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Filosofie licentiatavhandling

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1 Introduction

Much of mathematics evolves from trying to solve different kinds of equations: polynomial equations, differential equations and other kinds of equations. A differential equation is just a polynomial equation in the variables $D^n x$, $n \in \mathbb{N}$ over some ring R of functions. A partial differential equation is $p = 0$, where $p \in R[\partial^\alpha x]$, $\alpha \in \mathbb{N}^m$, $x = (x_1, ..., x_m)$.
One can also consider equations in the variables $\tau^n(x)$ where τ is an operator, injective ring endomorphism, in which case the equations are called difference equations. Such endomorphisms extend to the field of quotients and that field can be embedded in a bigger field, the inversive closure, where the endomorphism is surjective, an automorphism, see [11].
The structures where one studies these equations, difference fields, can be considered as fields on which $\mathbb{Z}$ acts by automorphism. The model theory of such structures have been thoroughly investigated in [24] and [8].

This thesis will describe structures suitable for studying equations in the variables gx, for g in a group G : Fields with a group action.

Generally when studying equations one is interested to know if a certain structure has solutions to as many equations it could possibly have or if one can find new solutions in an extension. Another question one would ask is if it is possible to say in an understandable way exactly which equations and systems of equations, that can have solutions.
In model theory we usually answer these questions by the concepts of existentially closed structures and model companion (the first order theory of the existentially closed structures if the class of such structures is elementary) respectively. Both concepts go back to Robinson and were inspired by the theory of algebraically closed fields, often called ACF.
ACF is the model companion of the theory of fields (and of the theory of integral domains) and it is a suitable place to work with polynomial equations. Robinson also showed that there is a model companion to the theory of differential fields called DCF, differentially closed fields.
Around 1990 a model companion for the theory of difference fields was found (by several people working together, most notably Angus Macintyre, see [24] and [8]). In recent years a model companion for the structures suitable for studying partial differential equations was found independently by Yaffe, [33], and McGrail, [23].

In what follows we will develop these themes for the theory of fields with a group action.
If G is a group, then the fields which G acts on, the structures we will be interested in, will be called G-fields, see definitions below.
The model theoretic properties of these theories will depend on the group. We begin by developing some basic field theoretic and model theoretic properties in section three and four. Being existentially closed has some strong implications on the structure of the fields, for instance on their Galois theory. We prove that the class of existentially closed structures has quantifier elimination in a suitably extended language and that the existential part of the class of existentially closed structures is decidable.
When it comes to model companions the situation is thus:
We first introduce a first order theory, $\mathbb{T}_G$, that properly contains the theory of G-fields and such that every existentially closed G-field is a $\mathbb{T}_G$-model.
It follows from a result of Hrushovski that if $\mathbb{Z} \times \mathbb{Z} \leqslant G$, then there is no model companion, the class of existentially closed structures is not an elementary class.
We prove here that if G is free, finite or if it satisfies a technical condition that will be explained later, there is a model companion. In the last case we do it for a slightly altered theory, the theory of G-fields that contain all the p-th roots of unity, for every prime p. The question about existence of model companions for other groups we leave open here, but will return to it some other time.

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2 Preliminaries

Let G be a group. When we say that G acts on a field K we mean that there is a group homomorphism $\rho : G \to \mathrm{AutK}$, the group of automorphisms on K, and we write $\rho(g); a \mapsto ga$. We immediately drop ρ from our notation, so it becomes $g; a \mapsto ga$, and sometimes, to emphasize the multiplicative nature of the action we will write $g; a \mapsto a^g$. A G-action is faithful if ρ is a monomorphism.
If A is a subset of field then $GA = \{ga : g \in G, a \in A\}$
Some group notation: other groups than G will be called H or some other upper case letter, and if A and B are two groups, then $A \leqslant B$ means that A is a subgroup of B, and if we are in the context of profinite groups it will mean that it is a closed subgroup. The identity element of G or any other group will be denoted 1.
A field on which G acts will be called a G-field. For technical reasons we do not assume that the action is faithful, so any field can be a G-field for any group G, but if we say that K is a G-field, it means that K comes with a specified G-action. However the G-action on an existentially closed G-field will be faithful automatically. If the action of G on a field K is not faithful, there will be an $h \in G$ such that $K \models \forall x(hx = x)$, but we can always add a new transcendental t and define a new action on an extension of $K(t)$ such that h moves t. For instance: let $C = K(t_g : g \in G)$ and define $g_i(t_{g_j}) = t_{g_i g_j}$ and the action on K is the same as before, then C is an extension of K and $\exists x(hx \neq x)$ is true in C, but not in K.
$L_{Ring} = \{+, \cdot, -; 0, 1\}$ is the language of rings. Let $L_G = L_{Ring} \cup \{g_L : g \in G\}$, where the g_L are new function symbols.
The theory of G-fields, T_G will be the theory of fields together with the sentences expressing that g_L are field automorphisms and the relations of G,

$$\forall x(g_L(h_L x) = (gh)_L x))$$

If the group G is finitely presented, see for instance [27], we can use another language, L_σ, that has the same expressive power but is more practical when working with finiteness conditions like if something is first order definable.
Let $G = \langle \sigma, R \rangle$, where $\sigma = (\sigma_1, ..., \sigma_n)$ is a tuple of generators for G and R is a set of words in $\sigma \cup \sigma^{-1}$, called relations. This shall be interpreted as follows: If F_σ is the free group on σ and $\bar{R}$ is the normal closure in F_σ of the group generated by the elements in R, then $G = F_\sigma / \bar{R}$, see [27].
Let $L_\sigma = L_{Ring} \cup \sigma_L \sigma_L^{-1}$, where $\sigma_L \sigma_L^{-1}$ is an $2n$-tuple of new function symbols (intended for the generators of G and their inverses). The theory of G-fields, T_G will be the theory of fields together with the sentences expressing that σ_{iL} are field automorphisms and a set R_L of equational relations interpreting the relations R of G. If $w \in R$, then R_L shall contain $\forall x(w(x) = x)$.
We will use both languages without mentioning when we switch between them. The subscript L on σ_L, g_L and R_L will henceforth be dropped; we use the same name for these objects both in the context of group theory and in the context of logic.
We will also use the tuple $\bar{\sigma} = (1, \sigma_1, ..., \sigma_n)$ at times.

The basic model theoretic concepts that we will discuss can be found in [15], but here are some of them for ease of reference:
If L is a language then $L_\exists$ means the set of all existential formulae in L. If A is any set (for instance a G-field) then $L(A)$ will denote the language we get from L by adding new constants for all elements of A. We will use the same name for those constants as for the elements.
A model M of a theory T is existentially closed (short: e.c.) if for any supermodel $N(\supseteq M)$ and any sentence $\phi \in L(M)_\exists$ $N \models \phi$ implies that $M \models \phi$
The model companion of a theory T is the first order theory of the e.c. models of T if the class of e.c. models is elementary.
A structure A is an amalgamation base if whenever $A \subseteq B$ and $A \subseteq C$, B and C can be jointly embedded in a bigger model D.
A theory, or a class of structures, is decidable if both the set of true sentences and its compliment are recursively enumerable.
A type p is a consistent (usually complete) set of formulae. $S(A)$ denotes the set of all complete types with parameters from A. $S(A) \subseteq \mathcal{P}(L(A))$
A theory T has quantifier elimination if for every substructure A of a T-model, $T \cup \mathrm{diagA}$ is a complete theory, where diagX denotes the set of true, quantifier free sentences in $L(X)$.
This is equivalent to if A is substructure of both B and C, then there is a T-model D such that $B \preceq D$ and $C \subseteq D$, and also equivalent to that every formula is equivalent to a quantifier free formula. See theorem 13.1 of [29]

Later, definition 7, we will introduce a big existentially closed G-field $\bar{\mathbf{C}}$ with nice properties and work inside it.
Lower case letters from the end of the alphabet $x, y, x_i, ..$ will be tuples of variables, lower case letters from the beginning of the alphabet $a, b, a_i, ..$ will be tuples of elements from a field (later on from $\bar{\mathbf{C}}$), $A, B, A_i, ..$ will be small ($< \left|\bar{\mathbf{C}}\right|$) subsets of G-fields (later $\subseteq \bar{\mathbf{C}}$), and $k, K, F, M, ..$ will be G-fields (models of T_G, later small sub-models of $\bar{\mathbf{C}}$) if not explicitly mentioned to be something else. We will usually use the same letter to denote the model and the underlying set even though we sometimes define different models on the same set, and hope this will not cause confusion. We will also use the same notation for the multiplicative identity of the field and the identity element of the group (namely 1).

$\twoheadrightarrow$ denotes an epimorphism and $\rightarrowtail$ a monomorphism of groups, fields or whatever.
If M is a field and $H \subseteq \mathrm{AutM}$, then $M^H = \{x \in M : hx = x, h \in H\}$, fixed field of H in M.
If k is a field, then $\bar{k}$ denotes the algebraic closure of k (as a field). And if A and B are two fields, with $A \leqslant B$ algebraic, then the Galois group of B/A is $\mathrm{Gal(B/A)} = \mathrm{Aut_AB}$, the group of field automorphisms of B fixing A point-wise. And the absolute Galois group of of a field F is $G(F) = \mathrm{Gal(\bar{F}/F)}$.

We will need to introduce some terminology and basic facts about G-equations. A G-equation over a G-field K is polynomial equation in the variables gx_i where g is in G.
$f = 0$ where $f \in K[gx_i : g \in G, 1 \leqslant i \leqslant m] = K[Gx]$, where $x = (x_1, ..., x_m)$
And $K[Gx]$ is a G-ring by:

$$hf = \Sigma h a_i (h\bar{g}_i x)^i$$

if

$$f = \Sigma a_i (\bar{g}_i x)^i$$

where $i \in \mathbb{N}^{<\omega}$, $\bar{g}_i x = g_{i1} x_1 \cdot ... \cdot g_{im} x_m$, $g_{ij} \in G$, $h \in G$
and $h\bar{g}_i x = hg_{i1} x_1 \cdot ... \cdot hg_{im} x_m$.
For $f \in K[x]$, $f = \Sigma a_i x^i$ and $\tau \in \mathrm{AutK}$, a field-automorphism, let $\tau f = \Sigma \tau(a_i) x^i$, so $f(b) = 0$ if and only if $\tau f(\tau(b)) = 0$.

If $I \subset K[x]$ is an ideal let $I^\tau = \{\tau f : f \in I\}$ and if $V \subset \mathbb{A}^m$ is a variety and $I_V = \{f \in K[x] : p \in V \Rightarrow f(p) = 0\}$, then $V^\tau = \{p \in \mathbb{A}^m : f \in I_V \Rightarrow \tau f(p) = 0\}$.

A quantifier-free formula is just a boolean combination of G-equations, but for many purposes one needs only consider a certain kind of those.
$K[\bar{\sigma} x]$ is the sub-ring (not a G-ring) of $K[Gx]$ consisting of polynomials in the variables $(x, \sigma_1(x), ..., \sigma_n(x)) = \bar{\sigma} x$.

Proposition 1. *For every quantifier-free formula ϕ there is a formula $\bar{\phi} = \bigwedge_i f_i = 0$, a finite conjunction of G-equations, where $f_i \in K[\bar{\sigma} x]$ and for every model M of T_G, $M \models \exists x \phi$ if and only if $M \models \exists y \bar{\phi}$. ($\bar{\phi}$ usually have more free variables than ϕ, $|x| \leq |y|$.)*

Proof: We may assume that ϕ is in conjunctive normal form, a finite conjunction of finite disjunctions of G-equations and G-inequations. For every inequation, $f(x) \neq 0$, replace it by $yf(x) - 1 = 0$, so we have a conjunction of disjunctions of G-equations, but every disjunction of equations is equivalent to a single equation ($f = 0 \vee g = 0 \Leftrightarrow fg = 0$), so we have a conjunction of G-equations.
Now let $f = 0$ be one of them, $f \in K[Gx]$:
Let $f = \Sigma_i a_i (g_i x)^i$, $i \in \mathbb{N}^m$, $g_i = (g_{i1}, ..., g_{im})$ and $g_{i_j} = \tau_{ij1} \cdot ... \cdot \tau_{ijl}$, where τ_{ijk} is either a generator, an inverse of a generator or 1. We assume that the presentations has been chosen so that they are as short as possible.
Now assume that the maximal length of a presentation $g_{i_j} = \tau_{ij1} \cdot ... \cdot \tau_{ijl}$, such that $g_{ij} x$ is a variable in f, is l.
Then for every g_{ij} that is not a generator or 1, change $g_{ij} x$ to $\tau_{ij1} y_j$ if τ_{ij1} is a generator and to y_j if τ_{ij1} is an inverse of a generator and add the new equation $y_j - \tau_{ij2} \cdot ... \cdot \tau_{ijl} x_j = 0$ or $\tau_{ij1} y_j - \tau_{ij2} \cdot ... \cdot \tau_{ijl} x_j = 0$ in the respective cases.
In the resulting conjunction f^1 every g_{ij} that occurs in a variable has a presentation shorter than l. Repeat this procedure until every g_{ij} that occurs in a variable is either a generator or 1. That will happen after less than l repetitions.
Let $f^l = \bar{f}$ and let $\bar{\phi}$ be $\bigwedge \bar{f}$ when ϕ is $\bigwedge f = 0$.
Then $\bar{\phi}$ is a conjunction of equations from $K[\bar{\sigma} xy]$ and there is a $x \in M$ such that $M \models \phi$ if and only if there is a $xy \in M$ such that $M \models \bar{\phi}$ ∎

In that way every q.f.-formula in $L_\sigma(K)$ is equivalent to a finitely generated ideal in $K[\bar{\sigma} x]$.

3 Field theoretic properties of e.c. G-fields

We will need some facts about profinite groups and pseudo-algebraically closed fields. Such facts can be found in [32] and/or [13], but we mention a few that will be vital in the sequel.
Profinite groups are inverse limits of finite groups, or equivalently compact, Hausdorff groups that have a basis for the neighborhoods of 1 consisting of normal subgroups. Galois groups are profinite and every profinite group occurs as a Galois group for some field extension.
An inverse limit of finite p-groups will be called a pro-p-group.

Definition 1. *If A is a*

group, then the profinite completion $\hat{A}$ of A is the inverse limit of all finite quotients of A.

$$\hat{A} = \varprojlim A/N$$

where N runs through all normal subgroups of finite index.

We will call the set of all such finite quotients Im(A). If $B \in$ Im(A), then there is an epimorphism from A to B.

A variety X over a field k is absolutely irreducible if $\bar{k}I_X$ is prime, that is if X is also irreducible as a variety over $\bar{k}$.

Definition 2. *A field A is pseudo algebraically closed, abbreviated PAC, if every absolutely irreducible variety over A has an A-rational point.*

Definition 3. *A profinite group G is projective if for every profinite A, B with $\alpha : A \twoheadrightarrow B$ and $\beta : G \to B$ there is a $\gamma : G \to A$ such that*

$$\begin{tikzcd} & G \arrow[dl, "\gamma"'] \arrow[d, "\beta"] \\ A \arrow[r, two heads, "\alpha"'] & B \end{tikzcd}$$

commutes.

Equivalently a profinite group is projective if all its p-Sylow subgroups are free pro-p-groups, for every prime p, or if its cohomological dimension is $\leqslant 1$ (Corollary 11.2.3 of [32]).
Contrary to the case of abstract groups, there are projective profinite groups that are not free.
Galois groups of PAC-fields are projective and every projective profinite group occurs as a Galois group of a PAC-field. Actually Ax, [1], proved that the Galois groups of PAC-fields has cohomological dimension $\leqslant 1$.
If B is a projective profinite group and $\alpha : A \twoheadrightarrow B$, then there is a closed subgroup $C \leqslant A$ such that $\alpha \restriction C$ is a bijection. To see this, let β be the identity in the above definition.

Define a partial order on the class of epimorphisms in the category of profinite groups by $\alpha \leqslant \beta$ if they have the same codomain and there is an epimorphism γ such that $\alpha\gamma = \beta$.

Definition 4. *A universal Frattini cover (also known as a projective cover) of group A,*

$$\tilde{\phi} : \tilde{A} \twoheadrightarrow A$$

is a smallest epimorphism from a projective profinite group to A.

Other characterizations equivalent to the one above is:
(i) $\tilde{\phi}$ is the biggest epimorphism to A with kernel in the Frattini subgroup $\Phi(\tilde{A})$, which is the intersection of all maximal open subgroups of $\tilde{A}$.
(ii) $\tilde{\phi}$ is an epimorphism to A with kernel in the Frattini subgroup $\Phi(\tilde{A})$ and the domain of $\tilde{\phi}$ is projective.
By common abuse of notation we will refer to both $\tilde{\phi}$ and $\tilde{A}$ of the above definition separately as "the" universal Frattini cover. It is unique up to homeomorphism.

Now we are ready to examine the existentially closed G-fields.

From now and to the end of this section let K be an existentially closed G-field and $F = K^G$, the fixed field of G in K.

Theorem 1. *K and F are perfect.*

Proof: Let $p = \mathrm{charK}$.
First let $a \in K \cap F^{p^{-\infty}}$. Then $a^{p^m} \in F$ for some $m \in \mathbb{N}$. If for any $g \in G$, $ga = b$, then $a^{p^m} = g(a^{p^m}) = (ga)^{p^m} = b^{p^m}$, so $0 = a^{p^m} - b^{p^m} = (a-b)^{p^m}$. Hence $a = b$ and therefore $a \in F$.
Now let $c^p \in K$. Let $\sigma = \{\sigma_i : i \in I\}$ be a, possibly infinite, set of generators for G, and $\varrho : \mathbf{F}_\sigma \to G$ a presentation. Extend the automorphisms σ to $\bar{K}$ any way you like; find $\dot{\sigma}_i \in \mathrm{Aut}\bar{\mathrm{K}}$ with $\dot{\sigma}_i \restriction K = \sigma_i$. Close $K(c)$ under the action of the $\dot{\sigma}_i$ and call the closure A. Let $\tau_i = \dot{\sigma}_i \restriction A$ and $H = \langle \tau_i : i \in I \rangle$. There is an epimorphism $\varrho : H \to G$, the restriction map. Now let a be any element in A and w any element in $\ker \varrho$ and assume that $wa = b$. $a^p \in K$ so

$$a^p = wa^p = b^p \Rightarrow 0 = a^p - b^p = (a-b)^p \Rightarrow a = b$$

So $w = 1$ and therefore ϱ is an isomorphism and since K is e.c. $A = K$. ■

Theorem 2. *F is pseudo algebraically closed.*

Proof: Let V be an absolutely irreducible variety over F. Define an action of G on $K \bigotimes_F F(V)$, where $F(V)$ is the function field, by $g \mapsto g \otimes 1$. This action extends uniquely to the field of fractions $K(V)$, which is a G-field that extends K. Because there is a $K(V)$-rational point, $x \in F[x]/I(V)$, there has to be a point in V(K). And since x is fixed by the G-action in $K(V)$ there has to be one in F. So F is PAC. ■

Theorem 3. *K is pseudo algebraically closed.*

Proof: Let V be an absolutely irreducible variety over K. Then if $g \in G$, V^g is also over K. $V^g = V(gI(V)) = \{p : gf(p) = 0,\ f \in I(V)\}$ and $gf \in K[x]$. In a large (saturated) algebraically closed field extending K, choose a_g for every $g \in G$ such that a_g is a generic point of V^g over $K(a_h : h \in G, h \neq g)$. Set

$M = K(a_g : g \in G)$, and define a G-action on M by $g(a_h) = a_{gh}$for every $g, h \in G$. That makes M a G-field, and $V(M) \neq \emptyset$ (a_1 is there) and K is e.c. so $V(K) \neq \emptyset$, so K is PAC. ■

A profinite group is small if it has, for every positive ineger n, only finitely many normal subgroups of index n.
A PAC-field is bounded if it has only finitely many separable extensions of each finite degree.
So a PAC-field is bounded if and only if it's Galois group is small.

From now on assume that G is finitely presented and $G = \langle \sigma, R \rangle$.

Theorem 4. $\mathrm{Gal}(\bar{\mathrm{F}} \cap \mathrm{K}/\mathrm{F})$ *is the profinite completion of* G

Proof: $K \cap \bar{F}/F$ is a normal and separable extension. Let $\mathcal{A} = \mathrm{Gal}(\mathrm{K} \cap \mathrm{F}^{\mathrm{s}}/\mathrm{F})$ and let H be finite quotient of G.
$\pi : G \twoheadrightarrow H$ and $|H| = m < \omega$
Embed H into S_m, $\epsilon : H \rightarrowtail S_m$. Let $f(x) = \prod_i (x - t_i)$ and $N = F(t_1, ..., t_m)$ and $M = N[x]/(f)$. The length of x is one here.
Let H act on N by $h(t_i) = t_{\epsilon(h)i}$, and let $F' = N^H$. Then the coefficients for f are in F' because they are symmetric polynomials in the t_i's, and the splitting field of f over F', $M = F'[x]/(f)$ has a Galois group isomorphic to H.
Let K' be the fraction field of $K \bigotimes_{\bar{F} \cap K} M$ and let G act on K' by $g \otimes \pi g$, so that K' is a G-field extending K.
Set $\phi = \exists y_0, ..., y_m, x\, (p(y, x) = 0 \bigwedge_{i,k} \sigma_i(y_k) = y_k \bigwedge_i$ "$\sigma_i(x) = \pi\sigma_i(x)''$), where $p(s(t), x) = f(x)$ and $s(t)$ are the appropriate symmetric polynomials in t and the last conjuncts are an abbreviation for that the relations of H are respected on x.
Then $K' \models \exists x \phi(x)$ and K is e.c. so ϕ is satisfied in K, but then F has a Galois extension with Galois group isomorphic to H inside K. Therefore H is a quotient of $\mathcal{A}$ and H was arbitrary, so:

$$\mathrm{Im(G)} = \mathrm{Im}(\hat{\mathrm{G}}) \subseteq \mathrm{Im}(\mathcal{G})$$

And $\mathrm{Im}(\mathcal{A}) \subseteq \mathrm{Im}(\hat{\mathrm{G}})$ trivially.
$\hat{G}$ is small since G is finitely generated, and two small profinite groups are homeomorphic if and only if they have the same finite quotients. ■

Theorem 5. *The absolute Galois group of* F *is the universal Frattini cover of the profinite completion of* G.

$$\tilde{\phi} : \tilde{G} \twoheadrightarrow \hat{G}$$

and $G(F) = \tilde{G}$

Proof: Let α be the restriction map

$$\alpha : G(F) \twoheadrightarrow \hat{G}(= \mathrm{Gal}(\mathrm{K} \cap \mathrm{F}^{\mathrm{s}}/\mathrm{F})); \mathrm{g} \mapsto \mathrm{g} \restriction \mathrm{K} \cap \mathrm{F}^{\mathrm{s}}$$

Now $G(F)$ projective by theorem two and the universal Frattini cover $\tilde{\phi} : \tilde{G} \twoheadrightarrow \hat{G}$ is the smallest projective cover of $\hat{G}$, so there is an epimorphism γ_1 with $\tilde{\phi}\gamma_1 = \alpha$ And since $\tilde{G}$ is projective there is a subgroup $\mathcal{G} \leqslant G(F)$ such that $\gamma_1 \restriction \mathcal{G}$ is a

homeomorphism.
Let $\gamma_2 = (\gamma_1 \upharpoonright \mathcal{G})^{-1}$ and $\iota : \mathcal{G} \rightarrowtail G(F)$ the embedding. Then

$$\begin{tikzcd} & & G(F) \arrow[dll, shift left, "\gamma_1"] \arrow[d, two heads, "\alpha"] \\ \tilde{G} \arrow[urr, shift left, "\iota\circ\gamma_2"] \arrow[rr, two heads, "\tilde{\phi}"'] & & \hat{G} \end{tikzcd}$$

commutes. So there is a split exact sequence:

$$\begin{tikzcd} \ker\gamma_1 \arrow[r, tail] & G(F) \arrow[r, two heads, "\gamma_2\circ\gamma_1"'] & \mathcal{G} \arrow[l, bend right, "\iota"'] \end{tikzcd}$$

Therefore $G(F) = \ker\gamma_1 \rtimes \mathcal{G}$, the semi-direct product, see for instance [27].

Then
$$F = \bar{F}^{G(F)} = \bar{F}^{\ker\gamma_1} \cap \bar{F}^{\mathcal{G}}$$
so $\bar{F}^{\ker\gamma_1}$ is linearly disjoint from $\bar{F}^{\mathcal{G}}$ over F, and $\bar{F} = \bar{F}^{\ker\gamma_1}\bar{F}^{\mathcal{G}}$, because $G(F)$ is the semi-direct product of $\ker\gamma_1$ and $\mathcal{G}$.
Since $K \cap F^s \subset \bar{F}^{\ker\gamma_1}$,because $\ker\gamma_1 \leqslant \ker\alpha$ and α is restriction to $K \cap F^s$. K is linearly disjoint from $\bar{F}^{\mathcal{G}}$ over F, so we can define an action of G on $K\bar{F}^{\mathcal{G}} = K \bigotimes_F \bar{F}^{\mathcal{G}}$ by $g \mapsto g \otimes 1$, but $K\bar{F}^{\mathcal{G}}$ is algebraic over K and K is e.c. so $K\bar{F}^{\mathcal{G}} = K$, which implies that $\bar{F}^{\mathcal{G}} = F$ and therefore

$$\bar{F}^1 = \bar{F} = \bar{F}^{\mathcal{G}}\bar{F}^{\ker\gamma_1} = \bar{F}^{\mathcal{G}} \bigotimes_F \bar{F}^{\ker\gamma_1} = F \bigotimes_F \bar{F}^{\ker\gamma_1} = \bar{F}^{\ker\gamma_1}$$

hence $\ker\gamma_1 = 1$, so γ_1 is a homeomorphism. ■

Theorem 6. *The Galois group of K is homeomorphic to the kernel of the universal Frattini cover of the profinite completion of G*
$\ker\tilde{\phi} \rightarrowtail \tilde{G} \twoheadrightarrow \hat{G}$,
$G(K) \cong \ker\tilde{\phi}$

Proof: From theorem 4 we know that $\mathrm{Gal}(\mathrm{F^s} \cap \mathrm{K/F}) \cong \hat{\mathrm{G}}$, the profinite completion of $\hat{G}$ and from theorem 5 we know that $G(F) \cong \tilde{G}$, the universal Frattini cover of $\hat{G}$.
By elementary Galois theory $G(F^s \cap K) \cong \ker\tilde{\phi}$.
For any Galois extension A/B we have

$$G(A) \rightarrowtail G(B) \twoheadrightarrow \mathrm{Gal(A/B)}$$

where the epimorphism is the restriction map.

Now K is regular over $F^s \cap K$ (see theorem 1) and therefore the restriction map $\mathrm{res} : \mathrm{G(K)} \to \mathrm{G(F^s \cap K)}$ is an epimorphism.

$G(F^s \cap K) \cong \ker\tilde{\phi}$ is projective since it is a Galois group of the PAC-field $\bar{F} \cap K$.

$$\begin{tikzcd}
& \bar{K} \arrow[dl, dash, "G(K)"'] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & F^s \arrow[dl, dash, "\ker\tilde{\phi}"] \\
& \bar{F} \cap K \arrow[d, dash, "\hat{G}"] & \\
& F &
\end{tikzcd}$$

So there is a subgroup $H \leqslant G(K)$ homeomorphic to $\ker\tilde{\phi}$, $\mathrm{res} \restriction \mathrm{H} : \mathrm{H} \xrightarrow{\sim} \ker\tilde{\phi}$.
Let $M = \bar{K}^H$.

$$\begin{tikzcd}
K^s \arrow[d, dash, "H"] \\
M \arrow[d, dash, "G(K)/H"] \\
K
\end{tikzcd}
\qquad
\begin{tikzcd}
G(K) \arrow[dd] & \\
& H \arrow[ul] \\
\ker\tilde{\phi} \arrow[ur] &
\end{tikzcd}$$

$$\sigma_i \in \mathrm{Aut_F K},\ ,\mathrm{R} \ni \mathrm{w} = \sigma_{\mathrm{i_1}}^{\mathrm{p_1}} ....\sigma_{\mathrm{i_k}}^{\mathrm{p_k}}\ , \mathrm{p_j} = 1, -1$$

Assume $M \neq K$. Then there is a $w \in R$ such that for every $\dot{\sigma} \in \mathrm{Aut_F M}$, such that $\dot{\sigma} \restriction K = \sigma$

$$\dot{\sigma}_{i_1}^{p_1} ... \dot{\sigma}_{i_k}^{p_k} = \dot{w} \neq 1$$

But then $\dot{w} \in \mathrm{Gal(M/K)} \cong \mathrm{G(K)/H}$, because $\dot{w} \restriction K = 1$.
One of the possible extensions of w to $\mathrm{Aut_F \tilde{K}}$ has to be in H since $w \in \mathrm{Gal(F^s \cap K)}$ and res $\restriction$ H is a homeomorphism. Say $\ddot{w} \in H$. From elementary Galois theory we know that there is an exact sequence:

$$H \xrightarrow{\alpha} G(K) \xrightarrow{\beta} G(M/K)$$

So $\beta \circ \alpha(\ddot{w}) = 1$ - A contradiction. So therefore $M = K$. ■

$\ker\tilde{\phi}$ is pro-nilpotent, its p-Sylow subgroups are free pro-p groups and $\ker\tilde{\phi} = \prod_p S_p$, where S_p is free pro-p and with index relatively prime to p.
So F is a bounded, perfect PAC-field and K is a perfect PAC-field that is fairly well behaved, has an understandable Galois group.
K is bounded if and only if $\ker\tilde{\phi}$ is small.

4 The category of e.c.-structures

Let $\mathcal{E}_G$ be the category of existentially closed G-fields with embeddings as morphisms, as in [25]. $\mathcal{E}_G$ is a full subcategory of $\mathcal{F}_G$ of G-fields and embeddings. The category $\mathcal{F}_G$ of G-fields is a subcategory (not full) of the category of fields and embeddings. We will be working in all those categories; to prove that something is true in an e.c. G-field K one has to find a G-field extending K where it is true, and for that we sometimes have to go to a pure field extending K and define a new G-action on it compatible with the action on K.
The category $\mathcal{E}_G$ will be the subject of this section and it will be important in the rest of this text.

When studying $\mathcal{E}_G$ we are primarily interested in existential formulae and we will need some more notation.
$A \equiv_1 B$ means that A and B satisfies the same existential sentences.
$A \preccurlyeq_1 B$ means that $A \leqslant B$ and if $a \in A$ and ϕ is any existential formula, then $B \models \phi(a)$ implies $A \models \phi(a)$.
An existential type p over a set A is a consistent set of existential formulae in $L(A)$. $\mathrm{etp_M(a/A)} = \{\phi \in \mathrm{L(A)_\exists : M} \models \phi(\mathrm{a})\}$ is the existential type of a over A in M. An existential type p is maximal if for any existential type q (over the same set) $p \subseteq q \Rightarrow p = q$. For instance, if M is e.c. and $a \in M$ then $\mathrm{etp_M(a)}$ is maximal.

Definition 5. *Let C be a G-field, then C is κ-existentially universal (κ-e.u.) if the following equivalent conditions holds.*
(i)For every subset A of C, $|A| < \kappa$ and every existential type p over A, if p is realized in an extension of C, then p is realized in C.
(ii)For every subset A of C, $|A| < \kappa$ and every existential type p over A, if every finite subset of p is realized in C, then p is realized in C.
(iii) For every $N, M \in \mathcal{E}_G$, if $M \subset C$, $M \hookrightarrow N$ and $|N| < \kappa$, then $N \hookrightarrow C$.

Facts: For every $K \in \mathcal{E}_G$, $|K| < \kappa$, there is a κ-existentially universal C, $|C| = \kappa$, such that $K \subset C$. See [26].
Any κ-existentially universal G-field is existentially closed, and it has the following special homogeneity property (equivalent to the defining properties): If A and B are small (cardinality $< \kappa$) subsets of a κ-e.u. G-field C and if $f : A \to B$ is any bijection such that $\mathrm{etp_C(a)} \subseteq \mathrm{etp_C(f(a))}$ for every tuple $a \in A$, then f extends to an automorphism of C. See [25].
A structure A that is $|A|$-e.u. will be called existentially universal, e.u.

A field extension K/k is regular if it is separable and k is relatively algebraically closed in K.
If the extension A/B is regular, we also say that A is regular over B.

Proposition 2. *If $A, B \in \mathcal{E}_G$ and $A \subseteq B$, then A and B have the same existential theory and B is regular over A.*

Proof: $A \preccurlyeq_1 B$ ■

If X is a set in the intersection of Y and Z, a X-morphism from Y to Z is a morphism that fixes X pointwise.

Theorem 7. *Let D be κ-e.u. G-field $A \subseteq D$, K regular over A, $|K| < \kappa$ and $K \models T_G$, then there is an A-embedding of K into D.*

Proof: Find an A-isomorphic copy K' of K,linearly disjoint from D over A, in a big algebraically closed field extending K and D. This is possible since K is regular over A. Now the field of fractions of $K' \bigotimes_A D$ is a G-field extending D, so every existential type over A realized in the field of fractions of $K' \bigotimes_A D$ is realized in D since D is existentially universal. That is especially true for $p = \{\phi : \phi \in L(A)_\exists \text{ and } K \models \phi\}$. ■

Definition 6. *If A is a subset of $K \in \mathcal{E}_G$, then $\langle GA \rangle = \langle \{g(a) : g \in G, a \in A\} \rangle$, where $\langle B \rangle$ means the smallest field containing the set B. Let $cl_K(A)$, the closure of A, be $cl(A) = \overline{\langle GA \rangle} \cap K$, the relative algebraic closure of $\langle GA \rangle$ in K.*

cl is a closure operator and it is easily seen that if $A = cl(A)$, then A is a G-field and any existentially closed G-field extending A is regular over it.

Proposition 3. *If $A = cl(A)$, then A is an amalgamation base for T_G-models.*

Proof: Let B and C be two extensions of A. Since they are both transcendental over A we can find an A-isomorphic copy B' of B linearly disjoint from C over A, and the field of quotients of $cl(B') \bigotimes_A cl(C)$ will be a common extension of B' and C. ■

Proposition 4. *If K and M are two G-fields with $Th(K)_\forall \cong Th(M)_\forall$, then there is an existentially universal D such that K and M can be jointly embedded into D.*

Proof: If K and M have the same universal theory, their intersections with the prime field P are isomorphic. Let

$$K \cap P = M \cap P = Q$$

Let K' and M' be existentially closed models extending K and M respectively. Then $cl(Q) \subseteq K'$ and $cl(Q) \subseteq M'$ and by the previous proposition K' and M' can be jointly embedded into a model, which in turn embeds into a existentially universal domain D. ■

Definition 7. *If K is any field and P is the prime field, then the absolute numbers of K,* $\mathrm{abs(K)}$ *are the elements in K algebraic over P.* $\mathrm{abs(K)} = \mathrm{K} \cap \bar{\mathrm{P}}$.

Theorem 8. *Let E, $F \in \mathcal{E}_G$, C a closed set ($C = cl_E(C) = cl_F(C)$) and $C \subseteq E \cap F$, then E and F satisfy the same existential sentences over C (abbreviated $E \equiv_{1,C} F$).*

Proof: Let D be e.u. G-field containing E. Since D is homogeneous, and E is regular over C, we can find an E' isomorphic to E over C (the isomorphism fixes C point-wise) and linearly disjoint from F over C. The field of quotients of $E' \bigotimes_C F$ is a G-field that contain both E' and F. Let M be an e.c. G-field extending the field of quotients of $E' \bigotimes_C F$, then since E' and F are e.c. we have $E', F \preccurlyeq_1 M$. So $E' \equiv_{1,C} F$. ■

Corollary 1. *If A and B are two e.c. G-fields and* $\mathrm{abs(A)} \cong \mathrm{abs(B)}$*, then $A \equiv_1 B$.*

$\mathcal{E}_G$ is a disjoint union of subcategories $\mathcal{E}_{GA}$, where

$$Ob(\mathcal{E}_{GA}) = \{K \in \mathcal{E}_G : \mathrm{abs(K)} \cong \mathrm{A}\}$$

for all possible isomorphism-types of the absolute numbers. Of course $G(A)$ must be smaller or equal to the Galois group of the kernel of the universal Frattini cover of the profinite completion of G and $G(A^G)$ must be smaller or equal to $\tilde{G}$, where the order of Galois groups is given by the epimorphisms. That is if $\mathcal{E}_{GA}$ is a nonempty subcategory of $\mathcal{E}_G$, then there are epimorphisms $\tilde{G} \twoheadrightarrow G(A^G)$ and $\mathrm{ker}\tilde{\phi} \twoheadrightarrow \mathrm{G(A)}$.

From now on we assume we are in one of those subcategories and we choose a cardinal κ bigger then any field we will be interested in (for instance κ can be inaccessible).

Definition 8. *Let $\bar{\mathbf{C}}$ be a κ-existentially universal G-field, such that every G-field we consider here is a sub-G-field of $\bar{\mathbf{C}}$.*

Definition 9. *Let*

$$\mathrm{etp(a/A)} = \{\phi(\mathrm{x}) \in \mathrm{L(A)}_\exists : \bar{\mathbf{C}} \models \phi(\mathrm{a})\}$$

and let

$$cl(A) = cl_{\bar{\mathbf{C}}}(A)$$

Theorem 9. *If $A = cl(A)$ and B is any G-field that extends A, $|B| < \left|\bar{\mathbf{C}}\right|$, then there are infinitely many linearly disjoint A-isomorphic copies of B in $\bar{\mathbf{C}}$.*

Proof: Find infinitely many copies $(B_i : i < \omega)$ of B, linearly disjoint over A, in some saturated algebraically closed field extending $\bar{\mathbf{C}}$. Let K be the field of quotients of $\bigotimes_A B_i$ and apply proposition 4. ■

Proposition 5. $\mathrm{etp(a/A)} = \mathrm{etp(b/A)}$ *if and only if there is an isomorphism $\alpha : cl(Aa) \to cl(Ab)$ that takes a to b and fixes A point-wise.*

We end this section with a definition that will be useful in the following sections.

Definition 10. *Let $\mathbb{T}_G$ be the first order theory that says that it's models are faithful, perfect, PAC G-fields, with the fixed fields PAC and perfect and that the Galois groups are as in the theorems of section three.*

That $\mathbb{T}_G$ is first order follows from [13].

We know from section 3 that all existentially closed G-fields satisfy $\mathbb{T}_G$, $\mathrm{Ob}(\mathcal{E}_\mathrm{G}) \subseteq \mathrm{Mod}\mathbb{T}_\mathrm{G}$, the class of $\mathbb{T}_G$-models, and if the kernel of the universal Frattini cover of $\hat{G}$ is small, then every extension between $\mathbb{T}_G$-models are regular, as we shall see from the next theorem.

Theorem 10. *If $k \models \mathbb{T}_G$ and the kernel of the universal Frattini cover of $\hat{G}$ is small, then there are no G-fields algebraic over k.*

Proof: For any faithful G-field F, $\mathrm{Gal(F{\cap}\bar{F}^G/F^G) = \hat{G}}$. Let $M = cl(F{\cap}\bar{F}^G)$, then $\mathrm{Gal(M/M^G) = \hat{G}}$ and

$$\begin{tikzcd}[column sep=small]
 & \bar{F} \arrow[d, dash, "\ker\tilde{\phi}"] & \\
 & M \arrow[dl, dash] \arrow[dr, dash, "\hat{G}"] & \\
F \cap \bar{F}^G \arrow[dr, dash, "\hat{G}"'] & & M^G \arrow[dl, dash] \\
 & F^G &
\end{tikzcd}$$

So $\mathrm{G(M) \leqslant G(F \cap \bar{F}^G)}$ and $\mathrm{G(M)} = \ker\tilde{\phi}$.
So if $\ker\tilde{\phi}$ has m normal subgroups of index p, for a prime p, then every faithful G-field has at least m extensions of degree p.
Let $H \leqslant \ker\tilde{\phi}$ be one of those normal subgroups, then the action of $\hat{G}$ can not be extended to a field containing M^H.
Let $M' = \{gm : g \in \mathrm{G(M^G), m \in M^H}\}$ and $H' = \{g \in \mathrm{G(M^G) : gm = m}\ if\ m \in M'\}$.
The fact that the action of $\hat{G}$ can not be extended to any field containing M^H is equivalent to that the following sequence does not split:

$$\mathrm{Gal(M'/M) \rightarrowtail Gal(M'/M^G) \twoheadrightarrow Gal(M/M^G) \cong \hat{G}}$$

Let $L = N \cap \bar{N}^G$ where N is any faithful G-field with $\mathrm{Abs(N) \rightarrowtail Abs(M)}$ (and $L \subseteq M$). Then $\mathrm{G(M^G) \rightarrowtail G(L^G)}$ and there is a field L' with $L \subseteq L' \subseteq \bar{L}$ such that the restriction-map $\mathrm{res : H' \to G(L')}$ is an isomorphism.
But then

$$\mathrm{Gal(L'/L) \rightarrowtail Gal(L'/L^G) \twoheadrightarrow Gal(L/L^G)}$$

does not split, so the action of $\hat{G}$ can not be extended to L'.
So every faithful G-field has m extensions of degree p to which the action of $\hat{G}$ can not be extended.
Assume $k \models \mathbb{T}_G$ and let $k' \neq k$ be a minimal algebraic extension of k. Since $\ker\tilde{\phi}$ is pro-nilpotent has to be cyclic of degree p for some prime p. But since k only have m those and every faithful G-field has to have at least m such extensions to which the G-action can not be extended, there are no G-field structure on any field containing k'.
So there are no G-fields extending k algebraic over k. ■

5 Quantifier elimination and Decidability

In this section assume that G is finitely presented and work in L_σ and let $\mathcal{U}_G$ be the category of existentially universal G-fields.
Then for $E, F \in \mathcal{U}_G$, $E \equiv F$ if $Th(E)_\forall = Th(F)_\forall$, see [14], and $Th(E)_\forall = Th(F)_\forall$ if and only if Abs(E) $\cong$ Abs(F). From these facts we can deduce decidability and a quantifier elimination in an extended language for $\mathcal{U}_G$.

Let $\mathcal{F}$ be the set of polynomials $f \in \mathbb{Z}[x]$, where x is of length one, that are monic, irreducible over $\mathbb{Q}$, the Galois group of the splitting field for f is a quotient of the universal Frattini cover of $\hat{G}$ and the splitting field is generated by one (any) root of f.

Obviously the equivalence type of the absolute numbers is determined by which of the polynomials in $\mathcal{F}$ that has roots and how the generators of G acts on those roots.
And in positive characteristics the only possible algebraic extensions of the prime field (of degree not divided by the characteristics) are splitting fields of polynomials of the form $x^m - 1$, if $m \neq$ chark.

To every $f \in \mathcal{F}$ we associate $r_i \in \mathbb{Q}[x]$, for $0 \leqslant i < deg(f)$, such that $f(a) = 0$ implies that $f(r_i(a)) = 0$ and $r_0(a), ..., r_{deg(f)-1}(a)$ are all the roots of f, $\mathrm{degr_i} < \mathrm{degf}$ and $r_0 = x$.
Let $\theta_{f,i,j}$ be $\exists x(f(x) = 0 \wedge \sigma_j(x) = r_i(x))$, for $f \in \mathcal{F}$
and let $\psi_{m,i,j}$ be $\exists x(x^m = 1 \wedge \sigma_j(x) = x^i)$, where $i \mid m$
and $\eta_{p,i,j}$ be $\exists x(x^p - x = 1 \wedge \sigma_j x = x + i)$
Let Θ be the set of boolean combinations of the $\theta_{f,i,j}$ and let Ψ be the set of boolean combinations of the $\psi_{m,i,j}$ and the $\eta_{p,i,j}$

First we restrict attention to existentially universal G-fields of characteristics zero. We will use ultraproducts to prove that modulo the class of e.u. G-fields of characteristics zero, every sentence in L_σ (or in L_G) is equalent to a member of Θ. For the theory of ultraproducts see [5].

Let A and B be e.u.G-fields, charA = charB = 0. We know that $A \equiv B$ if and only if $A \cap \bar{\mathbb{Q}} \cong B \cap \bar{\mathbb{Q}}$, and $A \cap \bar{\mathbb{Q}} \cong B \cap \bar{\mathbb{Q}}$ if and only if $A \models \theta_{f,i,j} \Leftrightarrow B \models \theta_{f,i,j}$, for all $\theta_{f,i,j}$.
Chose a representative S_i for each equivalence class of absolute numbers of objects in $\mathcal{U}_G$. Let M_i be the existentially universal closure of S_i, and let $M = \{M_i : i \in I\}$. Then M contains precisely one e.u. G-field of each equivalence class and $|M| \leqslant 2^{\aleph_0}$.
For every sentence $\phi \in L_\sigma$ let $P(\phi) = \{i \in I : M_i \models \phi\}$ and let $\mathbf{B}$ be the Boolean algebra generated by the $P(\theta_{f,i,j})$.

Proposition 6. *Any sentence in L_σ (or in L_G) is equivalent to a sentence i Θ modulo the class of e.u. G-fields of characteristics zero.*

Proof: Assume that we have a sentence $\phi \in L_G$ with $P(\phi) \notin \mathbf{B}$. Then from the general theory we know that we can find two ultra-filters $\mathcal{F}$ and $\mathcal{H}$ with the same $\mathbf{B}$-intersection, $\mathbf{B} \cap \mathcal{F} = \mathbf{B} \cap \mathcal{H}$, but with $P(\phi) \in \mathcal{F}$ and $P(\phi) \notin \mathcal{H}$.
Let $F = \prod M_i/\mathcal{F}$ and $H == \prod M_i/\mathcal{H}$. Then $F \equiv H$ since they agree on $\mathbf{B}$,

but $F \models \phi$ and $H \models \neg\phi$ - a contradiction. ■

For characteristics $p > 0$ the same is true for Ψ in place of Θ in exactly the same way.
So for every sentence ϕ of L_G there is a finite set π of primes and a particular prime p_0 such that for every $M \in \mathcal{U}_G$,

$$M \models \phi \leftrightarrow \bigvee_{p \in \pi} (p = 0 \wedge \psi) \vee \bigwedge_{p < p_0} (p \neq 0 \wedge \theta)$$

where $\psi \in \Psi$ and $\theta \in \Theta$.

It should now be clear that:

Theorem 11. *$\mathcal{U}_G$ is decidable.*

Proof: To decide a sentence ϕ we need to check, for a finite number of finite extensions of the prime field, if there is a $K \in \mathcal{U}_G$ that have (or don't have) these extensions with the prescribed G-action. Or more precisely for a finite number of polynomials $q \in P[x]$, where P is the prime field and x is a single variable, we have to decide if there is a G-field that don't have a solution to q, but has some sub-extension of the splitting field for q.
Let P_q be the splitting field for q over P, and let L_0, L_1 be two intermediate fields with $L_0 \subseteq L_1$. We need to check if there is a G-field K such that $K \cap P_q = L_1$ and $K^G \cap P_q = L_0$
And that is true if
$A = \mathrm{Gal(P_q/F_1)} \in \mathrm{Im(ker\tilde{\phi})}$,
$B = \mathrm{Gal(P_q/F_0)} \in \mathrm{Im(\tilde{G})}$ and
$C = \mathrm{Gal(F_1/F_0)} \in \mathrm{Im(\hat{G})}$
and there are morphisms α, β that makes the following diagram commute.

$$\begin{tikzcd}
\ker\tilde{\phi} \arrow[r, tail] \arrow[d, two heads] & \tilde{G} \arrow[r, two heads] \arrow[d, two heads] & \hat{G} \arrow[d, two heads] \\
A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

And since the groups and morphisms in top row are known all of this only involves finding finitely many Galois groups from the polynomials defining the extensions. ■

And we also have:

Theorem 12. *The existential part of $\mathcal{E}_G$ is decidable.*

Proof: Let $M, N \in \mathcal{E}_G$. Then $M \equiv_\exists N$ if and only if $\mathrm{Th(M)}_\forall \cong \mathrm{Th(N)}_\forall$, see [14], if and only if $\mathrm{Abs(M)} \cong \mathrm{Abs(N)}$. So for every existential sentence ϕ we have that there is a finite set of primes π and boolean combinations θ and ψ of sentences in Θ and Ψ respectively, such that in every $M \in \mathcal{E}_G$,

$$M \models \phi \leftrightarrow \bigvee_{p \in \pi} (p = 0 \wedge \psi) \vee \bigwedge_{p < p_0} (p \neq 0 \wedge \theta)$$

■

Further more by enriching the language we can obtain quantifier elimination.

Let $E_{f,g,i}$ be $\exists y(f(\bar{\sigma}xy) = 0 \wedge \sigma_i(y) = g(\bar{\sigma}xy)$
where $f \in \mathbb{Z}[v_0, ..., v_{nm+1}]$, lgy = 1, lgx = m and $g \in \mathbb{Q}[v]$. Where the total degree of g is strictly smaller than the total degree of f.

Let $L^E_\sigma = L_\sigma \cup \{E_{f,g,i} : f \text{ irreducible}\}$.

Theorem 13. *$\mathcal{E}_G$ has quantifier elimination in the language L^E_σ.*

Proof: First we claim that every L^E_σ-substructure of an object in $\mathcal{E}_G$ is closed in the sense of definition 5. To prove the claim take $M \in \mathrm{Ob}(\mathcal{E}_G)$ and a substructure $A \subseteq M$. Since A is substructure $GA \subseteq A$ so the only possible difference between A and clA comes from taking a relative algebraic closure, $\mathrm{clA} = \bar{\mathrm{A}} \cap \mathrm{M}$.
So assume $b \in \bar{A} \setminus A$ and let $f_b(x) \in A[x]$ be the minimal polynomial for b, $f_b = \sum a_i x^i$, $a_i \in A$.
Let $f'_b = \sum y_i x^i$ and let $\chi(y) = \bigwedge_g E_{f'_b gi}$, where g run through every polynomial such that $f_b(g(\bar{\sigma}ab)) = 0$. Then $\chi(a)$ is not true in A, since $b \notin A$, and it is a quantifier-free formula so we have:

$$A \models \neg\chi(a) \Rightarrow M \models \neg\chi(a) \Rightarrow b \notin M \Rightarrow A = \bar{A} \cap M \Rightarrow A = \mathrm{clA}$$

which proves the claim.
And from proposition 3 we know that closed sets are amalgamation bases and that is all we need to obtain quantifier elimination, see section 2. ■

So in L^E_σ every formula is equivalent to a quantifier-free formula modulo $\mathcal{E}_G$.

6 Finite Groups

If a finite group G acts on a field K, then K will be a finite Galois extension of K^G, by elementary Galois theory.
We start with an example. Let $G = \mathbb{Z}/(2)$, the group with two elements. Consider $\mathbb{C}$, the complex numbers and $\sigma; z \mapsto \bar{z}$, complex conjugation. Then $(\mathbb{C}, \sigma) \models T_G$, but $(\mathbb{C}, \sigma)$ is not existentially closed since $\mathbb{C}^G = \mathbb{R}$ is not PAC. For instance $x_1^2 + x_2^2 + x_3^2 + 1 = 0$ does not have an $\mathbb{R}$-rational solution. So let K be an extension of $\mathbb{C}$ existentially closed as a T_G-model.
The profinite completion of $\mathbb{Z}/(2)$ is $\mathbb{Z}/(2)$ and the universal Frattini cover of $\mathbb{Z}/(2)$ is $\mathbb{Z}_2$, the 2-adic integers (the inverse limit of $\mathbb{Z}/(2^k)$) and the kernel of the universal Frattini cover is $\mathbb{Z}_2 \to \mathbb{Z}_2; (a_i) \mapsto (2a_i)$. So $\mathrm{Gal(K)} \cong \mathrm{Gal(K^G)} \cong \mathbb{Z}_2$. So K is degree-two-extension of a field with no algebraic extension of odd degree, but K is not algebraically closed.

We know from section three of this thesis that if K is an e.c. G-field, then $K \models \mathbb{T}_G$. But, in fact, the other direction is also true here, so $\mathbb{T}_G$ is the model companion of T_G. This will be proved in the next theorem.

Theorem 14. *($|G| < \omega$)*
$\mathbb{T}_G$ is model complete.

Proof: Assume $K \models \mathbb{T}_G$ and that $\phi(x)$ is a quantifier-free formula in $L(K)$ satisfied in a G-field K' extending K
For some $a \in K'$, $K' \models \bar{\phi}(a)$, where $\bar{\phi}$ is as defined in proposition 1.
Set $F = K^G$ and $F' = K'^G$.
Then there is a finite $b \in K$ such that $K = F(b)$ and $K' = F'(b)$, and moreover b is a basis. Actually any basis for K over F will do.
Let c be the coordinates for a with respect to b, $a_i = \sum c_{ij} b_j$, and $c = (c_{11}, ..., c_{km})$, where $k = |a|$ and $m = |b|$
$\bar{\phi} = \bigwedge_r f_r = 0$ where $f_r = \sum d_s (\bar{\sigma} x)^{l_s}$ and $l_s \in \mathbb{N}^{nk}$.
Let $V = V(c/K)$, then $V(F') \neq \emptyset$ and F' is regular over F,which is PAC, so there has to be a K-rational point in V. Let $e \in V(F)$ be it. Then

$$0 = f_r(a) = \sum d_s (\bar{\sigma} a)^{l_s} = \sum d_s (\prod_{ki} \sum_j c_{ij} \sigma_k b_j)^{l_s} \Rightarrow$$

$$\sum d_s (\prod_{ki} \sum_j x_{ij} \sigma_k b_j)^{l_s} \in I(c/K) \Rightarrow$$

$$0 = \sum d_s (\prod_{ki} \sum_j e_{ij} \sigma_k b_j)^{l_s} = f_r(u)$$

where $u = (\sum c_{1j} b_j, ..., \sum c_{kj} b_j) \in K$.
So there is a solution in K and hence K is e.c. ■
We can also prove the following theorem that will be useful later.

Theorem 15. *If K is an e.c. G-field and $H \triangleleft G$ of finite index, then there is an algebraic extension M of K^H such that M is an e.c. G/H-field, and there is a basis b for K^H over K^G such that b is also a basis for M over $M^{G/H}$.*

Proof: K^H is perfect and PAC. It is a finite separable extension of K^G. $G(K^H)$ and $G(K^G)$ are projective. Let $\widetilde{G/H}$ be the universal Frattini cover of G/H, then

$$\begin{tikzcd}
G(K^G) \arrow[r, two heads] \arrow[d, dotted, "\exists"'] & \hat{G} \arrow[d] \\
\widetilde{G/H} \arrow[r, two heads] & G/H
\end{tikzcd}$$

because $\widetilde{G/H}$ is the smallest projective with an epi to G/H. And by Galois theory this means that K^G has an algebraic extension N with Galois group $\widetilde{G/H}$ and $N \cap K^H = K^G$. N and K^H are linearly disjoint over F_G, so if b is a basis for K^H/K^G, then b is still linearly independent over N. Set $M = N(b)$, the field extension of N by b, then Gal(M/N) = G/H and M is an e.c. G/H-field, because $M^{G/H}$ is perfect, PAC and has the right Galois group. ■

7 Free Groups

In this section $G = \mathbf{F}_n$, the free group on n generators. The profinite completion of $\mathbf{F}_n$ is $\hat{\mathbf{F}}_n$ the projective limit of all finite groups on $\leqslant n$ generators, which is

a free profinite group, so the universal Frattini cover is the identity. The kernel is 1, so the models are separably closed and we also know that they are perfect, so they are algebraically closed. In this section we may assume Therefore that $\bar{\mathbf{C}}$ is saturated for the theory ACF, which means that $\bar{\mathbf{C}}$ realizes every type in L_{ring}.
In [13] it is proved that if F is an algebraically closed field, then the set of n-tuples of automorphisms that (topologically) generates $\hat{\mathbf{F}}_n$ is dense in $(\mathrm{AutF})^{\mathrm{n}}$. So examples are abundant.

If Y, X_i are varieties such that $Y \subseteq X_1 \times ... \times X_m$ one says that Y projects dominantly if for every generic $p \in Y$ $\pi_i(p)$ is a generic point of X_i, where π_i is the usual projection. Or equivalently projections of open sets are open.

Definition 11. *Let $T^{\star}_{\mathbf{F}_n}$ be the union of $T_{\mathbf{F}_n}$, ACF and axioms saying that for all (absolutely irreducible) varieties V, W such that $W \subseteq V \times V^{\sigma_1} \times ... \times V^{\sigma_n}$ and W projects dominantly, there is a $p \in V$ such that $(p, \sigma_1(p), ..., \sigma_n(p)) \in W$.*

The fact that this is first order follows from the definable multiplicity property DMP of algebraically closed fields. The ω-stable theory ACF, algebraically closed fields, have the DMP, which means that the Morley degree (and rank) is definable, see [17]. Being absolutely irreducible is the same thing as having Morley degree one in ACF and projecting dominantly can be expressed just by using definability of dimension (Morley rank).
$T^{\star}_{\mathbf{F}_n}$ has elsewhere been called $ACFA_n$

The theorem below is not new; it easily follows from [22] and Hrushovski mentions it in [18], but it has not been proved in a published article.

Theorem 16. *$T^{\star}_{\mathbf{F}_n}$ is the model companion of $T_{\mathbf{F}_n}$.*

Proof: Let K be an e.c. $\mathbf{F}_n$-field, then we know from section three that K is algebraically closed.
Now let V and W be as in the axioms for $T^{\star}_{\mathbf{F}_n}$, ($W \subseteq V^{\bar{\sigma}}$, projecting dominantly).
Define partial isomorphisms τ_i in $K[x]/I(W)$ by $\tau_i \restriction K = \sigma_i$ and $\tau_i y_0 = y_i$, where $x = (y_0, ..., y_n)$ and the length of y_j is dimV. These partial isomorphisms extends to automorphisms of a saturated algebraically closed field extending $K(V)$, that make E into an $\mathbf{F}_n$-field. The point $y_0 \in V(E)$ satisfies $\bar{\sigma} y_0 \in W(E)$ and since K is e.c. and $K \subseteq E$, it has to be one such point i $V(K)$ to.
For the other direction: Assume that $K \models T^{\star}_{\mathbf{F}_n}$ and let M be a $\mathbf{F}_n$-field extending K and $M \models \exists x \phi(x)$ for a quantifier-free ϕ, then $M \models \exists y \bar{\phi}(y)$, where $\bar{\phi}$ is as in proposition 1.
So there is an $a \in M$ with $M \models \bar{\phi}(a)$
Now let $V = V(a)$ and $W = V(\bar{\sigma} a)$. Obviously W projects dominantly, so the axioms of $T^{\star}_{\mathbf{F}_n}$ gives us a point $b \in V(K)$ with $\bar{\sigma} b \in W(K)$.
So $K \models \bar{\phi}(b)$ and therefore by proposition 1 $K \models \exists x \phi(x)$. So K is e.c. ■

If K is an e.c. $\mathbf{F}_7$-field, then it also is an e.c. $\mathbf{F}_3$ by just ignoring four of the generators. Similarly any e.c. $\mathbf{F}_n$-field is an e.c. G-field for any free $G \leqslant \mathbf{F}_n$.

So the above proof is slightly redundant; actually the theorem follows from " $T^{\star}_{\mathbf{F}_2}$ is the model companion of $T_{\mathbf{F}_2}$", since $\mathbf{F}_2$ contain free subgroups on any number of generators. Also we can see that the fixed field of any element of $\mathbf{F}_n$ has a Galois group isomorphic to $\hat{\mathbb{Z}}$ and the fixed field of m elements that generate a free group has a Galois group homeomorphic to $\hat{\mathbf{F}}_m$.

8 The dihedral group

In this section we study a special case to see what the previous results can tell us about the structure of the e.c. models.
The infinite dihedral group $D_\infty = \langle \sigma_1, \sigma_2;\ \sigma_1^2, \sigma_2^2 \rangle$ can also be presented as $D_\infty = \langle x, y;\ xyx^{-1}y, y^2 \rangle$. That the two present the same group can be seen by the identifications $x = \sigma_1\sigma_2$ and $y = \sigma_1$.
$D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/(2)$, the semi-direct product, where $\mathbb{Z}/(2)$ acts on $\mathbb{Z}$ in the only possible way $n \mapsto -n$, see [27].
An example:
Let k be an algebraically closed field and let $\{a_i, b_j : i, j \in \mathbb{Z}\}$ be new constants (algebraically independent transcendentals).
Set $E = k(a_i, b_j : i, j \in \mathbb{Z})$, the field extension, and let D_∞ act on E by fixing k and:
$\sigma_1; a_i \mapsto b_i, b_i \mapsto a_i$
$\sigma_2; a_i \mapsto b_{i+1}, b_i \mapsto a_{i-1}$
This is a faithful action.
Consider the polynomial $g = x^2 - a_i + b_i$. If $g(d) = 0$, then

$$(\sigma_1 d)^2 = \sigma_1(d^2) = \sigma_1(a_i - b_i) = b_i - a_i = -d^2,$$

so $\sigma_1 d = di$, where i is a zero of $x^2 + 1$, and since $i \in k$ we have:

$$\sigma_1^2(d) = \sigma_1(i)\sigma_1(d) = i^2 d = -d \neq d$$

So there are no D_∞-field extending E that contain d. Actually there are a lot of polynomials over E that do not have solutions as we shall see below.

The profinite completion of D_∞ is $\hat{\mathbb{Z}} \rtimes \mathbb{Z}/(2) = (\prod \mathbb{Z}_p) \rtimes \mathbb{Z}/(2)$, and for $p \neq 2$ there is a unique p-Sylow subgroup, namely $\mathbb{Z}_p$, but a 2-Sylow subgroup is $\mathbb{Z}_2 \rtimes \mathbb{Z}/(2)$ which is topologically a two generator group.
We know from the theory of profinite groups that the 2-Sylow subgroup of $\tilde{D}_\infty$, the universal Frattini cover of $\hat{D_\infty}$, has to be $\mathbf{F}_2(2)$, the free pro-2-group on two generators, see [32].

To understand what the kernel of the Frattini cover is we study:

$$\pi : \mathbf{F}_2 \twoheadrightarrow D_\infty$$

$\ker \pi$ is the normal closure of the subgroup generated by the preimages of σ_1^2 and σ_2^2 in $\mathbf{F}_2$. If a and b are the generators of $\mathbf{F}_2$, then

$$\ker \pi = \langle ca^2c^{-1}, cb^2c^{-1}; c \in \mathbf{F}_2 \rangle \cong \mathbf{F}_\omega$$

the free group on countably many generators,
and taking pro-2-completion is in this context an exact functor from the category

of abstract groups to the category of pro-2-groups. We have:

$$\begin{tikzcd}
\mathbf{F}_\omega \arrow[r, tail] \arrow[d] & \mathbf{F}_2 \arrow[r, two heads] \arrow[d] & D_\infty \arrow[d] \\
\mathbf{F}_\omega(2) \arrow[r, tail] & \mathbf{F}_2(2) \arrow[r, two heads] & \mathbb{Z}_2 \rtimes \mathbb{Z}/(2)
\end{tikzcd}$$

So any existentially closed D_∞-field has the Galois group $\mathbf{F}_\omega(2)$.

If K is an e.c. D_∞-field, let $a \in K$ satisfy $a \neq \sigma_1 a \neq \sigma_2 a \neq a$ and let $a_i = (\sigma_1\sigma_2)^i a$, for $i \in \mathbb{Z}$. For $j = 1, 2$ let $b_{ij} = a_i - \frac{a_i - \sigma_j a_i}{2}$, then $b_{ij} \in K \setminus K^{\langle \sigma_j \rangle}$, but $b_{ij}^2 \in K^{\langle \sigma_j \rangle}$. The polynomials $p_{mij} = x^{2^m} - b_{ij}$ have no solutions i K. The Galois groups of the extensions corresponding to those polynomials (infinitely many for each m) are the cyclic 2-groups. And all extensions of K are built up from those.

9 $\mathbb{Z} \times \mathbb{Z}$ and uncompaniable theories

The group we shall study in this section is the commutative group on two generators $\mathbb{Z} \times \mathbb{Z} = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1} \rangle$ and we start with two examples:

1. Let $R = \prod \bar{\mathbb{F}}_p$, the product of the algebraic closures of $\mathbb{F}_p$ for all primes p. And let $\mathcal{U}$ be an ultra filter in $\mathcal{P}(\mathbb{P})$, the power-set of the primes, then the ultra product $K = R/\mathcal{U}$ is an algebraically closed field of characteristic 0. Define $\sigma_i = \prod(x \mapsto x^{p^{k_{ip}}})$, for $i = 1, 2$ in such a way that $\{p \in \mathbb{P} : (k_{1p}, k_{2p}) = 1, k_{ip} \neq 1\} \in \mathcal{U}$, where $(a, b) = 1$ means that a and b are relatively prime, for facts on ultra products see [5]. This is a faithful $\mathbb{Z} \times \mathbb{Z}$-action on K

2. Let k be algebraically closed and $K = k(x_1, x_2)$ and define $\sigma_i(x_i) = x_i + 1$ and $\sigma_i x_j = x_j$,if $i \neq j$, for $i, j = 1, 2$, then $K \models T_{\mathbb{Z}\times\mathbb{Z}}$.

Let K be as in any of the two examples above, then K is also a model of $T_{\mathbf{F}_2}$, so K embeds into an existentially closed model for $T_{\mathbb{Z}\times\mathbb{Z}}$ and a model M for $T^\star_{\mathbf{F}_2}$. Can we find an existentially closed model for $T_{\mathbb{Z}\times\mathbb{Z}}$ inside M?
For instance if $N \models T^\star_{\mathbf{F}_2}$, is the restriction of σ_1 and σ_2 to the fixed field, C, of the commutator subgroup of $\mathbf{F}_2$ existentially closed for $T_{\mathbb{Z}\times\mathbb{Z}}$? ($C = N^{[\mathbf{F}_2, \mathbf{F}_2]}$) It is obviously a model of $T_{\mathbb{Z}\times\mathbb{Z}}$, and every existential formula that has a solution in an extension of C also has one in N. Let us postpone the answer to that question for a moment and study the structure of the e.c. models of $T_{\mathbb{Z}\times\mathbb{Z}}$.

Let $\mathcal{N} = \{H : H \triangleleft \mathbb{Z} \times \mathbb{Z}, [\mathbb{Z} \times \mathbb{Z} : H] < \omega\} = \{n\mathbb{Z} \times m\mathbb{Z} : n, m \in \mathbb{Z}_+\}$.

So the profinite completion of $\mathbb{Z} \times \mathbb{Z}$ is:

$$\widehat{\mathbb{Z} \times \mathbb{Z}} = \varprojlim(\mathbb{Z}\times\mathbb{Z})/(n\mathbb{Z}\times m\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z} = \varprojlim \mathbb{Z}/n\mathbb{Z}\times\varprojlim \mathbb{Z}/m\mathbb{Z} = \hat{Z}\times\hat{Z}$$

And $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$, so $\hat{\mathbb{Z}} \times \hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p \times \prod_q \mathbb{Z}_q \cong \prod(\mathbb{Z}_p \times \mathbb{Z}_p)$. So the p-Sylow subgroup is $\mathbb{Z}_p \times \mathbb{Z}_p$, but we know that the p-Sylow subgroup of $\tilde{G}$ is $\mathbf{F}_2(p)$;

$\tilde{G} = \prod_p \mathbf{F}_2(p)$.
The universal Frattini cover is $\tilde{\phi} : \tilde{G} \twoheadrightarrow \hat{G}$; $\tilde{\phi} = \prod_p \psi_p$, where $\psi_p : \mathbf{F}_2(p) \twoheadrightarrow \mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathrm{ker}\tilde{\phi} = \prod_\mathrm{p} \mathrm{ker}\psi_\mathrm{p}$.

$$\mathrm{ker}\psi_\mathrm{p} \cong [\mathbf{F}_2(\mathrm{p}), \mathbf{F}_2(\mathrm{p})] \cong \mathbf{F}_\omega(\mathrm{p})$$

the free pro-p-group on countably many generators.

So the Galois group of any e.c. $\mathbb{Z} \times \mathbb{Z}$-field is $\prod_p \mathbf{F}_\omega(p)$, which is not small, so the e.c.-models of $T_{\mathbb{Z}\times\mathbb{Z}}$ are unbounded and therefore not simple.
In [25] Pillay shows that if T is a stable theory, then the category of e.c.-structures of T_σ, the theory of T-models with an automorphism, is simple. The present example ($\mathbb{Z} \times \mathbb{Z}$-fields) shows that his result can not be strengthen to simple instead of stable, because $\mathcal{E}_{\mathbb{Z}\times\mathbb{Z}}$ is equivalent to the category of e.c.-structures for ACFA_σ.
The comments on simplicity depends on a section omitted from this thesis, and may be ignored by anyone not interested in simplicity theory.

And now let us get back to the question above:

$$\begin{tikzcd} & N \arrow[dl, no head] \arrow[dd, no head] \\ C \arrow[dr, no head] & \\ & F \end{tikzcd}$$

$$F = N^{\langle \sigma_1, \sigma_2 \rangle}$$

The fixed field of $\mathbb{Z} \times \mathbb{Z}$, in C above, $C^{\mathbb{Z}\times\mathbb{Z}} = N^{\mathbf{F}_2} = F$, and $\mathrm{Gal(F)} \cong \hat{\mathbf{F}}_2$, but the Galois group of the fixed field of every existentially closed $\mathbb{Z} \times \mathbb{Z}$-field is $\prod_p \mathbf{F}_\omega(p))(\ncong \hat{\mathbf{F}}_2)$, so C is not e.c.

Hrushovski has proved (unpublished), in a different context that $T_{\mathbb{Z}\times\mathbb{Z}}$ doesn't have a model companion. We will present a version of his argument here, because we will need it later and it is not available in writing anywhere. The next theorem is entirely due to Hrushovski, except that any possible errors are of the present writers doing.

Theorem 17. *(Hrushovski) $T_{\mathbb{Z}\times\mathbb{Z}}$ does not have a model companion.*

Proof: $\mathbb{Z} \times \mathbb{Z} = \langle \gamma, \tau; \gamma\tau\gamma^{-1}\tau^{-1} \rangle$ Let $K \models T_{\mathbb{Z}\times\mathbb{Z}}$ be existentially closed and $\gamma(\zeta) = \tau(\zeta) = \zeta^2$ for a primitive third root ζ of 1. There are such models in $\mathcal{E}_{\mathbb{Z}\times\mathbb{Z}}$.
Let $\phi(x, y, z)$ be

$$\gamma(x) = x + z \wedge \tau(x) = x + z \wedge x = y^3 \wedge \tau(y) = \zeta\gamma(y)$$

Now assume there are $a, b, c \in K$ such that $K \models \phi(a, b, c)$ and $\sum_{k=0}^{n-1} \gamma(c) = 0$, then $\gamma^n(a) = a$, so $\gamma^n(b)$ has to be $\zeta^i b$ for some i and therefore

$$\gamma^n \tau(b) = \gamma^n \zeta \gamma(\gamma(b)) = \gamma^n(\zeta)\gamma(\zeta^i b) = \gamma^n(\zeta)\gamma(\zeta^i)\gamma(b)$$

and

$$\tau\gamma^n(b) = \tau(\zeta^i)\tau(b) = \gamma(\zeta^i)\zeta\gamma(b)$$

$\Rightarrow \gamma^n(\zeta) = \zeta$, so n is even.
Consider the partial type $p(z) = \{\neg\exists xy\phi(x, y, z), \sum_{k=0}^{n-1} \gamma(z) \neq 0; n \in \omega\}$.
p is finitely realized in K, so it is realized in an elementary extension K' by compactness. But then K' is not existentially closed because it doesn't have solution for $\phi(x, y, c)$ where c is the realization of p. Therefore the class of e.c.-structures is not elementary. ■

The same counterexample works whenever we have a group G with a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and there is also a version of this example on an elliptic curve, that is independent of the group action on the absolute numbers.

10 G-algebraic closedness

Much of the ideas in this section comes from Torsten Ekedahl, but the exact fomulations and every possible error are due to the author. A G-field K is G-algebraically closed, GAC, if there is no G-field $K' \supset K$ algebraic over K. It is easy to see that a G-field is GAC if and only if it is closed in the sense of section four, and every e.c. G-field is GAC.
As we saw earlier, theorem 10, $\mathbb{T}_G$-models are GAC if the kernel of the universal Frattini cover is small.
And if A and B are two G-fields with $A \leqslant B$, and A is G-algebraically closed, then B is regular over A.
We will now introduce a new class of G-fields on which GAC is a first order property for a larger class of groups.

Definition 12. *Let T_G^μ be the first order theory for G-fields that contain every p-th root of unity for every prime p.*

$$T_G^\mu = T_G \cup \bigcup_p \exists x(1 \neq x \wedge x^p = 1)$$

A model of T_G^μ will be called a μ-closed G-field.

The μ occurs because the group of p-th roots of unity is often called μ_p.
The category of μ-closed G-fields is a full sub-category of the category of G-fields. Let $\mathcal{E}_G^\mu$ be the category of e.c. μ-closed G-fields. Everything proved about $\mathcal{E}_G$ in section four transfers immediately to $\mathcal{E}_G^\mu$. In fact for the disjoint sub-categories $\mathcal{E}_{GA}^\mu = \mathcal{E}_{GA}$, where A is the absolute numbers of a certain e.c. μ-closed G-field.
To see that $\mathcal{E}_G^\mu \neq \mathcal{E}_G$ consider the following example:
Let $G = \mathbb{Z}/(2)$ and $\zeta = e^{i\frac{2}{5}\pi} + e^{i\frac{8}{5}\pi}$ and define an action on $\mathbb{Q}(\zeta)$ by $\zeta \mapsto e^{i\frac{4}{5}\pi} + e^{i\frac{6}{5}\pi}$.
This action can not be extended to any field containing $\mathbb{Q}(\mu_5)$, so any e.c. $\mathbb{Z}/(2)$-field k extending $\mathbb{Q}(\zeta)$ will not be μ-closed.

But if G is free, then $\mathcal{E}^{\mu}_G = \mathcal{E}_G$.

We will now start to define a fist order sub-category of the category of μ-closed G-fields that will contain all e.c. μ-closed G-fields and we will further refine it in the next section to get some more results on model companions.

Let $R = \mathbb{Z}[G]$, the group-ring, and $\mathcal{K}_p$, for a prime p, be the set of all ideal I in R such that $pR \subseteq I$.
Define for every prime $p \neq \mathrm{char(k)}$ and every $I \in \mathcal{K}_p$, $S_{p,I}$ to be the axiom:
For every ideal $J \subseteq R$ with $J/I \cong \mu_p$, if we have

$$\begin{tikzcd} & & I \arrow[r, tail] \arrow[d, "\phi"'] & J \arrow[d, "\chi"] & \\ 1 \arrow[r] & \mu_p \arrow[r] & k^* \arrow[r, "(-)^p"] & k^{*p} \arrow[r] & 1 \end{tikzcd}$$

and there are no G-homomorphism $\psi : J \to k^*$ that commutes with the above diagram, then there is an element $a \in k$ such that $\phi(i) = a^i$, for every $i \in I$.

χ always exists, actually there is the restriction of the map $(1 \mapsto \phi(p))$ from R to k^{*p} that would fit in χ's place.
If $p = \mathrm{char(k)}$ then $S_{p,I}$ is similar but with the morphism $x \mapsto x^p - x$ instead of $x \mapsto x^p$.

Theorem 18. *If R is notherian, then $S_{p,I}$ is a first order sentence.*

Proof: If R is noetherian, then all ideals are finitely presented and there can only be finitely many J with $I \subseteq J \subseteq R$ and $J/I \cong \mu_p$.

Let $R^m \xrightarrow{(a_{ij})} R^k \xrightarrow{r_i} I$ be a presentation of I.

Then the existence of ϕ is equivalent to

$$\exists x \bigwedge_i (x_i \neq 1 \wedge \bigwedge_j x_i^{a_{ij}} = 1) = S_\phi$$

for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$ Similarly for χ, then called S^J_χ.
Let J be generated by the same generators as I plus one extra called r_{k+1}.
Then the claim that there does not exist a $\psi : J \to k^*$ that commutes with the diagram can be expressed like

$$\neg\exists x_{k+1}(x_{k+1} \neq 1 \wedge \bigwedge_j x_{k+1}^{b_{(k+1)j}} = 1 \wedge x_{k+1}^p = \chi(r_{k+1})) = N^J_\psi$$

where the b_{ij} is from the presentation of J and $b_{ij} = a_{ij}$ if $i \leqslant k$ and $j \leqslant m$.
Then $S_{p,I}$ can be written $S_\phi \wedge \bigwedge_J S^J_\chi \wedge N^J_\psi$. ■

Let S be the scheme of axioms $\bigcup S_{p,I}$ for every prime p and every $I \in \mathcal{K}_p$.

Definition 13. *Let the theory $\mathbb{T}^\mu_G = T^\mu_G \cup \mathbb{T}_G \cup S$*

So the $\mathbb{T}^\mu_G$-models are μ-closed, perfect, PAC G-fields that have the Galois group $\ker \tilde{\phi}$ and the fixed fields have $\tilde{G}$ for a Galois group.

We will prove that for certain groups G all $\mathbb{T}_G^\mu$-models are G-algebraically closed and that $\mathrm{Ob}(\mathcal{E}_{\mathrm{G}}^\mu) \subseteq \mathrm{Mod}(\mathbb{T}_{\mathrm{G}}^\mu)$, but we start with a technical lemma.

Definition 14. *If A is $\mathbb{Z}[G]$-module let IA be the injective hull of A. We say that a group G has the property T if for every p-primary $\mathbb{Z}[G]$-module M, the socle of IM/M is finite, for every prime p.*

For instance abelian groups has property T.

To prove the following lemma we need the fact that short exact sequences of modules give rise to long exact sequences in Ext-groups and that the elements of $\mathrm{Ext}^1(\mathrm{A}, \mathrm{B})$ corresponds to equivalence classes of extensions of A by B (0 is the split extension). That can be found in any book in homological algebra or in [4].

Lemma 1. *If G has the property T and if*

$$\begin{tikzcd}
 & & I \arrow[r, tail] \arrow[d, "\phi"'] & J \arrow[d, "\chi"] & \\
1 \arrow[r] & \mu_p \arrow[r] & k^* \arrow[r, "(-)^p"] & k^{*p} \arrow[r] & 1
\end{tikzcd}$$

and for every ideal N with $I \subsetneq N \subseteq J$ there are no G-homomorphism $N \to k^*$ that commutes with the above diagram, then there is a $\mu_p \rightarrowtail J/I$.

Proof: Let M be the fibre product $k^* \times_{k^{*p}} J$, then

$$\begin{tikzcd}
 & & I \arrow[r, tail] \arrow[d] & J \arrow[d, equal] & \\
 & & M \arrow[r] \arrow[d] & J \arrow[d] & \\
1 \arrow[r] & \mu_p \arrow[r] & k^* \arrow[r, "(-)^p"] & k^{*p} \arrow[r] & 1
\end{tikzcd}$$

and there is a non-split extension: $\mu_p \rightarrowtail M \twoheadrightarrow J$ (= a non-zero element of $\mathrm{Ext}^1(\mathrm{J}, \mu_\mathrm{p})$), which by the long exact Ext-sequence

$$... \to \mathrm{Ext}^1(\mathrm{J/I}, \mu_\mathrm{p}) \to \mathrm{Ext}^1(\mathrm{J}, \mu_\mathrm{p}) \to \mathrm{Ext}^1(\mathrm{I}, \mu_\mathrm{p}) \to ...$$

comes from a nonzero element in $\mathrm{Ext}^1(\mathrm{J/I}, \mu_\mathrm{p})$, since the image of it in $\mathrm{Ext}^1(\mathrm{I}, \mu_\mathrm{p})$ is 0, a split extension of I by μ_p (by the lifting ϕ).
So we have a non-split extension:

$$\mu_p \rightarrowtail A \twoheadrightarrow J/I$$

Call it $*$. And since we are searching for submodules we may assume that J/I is as small as possible. Let $J = I + aR$, for some $a \in R$. Then $J/I \cong R/K$, for an ideal K; let $\eta : R \to J/I; 1 \mapsto a$, then $K = \ker \eta$.
By

$$... \to \mathrm{Hom}(\mathrm{K}, \mu_\mathrm{p}) \to \mathrm{Ext}^1(\mathrm{R/K}, \mu_\mathrm{p}) \to \mathrm{Ext}^1(\mathrm{R}, \mu_\mathrm{p}) = 0$$

any non-split extension of $R/K(\cong J/I)$ comes from a non-zero homomorphism $\alpha : K \to \mu_p$, and if it is non-zero it has to be an epimorphism since μ_p is simple. So $\mu_p \cong K/L$, where $L = \ker \alpha$.
Actually we can see that $A \cong R/L$ and the above non-split extension,$*$, is equivalent to

$$K/L \rightarrowtail R/L \twoheadrightarrow R/K$$

Claim: K/L is essential in R/L
Proof of claim: Assume there is a $0 \neq B \subseteq R/L$ such that $K/L \cap B = 0$, then we have a split extension

$$\mu_p \cong K/L \rightarrowtail B \oplus K/L \twoheadrightarrow B$$

and this extension comes from $K/L \rightarrowtail R/L \twoheadrightarrow R/K$, because there is a $B \rightarrowtail R/K$, the kernel the of composition $B \rightarrowtail R/L \twoheadrightarrow R/K$ is $K/L \cap B = 0$. The assumptions in the theorem implies that for any C, with $0 \neq C \subseteq J/I$, $*$ is not in the kernel of $\mathrm{Ext^1(J/I, \mu_p) \to Ext^1(C, \mu_p)}$, but

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mu_p \arrow[r] \arrow[d, "\sim"] & A \arrow[r] \arrow[d, "\sim"] & J/I \arrow[r] \arrow[d, "\sim"] & 0 \arrow[d, equal] \\
0 \arrow[r] & K/L \arrow[r] & R/L \arrow[r] & R/K \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, equal] & K/L \arrow[r] \arrow[u, equal] & B \bigoplus K/L \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & 0 \arrow[u, equal] \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$B \rightarrowtail J/I$ and the bottom-line is split, so $* \mapsto 0$ in $\mathrm{Ext^1(J/I, \mu_p) \to Ext^1(B, \mu_p)}$, a contradiction!
So K/L is essential in R/L.

Let E be the injective hull of K/L. Then

$$\begin{tikzcd}
K/L \arrow[r, tail] \arrow[d, tail] & R/L \arrow[dl] \\
E &
\end{tikzcd}$$

because of injectivity and essentiality. So we have

$$\begin{tikzcd}
K/L \arrow[r, tail] \arrow[d, "\sim"] & R/L \arrow[r, two heads] \arrow[d, tail] & R/K \arrow[d, tail] \\
\mu_p \arrow[r, tail] & E \arrow[r, two heads] & E/\mu_p
\end{tikzcd}$$

But the socle of E/μ_p is essential in E/μ_p, so $R/K \cap \mathrm{soc(E/\mu_p)} \neq 0$.
Now G has property T, so $\mathrm{soc(E/\mu_p)} = \mu_{\mathrm{P}}^{\mathrm{m}}$, for some $m \in \mathbb{N}$.

So therefore there must be a $\mu_p \rightarrowtail R/K \cong J/I$. ■

We are now ready to prove:

Theorem 19. *If G has property T and $k \models \mathbb{T}_G^\mu$, then k is GAC.*

Proof: Assume k is not G-algebraically closed and $k \models T_G^\mu \cup \mathbb{T}_G$ and we proceed to prove that $k \not\models S$.
Let $L(\neq k)$ be a minimal G-field, algebraic over k. Since $G(k)$ is pro-nilpotent L must be a Kummer-extension, actually $L = k(a^G)$ for some $a \in \bar{k}$ with $a^p \in k$, for some prime p.
Let $I = \{r \in R : a^r \in k\}$, then $pR \subseteq I \neq R$.
Let $J \subseteq R$ be an ideal such that $I \subseteq J$ and $J/I \cong \mu_p$.
Assume that there a $\psi : J \to k^*$ that commutes with

$$\begin{tikzcd} & & I \arrow[r, tail] \arrow[d, "\phi"'] & J \arrow[d, "\chi"] & \\ 1 \arrow[r] & \mu_p \arrow[r] & k^* \arrow[r, "(-)^p"] & k^{*p} \arrow[r] & 1 \end{tikzcd}$$

and let $k^* \ni b = \psi(j)$ for a $j \in J \setminus I$.
Then $b^p = \psi(j)^p = \psi(jp) = \psi(pj) = \psi(p)^j = a^{pj}$, so $b = \zeta a^j$, where $\zeta^p = 1$, but $a^j \notin k$, since $j \notin I$ so $b \notin k$.
Therefore ϕ does not extend to J and since $a \notin k$, S_I is not true in k. ■

Theorem 20. *If $k \models T_G^\mu \cup \mathbb{T}_G$ and k is G-algebraically closed, then $k \models S$.*

Proof: Assume S is not true in k and $k \models T_G^\mu \cup \mathbb{T}_G$, and let I be maximal with $k \models \neg S_I$.
For every J with $J/I \cong \mu_p$ we have

$$\begin{tikzcd} & & I \arrow[r, tail] \arrow[d, "\phi"'] & J \arrow[d, "\chi"] & \\ 1 \arrow[r] & \mu_p \arrow[r] & k^* \arrow[r, "(-)^p"] & k^{*p} \arrow[r] & 1 \end{tikzcd}$$

and ϕ can not be extended to J.
Let M be any ideal with $I \subseteq M \subseteq R$, then by the lemma either ϕ extends to M or there is a $\mu_p \rightarrowtail M/I$.
So assume that M/I doesn't contain a copy of μ_p, then ϕ extends to M.
Let N be an ideal such that $N/M \cong \mu_p$, then $N = M + rR$, for some $r \in R$.
Let $J = I + rR$, with the same r,then $J/I \rightarrowtail N/M \cong \mu_p$ and if $J/I = 0$ then $r \in I$, but then $N/M = 0$, and since μ_p is simple the inclusion has to be an isomorphism, $J/I \cong \mu_p$. So ϕ can not be extended to r, which implies that S_M is false, and by the maximality of I we must have $M = I$.
So ϕ can not be extended to any ideal K with $I \subsetneq K \subseteq R$.
Let $L = k[x^R]/(\phi(i) - x^i, i \in I)$
By definition L is a G-algebra.
Now any ideal in $k[x^R]$ is generated by elements on the form $a - x^r$, for some $a \in k$ and $r \in R(= \mathbb{Z}[G])$. If $A = (\phi(i) - x^i, i \in I)$ is not maximal, then there is a B with $A \subsetneq B \subsetneq R$.
Let $b - x^s \in B \setminus A$, then ϕ could be extended to $I + sR$ by $\phi(s) = b$, a

contradiction.
So A is a maximal ideal and therefore L is a field algebraic over k (actually a Kummer extension since $pR \subseteq I$). ■

Corollary 2. $\mathrm{Ob}(\mathcal{E}^{\mu}_{\mathrm{G}}) \subseteq \mathrm{Mod}(\mathbb{T}^{\mu}_{\mathrm{G}})$

11 Model companions

In this section we prove one last theorem on the existence of model companions. Let G be a group that has property T and has a normal subgroup of finite index isomorphic to $\mathbb{Z}$. Let $G = \langle \tau\sigma; R\rangle$ where τ is a single generator such that $\langle\tau\rangle \cong \mathbb{Z}$.
Let V_G be the scheme of axioms expressing that for every absolutely irreducible variety over $k^{G/\mathbb{Z}} = \{a \in k : \sigma_i a = a\}$, If $Y \subseteq X \times X^{\tau}$ projects dominantly, then there is a point $p \in X(k^{G/\mathbb{Z}})$ such that $p\tau p \in Y$.
Let $T^{\star}_G = \mathbb{T}^{\mu}_G \cup V_G$
That V_G is first order follows from the facts that dimension and being absolutely prime is first order definable, and that is shown in [12].

Theorem 21. *If G is a group that has property T and has a normal subgroup of finite index isomorphic to $\mathbb{Z}$, then $T^{\star}_G$ is the model companion of T^{μ}_G.*

Proof: We know that if k is an e.c. μ-closed G-field, then it satisfies $\mathbb{T}^{\mu}_G$. So let Y and X be varieties over $k^{G/\mathbb{Z}}$ with $Y \subseteq X \cup X^{\tau}$ dominantly. Extend the action of G to $k(Y)$ by $\tau x_i = x_{dimX+i}$, $\sigma_j x_i = x_i$. τ may no longer be an automorphism, so go to the inversive closure M of $k(Y)$. Then $M \models T^{\mu}_G$ and there is a point $p \in X(M)$ with $(p, \tau p) \in Y$. If $I(Y) = (f_1, ...f_r)$ and $\phi = \bigwedge_i f_i(x) = 0 \wedge \tau x_i = x_{dimX+i}$, then $\exists x\phi$ is a first order existential sentence true in M, but k is e.c. so $\exists x\phi$ is true in k as well. That gives us a point $q \in X(k)$ with $(q\tau q) \in Y$. So $k \models T^{\star}_G$.
Now assume $k \models T^{\star}_G$ and there is a G-field $K(\supseteq k)$ with $K \models \exists x\bar{\phi}(x)$, where $\bar{\phi}$ is as in proposition 1.
Let $a \in K$ be an element that realizes $\bar{\phi}$, $K \models \bar{\phi}(a)$
Since k is perfect and GAC, K is regular over k. Therefore there is a finite basis $b \in k$ such that $k = k^{G/\mathbb{Z}}(b)$ and $K = K^{G/\mathbb{Z}}(b)$
Let c be the coordinates of a, that is $a = (a_1, ..., a_s)$, $a_i = \sum c_{ij} b_j$ and $c = (c_{11}, c_{12}, ...c_{sr})$.
Let $X = V(c)$ and $Y = V(c\tau c)$. Then $Y \subseteq X \cup X^{\tau}$ projects dominantly so there is a point $p \in X(k^{G/\mathbb{Z}})$ with $(p\tau p \in)Y$, but then the element $e = (e_1, ..., e_s)$, where $e_i = \sum p_{ij} bj$, satisfies $\bar{\phi}$, $k \models \bar{\phi}(e)$, just as in theorem 15 , and therefore k is existentially closed. ■

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