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Dolbeault cohomology and locally residual currents on projective varieties

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LOCALLY RESIDUAL CURRENTS AND DOLBEAULT COHOMOLOGY ON PROJECTIVE MANIFOLDS

BY
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ABSTRACT. - Let be X a projective manifold of dimension n, and n hypersurfaces $Y_i (1 \leq i \leq n)$ on X, defining ample line bundles, and intersecting properly. After introducing sheaves of locally residual currents, we enunciate the following two main theorems. First, for any positive integer i, the Dolbeault cohomology group $H^i(\Omega^q_X)$ of the sheaf of holomorphic $q-$forms on X can be computed as the $i-$th cohomology group of some complex of global sections of locally residual currents on X. We get from this the theorem of [3] that any locally residual current on X which is $\overline{\partial}-$exact is globally residual. Secondly, for $q = n$, we get another exact sequence computing $H^i(\Omega^n_X)$ by restricting to residual currents obtained from meromorphic forms with simple poles on the Y_i. We deduce from this a reformulation of the main theorem of [10], saying that we can compute the cohomology groups $H^i(\Omega^n_X)$ by the cohomology of a complex of principal value currents with simple poles. We also deduce from this the result from [7] that if $Y_1, \ldots, Y_n$ intersect transversally in a finite set of points $P_i (1 \leq i \leq s)$, then for any sequence of s complex numbers $c_i (1 \leq i \leq s)$, there is a global meromorphic $n-$form Ψ with simple poles on each Y_i such that:

$$(\forall i, 1 \leq i \leq s) Res^{P_i}_{Y_1,\ldots,Y_n} \Psi = c_i$$

iff $\sum_{i=1}^{s} c_i = 0$. In the second part, we give proofs of the theorems by mean of several exact sequences of sheaves of locally residual currents. We conclude by giving two directions for further developments: one direction is related to the Hodge conjecture, the other is concerning the Abel-Radon transform.

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1 Main results

Let us introduce the main objects used in the results, that is locally residual currents, by giving a brief account of their construction. Further details can be found in [2], or [1].

Let X be a complex manifold of dimension n. We denote $\mathcal{O}_X$ the sheaf of holomorphic functions, Ω^q_X the sheaf of holomorphic $q-$forms (in particular, $\Omega^0 = \mathcal{O}_X$) , $\mathcal{M}^q_X$ the sheaf of meromorphic $q-$forms. First, there is, canonically associated to any meromorphic $q-$form Ψ, a current of bidegree $(q, 0)$, denoted $P(\Psi)$ or $[\Psi]$, and called the *principal value* of Ψ, which satisfies the following properties ([2]):

1. if Ψ is holomorphic, $[\Psi]$ coincides with the classical current associated to Ψ:
$$[\Psi](\phi) = \int_X \phi \wedge \Psi.$$

2. Let us denote $\mathcal{C}^{q,0}_X(\star)$ the sheaf of currents which can be locally written as $[\omega]$, with ω a meromorphic $q-$form, and $\mathcal{C}^{q,0}_X$ the subsheaf of those which are $\overline{\partial}-$closed. Then, P induces an isomorphism of $\mathcal{O}_X-$modules, $\mathcal{P} : \mathcal{M}^q_X \to \mathcal{C}^{q,0}_X(\star)$, and also an isomorphism from Ω^q_X to $\mathcal{C}^{q,0}_X$. Moreover, the natural operator $\partial : \mathcal{M}^q_X \to \mathcal{M}^{q+1}_X$ commutes with $\mathcal{P}$: $[\partial\Psi] = \partial[\Psi]$.

Now, let on an open subset $U \subset X$, $p + 1$ hypersurfaces $Y_0, \ldots , Y_p$ hypersurfaces intersecting properly. There is (cf. [1]), for a meromorphic form with poles contained in $Y_0 \cup \ldots \cup Y_p$, a currents on U denoted $\mathbf{Res}_{Y_1,\ldots,Y_q}(\Psi)$, and called *residual* current, or *Coleff-Herrera* current. We have:

$$\mathbf{Res}_{Y_0,\ldots,Y_q}(\Psi) = \overline{\partial}\mathbf{Res}_{Y_1,\ldots,Y_q}(\Psi).$$

If the Y_i are defined by holomorphic functions f_i, we also denote:

$$\mathbf{Res}_{Y_1,\ldots,Y_q}(\omega/f_0 \ldots f_p) = \omega/f_0 \wedge \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p),$$

so that:

$$\overline{\partial}(1/f_0\overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p)) = \overline{\partial}(1/f_0) \wedge \ldots \wedge \overline{\partial}(1/f_p).$$

A *locally residual current* is a current which can be written locally as a residual current, thus also as a current $\omega/f_0 \wedge \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p)$, with

$(f_0, \dots, f_p)$ a regular sequence of holomorphic functions and ω an holomorphic $q-$form. We do not assume here (as for instance in [4]) that a locally residual current is $\overline{\partial}-$closed.

Now let Z an analytic subset of pure codimension p. Then, let us denote $\mathcal{C}_Z^{q,p}$ the sheaf which associate to any subset U the set $\mathcal{C}_Z^{q,p}(U)$ of $\overline{\partial}-$closed locally residual currents of bidegree (q,p), with support contained in Z. If Y is an hypersurface of Z, we denote $\mathcal{C}_Z^{q,p}(\star Y)$ the sheaf of locally residual currents of bidegree (q,p) supported in Z, $\overline{\partial}-$closed outside Y.

If Y is an hypersurface in X intersecting Z properly, i.e. such that the analytic subset $Z \cap Y$ is of pure codimension $p+1$, we will denote $\mathcal{C}_Z^{q,p}(\star Y)$ for $\mathcal{C}_Z^{q,p}(\star(Y \cap Z))$. Since

$$\overline{\partial}(1/f_0\overline{\partial}(1/f_1) \wedge \dots \wedge \overline{\partial}(1/f_p)) = \overline{\partial}(1/f_0) \wedge \dots \wedge \overline{\partial}(1/f_p),$$

we then have a natural map: $\overline{\partial} : \mathcal{C}_Z^{q,p}(\star Y) \to \mathcal{C}_{Z\cap Y}^{q,p+1}$.

We associate to a current T and a smooth form ω the current $\omega \wedge T(\phi) = T(\phi \wedge \omega)$, so that we can write formally: $T(\phi) = \int_X \phi \wedge T$.

First, let us remark that a special kind of locally residual currents are given by the following lemma (cf. [2]):

Lemma 1 *Let Z be an analytic subset of pure codimension p, and ω a meromorphic $r-$form on Z. There is a natural way to associate to ω a current of bidegree $(r+p,p)$ on X, denoted $\omega \wedge [Z]$, and called the principal value of ω (on X), which coincides with the classical current $\omega \wedge [Z](\phi) = \int_Z \phi \wedge \omega$ if ω is holomorphic on Z. $\omega \wedge [Z]$ is a locally residual current.*

The currents locally written as $\omega \wedge [Z]$ are called *principal value currents.* If $Z' \subset Z$ is the polar hypersurface of ω (outside which ω is $\overline{\partial}-$closed), $\overline{\partial}(\omega \wedge [Z])$ is a locally residual current of bidegree $(r+p, p+1)$ with support in Z'. We say that ω has *logarithmic pole* if this current can still be written $\omega' \wedge [Z']$, with ω' a meromorphic $(r-1)-$form on Z'.

We denote $\mathcal{C'}_Z^{r+p,p}$ the subsheaf of $\mathcal{C}_Z^{r+p,p}$ of those currents which can be locally written $\omega \wedge [Z \cap U]$, for ω a meromorphic $r-$form. The maps $\omega \to \omega \wedge [Z]$ thus gives an isomorphism between the Barlet's sheaves Ω_X^q on Z of abelian differential $q-$forms and the sheaves $\mathcal{C'}_Z^{q+p,p}$. Let Y be an hypersurface cutting Z properly. We denote $\mathcal{C'}_Z^{n,p}(Y)$ the subsheaf of $\mathcal{C}_Z^{n,p}(\star Y)$, given by those meromorphic forms of maximal degree on Z, having logarithmic poles on Y.

Let us now assume that X is a compact complex manifold, and that $Y_1, \dots, Y_n$ are analytic hypersurfaces. We assume that the Y_i are *positive*, in the sense that the corresponding Cartier divisors are ample. By the theorem of Kodaira, it implies that X is projective. We also assume that the Y_i intersect properly, so that the intersection is a finite set of points.

Theorem 1 *1. The complex:*

$$0 \to \Omega^q_X \to \mathcal{C}^{q,0}(\star Y_1) \xrightarrow{\overline{\partial}} \mathcal{C}^{q,1}_{Y_1}(\star Y_2) \xrightarrow{\overline{\partial}} \cdots$$
$$\xrightarrow{\overline{\partial}} \mathcal{C}^{q,n-1}_{Y_1\cap\cdots\cap Y_{n-1}}(\star Y_n) \xrightarrow{\overline{\partial}} \mathcal{C}^{q,n}_{Y_1\cap\cdots\cap Y_n} \to 0$$

is an acyclic resolution of Ω^q_X by $\mathcal{O}_X-$modules, and thus we have a canonical isomorphism for all $i, 0 \leq i \leq n$:

$$H^i(\Omega^q_X) \simeq H^0(\mathcal{C}^{q,i}_{Y_1\cap\cdots\cap Y_i})/\overline{\partial} H^0(\mathcal{C}^{q,i-1}_{Y_1\cap\cdots\cap Y_{i-1}}(\star Y_i))$$

2. *Moreover, an element $T = H^0(\mathcal{C}^{q,i-1}_{Y_1\cap\cdots\cap Y_{i-1}}(\star Y_i))$ can be written as a global residue: $T = \mathbf{Res}_{Y_i,\ldots,Y_{i-1}}(\Psi)$, with Ψ a meromorphic $q-$form with poles contained in $Y_1 \cup \ldots \cup Y_i$.*

3. *Moreover, T is $\overline{\partial}-$exact iff we can choose Ψ with poles in $Y_1\cup\ldots\cup Y_{i-1}$.*

Let us assume now $q = n$. Let us notice that the operators: $\overline{\partial}$: $\mathcal{C'}^{n,p-1}_{Y_1\cap\cdots\cap Y_{p-1}}(Y_p) \to \mathcal{C'}^{n,p}_{Y_1\cap\cdots\cap Y_p}$ define a subcomplex of the preceding one. We also assume now that the $Y_i (1 \leq i \leq n)$ intersect *transversally* in s distinct points.

Then we have the following variant of the preceding theorem:

Theorem 2 *1. The complex:*

$$0 \to \Omega^n_X \to \mathcal{C'}^{n,0}(Y_1) \xrightarrow{\overline{\partial}} \mathcal{C'}^{n,1}_{Y_1}(Y_2) \to \cdots$$
$$\xrightarrow{\overline{\partial}} \mathcal{C'}^{n,n-1}_{Y_1\cap\cdots\cap Y_{n-1}}(Y_n) \xrightarrow{\overline{\partial}} \mathcal{C'}^{n,n}_{Y_1\cap\cdots\cap Y_n} \to 0$$

is an acyclic resolution of Ω^n_X by $\mathcal{O}_X-$modules, and thus we have a canonical isomorphism for all $i, 0 \leq i \leq n$:

$$H^i(\Omega^n_X) \simeq H^0(\mathcal{C'}^{n,i}_{Y_1\cap\cdots\cap Y_i})/\overline{\partial} H^0(\mathcal{C'}^{n,i-1}_{Y_1\cap\cdots\cap Y_{i-1}}(Y_i))$$

2. *Moreover, an element $T = H^0(\mathcal{C'}^{n,i-1}_{Y_1\cap\cdots\cap Y_{i-1}}(Y_i))$ can be written as a global residue: $T = \mathbf{Res}_{Y_i,\ldots,Y_{i-1}}(\Psi)$, with Ψ a meromorphic closed $n-$form with poles in $Y_1 \cup \ldots \cup Y_i$.*

3. *Moreover, T is $d-$exact iff we can choose Ψ with poles in $Y_1\cup\ldots\cup Y_{i-1}$.*

Remarks.

1. For $q < n$, it is not true in general that the complex with logarithmic poles computes the Dolbeault cohomogy groups, since it is not in general acyclic. The acyclicity for the logarithmic poles, for $q = n$, comes from the Kodaira annihilation theorem.

2. In each of the two preceding theorems, the first part is a variant of Dolbeault's theorem, representing cohomology classes by $\overline{\partial}-$closed currents with fixed supports. The theorems would remain true if we don't fix the supports in given complete intersections, but consider more general complexes of locally residual currents (resp. of principal value currents) with any supports. In fact, let us denote for instance $\mathcal{C}'^{n,i}$ (resp. $\mathcal{C}'^{n,i}(*)$) the sheaves of $\overline{\partial}-$closed principal value currents of bidegree (n, i) (resp. with logarithmic poles). Then, we have by Dolbeault's theorem a natural morphism:

$$H^0(\mathcal{C}'^{n,i})/\overline{\partial} H^0(\mathcal{C}'^{n,i-1}(*)) \to H^i(\Omega_X^n).$$

This morphism is clearly surjective, since by the preceding theorem we know that we even can fix the supports. But it is also injective: in fact, if the image of $T \in H^0(\mathcal{C}'^{n,i})$ is zero, we know that by definition of the morphism, the current T is $\overline{\partial}-$exact; and we can include the support of T, which is of pure codimension i, in a complete intersection of i positive hypersurfaces $Y_1, \ldots, Y_i$. Thus we can apply again the preceding theorem, to write T in the form $\overline{\partial}\omega' \wedge [Y']$, with $Y' = Y_1 \cap \ldots \cap Y_{i-1}$, and ω' with logarithmic pole on Y_i. Thus we get, expressed in another way, the main theorem of [10].

3. In the first theorem, we could enonce the same theorem, assuming X is compact algebraic and the complements $X \backslash Y_i$ are affine.

We get as corollary a theorem of P. Griffiths ([7]):

Corollary 1 *Let the n positive hypersurfaces $Y_1, \ldots, Y_n$ intersect transversally in s distinct points $P_1, \ldots, P_s$, and let be $c_1, \ldots, c_s$ s complex numbers. A necessary and sufficient condition for the existence of a meromorphic $n-$form Ψ, with simple pole contained in $Y_1 \cup \ldots \cup Y_n$, and*

$$(\forall i, 1 \leq i \leq s) Res_{Y_1,\ldots,Y_n}^{P_i} \Psi = c_i,$$

is that $\sum_{i=1}^s c_i = 0$.

Proof.
The existence of Ψ is equivalent of the existence of Ψ such that: $Res_{Y_1,\ldots,Y_n}(\Psi) = \sum_{i=1}^s c_i[P_i]$. Thus, the existence of Ψ imply that the "evaluation" current $T = \sum_{i=1}^s c_i[P_i]$, which associate to a function f the sum $\sum_{i=1}^s f(P_i)$, is

$\overline{\partial}-$exact, and thus annihilates on 1, which means $\sum_{i=1}^{s} c_i = 0$. Reciprocally, if the sum is zero, then the current $T = \sum_{i=1}^{s} c_i[P_i]$ is exact, since $H^{n,n}(X) \simeq H^{0,0}(X) \simeq H^0(\mathcal{O}_X) \simeq \mathbb{C}$ since X is smooth, compact and connected. Thus by the last theorem, it can be written as a global residue $T = \mathbf{Res}_{Y_1,\ldots,Y_n}\Psi$, with Ψ having simple pole on $Y_1 \cup \ldots \cup Y_n$. ∎

2 Secondary results and proofs of the main theorems

2.1 Topological residue operator

We recall here the construction of the topological residue operator, as given in [8], and deduce a composed residue operator on local cohomology classes. The reason for this is that the residue operator on *moderate* cohomology classes will be constructed of the same model.

Let be X be a topological space. Let be $\mathcal{F}$ an abelian sheaf on X, and Y a locally closed subset in X. Then the sheaf $\Gamma_Y\mathcal{F}$ is defined as follows. Let us assume that Y is a closed subset of the open subset V of X. Then $\Gamma_Y\mathcal{F}(U)$ is the subgroup of $\Gamma(V \cap U, \mathcal{F})$ consisting of sections with support contained in Y. If V' is another open subset such that Y is a closed subset of V', the two subgroups obtained are isomorphic; so we can identify them both with the inductive limit, for all the open neighborhoods of Y. Thus, we obtain as sheaf $\Gamma_Y\mathcal{F}$ the sheaf which fiber is zero for $x \notin Y$, and whose fiber at $x \in Y$ is the germs of sections of $\mathcal{F}$ at x, with support in Y. This is also an abelian sheaf, and thus we get a functor $\Gamma_Y : \mathcal{F} \to \Gamma_Y\mathcal{F}$ in the category of abelian sheaves. This functor is left-exact; we denote $\mathcal{H}_Y^i$ the right derived functor of Γ_Y.

We have:

$$\Gamma_Y \circ \Gamma_{Y'} = \Gamma_{Y\cap Y'}$$

Let us denote $\mathbf{\Gamma}_Y(U,\bullet) = \Gamma(U,\bullet) \circ \Gamma_Y$ (resp. $\mathbf{\Gamma}_Y = \Gamma \circ \Gamma_Y$); thus $\Gamma_Y\mathcal{F}(U) = \mathbf{\Gamma}_Y(U,\mathcal{F})$. The functor $\mathbf{\Gamma}_Y(U,\bullet)$ (resp. $\mathbf{\Gamma}_Y$) is left-exact. We denote $H_Y^i(U,\bullet)$ (resp. H_Y^i) the $i-$th right derived functor of $\mathbf{\Gamma}_Y(U,\bullet)$ (resp. $\mathbf{\Gamma}_Y$). The groups $H_Y^i(\mathcal{F})$ are called the cohomology groups of $\mathcal{F}$ with support in Y.

Let us suppose Y is closed. Then, we have natural the exact sequence:

$$0 \to \Gamma_Y\mathcal{F} \to \mathcal{F} \to \Gamma_{X\setminus Y}\mathcal{F},$$

and for any open set U the exact sequence:

$$0 \to \Gamma_Y \mathcal{F}(U) \to \mathcal{F}(U) \to \Gamma_{X\setminus Y}\mathcal{F}(U)$$

with a zeros at the right if the sheaf $\mathcal{F}$ is flabby.

Let be Z two another closed subsets of X. If $\mathcal{F}$ is flabby, so is still $\Gamma_Z\mathcal{F}$, and thus we have also the short exact sequences:

$$0 \to \Gamma_{Z\cap Y}\mathcal{F} \to \Gamma_Z\mathcal{F} \to \Gamma_{Z\setminus Y)}\mathcal{F} \to 0,$$
$$0 \to \Gamma_{Z\cap Y}\mathcal{F}(U) \to \Gamma_Z\mathcal{F}(U) \to \Gamma_{Z\setminus Y)}\mathcal{F}(U) \to 0.$$

Let us consider an injective (thus flabby) resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

Since the $\mathcal{I}^i$ are flabby, we get short exact sequences of complexes:

$$0 \to \Gamma_{Z\cap Y}\mathcal{I}^\bullet \to \Gamma_Z\mathcal{I}^\bullet \to \Gamma_{Z\setminus Y}\mathcal{I}^\bullet \to 0,$$

$$0 \to \Gamma_{Z\cap Y}\mathcal{I}^\bullet(U) \to \Gamma_Z\mathcal{I}^\bullet(U) \to \Gamma_{Z\setminus Y}\mathcal{I}^\bullet(U) \to 0,$$

By the classical snake lemma, we get from this short exact sequences of complexes the following long exact sequences:

$$\begin{gathered} 0 \to \mathcal{H}^0_{Z\cap Y}(\mathcal{F}) \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{H}^0_{Z\setminus Y}(\mathcal{F}) \stackrel{Res_Y}{\to} \mathcal{H}^1_{Z\cap Y}(\mathcal{F}) \\ \to \cdots \to \mathcal{H}^i_{Z\cap Y}(\mathcal{F}) \to \mathcal{H}^i_Z(\mathcal{F}) \to \mathcal{H}^i_{Z\setminus Y}(\mathcal{F}) \\ \stackrel{Res_Y}{\to} \mathcal{H}^{i+1}_{Z\cap Y}(\mathcal{F}) \to \cdots \end{gathered}$$

and

$$\begin{gathered} 0 \to H^0_{Z\cap Y}(U,\mathcal{F}) \to H^0_Z(U,\mathcal{F}) \to H^0_{Z\setminus Y}(U,\mathcal{F}) \stackrel{\mathbf{Res_Y}}{\to} H^1_{Z\cap Y}(U,\mathcal{F}) \\ \to \cdots \to H^i_{Z\cap Y}(U,\mathcal{F}) \to H^i_Z(U,\mathcal{F}) \to H^i_{Z\setminus Y}(U,\mathcal{F}) \\ \stackrel{\mathbf{Res_Y}}{\to} H^{i+1}_{Z\cap Y}(U,\mathcal{F}) \to \cdots \end{gathered}$$

The natural maps in the preceding exact sequence

$$Res_Y : \mathcal{H}^i_{Z\setminus Y}(\mathcal{F}) \to \mathcal{H}^{i+1}_{Z\cap Y}(\mathcal{F})$$

and

$$\mathbf{Res_Y} : H^i_{Z\setminus Y}(U,\mathcal{F}) \to H^{i+1}_{Z\cap Y}(U,\mathcal{F})$$

are called the *topological* residue operators.

Let be $Y_1, \ldots, Y_p$ be closed subsets. We define the *composed* topological residue operator as:

$$Res_{Y_1,\cdots,Y_i} := Res_{Y_1} \circ \cdots \circ Res_{Y_i} : \mathcal{H}^k_{Z\backslash(Y_1\cup\cdots\cup Y_i)}(\mathcal{F}) \to \mathcal{H}^{i+k}_{Z\cap Y_1\cap\cdots\cap Y_i}(\mathcal{F}),$$

for any locally closed subset Z.

Example. Let us consider n hypersurfaces $Y_i (1 \leq i \leq n)$ intersecting in a point P. Let us take Ψ be an holomorphic $n-$form in $U\backslash Y_1 \cup \cdots \cup Y_n$, for some open neighborhood U of P, such that $Y_1 \cap \ldots \cap Y_n \cap U = P$. Let us denote $H^n_P(\Omega^n_X) := \mathcal{H}^n_P(\Omega^n_X)_P$. Then $\mathbf{Res}_{Y_1,\ldots,Y_n}(\Psi) \in H^n_P(U, \Omega^n_X) \simeq H^n_P(\Omega^n_X)$. Any element $H^n_P(\Omega^n_X)$ is written like this for a such Ψ in some open neighborhood U of P.

Let us consider, for a germ at P of holomorphic $n-$form Ψ in the complement of $Y_1 \cup \cdots \cup Y_n$, and a germ of holomorphic function g, the mapping $(\Psi, g) \mapsto \int_{|f_i|=\epsilon_i(1\leq i\leq n)} g\Psi$, the f_i being a local defining function for Y_i at P, and ϵ_i sufficiently small. We can check that his defines a perfect pairing $H^n_P(\Omega^n_X) \times \mathcal{O}_{X,P} \to \mathbb{C}$, such that $Res_{Y_1,\cdots,Y_n}(\Psi)$ can be seen as a linear form on the germs of holomorphic functions in a neighborhood of P, continuous for an appropriate topology. We will denote $Res^P_{Y_1,\cdots,Y_n}(\Psi)$ the value at 1 of this linear form, that is the integral: $\int_{|f_i|=\epsilon_i(1\leq i\leq n)} \Psi$. This is the *punctual residue.*

2.2 Moderate cohomogical residue operator

Let us now assume that X is a complex manifold of dimension n.

Let Z a closed analytic subset on X, defined by the sheaf of ideals $\mathcal{I}_Z$, and $Z' \subset Z$ another closed analytic subset. We define the following functor, in the category of $\mathcal{O}_X-$modules:

$$\Gamma_{[Z\backslash Z']}(\mathcal{F}) := \overrightarrow{lim}_k \mathcal{H}om(\mathcal{I}^k_{Z'}/\mathcal{I}^k_Z, \mathcal{F}).$$

This is a left-exact functor; we define $\mathcal{H}^i_{Z\backslash Z'}$ as the right derived cohomology of this functor.

Let us assume Y is an hypersurface. Then, $\Gamma_{[X\backslash Y]}(\mathcal{F})_x$ can be identified by the subset of $\mathcal{F}_x$, which multiplied by a power of f, extend to an element of $\mathcal{F}_x$; we identify two elements if their difference is annihilated by a power of f. We denote also $\mathcal{F}(\star Y) := \Gamma_{X\backslash Y}(\mathcal{F})$. Then we have:

Lemma 2 *$\mathcal{F} \to \mathcal{F}(\star Y)$ is an exact functor of $\mathcal{O}_X-$ modules.*

Proof.
We have that $\mathcal{I}^k_Y$ is coherent, and locally free, since it is generated by one

element f^k at each point, where f is the defining function for Y. Thus, since for a free $A-$module M, the functor $Hom_A(M,\bullet)$ is exact, we have that $\mathcal{F} \to \mathcal{H}om(\mathcal{I}_Y^k,\mathcal{F})$ is exact. By the injective (directed) limit, we get that $\Gamma_{X\setminus Y}$ is also exact. ∎

In particular: $\mathcal{H}^i_{Z\setminus Y}(\mathcal{F}) = \mathcal{H}^i_Z(\mathcal{F})(\star Y)$.

Lemma 3 *Let Z be an analytic subset, and $Z' \subset Z$. Then we have a natural long exact sequence:*

$$\begin{gathered}0 \to \mathcal{H}^0_{[Z']}(\mathcal{F}) \to \mathcal{H}^0_{[Z]}(\mathcal{F}) \to \mathcal{H}^0_{[Z\setminus Z']}(\mathcal{F}) \stackrel{\delta_0}{\to} \mathcal{H}^1_{[Z']}(\mathcal{F}) \\ \to \cdots \to \mathcal{H}^i_{[Z']}(\mathcal{F}) \to \mathcal{H}^i_{[Z]}(\mathcal{F}) \to \mathcal{H}^i_{[Z\setminus Z']}(\mathcal{F}) \\ \stackrel{\delta_i}{\to} \mathcal{H}^{i+1}_{[Z']}(\mathcal{F}) \to \cdots\end{gathered}$$

Proof.
Let us consider a resolution $0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \ldots$ of $\mathcal{F}$ by injective $\mathcal{O}_X-$modules. Then we have, applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\bullet,\mathcal{I}_i)$ to the short exact sequence $0 \to \mathcal{I}^k_{Z'}/\mathcal{I}^k_Z \to \mathcal{O}_X/\mathcal{I}^k_Z \to \mathcal{O}_X/\mathcal{I}^k_{Z'} \to 0$, an exact sequence of complexes:

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_{Z'},\mathcal{I}_i) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z,\mathcal{I}_i) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^k_{Z'}/\mathcal{I}^k_Z,\mathcal{I}_i) \to 0$$

For a given i, we have an injective system of sheaves, indexed by the integers k; moreover, this system is directed. Thus, the direct limit on k remains a short exact sequence, and we obtain an exact sequence of complexes:

$$0 \to \Gamma_{[Z']}(\mathcal{I}_i) \to \Gamma_{[Z]}(\mathcal{I}_i) \to \Gamma_{[Z\setminus Z']}(\mathcal{I}_i) \to 0.$$

Since the cohomologies of these complexes compute, by definition, the moderate cohomology sheaves, the long exact sequence associated to this short exact sequence give us the wanted long exact sequence. ∎

Let be Z an analytic subvariety of pure codimension p, and Y, Y' two hypersurfaces intersecting properly on Z. By taking $Z' := Z \cap Y$, we get in the preceding long exact sequence, as connection operator, a *moderate residue* operator:

$$Res_{[Y]} : \mathcal{H}^i_{[Z\setminus Y]}(\Omega^q_X) \to \mathcal{H}^{i+1}_{[Z\cap Y]}(\Omega^q_X).$$

By applying the exact functor $\Gamma_{[X\setminus Y']}$ we deduce another operator:

$$\mathcal{H}^i_{[Z\setminus(Y\cup Y')]}(\Omega^q_X) \to \mathcal{H}^{i+1}_{[(Z\cap Y)\setminus Y']}(\Omega^q_X).$$

Let us consider the resolution:

$$0 \to \Omega_X^q \to \mathcal{D}^{q,0} \overset{\overline{\partial}}{\to} \mathcal{D}^{q,1} \overset{\overline{\partial}}{\to} \mathcal{D}^{q,2} \to \dots$$

by currents.

We know the following (see [9]):

Lemma 4 *The sheaves $\mathcal{D}^{q,p}$ have $\mathcal{O}_{X,x}-$injective fibers.*

We thus deduce the short exact sequences:

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_{Z\cap Y}, \mathcal{D}^{q,p}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{D}^{q,p}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^k_{Z\cap Y}/\mathcal{I}^k_Z, \mathcal{D}^{q,p}) \to 0,$$

thus by direct limit:

$$0 \to \Gamma_{[Z\cap Y]}(\mathcal{D}^{q,p}) \to \Gamma_{[Z]}(\mathcal{D}^{q,p}) \to \Gamma_{[Z\setminus Y]}(\mathcal{D}^{q,p}) \to 0;$$

these exact sequences commute with the operator $\overline{\partial}$, thus we get an exact sequence of complexes:

$$0 \to \Gamma_{[Z\cap Y]}(\mathcal{D}^{q,\bullet}) \to \Gamma_{[Z]}(\mathcal{D}^{q,\bullet}) \to \Gamma_{[Z\setminus Y]}(\mathcal{D}^{q,\bullet}) \to 0;$$

Thus we get a long exact sequence on the cohomologies of these complexes. But we have:

Lemma 5 *These complexes compute the moderate cohomology sheaves:*

$$\mathcal{H}^i_{[Z\cap Y]}(\Omega^q_X), \mathcal{H}^i_{[Z]}(\Omega^q_X), \mathcal{H}^i_{[Z\setminus Y]}(\Omega^q_X).$$

Proof.
Let us restrict to show $H^i_{[Z]}(\mathcal{D}^{q,p}) = 0$. Since the sheaves $\mathcal{D}^{q,p}$ have injective fibers, the $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{D}^{q,p})$ are zero, since $\mathcal{O}_X/\mathcal{I}^k_Z$ being coherent, the fibers commute with the Ext. But since the injective system indexed by the integers k is directed, the cohomology of the direct limit coincide with the direct limit of the cohomology, that is, $\overrightarrow{lim}_k \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{D}^{q,p}) = \mathcal{H}^p_{[Z]}(\mathcal{D}^{q,p})$, which is thus zero. Thus we have also $H^i_{[Z\cap Y]}(\mathcal{D}^{q,p}) = 0$, and by the long exact sequence also $H^i_{[Z\setminus Y]}(\mathcal{D}^{q,p}) = 0$. ∎

Thus, in the long exact sequence:

$$0 \to \mathcal{H}^0_{[Z\cap Y]}(\mathcal{F}) \to \mathcal{H}^0_{[Z]}(\mathcal{F}) \to \mathcal{H}^0_{[Z]}(\mathcal{F})(\star Y) \overset{Res_{[Y]}}{\to} \mathcal{H}^1_{[Z\cap Y]}(\mathcal{F}) \to \dots$$

the residues $Res_{[Y]}$ can in fact, if we compute the cohomogy groups by currents, be computed as $\overline{\partial}-$operator.

2.3 Relations with the sheaves of locally residual currents

Let us know consider Z an analytic subset of pure codimension p, and Y an hypersurface inyersecting Z properly. We have by the preceding a natural application $\phi : \mathcal{C}_Z^{q,p} \to \mathcal{H}_{[Z]}^p(\Omega_X^q)$, by associating to a $\overline{\partial}-$closed residual current with support in Z his class. We have, by [4]:

Lemma 6 *ϕ is bijective.*

This is also true for the maps:

$$\phi' : \mathcal{C}_{Z\setminus Y}^{q,p} \to \mathcal{H}_{[Z\setminus Y]}^p(\Omega_X^q), \phi'' : \mathcal{C}_{Z\cap Y}^{q,p} \to \mathcal{H}_{[Z\cap Y]}^p(\Omega_X^q).$$

Let us assume Z locally complete intersection. Then we have $\mathcal{H}_{[Z]}^{p+1}(\Omega_X^q) = 0$, and by the long exact sequence, since also $\mathcal{H}_{[Z\cap Y]}^p(\Omega_X^q)$ as Y intersects Z properly, an exact sequence:

$$0 \to \mathcal{H}_{[Z]}^p(\Omega_X^q) \to \mathcal{H}_{[Z]}^p(\Omega_X^q)(\star Y) \to \mathcal{H}_{[Z\cap Y]}^{p+1}(\Omega_X^q) \to 0,$$

where the second operator is given by $\overline{\partial}$.

Thus, we have in particular that in the sequence $\mathcal{C}_Z^{q,p} \to \mathcal{C}_Z^{q,p}(\star Y) \to \mathcal{C}_{Z\cap Y}^{q,p+1}$, the first map is injective, and the second map is surjective.

We also have:

Lemma 7 *$\mathcal{C}_Z^{q,p}(\star Y)$ is an $\mathcal{O}_X-$module.*

Proof.
We have to show the stability for the sum. Let us remark that if $T \in \mathcal{C}_{Z,x}^{q,p}(\star Y)$ for some $x \in Y$, then if f is an holomorphic function in an open neighborhood U_x of x such that $U_x \cap Y \subset U_x \cap Z \cap \{f = 0\}$, we have that for some integer k, $f^k T$ extends to a germ of $\mathcal{C}_{Z,x}^{q,p}$. Thus, it suffices to show that the subsheaf $\mathcal{C}_Z^{q,p}$ is stable for the sum. Since we assume Z is locally complete intersection, let us consider a reduced complete intersection $Z_x = Y_{1,x} \cap \ldots \cap Y_{p,x}$, define by a regular sequence $(f_1, \ldots, f_p)$, and $\alpha \in \mathcal{C}_{Z,x}^{q,p}$.

Then we can write $\alpha = \overline{\partial}\alpha_1$, with:

$$\alpha_1 \in \mathcal{C}_{Y_{1,x}\cap\ldots\cap Y_{p-1,x}}^{q,p-1}(\star Y_p),$$

thus $\alpha_1 = \beta_1/f_p^{k_p}$; similarly, $\beta_1 = \overline{\partial}(\beta_2/f_{p-1}^{k_{p-1}})$. Finally, since

$$\overline{\partial}(1/f_0 \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_i)) = \overline{\partial}(1/f_0) \wedge \ldots \wedge \overline{\partial}(1/f_i),$$

we get

$$\alpha = \omega \wedge \overline{\partial}(1/f_1^{k_1}) \wedge \ldots \wedge \overline{\partial}(1/f_p^{k_p}),$$

with an holomorphic $q-$form ω. ∎

Thus, if Z is locally complete intersection of codimension p, the bijection:

$$\phi : \mathcal{C}^{q,p}_{Z,x} \to \mathcal{H}^p_{[Z],x}(\Omega^q_X), \alpha \mapsto cl(\alpha)$$

is an isomorphism of $\mathcal{O}_{X,x}-$modules. In particular, we get, if Y cuts Z properly, an exact sequence of $\mathcal{O}_X-$modules:

$$0 \to \mathcal{C}^{q,p}_Z \to \mathcal{C}^{q,p}_Z(\star Y) \to \mathcal{C}^{q,p+1}_{Z\cap Y} \to 0$$

More generally, if Y, Y' are two hypersurfaces in X intersecting properly on Z, we deduce an operator:

$$Res_Y : \mathcal{C}^{q,p}_Z(\star(Y \cup Y')) \to \mathcal{C}^{q,p+1}_{Z\cap Y}(\star Y').$$

If Y' is empty, Res_Y coincide with $\overline{\partial}$. We defined the *composed residue* as $Res_{Y_1,\ldots,Y_p} := Res_{Y_1} \circ \ldots \circ Res_{Y_p}$. The associate residue operator on global sections coincide with the classical Coleff-Herrera residue, so that we also write it as $\mathbf{Res}_{Y_1,\ldots,Y_p}$.

Let us assume $(f_0, \ldots, f_p)$ a regular sequence of germs of holomorphic functions, and ω a germ of holomorphic $q-$form, at a point $x \in X$. We have the following fundamental lemma for the residual currents (cf.[2]):

Lemma 8 *If we have:*

$$\omega/f_0 \wedge \overline{\partial}(1/f_1) \wedge \ldots \wedge \overline{\partial}(1/f_p) = 0,$$

then we can write: $\omega_i = \sum_{i=1}^p f_i\omega_i$*, with* ω_i *germs of holomorphic* $q-$*forms.*

Moreover:

Lemma 9 *If* $\mathbf{Res}_{Y_1,\ldots,Y_p}\omega/(f_0 \ldots f_p)$ *is* $\overline{\partial}-$*closed, then it can be written* $Res_{Y_1,\ldots,Y_p}\Psi$*, with* $Pol(\Psi) \subset Y_1 \cup \ldots \cup Y_p$.

Proof.
Let us assume that $Res_{Y_1,\ldots,Y_p}\omega/(f_0 \ldots f_p)$ is $\overline{\partial}-$closed. This means that $\omega \wedge \overline{\partial}(1/f_0) \wedge \ldots \wedge \overline{\partial}(1/f_p) = 0$. Thus, by the lemma 8, we have: $\omega = \sum_{i=0}^p f_i\omega_i$, with ω_i holomorphic, and thus:

$$Res_{Y_1,\ldots,Y_p}\omega/(f_0 \ldots f_p) = Res_{Y_1,\ldots,Y_p}\omega_0/(f_1 \ldots f_p).$$

∎

2.4 Exact sequences of sheaves of locally residual currents

Now, let be $Y_1, \ldots, Y_p$ analytic hypersurfaces, intersecting properly, that is, $Y_{i_1} \cap \ldots \cap Y_{i_k}$ is of pure codimension k for $k \leq p$, and at each point a reduced complete intersection. Thus, for $i < p$, the hypersurfaces $Y_1, \ldots, Y_i$ also intersect properly. By the preceding, the operator $\overline{\partial}$ maps $\mathcal{C}^{q,i}_{Y_1 \cap \ldots \cap Y_i}(\star Y_{i+1})$ onto $\mathcal{C}^{q,i+1}_{Y_1 \cap \ldots \cap Y_{i+1}}$.

Lemma 10 *We have the following exact sequence of sheaves:*

$$0 \to \Omega^q_X \to \mathcal{C}^{q,0}(\star Y_1) \to \mathcal{C}^{q,1}_{Y_1}(\star Y_2) \to \cdots \to$$
$$\mathcal{C}^{q,p-1}_{Y_1 \cap \cdots \cap Y_{p-1}}(\star Y_p) \to \mathcal{C}^{q,p}_{Y_1 \cap \cdots \cap Y_p} \to 0$$

with as morphisms the $\overline{\partial}-$*operator on currents.*

Proof.
The exactitude of the preceding complex comes from the fact that locally, when the locally residual current $T \in \mathcal{C}^{q,k-1}_{Y_1 \cap \cdots \cap Y_{k-1}}(\star Y_k)_x$ is $\overline{\partial}-$closed, it belongs to $\mathcal{C}^{q,k-1}_{Y_1 \cap \cdots \cap Y_{k-1},x}$; this comes from the above exact sequence:

$$0 \to \mathcal{C}^{q,k-1}_{Y_1 \cap \cdots \cap Y_{k-1}} \to \mathcal{C}^{q,k-1}_{Y_1 \cap \cdots \cap Y_{k-1}}(\star Y_k) \to \mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k} \to 0.$$

It suffices then to remark that

$$\overline{\partial} : \mathcal{C}^{q,p-1}_{Y_1 \cap \cdots \cap Y_{k-1}}(\star Y_k) \to \mathcal{C}^{q,p-1}_{Y_1 \cap \cdots \cap Y_k}$$

is surjective. ∎

Lemma 11 *Let* $k \leq p-1$*. Let us denote* $Y^i = Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_k$*. Then, the following complex:*

$$0 \to \Omega^n_X \to \oplus_{i=1}^k \Omega^n_X(\star Y_i) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{k-2}} \oplus_{i=1}^k \Omega^q_X(\star Y^i)$$
$$\xrightarrow{\delta^{k-1}} \Omega^q_X(\star(Y_1 \cup \cdots \cup Y_k)) \xrightarrow{Res_{Y_1,\ldots,Y_k}} \mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k} \to 0$$

where the last operator is the composed residue operator, and the other operators δ^i *are the alternate sums, is exact. By allowing poles on* Y_{k+1}*, we get another exact sequence:*

$$0 \to \Omega^q_X(\star Y_{k+1}) \to \oplus_{i=1}^k \Omega^q_X(\star(Y_i \cup Y_{k+1})) \to \cdots \to \oplus_{i=1}^k \Omega^q_X(\star(Y^i \cup Y_{k+1}))$$
$$\to \Omega^q_X(\star(Y_1 \cup \cdots \cup Y_{k+1})) \to \mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k}(\star Y_{k+1}) \to 0$$

Proof.
First, it is clear that this is a complex: the succession of two alternate sums is zero, and the composed residue $Res_{Y_1,\dots,Y_k}$ is zero on a meromorphic form with pole contained in $Y_1\cup\ldots\cup Y_{k-1}$. The exactitude on the last step comes from a preceding lemma: $\omega\wedge\overline{\partial}(1/f_1)\wedge\ldots\wedge\overline{\partial}[1/f_k]=0$ implies that ω can be written as follows: $\omega=\sum_{i=1}^{k}f_i\omega_i$ with ω_i holomorphic forms. The exactitude on the first step is clear, for if $\omega_i=\omega_j$, then we can glue them together, and by Hartogs, we can extend them through $Y_i\cap Y_j$, which is of codimension two. At the other steps, the exactitude comes the Cech complex. In fact, we can consider on $X':=X\backslash(Y_1\cap\ldots\cap Y_k)$ the sheaf Ω'^q, whose sections on an open set U are meromorphic $q-$forms on X', which are holomorphic on U. Then, if we consider the covering of X' by the open subsets $U_i=X\backslash Y_i$, the exactitude of the Cech complex associated to Ω'^q relative to the covering (U_i) gives us the searched exactitude. We can modify the preceding exact sequence by allowing poles on Y_{k+1}, which is equivalent to apply the exact functor $(\star Y_{k+1})$. ∎

We deduce from the preceding that any germ of locally residual current $T=H^0(\mathcal{C}^{q,p}_{Y,x})$ can be written as a residue: $T=Res_{Y_1,\dots,Y_p}(\Psi)=\overline{\partial}Res_{Y_i,\dots,Y_{p-1}}(\Psi)$, with Ψ a germ of meromorphic $q-$form with poles in $Y_1\cup\ldots\cup Y_p$.

Remark. Let us assume that X is a Stein manifold, and that $Y_1,\dots,Y_p$ are analytic hypersurfaces intersecting properly. Since X is Stein, any cohomology group of positive degree vanishes automatically for any coherent sheaf. If $\Omega^q_X(kY)$ denotes the sheaf of meromorphic $q-$forms with poles on Y of order $\leq k$, it is coherent and thus acyclic. We conclude by passage to the direct limit that $\Omega^q_X(\star Y)$ is acyclic for any hypersurface Y. Thus, by the preceding exact sequence of lemma (10), the sheaves $\mathcal{C}^{q,k}_{Y_1\cap\cdots\cap Y_k}(\star Y_{k+1})$ are acyclic on X. By the exact sequence of lemma 11, we deduce that any locally residual current $T=H^0(\mathcal{C}^{q,p}_Y)$ can be written as a global residue: $T=\mathbf{Res}_{Y_1,\dots,Y_p}(\Psi)=\overline{\partial}Res_{Y_i,\dots,Y_{p-1}}(\Psi)$, with Ψ a meromorphic $q-$form with poles in $Y_1\cup\ldots\cup Y_p$.

Now, let us assume that X is compact, and the hypersurfaces $Y_1,\dots,Y_n$ are positive. Then we have the following:

Lemma 12 *For any $s\leq k, 1\leq j_1<\ldots<j_s\leq k$, the sheaves $\Omega^q_X(\star Y_{j_1}\cup\ldots\cup Y_{j_s}\cup Y_{k+1})$ are acyclic. From the exact sequence of lemma 11, the sheaf $\mathcal{C}^{q,k}_{Y_1\cap\cdots\cap Y_k}(\star Y_{k+1})$ is also acyclic.*

Proof.
Since the hypersurfaces Y_i are positive, we have by Serre's theorem, if we

denote L_i the corresponding Cartier divisor on X, that: $H^j(\mathcal{F}(L_i^k)) = 0$ for $k >> 0$ and $j > 0$, if $\mathcal{F}$ is a coherent sheaf on X. Since X is compact, we can do the passage to the limit when $k \to \infty$ ([6]), and we get: $H^j(\mathcal{F}(\star Y_i)) = 0$. Since the sheaves $\Omega_X^q(\star Y_{j_1} \cup \ldots \cup Y_{j_s})$ are not coherent, we can not use directly the positivity assumption on Y_{k+1}. But let us first take a limitation of the order of the poles: we get subsheaves $\Omega_X^q(Y_{j_1}^t \cup \ldots \cup Y_{j_s}^t)$, which have poles of order $\leq t$ along the Y_i. These subsheaves are coherent, thus we can apply the positivity assumption; we get thus: $H^j(\Omega_X^q(Y_{j_1}^t \cup \ldots \cup Y_{j_s}^t)(\star Y_{k+1})) = 0$ for $j > 0$; by taking the limit for t going to infinity, we get: $H^j(\Omega_X^q(\star Y_{j_1} \cup \ldots \cup Y_{j_s} \cup Y_{k+1})) = 0$ for $j > 0$. So get have the acyclicity of all the terms in the preceding exact sequence, except the last one. But then the acyclicity of $\mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k}(\star Y_{k+1})$ follows from the fact that a sheaf with an "acyclic" resolution is itself acyclic. ∎

Proof of the first theorem.

Since by lemma 12 the given resolution of Ω_X^q is acyclic, it computes its cohomology groups:

$$H^i(\Omega_X^q) \simeq H^0(\mathcal{C}^{q,i}_{Y_1 \cap \cdots \cap Y_i}) / \overline{\partial} H^0(\mathcal{C}^{q,i-1}_{Y_1 \cap \cdots \cap Y_{i-1}}(\star Y_i)).$$

We thus get the first part of theorem 1.

Moreover, let us consider the exact sequence:

$$\begin{aligned} 0 \to \Omega_X^q(\star Y_{k+1}) \to \oplus_{i=1}^k \Omega_X^q(\star(Y_i \cup Y_{k+1})) \to \cdots \to \oplus_{i=1}^k \Omega_X^q(\star(Y^i \cup Y_{k+1})) \\ \to \Omega_X^q(\star Y_1 \cup \cdots \cup Y_{k+1}) \to \mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k}(\star Y_{k+1}) \to 0 \end{aligned}$$

Since the functor of global section Γ is exact on acyclic sheaves, we get a surjective map:

$$\mathbf{Res}_{Y_1, \ldots, Y_k} : H^0(\Omega_X^q(\star Y_1 \cup \cdots \cup Y_{k+1})) \to H^0(\mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k}(\star Y_{k+1}))$$

which gives us the second part of the theorem 1.

Then, if $T \in H^0(\mathcal{C}^{q,k}_{Y_1 \cap \cdots \cap Y_k}(\star Y_{k+1}))$ can be written as $\mathbf{Res}_{Y_1, \ldots, Y_k}(\Psi)$, it is also: $T = \overline{\partial}(\mathbf{Res}_{Y_1, \ldots, Y_{k-1}}(\Psi))$, and thus T is $\overline{\partial}-$exact. Reciprocally, if T is $\overline{\partial}-$exact, then its class in $H^k(\Omega_X^q)$ is zero, and thus by the first part of the theorem, it is $T = \overline{\partial} T'$, with $T' \in H^0(\mathcal{C}^{q,k-1}_{Y_1 \cap \ldots \cap Y_{k-1}}(\star Y_k))$; but then, $T' = \mathbf{Res}_{Y_1, \ldots, Y_{k-1}}(\Psi)$, with Ψ a meromorphic $q-$form with poles contained in $Y_1 \cup \ldots \cup Y_k$; and thus, $T = Res_{Y_1, \ldots, Y_k}(\Psi)$. This achieves the proof of the theorem 1. ∎

Proof of the second theorem.

Lemma 13 *The following complex is also an exact resolution of Ω^n_X:*

$$0 \to \Omega^n_X \to \mathcal{C}'^{n,0}(Y_1) \to \mathcal{C}'^{n,1}_{Y_1}(Y_2) \to \cdots \to$$
$$\mathcal{C}'^{n,p-1}_{Y_1\cap\cdots\cap Y_{n-1}}(Y_n) \to \mathcal{C}'^{n,n}_{Y_1\cap\cdots\cap Y_n} \to 0$$

Proof.
It suffices to show that, if U is a Stein open subset and $X_1, \ldots, X_{i+1}$ are analytic hypersurfaces intersecting properly in U, a $\overline{\partial}-$closed current $\omega \wedge [Y]$, with $Y = X_1 \cap \ldots \cap X_{i+1}$ and ω a meromorphic $r-$form of maximal degree, can be written $\overline{\partial}\omega' \wedge [Y']$, with $Y' = X_1 \cap \ldots \cap X_i$ and ω' a meromorphic $(r+1)-$form on Y'. The case $i = 0$ is classical; the general case is shown by induction. ∎

Lemma 14 *We have similar exact sequence as above for logarithmic poles:*

$$0 \to \Omega^n_X \to \oplus_{i=1}^k \Omega^n_X(Y_i) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i)$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_k) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k} \to 0$$

and

$$0 \to \Omega^n_X(Y_{k+1}) \to \oplus_{i=1}^k \Omega^n_X(Y_i \cup Y_{k+1}) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i \cup Y_{k+1})$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_{k+1}) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k}(Y_{k+1}) \to 0$$

The sheaves in the left of the above exact sequences:

$$0 \to \Omega^n_X \to \oplus_{i=1}^k \Omega^n_X(Y_i) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i)$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_k) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k} \to 0$$

and

$$0 \to \Omega^n_X(Y_{k+1}) \to \oplus_{i=1}^k \Omega^n_X(Y_i \cup Y_{k+1}) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i \cup Y_{k+1})$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_{k+1}) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k}(Y_{k+1}) \to 0$$

are here coherent, and the amplitude of the divisors suffices here, by Kodaira vanishing theorem, to show that the sheaves $\Omega^n_X(Y)$, for Y some union of the Y_i (which are also positive), are acyclic, and thus also the sheaves $\mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k}(Y_{k+1})$ are acyclic. Thus the exact sequence:

$$0 \to \Omega^n_X \to \mathcal{C}'^{n,0}(Y_1) \to \mathcal{C}'^{n,1}_{Y_1}(Y_2) \to \cdots \to$$
$$\mathcal{C}'^{n,n-1}_{Y_1\cap\cdots\cap Y_{p-1}}(Y_p) \to \mathcal{C}'^{n,n}_{Y_1\cap\cdots\cap Y_n} \to 0$$

is an acyclic resolution of Ω^n_X, and thus the first part of the theorem. The acyclic exact sequence:

$$0 \to \Omega^n_X(Y_{k+1}) \to \oplus_{i=1}^k \Omega^n_X(Y_i \cup Y_{k+1}) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i \cup Y_{k+1})$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_{k+1}) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k}(Y_{k+1}) \to 0$$

also shows that any $T \in H^0(\mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k}(Y_{k+1}))$ can be written as a global residue of a meromorphic form with simple poles. Finally, the exact sequence:

$$0 \to \Omega^n_X \to \oplus_{i=1}^k \Omega^n_X(Y_i) \to \cdots \to \oplus_{i=1}^k \Omega^n_X(Y^i)$$
$$\to \Omega^n_X(Y_1 \cup \cdots \cup Y_k) \to \mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k} \to 0$$

shows that if the current $T \in H^0(\mathcal{C}'^{n,k}_{Y_1\cap\cdots\cap Y_k})$ gives a zero class in $H^k(\Omega^k)$, it can be written as a global residue with simple poles. ∎

3 Prolongations

3.1 Hodge conjecture

Let X be a projective manifold. The Hodge conjecture for the bidegree (p,p) says the following:

Conjecture 1. If T is a closed form of bidegree (p,p) with *rational class* (which means that the integral of T on any real cycle of dimension $2p$ is rational), then T is cohomologous, as current, to an integration current $\sum_i c_i[Y_i]$ with Y_i analytic subvarities and c_i rational coefficients.

Proposition. The Hodge conjecture on X, for bidegree $(n-p, n-p)$, is equivalent to the following:

Conjecture 1'. Any closed current of bidegree (p,p), with integral class, is cohomologous to a $d-$closed locally residual current of bidegree (p,p).

Proof.
First, Hodge conjecture implies that any closed current of bidegree (p,p) with *rational cohomology class* is cohomologous to integration current $\sum_i c_i[Y_i]$ with rational coefficients, and thus conjecture 1.

Reciprocally, let us assume that conjecture 1 is true. First, we know that the Hodge conjecture for bidegree $(n-p,n-p)$ is equivalent to the following: For a given closed current T of bidegree (p,p), with rational cohomology class, then: if $\int_Y T = (Y,T) = 0$ for any *complex* subvariety Y of dimension p, then the current is exact. In fact, if Hodge conjecture is true for bidegree $(n-p,n-p)$, then it follows from the condition: $\int_Y T = (Y,T) = 0$ for any *complex* subvariety Y of dimension p, that $(\mu, T) = 0$ for any μ of degree $2p$ with rational class, and thus that T is exact. And conversely, if this condition implies exactitude of T, then the $\mathbb{Q}-$vector spaces of finite dimension in $H^{2n-2p}(X,\mathbb{Q})$ generated respectively by integration currents $[Y]$ or by $(n-p, n-p)-$currents of rational class have the same orthogonal in $H^{2p}(X,\mathbb{Q})$, and thus coincide.

Now, let us remark that we could deduce from conjecture 1 that any such current T has a representative as *integration current*, of the form $\sum_i c_i[Z_i]$, with Z_i irreducible analytic subvarieties of codimension p and c_i complex coefficients. In fact, let us consider a $d-$closed locally residual current T of bidegree (p,p). We can include the support of T in a (non-irreducible) generically reduced complete intersection $Z = Y_1 \cap \ldots \cap Y_p$, and write locally T, in a neighborhood of a point $x \in Z$, as $Res_{Y_1,\ldots,Y_p}\Omega_X$ for some meromorphic closed $p-$form ω with $Pol(\omega) \subset Y_1 \cup \ldots \cup Y_p$. Then we associate to T at x the number $\int_{|f_1|=\epsilon_1,\ldots,|f_p|=\epsilon_p} \Omega_X$, if f_i are the local defining functions for Y_i. These numbers do not depend of the choices of ω or of the f_i, and are a constant c_i along an irreducible component Z_i of Z. We thus associate to T a current $\sum_i c_i[Z_i]$ with Z_i the irreducible components of Z and c_i the numbers just defined. This current is homologous to T, since it gives the same integration on any real oriented $2p-$dimensional subvariety.

But now, we can show that since$T$ has rational class, we can find such a representative $\sum_{i=1}^k c_i[Z_i]$, with *rational coefficients*. In fact, we can first assume that the $[Z_i]$ give independant cohomology classes over $\mathbb{Q}$. Then, we can find rational cohomology classes $\phi_i \in H^{2n-2p}(X,\mathbb{Q})(1 \leq i \leq k)$ such that $[Z_i](\phi_j) = \delta_{ij}$. Thus, we get: $T(\phi_j) = c_j$, and thus the coefficients c_j are rational since T has rational class. ∎

The Hodge conjecture implies the following (also classical) weaker one: For a cycle $\sum_i c_i Y_i$ with *rational coefficients*, numerical and cohomological equivalence coincide. We would have the following stronger one, without assuming that the coefficients are rational:

Conjecture 2. If a $d-$closed locally residual current T of bidegree (p,p) has zero integral on any complex subvariety Y of dimension p, it is exact.

Notice that the conjecture would not be true if we replace $d-$closed by $\overline{\partial}-$closed. We have by the preceding that on any complex variety Y of dimension p, the "restricted" residual current of bidegree (p,p) on Y is globally residual, thus residue of a closed meromorphic $p-$form on Y. We should use the closed property of T to show that we can choose these meromorphic $p-$forms in such a way that they glue together.

Let us notice that, by the preceding, it would suffice to show this for

T being an integration current $\sum_i c_i[Y_i]$, with Y_i complex subvarieties of codimension p, and the c_i *real* coefficients.

The conjecture could still be generalized for other bidegrees. In fact, these integrals on complex cycles could be viewed as a kind of *Abel-Radon transform*. Thus we have the more general conjecture:

Conjecture 3. If a $d-$closed locally residual current of bidegree $(q,p), q \geq p$ has zero Abel-Radon transform on the space of $p-$dimensional complex cycles, it is exact (or, what is the same, globally residual).

3.2 Abel-Radon transform

The first theorem could be generalized to the case when X is the complement of an $(n-p)-$complete compact subset in a projective variety Z, and the hypersurfaces Y_i on X extend to positive hypersurfaces on Z. We will use this generalization in a future article to show the a generalization in arbitrary codimension of the result concerning the Abel-Radon transform shown in [5] for codimension 1. This generalization would assert that when the Abel-Radon transform of a locally residual current T of bidegree $(q,p), q > p$, in an open subset of the projective space which is reunion of a continuous family of $p-$planes, is zero, then the locally residual current T in this open subset extend to a global residual current on the projective space. This could also be extended on other projective varieties than the projective space.

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