \mathrm{diag}(1,1,0)$, $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $C = I$, $F_0 = F_1 = F_2 = \mathrm{diag}(1,0,0)$, $t_1 = \frac{1}{2}$, $y_0 = (1,0,0)^T$, $y_1 = y_2 = (0,0,0)^T$.

It is not difficult to see that the last system for $\psi(t)$ is not solvable as the equality (3.16), when $t = 1$, contradicts to the relations (3.17), (3.18).

Thus the considered optimal control problem has a solution but the problem for the adjoint variable from Theorem 3.1 is not solvable, i.e. in this example the necessary control optimality condition from Theorem 3.2 is not valid.

It should be noted that for this problem (also, as in section 2.5) the condition II from Theorem 3.2 (necessary control optimality condition) is not valid.

4. Problem with given intermediate points

4.1. Problem statement

Let us consider the problem of minimizing the functional

$$J(u,y) = \frac{1}{2}\langle y(T), Fy(T)\rangle + \frac{1}{2}\int_0^T \langle u(t), Ru(t)\rangle dt \tag{4.1}$$

with respect to trajectories of the system

$$\frac{d(Ex(t))}{dt} = Ax(t) + Bu(t), \tag{4.2}$$

$$Ex(0) = x^0, \tag{4.3}$$

$$y(t) = Cx(t), \tag{4.4}$$

$$y(t_j) = y_j, \quad j = 1, ..., N, \tag{4.5}$$

where the element $x^0 \in Im\ E$ is given; $F \in \mathcal{L}(Y)$ is a symmetric non-negative definite operator.

4.2. Sufficient control optimality conditions

Theorem 4.1 (sufficient control optimality condition). *The control $u_*(\cdot)$, given by the formula*

$$u_*(t) = R^{-1}B^*\psi(t), \tag{4.6}$$

where $\psi(\cdot)$ *is a solution of the problem*

$$E^* \frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \quad j = 1, ..., N, \tag{4.7}$$

$$E^*\psi(T) = -C^* F y_*(T), \tag{4.8}$$

such that

$$E^*(\psi(t_j - 0) - \psi(t_j + 0)) \in Im\, C^*, \tag{4.9}$$

$y_*(t) = Cx_*(t)$, $x_*(\cdot)$ *is a trajectory of system (4.2)-(4.5) corresponding to the control* $u_*(\cdot)$, *is an optimal control for problem (4.1) - (4.5).*

Proof. Let $u(\cdot)$ be an arbitrary admissible control and $x(\cdot)$ be a corresponding trajectory, $y(t) = Cx(t)$. Taking into account relation (4.1), we have

$$J(u, y) - J(u_*, y_*) = \frac{1}{2}\langle y(T) - y_*(T), F(y(T) - y_*(T))\rangle$$

$$+\frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt + \Delta, \tag{4.10}$$

where

$$\Delta = \langle F y_*(T), y(T) - y_*(T)\rangle + \int_0^T \langle R u_*(t), u(t) - u_*(t)\rangle dt.$$

Now we transform the expression for Δ using equalities (4.2) - (4.9) and the formula of integration by parts

$$\begin{aligned}
\Delta &= -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \int_0^T \langle B^*\psi(t), u(t) - u_*(t)\rangle dt \\
&= -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \int_0^T \langle \psi(t), \frac{d(Ex(t))}{dt} - \frac{d(Ex_*(t))}{dt} - Ax(t) + Ax_*(t)\rangle dt \\
&= -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \int_0^T \langle \psi(t), A(x_*(t) - x(t))\rangle dt \\
&\quad + \sum_{j=0}^{N} \langle \psi(t), E(x(t) - x_*(t))\rangle\big|_{t_j+0}^{t_{j+1}-0} - \int_0^T \langle E^* \frac{d\psi(t)}{dt}, x(t) - x_*(t)\rangle dt \\
&= \int_0^T \langle \psi(t), A(x_*(t) - x(t))\rangle dt + + \sum_{j=1}^{N} \langle \psi(t_j - 0) - \psi(t_j + 0), E(x(t_j) - x_*(t_j))\rangle \\
&\quad + \textstyle\int_0^T \langle A^*\psi(t), x(t) - x_*(t)\rangle dt = 0.
\end{aligned}$$

As F, R are non-negative definite, then the optimality $u_*(\cdot)$ follows from (4.10). □

Remark 4.1. If we will consider the analogous problem of minimizing the functional

$$J(u, y) = \frac{1}{2}\int_0^T \langle u(t), Ru(t)\rangle dt \tag{4.11}$$

with respect to trajectories of the system (4.2) under the given conditions

$$y(t_j) = Cx(t_j) = y_j, \quad j = 0, ..., N + 1, \tag{4.12}$$

then we have the following Theorem 4.2 the proof of which is similar to the Theorem 4.1 proof.

Theorem 4.2 (sufficient control optimality condition). *The control* $u_*(\cdot)$, *given by the formula (4.6), where* $\psi(\cdot)$ *is a solution of problem (4.7), (4.9) such that* $E^*\psi(0), E^*\psi(T) \in Im\, C^*$, $y_*(t) = Cx_*(t)$, $x_*(\cdot)$ *is a trajectory of system (4.2), (4.12) corresponding to the control* $u_*(\cdot)$, *is an optimal control for problem (4.11),* *(4.2), (4.12).*

4.3. Necessary control optimality conditions

Theorem 4.3 (necessary control optimality condition). *Let $u_*(\cdot)$ be an optimal control for problem (4.1)-(4.5), condition II is satisfied, system (4.7)-(4.9), where $y_*(t) = Cx_*(t)$, $x_*(\cdot)$ is an optimal trajectory, has a solution $\psi(\cdot)$ such that the control $R^{-1}B^*\psi(\cdot)$ is admissible. Then $u_*(\cdot)$ is defined by formula (4.6).*

Proof. Let $u(\cdot)$ be an arbitrary admissible control and $x(\cdot)$ be a corresponding trajectory, $y(t) = Cx(t)$. We will find the expression for the difference $J(u, y) - J(u_*, y_*)$. Using relations (4.10), (4.4), (4.8), (4.3), (4.7), (4.2), (4.9), we have the following sequence of the equalities

$$
\begin{aligned}
&J(u, y) - J(u_*, y_*) = -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt \\
&+\frac{1}{2}\langle y(T) - y_*(T), F(y(T) - y_*(T))\rangle + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt \\
&= -\langle E^*\psi(T), x(T) - x_*(T)\rangle + \sum_{j=0}^{N}\int_{t_j}^{t_{j+1}} \frac{d}{dt}\langle \psi(t), E(x(t) - x_*(t))\rangle dt \\
&-\int_0^T \frac{d}{dt}\langle \psi(t), E(x(t) - x_*(t))\rangle dt + \int_0^T \langle Ru_*(t), u(t) - u_*(t)\rangle dt \\
&+\frac{1}{2}\langle y(T) - y_*(T), F(y(T) - y_*(T))\rangle + \frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt \\
&= \int_0^T \langle Ru_*(t) - B^*\psi(t), u(t) - u_*(t)\rangle dt + \frac{1}{2}\langle y(T) - y_*(T), F(y(T) - y_*(T))\rangle \\
&+\frac{1}{2}\int_0^T \langle u(t) - u_*(t), R(u(t) - u_*(t))\rangle dt.
\end{aligned}
$$

As $u_*(\cdot)$ is an optimal control then from the last equality it follows that for an admissible control $u(\cdot)$ sufficiently close to $u_*(\cdot)$ we must have

$$\int_0^T \langle Ru_*(t) - B^*\psi(t), u(t) - u_*(t)\rangle dt \geq 0.$$

It is not difficult to see from the formula for the solution of a linear non-homogeneous differential equation that if $u(\cdot) = u_*(\cdot) + \delta u(\cdot)$ is an admissible control, then $u_*(\cdot) + k\delta u(\cdot)$ is also the admissible control for any real number k. Taking the positive k and the negative k, we receive the contradiction with the optimality of $u_*(\cdot)$, if $Ru^*(t) \neq B^*\psi(t)$. Hence the theorem is proved. □

4.4. Solvability of problem following from control optimality condition

As earlier we will use the notation $S = BR^{-1}B^*$.

Let us assume that the following condition I" is valid.

I". $ImC^* = ImE^*$.

Then the inclusion (4.9) is always true.

Theorem 4.4 *If conditions I", II are satisfied, then the system*

$$\frac{d(Ex(t))}{dt} = Ax(t) + S\psi(t), \tag{4.11}$$

$$Ex(0) = x^0 \tag{4.12}$$

$$Cx(t_j) = y_j, \tag{4.13}$$

$$E^*\frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \quad j = 1, ..., N, \tag{4.14}$$

$$E^*\psi(T) = -C^*FCx(T) \tag{4.15}$$

has an unique solution.

Proof. At first, we establish the uniqueness of the solution, i.e. we prove the absence of a non-trivial solution for system (4.11) - (4.15), when $x^0 = 0, \quad y_j = 0, \quad j = 1, ..., N$.

So, we assume now that $x^0 = 0, \quad y_j = 0, \quad j = 1, ..., N$. We multiply equation (4.11) scalarly by $\psi(t)$ and we multiply equation (4.14) scalarly by $x(t)$. After adding the obtained results we have

$$\frac{d}{dt}\langle Ex(t), \psi(t)\rangle = \langle S\psi(t), \psi(t)\rangle, \quad t \neq t_j.$$

Integrating this equality consecutively with respect to the segments $[t_j, t_{j+1}], \quad j = 0, ..., N$, and adding the obtained results, we have

$$\sum_{j=0}^{N}\int_{t_j}^{t_{j+1}}\frac{d}{dt}\langle Ex(t), \psi(t)\rangle dt = \int_0^T\langle S\psi(t), \psi(t)\rangle dt.$$

From here, using equalities (4.12), (4.15), we obtain

$$\begin{aligned}
&\sum_{j=0}^{N}\langle Ex(t), \psi(t)\rangle\Big|_{t_j+0}^{t_{j+1}-0} = \sum_{j=0}^{N}(\langle x(t_{j+1}), E^*\psi(t_{j+1}-0)\rangle - \langle x(t_j), E^*\psi(t_j+0)\rangle)\\
=&\sum_{j=0}^{N}\langle x(t_{j+1}), E^*\psi(t_{j+1}-0)\rangle - \sum_{j=0}^{N}\langle x(t_j), E^*\psi(t_j+0)\rangle\\
=&\sum_{j=1}^{N+1}\langle x(t_j), E^*\psi(t_j-0)\rangle - \sum_{j=0}^{N}\langle x(t_j), E^*\psi(t_j+0)\rangle\\
=&\langle x(T), E^*\psi(T)\rangle + \sum_{j=1}^{N}\langle x(t_j), E^*(\psi(t_j-0) - \psi(t_j+0))\rangle\\
=&-\langle x(T), C^*FCx(T)\rangle = \int_0^T\langle S\psi(t), \psi(t)\rangle dt.
\end{aligned}$$

As F, S are non-negative definite, from the last relation the next equalities follow

$$FCx(T) = 0, \qquad S\psi(t) \equiv 0.$$

Then we obtain from (4.11), (4.12) the system

$$\frac{d(Ex(t))}{dt} = Ax(t), \quad Ex(0) = 0,$$

which has, in view of condition II, the unique solution

$$x(t) \equiv 0.$$

Now, from (4.14), (4.15), we obtain the system

$$E^*\frac{d\psi(t)}{dt} = -A^*\psi(t), \quad t \neq t_j, \quad j = 1, ..., N, \quad E^*\psi(T) = 0,$$

which has, in view of condition II, the unique solution

$$\psi(t) \equiv 0.$$

The uniqueness of the system (4.11) - (4.15) solution is proved. In view of condition II, system (4.11) - (4.15) is reduced to the linear system resolved with respect to the derivative. Namely, we obtain from this system the relations

$$\frac{d(I-P)x}{dt} = E_+(I-Q)APx + E_+(I-Q)A(I-P)x$$

$$+E_+(I-Q)SQ\psi + E_+(I-Q)S(I-Q)\psi, \tag{4.16}$$

$$0 = QAPx + QA(I-P)x + QSQ\psi + QS(I-Q)\psi, \tag{4.17}$$

$$(I-P)x(0) = E_+(I-Q)x^0, \tag{4.18}$$

$$C(I-P)x(t_j) = y_j, \tag{4.19}$$

$$(I-P)E^*\frac{d(I-Q)\psi}{dt} = -(I-P)A^*Q\psi - (I-P)A^*(I-Q)\psi, \tag{4.20}$$

$$0 = PA^*Q\psi + PA^*(I-Q)\psi, \quad t \neq t_j, \quad j = 1, ..., N, \tag{4.21}$$

$$(I-P)E^*(I-Q)\psi(T) = -(I-P)C^*FC(I-P)x(T). \tag{4.22}$$

Here we have used the relation $CP = 0$, following from condition I". For brevity, the argument t is omitted in (4.16), (4.17), (4.20), (4.21). In view of condition II, we can express $Q\psi$ by $(I-Q)\psi$ from (4.21) and Px by $(I-P)x, Q\psi, (I-Q)\psi$ from (4.17). Substituting the expressions for $Q\psi, Px$ into (4.16), (4.19), (4.20), we obtain from (4.16) - (4.22) for $v = (I-P)x, \quad w = (I-P)E^*(I-Q)\psi$ in the space $\mathrm{Im}A^* \times \mathrm{Im}A^*$ the system of the form

$$\frac{dv}{dt} = \tilde{A}v + \tilde{S}w,$$

$$\frac{dw}{dt} = -\tilde{A}^*w, \quad t \neq t_j,$$

$$v(0) = v^0, \quad Cv(t_j) = y_j, \quad j = 1, ..., N, \quad w(T) = -\tilde{F}v(T).$$

The last system has the unique solution as the corresponding homogeneous system has the trivial solution only (We have shown it earlier). The theorem is proved. □

4.5. Illustrative example

We consider the problem of minimizing the performance index

$$J(u) = \frac{1}{2}\int_0^1 u^2(t)dt$$

on trajectories of the system

$$\frac{dx_1}{dt} = x_2 + u,$$

$$0 = -x_2 + u, \tag{4.23}$$

which satisfy the boundary values

$$x_1(0) = 1, \quad x_1(\frac{1}{2}) = x_1(1) = 0.$$

The descriptor system (4.23) is regular with index 1.

Consider $\psi(\cdot) \in \mathrm{PC}([0,1], \mathbb{R}^2)$ to be the adjoint variable for the considered problem. We write the problem for the adjoint variable from Theorem 4.2

$$\frac{d\psi_1(t)}{dt} = 0, \quad 0 = -\psi_1(t) + \psi_2(t), \quad t \neq \frac{1}{2}. \tag{4.24}$$

Here we have $N = 1$, $T = 1$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $R = 1$, $t_1 = \dfrac{1}{2}$.

We take the optimal control in the form (see Theorem 4.2) $u_*(t) = \psi_1(t) + \psi_2(t)$.

We obtain from (4.24) the relation

$$\psi_1(t) = \begin{cases} c_1, & 0 \le t < \frac{1}{2}, \\ c_2, & \frac{1}{2} \le t \le 1, \end{cases}$$

where c_1 and c_2 are constants.

Then the trajectory $x_{1*}(\cdot)$ is a continuous solution of the system

$$\frac{dx_{1*}}{dt} = \begin{cases} 4c_1, & 0 \le t < \frac{1}{2}, \\ 4c_2, & \frac{1}{2} < t \le 1. \end{cases}$$

As the function x_{1*} must satisfy given boundary values, then from the last system we have

$$c_1 = -\frac{1}{2}, \qquad c_2 = 0.$$

Hence, we obtain the solution of the considered problem

$$x_{1*}(t) = \begin{cases} -2t + 1, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$x_{2*}(t) = u_*(t) = \begin{cases} -1, & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$J(u_*) = \frac{1}{4}.$$

5. On splines

5.1. Connection between problems from sections 3, 4 and problem on splines

The problem on smoothing splines concluding in minimizing the functional

$$\sum_{j=0}^{n} p_j (f(t_j) - f_j)^2 + p \int_0^T (f^{(m)}(t))^2 dt$$

is considered, for example, in [10]. We will show that this problem may be presented in the form of the problem from section 3.

Let us introduce the notations $u(t) = f^{(m)}(t)$, $x_1(t) = f(t)$, $x_2(t) = f'(t)$, ..., $x_m(t) = f^{(m-1)}(t)$ and $x(t) = (x_1(t), ..., x_m(t))^T$. Then the previous problem has the form (3.1)-(3.3), where $X = \mathbb{R}^m$, $U = Y = \mathbb{R}$, $N = n - 1$, $R = 2p$, $F_j = 2p_j$,

$x = (x_1, ..., x_m)^T$, $E = I$, $A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$,

$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$, $y_j = f_j$, i.e. we have the problem of minimizing the functional

$$J(u) = \sum_{j=0}^{N+1} p_j (x_1(t_j) - f_j)^2 + p \int_0^T u^2(t) dt \tag{5.1}$$

with respect to trajectories of the system

$$\frac{dx_i(t)}{dt} = x_{i+1}(t), \quad i = 1, ..., m-1, \tag{5.2}$$

$$\frac{dx_m(t)}{dt} = u(t). \tag{5.3}$$

In this case the problem (3.5)-(3.8) for the adjoint variable $\psi(\cdot)$ ($\psi(t) \in \mathbb{R}^m$) takes the form

$$\frac{d\psi_1(t)}{dt} = 0, \quad t \neq t_j, \tag{5.4}$$

$$\frac{d\psi_i(t)}{dt} = -\psi_{i-1}(t), \quad i = 2, ..., m, \quad t \neq t_j, \tag{5.5}$$

$$\psi_1(0) = 2p_0(x_1(0) - f_0), \tag{5.6}$$

$$\psi_i(0) = 0, \quad i = 2, ..., m, \tag{5.7}$$

$$\psi_1(t_j - 0) - \psi_1(t_j + 0) = -2p_j(x_1(t_j) - f_j), \tag{5.8}$$

$$\psi_i(t_j - 0) - \psi_i(t_j + 0) = 0, \quad j = 1, ..., n, \quad i = 2, ..., m, \tag{5.9}$$

$$\psi_1(T) = -2p_{N+1}(x_1(T) - f_{N+1}), \tag{5.10}$$

$$\psi_i(T) = 0, \quad i = 2, ..., m. \tag{5.11}$$

The relation (3.4) for the control in Theorem 3.1 has the following form

$$u(t) = \frac{1}{2p}\psi_m(t) \tag{5.12}$$

(We suppose that $p > 0$).

It follows from the equalities (5.12), (5.4), (5.5), (5.9) that the control and, hence, $f^{(m)}(\cdot)$ are continuous.

It is obvious from relations (5.12), (5.2)-(5.5) that only the spline of the degree $2m-1$ may be as solution $x_1(\cdot)$ of problem (5.1)-(5.3). This result corresponds to the classical theory.

It should be noted that condition III is not valid for this problem.

Using the analogous arguments, it is not difficult to see that the problem on interpolating splines is the particular case of the problem considered in section 4.

5.2. Illustrative example

Let us consider the problem (5.1)-(5.3) when $m = 2$, $T = 1$, $p_0 = p_1 = p_2 = p = \frac{1}{2}$, $N = 1$, $t_1 = \frac{1}{2}$, $f_0 = 1$, $f_1 = f_2 = 0$, i. e. we will consider the problem of minimizing the functional $\frac{1}{2}((f(0)-1)^2 + (f(\frac{1}{2}))^2 + (f(1))^2) + \frac{1}{2}\int_0^1 (f''(t))^2 dt.$

We obtain from system (5.2)-(5.12) the relations

$$\begin{aligned}
&\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \psi_2,\\
&\frac{d\psi_1}{dt} = 0, \quad \frac{d\psi_2}{dt} = -\psi_1, \quad t \neq \frac{1}{2},\\
&\psi_1(0) = x_1(0) - 1, \quad \psi_2(0) = 0,\\
&\psi_1(\frac{1}{2} - 0) - \psi_1(\frac{1}{2} + 0) = -x_1(\frac{1}{2}), \quad \psi_2(\frac{1}{2} - 0) - \psi_2(\frac{1}{2} + 0) = 0,\\
&\psi_1(1) = -x_1(1), \quad \psi_2(1) = 0.
\end{aligned}$$

(For brevity, the argument t is omitted.)
From the last system we have

$$\begin{aligned}
&\psi_1(t) = \begin{cases} c_1, & 0 \le t < \frac{1}{2},\\ c_2, & \frac{1}{2} \le t \le 1, \end{cases}\\
&\psi_2(t) = \begin{cases} -c_1 t + d_1, & 0 \le t < \frac{1}{2},\\ -c_2 t + d_2, & \frac{1}{2} \le t \le 1, \end{cases}\\
&c_1 = x_1(0) - 1,\\
&d_1 = 0,\\
&c_1 - c_2 = -x_1(\frac{1}{2}),\\
&-c_1 \cdot \frac{1}{2} + d_1 + c_2 \cdot \frac{1}{2} - d_2 = 0,\\
&c_2 = -x_1(1),\\
&-c_2 + d_2 = 0.
\end{aligned}$$

From here, it follows

$$c_2 = -c_1,$$

$$c_1 = x_1(0) - 1, \tag{5.13}$$

$$2c_1 = -x_1(\frac{1}{2}), \tag{5.14}$$

$$c_1 = x_1(1). \tag{5.15}$$

Further,

$$x_2(t) = \begin{cases} -c_1 \cdot \dfrac{t^2}{2} + c_3, & 0 \le t \le \frac{1}{2},\\ c_1 \cdot \dfrac{t^2}{2} - c_1 t + c_4, & \frac{1}{2} \le t \le 1. \end{cases}$$

As $x_2(\cdot)$ must be continuous, when $t = \dfrac{1}{2}$, we obtain $c_4 = \dfrac{c_1}{4} + c_3$.

Using the continuity for $x_1(\cdot)$, when $t = \dfrac{1}{2}$, we have

$$x_1(t) = \begin{cases} -c_1 \cdot \dfrac{t^3}{6} + c_3 t + c_5, & 0 \le t \le \frac{1}{2},\\ c_1 \cdot \dfrac{t^3}{6} - c_1 \cdot \dfrac{t^2}{2} + (c_3 + \dfrac{c_1}{4})t + c_5 - \dfrac{c_1}{24}, & \frac{1}{2} \le t \le 1. \end{cases} \tag{5.16}$$

Taking into account relations (5.13)-(5.16), we obtain for defining c_1, c_3, c_5 the system, from which we find $c_1 = -\dfrac{12}{73}, \quad c_3 = -\dfrac{149}{146}, \quad c_5 = \dfrac{61}{73}$.

Thus, the optimal trajectory $x_{1*}(\cdot)$ has the form

$$x_{1*}(t) = \begin{cases} \dfrac{2}{73}t^3 - \dfrac{149}{146}t + \dfrac{61}{73}, & 0 \le t \le \frac{1}{2}, \\ -\dfrac{2}{73}t^3 + \dfrac{6}{73}t^2 - \dfrac{155}{146}t + \dfrac{123}{146}, & \frac{1}{2} \le t \le 1. \end{cases}$$

We have obtained the cubic spline of deficiency 1, which coincides with the spline constructed by the standard method.

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