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Alexander Berglund

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Department of Mathematics
Stockholm University

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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

POINCARÉ SERIES AND HOMOTOPY LIE ALGEBRAS OF MONOMIAL RINGS

ALEXANDER BERGLUND

Abstract. This thesis comprises an investigation of (co)homological invariants of *monomial rings*, by which is meant commutative algebras over a field whose minimal relations are monomials in a set of generators for the algebra, and of combinatorial aspects of these invariants. Examples of monomial rings include the 'Stanley-Reisner rings' of simplicial complexes. Specifically, we study the *homotopy Lie algebra* $\pi(R)$, whose universal enveloping algebra is the Yoneda algebra $\operatorname{Ext}_R(k,k)$, and the *multigraded Poincaré series* of R,

$$\mathrm{P}_R(\mathbf{x},z) = \sum_{i\geq 0, \alpha\in\mathbb{N}^n} \dim_k \operatorname{Ext}^i_R(k,k)_\alpha x^\alpha z^i.$$

To a set of monomials M we introduce a finite lattice K_M, and show how to compute the Poincaré series of an algebra R, with minimal relations M, in terms of the homology groups of lower intervals in this lattice. We introduce a finite dimensional L_∞-algebra $\mathfrak{L}_\infty(M)$, and compute the Lie algebra $\pi^{\geq 2}(R)$ in terms of the cohomology Lie algebra $\mathrm{H}^*(\mathfrak{L}_\infty(M))$. Applications of these results include a combinatorial criterion for when a monomial ring is Golod.

Analysis of the combinatorics involved leads us to introduce a new class of finite lattices, called *complete lattices*, which contain all geometric lattices and is closed under direct products. Completeness of a lattice L is characterized by the property that the higher operations of $\mathfrak{L}_\infty(M)$ are trivial, where M is the 'minimal realization' of L. We show how to interpret K_M as the intersection lattice of a certain real subspace arrangent $\mathcal{A}_M$ and, via the Goresky-MacPherson formula, we are able to give a new proof of a result relating the cohomology of the complement of the arrangement to the graded vector space $\operatorname{Tor}^R_*(k,k)$.

Department of Mathematics, Stockholm University
SE-106 91 Stockholm, Sweden
E-mail address: `alexb@math.su.se`

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Introduction

Classical cohomology theory assigns to a space X and a commutative ring k a graded k-algebra $\mathrm{H}^*(X;k) = \bigoplus_{n\geq 0} \mathrm{H}^n(X;k)$. Naively, the n:th graded component of this cohomology algebra detects n-dimensional 'holes' in the space X. For an algebraist, the natural objects to study are algebras, rather than spaces. What is the correct cohomology theory for algebras, or, in naive terms, what is the analogue of a 'hole' in an algebra? One cohomology theory for augmented k-algebras R is the Hochschild cohomology, $\mathrm{H}^*(R;k)$. In the case when k is a field, $\mathrm{H}^n(R;k)$ is isomorphic to $\mathrm{Ext}^n_R(k,k)$, the value at k of the nth right derived functor of $\mathrm{Hom}_R(-,k)$. Furthermore, the Yoneda interpretation of $\mathrm{Ext}^n_R(k,k)$ as equivalence classes of exact sequences $0 \to k \to E_n \to \ldots \to E_1 \to k \to 0$ of R-modules enables one to define a multiplication on $\mathrm{Ext}^*_R(k,k)$ by 'splicing' sequences, cf. [28], so that $\mathrm{H}^*(R;k)$ becomes a graded k-algebra.

If R is a commutative noetherian augmented k-algebra, where k is a field, then $\mathrm{Ext}^*_R(k,k)$ is the universal enveloping algebra of a graded Lie algebra $\pi(R) = \bigoplus_{i\geq 1} \pi^i(R)$, called the *homotopy Lie algebra* of R, cf. [4]. The name comes from an analogy with rational homotopy theory. For a simply connected based topological space X, the collection of rational homotopy groups $\pi^n(\Omega X)\otimes\mathbb{Q}$ of the loop space

ΩX form a graded Lie algebra with bracket induced from the Whitehead products. It is called the *rational homotopy Lie algebra* of X, and its universal enveloping algebra is isomorphic to the homology algebra $\mathrm{H}_*(\Omega X;\mathbb{Q})$. See [6] for an elaboration of this analogy.

There are few examples of algebras R where $\mathrm{Ext}_R(k,k)$, its dual $\mathrm{Tor}^R(k,k)$, or equivalently the Lie algebra $\pi(R)$, have been described explicitly, say in terms of a presentation of R. Even the enumerative problem of determining the graded vector space structure of $\mathrm{Ext}_R(k,k)$, or equivalently, determining the *Poincaré series* of R,

$$\mathrm{P}_R(z)=\sum_{i\geq 0}\dim_k \mathrm{Ext}^i_R(k,k)z^i,$$

is in general difficult. In practice, it amounts to constructing a minimal free resolution of k as an R-module, which will be infinite unless R is a regular ring.

In this thesis we focus on the problem of describing these objects for the class of *monomial rings*, by which we mean commutative algebras R whose minimal relations are monomials in the minimal set of generators for R.

There are at least two motivations for studying monomial rings. If $R=S/I$ for some homogeneous ideal I in the polynomial ring $S=k[x_1,\dots,x_n]$, then one can form the monomial ideal $\mathrm{in}(I)$ which is generated by the initial terms of the elements of I with respect to some term order for the monomials in S. Letting $A=S/\mathrm{in}(I)$, there is a convergent spectral sequence

$$\mathrm{Tor}^A_{*,*}(k,k)\Longrightarrow \mathrm{Tor}^R_{*,*}(k,k),$$

so in this sense the homological behaviour of R is approximated by that of A, cf. [2]. Secondly, numerous examples of monomial rings appear in algebraic combinatorics under the name of 'Stanley-Reisner rings', or 'face rings', of simplicial complexes. Computations of algebraic invariants of face rings have led to interesting results in combinatorics. For instance, the local cohomology of a face ring $k[\Delta]$ is computable in terms of the homology of the links of the simplicial complex Δ, and as a consequence one can derive topological criteria for when $k[\Delta]$ is a Cohen-Macaulay ring, cf. [13] or [32]. So a combinatorial description of the algebra $\mathrm{Ext}_{k[\Delta]}(k,k)$ could lead to new results in combinatorics. For instance, in simple-minded comparison with the case of local cohomology, such a description would in principle yield a combinatorial criterion for when a face ring is Golod (cf. Section 10).

A finitely generated monomial ring is of the form $R=S/I$, where S is the polynomial ring $k[x_1,\dots,x_n]$ and $I\subseteq S$ is an ideal generated by monomials. The algebra R inherits the natural $\mathbb{N}^n$-grading of S, and $\mathrm{Ext}_R(k,k)$ can be equipped with an $\mathbb{N}^n$-grading by considering $\mathbb{N}^n$-graded resolutions of k over R. Backelin [7] proved that the multigraded Poincaré series

$$\mathrm{P}_R(\mathbf{x},z)=\sum_{i\geq 0,\alpha\in\mathbb{N}^n}\dim_k \mathrm{Ext}^i_R(k,k)_\alpha x^\alpha z^i\in\mathbb{Z}[\![x_1,\dots,x_n,z]\!]$$

is the Taylor series expansion of a rational function of the form

$$\frac{\prod_{i=1}^n(1+x_iz)}{b_R(\mathbf{x},z)},$$

for some polynomial $b_R(\mathbf{x},z)\in\mathbb{Z}[x_1,\dots,x_n,z]$. This result was subsequently generalized by Backelin and Roos [8], who proved that the double Yoneda algebra

$\mathrm{Ext}_{\mathrm{Ext}_R(k,k)}(k,k)$ is noetherian. From this it follows that $\mathrm{Ext}_R(k,k)$, or equivalently $\pi(R)$, is finitely generated.

These qualitative results notwithstanding, only recently has one realized what combinatorial structures govern the (co)homological behaviour of R. For a monomial ideal I, with minimal set of generators M, let L_I denote the set $\{m_S \mid S \subseteq M\}$ partially ordered by divisibility, where m_S denotes the least common multiple of the monomials in S. It is called the *lcm-lattice* of I, cf. [21]. In addition to the partial order, L_I is the vertex set of a graph whose edges are pairs of monomials that have a non-trivial common factor. Avramov [5] proved that most of the homotopy Lie algebra $\pi(R)$ is determined by the combinatorial data encoded in the partially ordered graph L_I. Indeed, if I and J are two monomial rings in the polynomial rings S and T respectively, we say that I is equivalent to J if there is a bijection $f\colon L_I \to L_J$ which is both an isomorphism of graphs and of partial orders. With $Q = S/I$ and $R = T/J$, Avramov's result says that if I and J are equivalent, then there is an isomorphism of graded Lie algebras

$$\pi^{\geq 2}(Q) \cong \pi^{\geq 2}(R).$$

Here $\pi^{\geq 2}(Q)$ is the sub Lie algebra $\bigoplus_{i\geq 2}\pi^i(Q)$ of $\pi(Q)$. Recently, Charalambous [15] showed that this isomorphism behaves as expected with respect to multidegrees.

This equivalence relation on monomial ideals first made its appearance in [21], where it was proved that R is a Golod ring if and only if Q is. This follows also from the above isomorphism of Lie algebras, because of the general fact that R is Golod if and only if $\pi^{\geq 2}(R)$ is a free Lie algebra.

Results. This thesis is an expanded and modified version of the paper [9]. Also, parts of the material from the section 'Combinatorics' will appear in [10].

The thesis contains two main results. The first is the computation of the Poincaré series of a monomial ring R and the second is the computation of the graded Lie algebra $\pi^{\geq 2}(R)$.

We introduce a finite lattice K_M associated to a monomial set M — the lattice of 'saturated subsets' of M — and we prove (Theorem 5) that for a monomial ideal I in $S = k[x_1,\ldots,x_n]$ with minimal set of generators M, the denominator of the Poincaré series of $R = S/I$ can be computed by the formula

$$b_R(\mathbf{x},z) = 1 + \sum_{\emptyset\neq S\in K_M} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset,S);k)(z).$$

Here, $c(S)$ denotes the number of connected components of S with respect to the graph structure given by connecting monomials with non-trivial common factors, $(\emptyset,S)$ is the open interval between $\emptyset$ and S in the poset K_M, and $\widetilde{\mathrm{H}}((\emptyset,S);k)(z)$ denotes the generating function of the dimensions of the reduced homology groups of the poset $(\emptyset,S)$ with coefficients in the field k.

To a set of monomials M we associate a combinatorially defined finite dimensional $\mathbb{N}^n$-graded L_∞-algebra, $\mathfrak{L}_\infty(M)$, and we show how to compute the graded Lie algebra $\pi^{\geq 2}(R)$ in terms of the cohomology Lie algebra $\mathrm{H}^*(\mathfrak{L}_\infty(M))$. In fact, there is a functor F on the category of multigraded Lie algebras, defined by

$$FL = \frac{\mathbb{L}(L)}{\langle [\![x,y]\!] - [x,y] \mid x\perp y\in L\rangle},$$

where $[\![x,y]\!]$ denotes the bracket in the free Lie algebra $\mathbb{L}(L)$ and $[x,y]$ the bracket in L, and $x\perp y$ means that the multidegrees of x and y have disjoint supports, and we show (Theorem 6) that there is an isomorphism of multigraded Lie algebras

$$\pi^{\geq 2}(R) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(M)).$$

By construction, the L_∞-algebra $\mathfrak{L}_\infty(M)$ depends (up to an isomorphism which preserves the $\perp$-relation) only on the partially ordered graph L_M, so this could be seen as a refinement of Avramov's result.

As a consequence of our results we solve a problem posed in [5] of determining an upper bound for the degree of the denominator polynomial of the Poincaré series in terms of the number of minimal relations (Corollary 1). Also, the formula in [21] for the Betti numbers of R is coupled with our formula for the Poincaré series to obtain a combinatorial criterion for when a monomial ring is Golod (Theorem 7).

We introduce a new class of finite lattices, called *complete* lattices, which is closed under direct products and contains all geometric lattices. The main feature of this class is that monomial sets whose lcm-lattices are complete define Golod rings if and only if their corresponding graphs are complete. This generalizes the previously known result that this holds if the lcm-lattice is boolean.

In the last part we note how to interpret the lattice K_M as the intersection lattice of a certain real subspace arrangement. Our formula combined with the Goresky-MacPherson formula for the cohomology of the complement of such an arrangement results in a new proof of a result of [31] relating this cohomology to the graded vector space $\mathrm{Tor}^R_*(k,k)$ for a certain monomial ring R.

Conventions, notations

In this section we list the conventions and notations that should be kept in mind at all times.

Base ring. We work over a field k of arbitrary characteristic. Often, S will denote the polynomial ring $k[x_1,\dots,x_n]$.

Multigraded vector spaces. Our work takes place in the category of $\mathbb{N}\times\mathbb{N}^n$-graded vector spaces. The objects, referred to as 'multigraded vector spaces', or sometimes simply 'vector spaces', are collections $V=\{V_{i,\alpha}\}_{i,\alpha}$ of vector spaces $V_{i,\alpha}$ over k, indexed by $(i,\alpha)\in\mathbb{N}\times\mathbb{N}^n$. We write $|x|=i$, $\deg(x)=\alpha$ if $x\in V_{i,\alpha}$. In this case $|x|$ is called the *homological degree* and $\deg(x)$ is called the *multidegree* of x. The reason for distinguishing the homological grading from the other $\mathbb{N}$-gradings is that it governs signs in formulas.

If $\alpha\in\mathbb{N}^n$, then the *support* of α is the set $\mathrm{supp}(\alpha)=\{i \mid \alpha_i\neq 0\}$. Write $x\perp y$ if $\mathrm{supp}(\deg(x))\cap\mathrm{supp}(\deg(y))=\emptyset$. By coarsening the multigrading, our objects are $\mathbb{N}\times\mathbb{N}$-graded: an element x has degree (i,j) if $|x|=i$ and $|\deg(x)|=j$. Here $|\alpha|=\alpha_1+\ldots+\alpha_n$ if $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$. In order to avoid confusion with the homological $\mathbb{N}$-grading we call $|\deg(x)|$ the *weight* of the element x.

A linear map $f\colon V\to W$ of degree (j,β) is a collection of linear maps $f_{i,\alpha}\colon V_{i,\alpha}\to W_{i+j,\alpha+\beta}$. Most often, our maps are homogeneous with respect to the multigrading, so when we say 'f has degree n' it means that f has degree $(n,0)$. Morphisms of multigraded vector spaces are linear maps of degree 0.

Of course, direct sums of multigraded vector spaces are defined by taking degreewise sums. The tensor product $V\otimes W$ has $(V\otimes W)_{i,\alpha}=\bigoplus V_{j,\beta}\otimes W_{l,\gamma}$, where summation is over $(j,\beta)+(l,\gamma)=(i,\alpha)$, and all tensor products are over k, i.e., $\otimes=\otimes_k$.

If V is a vector space with $V_{i,\alpha}$ finite dimensional for all i, α, then the *generating function* of V is the formal power series

$$V(z, x_1, \dots, x_n) = \sum_{i \geq 0, \alpha \in \mathbb{N}} \dim_k(V_{i,\alpha}) z^i x^\alpha.$$

Forgetting multidegrees, $V(z)$ will denote $V(z, 1, \dots, 1)$, provided all but finitely many $V_{i,\alpha}$ are zero for i fixed.

The *suspension* sV of a vector space V has $(sV)_{i,\alpha} = V_{i-1,\alpha}$. Thus $|sx| = |x| + 1$ and $\deg(sx) = \deg(x)$ for $x \in V$. The *dual* of V is the vector space V^* where $(V^*)_{i,\alpha} = \mathrm{Hom}_k(V_{i,\alpha}, k)$ is the space of linear maps $V_{i,\alpha} \to k$.

$V_{\geq n}$ denotes the vector space with $(V_{\geq n})_{i,\alpha} = V_{i,\alpha}$, if $i \geq n$, and 0 otherwise. Similarly define $V_{\leq n}$ and $V_{>n}$. For a non-negative integer i, we denote by V_i the vector space $V_{\geq i} \cap V_{\leq i}$.

If V is a vector space, and if S is a subset of $\mathbb{N}^n$, then V_S is the graded vector space with $(V_S)_{i,\alpha} = V_{i,\alpha}$ if $\alpha \in S$ and $(V_S)_{i,\alpha} = 0$ otherwise. The space $V_\tau := V_{\{0,1\}^n}$ is called the *truncation* of V, and V is called *truncated* if $V = V_\tau$.

DG-algebras. A complex is a vector space $V = \{V_{i,\alpha}\}_{i,\alpha}$ together with a map $d \colon V \to V$ of homological degree -1, such that $d^2 = 0$. A differential graded algebra, or a *dg-algebra* for brevity, is a complex A together with a morphism of complexes $A \otimes A \to A$, denoted $a \otimes b \mapsto ab$. In this context, this means that $A_{i,\alpha} \cdot A_{j,\beta} \subseteq A_{i+j,\alpha+\beta}$, and d is a derivation, i.e., $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Note that A and the homology $\mathrm{H}(A) = \operatorname{Ker} d / \operatorname{Im} d$ become graded A_0-algebras, where $A_0 = \{A_{0,\alpha}\}_\alpha$. We say that A is commutative if $xy = (-1)^{|x||y|} yx$ for homogeneous $x, y \in A$. A morphism of dg-algebras is called a *quasi-isomorphism* if the induced map on homology is an isomorphism. An algebra A is called *connected* if $A_0 = k$.

When convenient, we assume our algebras to be augmented with a morphism of algebras $\epsilon \colon A \to k$ such that the augmentation ideal $\operatorname{Ker} \epsilon$ is concentrated in positive weight. This is to ensure that the results available for local rings are to be valid in our situation as well. Many results are cited from the exposition [4] which deals exclusively with local rings.

Graded Lie algebras. Following [4], by a graded Lie algebra we will mean a vector space $L = \{L_{i,\alpha}\}$ concentrated in positive homological degrees, i.e., $L = L_{\geq 1}$, together with a linear map $L \otimes L \to L$, $x \otimes y \mapsto [x, y]$ of degree 0 and a squaring operation $L_{i,\alpha} \to L_{2i,2\alpha}$, $x \mapsto x^{[2]}$, for odd i, such that

$$\begin{array}{lcll}
[x, y] & = & -(-1)^{|x||y|}[y, x] & \\
[x, [y, z]] & = & [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] & \\
(x + y)^{[2]} & = & x^{[2]} + y^{[2]} + [x, y], & \text{if } |x| = |y| \text{ is odd} \\
(cx)^{[2]} & = & c^2 x^{[2]}, & \text{if } |x| \text{ is odd and } c \in k \\
[x^{[2]}, y] & = & [x, [x, y]], & \text{if } |x| \text{ is odd}
\end{array}$$

Furthermore it is required that $[x, x] = [y, [y, y]] = 0$ for even $|x|$ and odd $|y|$.

The *universal enveloping algebra* UL of a graded Lie algebra L is defined as the quotient of the tensor algebra $T(L)$ by the relations $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$ for $x, y \in L$ and $x \otimes x - x^{[2]}$ for $x \in L$ with $|x|$ odd.

If L is a graded Lie algebra, then $I = L_{\mathbb{N}^n - \{0,1\}^n}$ is a homogeneous ideal in L, and L_τ is naturally a graded Lie algebra by identifying it with the quotient L/I.

Furthermore, if J is any homogeneous ideal of L, then J_τ is an ideal of L_τ and $(L/J)_\tau \cong L_\tau/J_\tau$ as graded Lie algebras.

Monomial sets. Let $x_1, \ldots, x_n$ be variables. If $\alpha \in \mathbb{N}^n$, then we write x^α for the monomial $x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. The multidegree of x^α is $\deg(x^\alpha) = \alpha$. If $\alpha \in \{0,1\}^n$, then both α and x^α are called *squarefree*.

To a set M of monomials we associate an undirected graph, with vertices M, whose edges go between monomials having a non-trivial common factor. This is the graph structure referred to when properties such as connectedness etc., are attributed to monomial sets. Thus, for instance, a monomial set is called *independent* if the monomials therein are pairwise without common factors. By $D(M)$ we denote the set of non-empty independent subsets of M, and the *independence number* of M is the largest size of an independent subset of M. A connected component of M is a maximal connected subset. Any monomial set M has a decomposition into connected components $M = M_1 \cup \ldots \cup M_r$, and we let $c(M) = r$ denote the number of such.

If I is an ideal in a polynomial ring generated by monomials there is a uniquely determined minimal set of monomials generating I. This minimal generating set, denoted $\mathrm{Gen}(I)$, is characterized by being an *antichain* with respect to divisibility, that is, for all $m, n \in \mathrm{Gen}(I)$, $m|n$ implies $m = n$.

If S is a finite set of monomials, then m_S denotes the least common multiple of all elements of S. By convention $m_\emptyset = 1$. The set $L_M = \{m_S \mid S \subseteq M\}$ partially ordered by divisibility is a lattice with lcm as join, called the *lcm-lattice* of the set M. If I is a monomial ideal, then $L_I := L_{\mathrm{Gen}(I)}$ is called the lcm-lattice of I. By our general convention that monomial sets are graphs, L_I has also a graph structure. The *gcd-graph* of I, studied in [5], is the complement of the graph L_I, i.e., it has the same vertices L_I, but its edges go between monomials that are relatively prime.

Two monomial sets M and N are said to be *separated* if $\gcd(m_M, m_N) = 1$.

If M, N are two sets of monomials then M_N denotes the set of those monomials in M which divide some monomial in N. Write $M_m = M_{\{m\}}$, and $M_\alpha = M_{x^\alpha}$.

Sign convention. The following sign convention will be used. If X is a totally ordered set, let $\{v_x\}_{x\in X}$ be anti-commuting variables indexed by X. For a subset $S = \{x_1, \ldots, x_n\}$ of X, where $x_1 < \ldots < x_n$, let $v_S = v_{x_1} \wedge \ldots \wedge v_{x_n}$. If $S = S_1 \cup \ldots \cup S_r$ is a partition of S, then define the sign $\mathrm{sgn}(S_1, \ldots, S_r) \in \{-1, 1\}$ by

$$v_S = \mathrm{sgn}(S_1, \ldots, S_r) v_{S_1} \wedge \ldots \wedge v_{S_r}.$$

Set $\mathrm{sgn}(S_1, \ldots, S_r) = 0$ if $S_i \cap S_j \neq \emptyset$ for some $i \neq j$.

Minimal models and strongly homotopy Lie algebras

1. Free commutative dg-algebras

We here collect some well known facts about free commutative algebras.

The free graded commutative algebra on a graded vector space V is the algebra

$$\Lambda V = \text{exterior algebra}(V_{\text{odd}}) \otimes_k \text{symmetric algebra}(V_{\text{even}}).$$

There is a unique comultiplication in ΛV making it into a Hopf algebra in which V is the space of primitive elements. If X is a well ordered basis for V, then a basis for ΛV is given by the monomials $x_1^{a_1} \ldots x_r^{a_r}$, where $r \geq 0$, $x_1 < \ldots < x_r$

and $a_i = 1$ if $|x_i|$ is odd. The subspace $\Lambda^n V$ has basis all such monomials with $a_1 + \ldots + a_r = n$, and elements thereof have *word length* n.

A graded vector space $V = \{V_i\}_{i\geq 1}$ is called locally finite if V_i is finite dimensional for each i. Note that if V is locally finite, then so is ΛV. In this case the dual Hopf algebra of ΛV is isomorphic to the divided power algebra $\Gamma(V^*)$ on V^*, cf. [24]. This Hopf algebra can be described as follows. If X is an ordered basis for V, and X^* is a dual basis, then $\Gamma(V^*)$ has a basis of 'divided monomials' dual to the monomial basis for ΛV:

$$\xi_r^{(a_r)} \ldots \xi_1^{(a_1)},$$

where $r \geq 0$, $\xi_i \in X^*$ is the dual basis element of $x_i \in X$, $x_1 < \ldots < x_r$ and $a_i = 1$ if $|x_i|$ is odd. $\Gamma^n(V^*)$ has basis all such monomials with $a_1 + \ldots + a_r = n$. The multiplication of $\Gamma(V^*)$ is graded commutative and satisfies

$$\xi^{(i)}\xi^{(j)} = \binom{i+j}{i}\xi^{(i+j)}.$$

The comultiplication Δ is a morphism of algebras and

$$\Delta(\xi^{(n)}) = \sum_{i+j=n} \xi^{(i)} \otimes \xi^{(j)}.$$

The following two properties of $\Gamma(V)$ will be needed. The map $p_V \colon \Gamma(V) \to V$ is defined to be the identity on $\Gamma^1(V)$ and zero on $\Gamma^n(V)$ if $n \neq 1$.

- A linear map $f \colon V \to W$ of degree 0 extends uniquely to a morphism of coalgebras $\tilde{f} \colon \Gamma(V) \to \Gamma(W)$ such that $fp_V = p_W\tilde{f}$. The extension is given by
$$\tilde{f}(x_1^{(a_1)} \ldots x_n^{(a_n)}) = (f(x_1))^{(a_1)} \ldots (f(x_n))^{(a_n)}$$
on basis elements.
- A linear map $\delta \colon \Gamma^n(V) \to V$ extends uniquely to a coderivation $\tilde{\delta}$ on $\Gamma(V)$ which decreases word length by $n-1$. On a basis element it is given by

$$\tilde{\delta}(x_1^{(a_1)} \ldots x_r^{(a_r)}) = \sum_{\substack{i_1+\ldots+i_r=n \\ 0\leq i_j\leq a_j}} \pm\delta(x_1^{(i_1)} \ldots x_r^{(i_r)})x_1^{(a_1-i_1)} \ldots x_r^{(a_r-i_r)}, \tag{1}$$

where the sign is determined by the Koszul sign convention.

Denote by (V) the ideal generated by V in ΛV. A homomorphism $f \colon \Lambda V \to \Lambda W$ of graded algebras with $f(V) \subseteq (W)$ induces a linear map $Lf \colon V \to W$, called the *linear part* of f, which is defined by the requirement $f(v) - Lf(v) \in (W)^2$ for all $v \in V$.

If $x_1, \ldots, x_n$ is a basis for V_0, then $\Lambda V = S \otimes_k \Lambda(V_+)$, where $S = k[x_1, \ldots, x_n]$ and V_+ is the sum of all V_i for positive i. Therefore ΛV may be regarded as an S-module and each $(\Lambda V)_n$ is a finitely generated free S-module. Let $\mathfrak{m} \subseteq S$ be the maximal ideal generated by V_0 in S. Note that $(V) = (V_+) + \mathfrak{m}$ as vector spaces. The following basic lemma is a weak counterpart of Lemma 14.7 in [17] and of Lemma 1.8.7 in [23].

Lemma 1. *Let $f \colon \Lambda U \to \Lambda V$ be a homomorphism of graded algebras such that $f_0 \colon \Lambda(U_0) \to \Lambda(V_0)$ is an isomorphism and the linear part, $Lf \colon U \to V$, is an isomorphism of graded vector spaces. Then f is an isomorphism.*

Proof. Identify $S = \Lambda(U_0) = \Lambda(V_0)$ via f_0. Since Lf is an isomorphism, ΛU and ΛV are isomorphic. Thus to show that f is an isomorphism it is enough to show that $f_n\colon (\Lambda U)_n \to (\Lambda V)_n$ is surjective in each degree n, because f_n is a map between finitely generated isomorphic free S-modules. We do this by induction. The map f_0 is surjective by assumption. Let $n \geq 1$ and assume that f_i is surjective for every $i < n$. Then since Lf is surjective we have

$$(\Lambda V)_n \subseteq f((\Lambda U)_n) + ((V_+)^2)_n + \mathfrak{m}(\Lambda V)_n.$$

$((V_+)^2)_n$ is generated by products vw, where $|w|, |v| < n$, so by induction $((V_+)^2)_n \subseteq f((\Lambda U)_n)$. Hence

$$(\Lambda V)_n \subseteq f((\Lambda U)_n) + \mathfrak{m}(\Lambda V)_n.$$

$(\Lambda V)_n$ and $f((\Lambda U)_n)$ are graded S-modules, so it follows from the graded version of Nakayama's lemma that $(\Lambda V)_n = f((\Lambda U)_n)$. $\square$

By a *free commutative dg-algebra*, we will mean a dg-algebra of the form $(\Lambda V, d)$, for some graded vector space V, where the differential d satisfies $dV \subseteq (V)$. The linear part Ld of d on ΛV is a differential on V, and will be denoted d_0. A free commutative dg-algebra $(\Lambda V, d)$ is called *minimal* if $dV \subseteq (V)^2$. Thus $(\Lambda V, d)$ is minimal if and only if $d_0 = 0$.

The following 'lifting lemma' is often useful.

Lemma 2. *Let $\Lambda V = (\Lambda V, d)$ be a free commutative dg-algebra and let $p\colon A \to B$ be a surjective quasi-isomorphim of commutative dg-algebras. Then any map of dg-algebras $f\colon \Lambda V \to B$ lifts to A making the diagram commutative*

$$\begin{tikzcd} & A \arrow[d, "\sim"] \\ \Lambda V \arrow[ur, dotted, "\eta"] \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. ΛV is the union of sub dg-algebras $\Lambda(V_{<n})$, and we define η by induction over these 'skeleta'. Suppose $\eta_n\colon \Lambda(V_{<n}) \to B$ has been defined so that $p\eta_n = f$ and $d\eta_n = \eta_n d$. Choose a basis X for V_n. For $v \in X$, choose an $a \in A$ with $p(a) = f(v)$. Since $dv \in \Lambda(V_{<n})$, $\eta_n(dv)$ is a cycle in A and by assumption $p\eta_n(dv) = f(dv) = df(v) = dp(a) = p(da)$, so $\eta_n(dv) - da$ is a cycle in $\operatorname{Ker} p$. Because p is a quasi-isomorphism, the long exact homology sequence derived from $0 \to \operatorname{Ker} p \to A \to B \to 0$ shows that $\operatorname{Ker} p$ has trivial homology. Therefore $\eta_n(dv) - da = dy$ for some $y \in \operatorname{Ker} p$. Set $w_v = a + y$. Then $p(w_v) = f(v)$ and $dw_v = \eta_n(dv)$. Since A is commutative, the map $v \mapsto w_v$ from X to A extends to a map of dg-algebras $\eta_{n+1}\colon \Lambda(V_{\leq n})$ such that $p\eta_{n+1} = f$. The induction starts with the structure maps from $k = \Lambda(V_{<0})$. $\square$

2. The homotopy Lie algebra of a free commutative dg-algebra

Let $(\Lambda V, d)$ be a free commutative dg-algebra with $V_0 = 0$. The differential d splits as $d = d_0 + d_1 + d_2 + \ldots$, where d_i raises word length by i, i.e., $d_i(\Lambda^n V) \subseteq \Lambda^{n+i} V$. It is easy to check that each d_i is a derivation on ΛV. Furthermore, a look at the homogeneous components of the relation $d^2 = 0$ with respect to word length yields the sequence of relations

$$\begin{aligned} d_0^2 &= 0 \\ d_0d_1 + d_1d_0 &= 0 \\ d_0d_2 + d_1^2 + d_2d_0 &= 0 \\ &\vdots \end{aligned}$$

Let $sL = V^*$ (or equivalently $L = (sV)^*$). Dualizing $(\Lambda V, d)$, we obtain $(\Gamma(sL), \delta)$, where $\delta = d^*$ is a coderivation of degree 1 such that $\delta^2 = 0$. The coderivation δ splits as $\delta = \delta_0 + \delta_1 + \delta_2 + \ldots$, where $\delta_i = d_i^*$ is a coderivation decreasing word length by i, i.e., $\delta_i(\Gamma^n(sL)) \subseteq \Gamma^{n-i}(sL)$. By definition, the data of a degree 1 coderivation on $\Gamma(sL)$ of square zero determines the structure of an *L_∞-algebra*, or a *strongly homotopy Lie algebra*, on L, cf. [26] and [27]. If the base field has characteristic zero, then, as is shown in [27], this structure is equivalent to a sequence of anti-symmetric brackets $L^{\otimes r} \to L$, $x_1 \otimes \ldots \otimes x_r \mapsto [x_1, \ldots, x_r]$, of degree $2-r$, for $r \geq 1$, satisfying a the 'generalized Jacobi identities' ([26], Definition 2.1)

$$\sum_{i=1}^{n} \sum_{\sigma} \chi(\sigma)(-1)^{i(n-i)} [[x_{\sigma(1)}, \ldots, x_{\sigma(i)}], x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}] = 0$$

for $n \geq 1$. Here the second sum is over all permutations σ of $\{1, 2, \ldots, n\}$ such that $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$, and $\chi(\sigma) = \pm 1$ is the sign for which the equality

$$[x_1, \ldots, x_n] = \chi(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]$$

is implied by the anti-symmetry condition, e.g., $\chi(231) = (-1)^{|x_1||x_3| + |x_1||x_2|}$. The brackets are defined in terms of the coderivations δ_{r-1} as follows:

$$\delta_{r-1}(sx_1 \ldots sx_r) = (-1)^{\epsilon} s[x_1, \ldots, x_r].$$

Here $\epsilon = |x_1| + |x_3| + \ldots + |x_{r-1}|$ if r is even and $\epsilon = 1 + |x_2| + |x_4| + \ldots + |x_{r-1}|$ if r is odd. The definition of the brackets uses only the subalgebra of $\Gamma(sL)$ generated by sL. In characteristic zero, this is of course the whole algebra, but in positive characteristics, the inclusion is strict. Several non-linear operations can be defined on L, for instance a 'reduced r-th power', $x \mapsto x^{[r]}$, given by $\delta_{r-1}((sx)^{(r)}) = \pm sx^{[r]}$, and these can not be recovered from the multilinear brackets above.

An L_∞-algebra is a Lie algebra 'up to homotopy' in the following sense. The coderivations δ_0 and δ_1 give rise to a degree 1 map $d\colon L \to L$, a bracket $[\cdot, \cdot]\colon L^{\otimes 2} \to L$, and for elements x of odd degree, a squaring operation $x \mapsto x^{[2]}$:

$$\delta_0(sx) = -sd(x), \quad \delta_1(sxsy) = (-1)^{|x|} s[x, y], \quad \delta_1((sx)^{(2)}) = -sx^{[2]}.$$

The graded commutativity of $\Gamma(sL)$ results in anti-commutativity of the bracket:

$$[x, y] = -(-1)^{|x||y|}[y, x].$$

The relations $\delta_0^2 = 0$ and $\delta_0\delta_1 + \delta_1\delta_0 = 0$ translate via (1) to $d^2 = 0$, $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$ and $d(x^{[2]}) = d(x)x$. Define $h\colon L^{\otimes 3} \to L$ by

$$\delta_2(sxsysz) = -(-1)^{|y|} sh(x, y, z),$$

and let $J(x, y, z) = [[x, y], z] - [x, [y, z]] - (-1)^{|y||z|}[[x, z], y]$ be the Jacobian of the bracket in L. Using (1), it is an easy exercise to verify that

$$\delta_1^2(sxsysz) = (-1)^{|y|} sJ(x, y, z).$$

The relation $\delta_0\delta_2 + \delta_1^2 + \delta_2\delta_0 = 0$ therefore shows that

$$J(x, y, z) = dh(x, y, z) + h(dx, y, z) + (-1)^{|x|}h(x, dy, z) + (-1)^{|x|+|y|}h(x, y, dz),$$

or in other words, $J = dh + hd$, so h is a contracting homotopy for the Jacobian J. In this sense the Jacobi identity is satisfied up to homotopy in L. Applying δ_1 to the relation $(sx+sy)^{(2)}-(sx)^{(2)}-(sy)^{(2)} = sxsy$, shows that $(x+y)^{[2]}-x^{[2]}-y^{[2]} = [x, y]$, for odd $x, y \in L$. Similarly the rest of the axioms for a graded Lie algebra hold strictly or up to homotopy in L.

Relations that hold up to homotopy in L become strict relations in the cohomology algebra $\mathrm{H}^*(L)$. Thus if L is an L_∞-algebra, then $\mathrm{H}^*(L)$ is a graded Lie algebra. Of course we do not use the full L_∞-structure of L in order to exhibit $\mathrm{H}^*(L)$ as a graded Lie algebra. We use only the differential, the binary bracket and the squaring operation — the higher operations merely ensure that passage to cohomology produces a strict Lie algebra.

Definition 1. Let $(\Lambda V, d)$ be a free commutative dg-algebra with $V_0 = 0$. The *homotopy Lie algebra* of $(\Lambda V, d)$ is the graded Lie algebra $\mathfrak{L}(\Lambda V, d) = \mathrm{H}^*(L)$, where $L = (sV)^*$ is the L_∞-algebra associated to $(\Lambda V, d)$.

With serious abuse of notation, the homotopy Lie algebra $\mathfrak{L}(\Lambda V, d)$ of $(\Lambda V, d)$ will sometimes be denoted $\mathfrak{L}_V$. The homotopy Lie algebra construction is a contravariant functor in $(\Lambda V, d)$: A morphism of dg-algebras $f\colon (\Lambda V, d) \to (\Lambda W, d')$ such that $f(V) \subseteq (W)$ has a linear part f_0 specified by $f(v) - f_0(v) \in \Lambda^{\geq 2}(W)$ for all $v \in V$. Inspection of the linear parts of the relations $f(xy) = f(x)f(y)$ and $d'f = fd$ shows that f_0 is a homomorphism dg-algebras $(\Lambda V, d_0) \to (\Lambda W, d_0')$ with $f_0(V) \subseteq W$. Thus the dual map $f_0^*\colon \Gamma(sL_W) \to \Gamma(sL_V)$ is a morphism of coalgebras mapping W^* into V^*, and therefore $f_0^*((sx_1)^{(a_1)}\ldots(sx_n)^{(a_n)}) = (f_0^*(sx_0))^{(a_0)}\ldots(f_0^*(sx_n))^{(a_n)}$ for all $x_i \in L$. The induced map $\hat{f}\colon L_W \to L_V$ is given by $f_0^*(sx) = s\hat{f}(x)$, and it is a map of complexes. From the relation $\delta_1 f_0^* - f_0^*\delta_1' = f_1^*\delta_0' - \delta_0 f_1^*$, we see that $\hat{f}$ is a morphism of algebras up to homotopy. Thus, passing to cohomology, $\mathfrak{L}_f := \mathrm{H}^*(\hat{f})$ is seen to be a morphism of Lie algebras $\mathfrak{L}_W \to \mathfrak{L}_V$. It is trivial that $\mathfrak{L}_{fg} = \mathfrak{L}_g\mathfrak{L}_f$ and that $\mathfrak{L}_1 = 1$. So we have a contravariant functor from free commutative dg-algebras to graded Lie algebras.

3. Models

Let (A, d_A) be a commutative dg-algebra. A *model* for (A, d_A) is a free commutative dg-algebra $(\Lambda V, d)$, where V is a non-negatively graded vector space, together with a quasi-isomorphism of dg-algebras

$$(\Lambda V, d) \xrightarrow{\sim} (A, d_A)$$

The model is called *minimal* if $(\Lambda V, d)$ is a minimal, i.e., if $d(V) \subseteq (V)^2$. We focus on two particular species of dg-algebras (A, d_A).

- If $A = A_0$ is a commutative algebra with trivial differential, then a model $(\Lambda V, d)$ of A satisfies $A = \mathrm{H}_0(\Lambda V, d) = \Lambda(V_0)/(\mathrm{Im}\, d)_0$ and $\mathrm{H}_i(\Lambda V, d) = 0$ for $i > 0$. In particular, A is a module over the polynomial ring $S = \Lambda(V_0)$ and $(\Lambda V, d)$ is a resolution of A by free S-modules

$$\cdots \to (\Lambda V)_n \to (\Lambda V)_{n-1} \to \cdots \to (\Lambda V)_1 \to S \to A \to 0$$

- If $\mathrm{H}_0(A) = k$, then the map $A \to \mathrm{H}_0(A) = k$, defined by identifying $\mathrm{H}_0(A)$ with $A_0/(\operatorname{Im} d)_0$, is a morphism of dg-algebras, so A is augmented. In this case it is always possible to choose a model $(\Lambda V, d)$ of A with $V_0 = 0$. If the model is minimal, then necessarily $V_0 = 0$.

By using dg-algebra resolutions with free S-modules, one can bridge the gap between commutative algebras and connected augmented dg-algebras.

Lemma 3. *Let $(\Lambda V, d)$ be a model for a commutative k-algebra R concentrated in (homological) degree 0. Let S be the polynomial ring $\Lambda(V_0)$, and suppose that $F \to R$ is a surjective quasi-isomorphism, where F is a dg-algebra with $F_0 = S$ and each F_i is a free S-module. Then the algebra $\Lambda V \otimes_S k \cong \Lambda(V_{\geq 1})$ with differential $\bar{d}$ induced from d is a model of the connected dg-algebra $F \otimes_S k$. Furthermore, if $(\Lambda V, d)$ is minimal, then so is $(\Lambda(V_{\geq 1}), \bar{d})$.*

Proof. By Lemma 2 there is a map of dg-algebras $\eta\colon \Lambda V \to F$ making the diagram below commutative.

$$\begin{tikzcd} & F \arrow[d, "\sim"] \\ \Lambda V \arrow[ur, dotted, "\eta"] \arrow[r, "\sim"] & R \end{tikzcd}$$

The map η is then necessarily a quasi-isomorphism. Both ΛV and F are semi-free S-modules, so there results a quasi-isomorphism $\eta \otimes_S 1_k \colon \Lambda V \otimes_S k \to F \otimes_S k$. This exhibits $(\Lambda V \otimes_S k, \bar{d}) = (\Lambda(V_{\geq 1}), \bar{d})$ as a model of the dg-algebra $F \otimes_S k$. It is clear that minimality of $(\Lambda V, d)$ implies minimality of $(\Lambda(V_{\geq 1}), \bar{d})$. □

A minimal model for R always exists, and is unique up to (non-canonical) isomorphism, cf. [4] Proposition 7.2.4.

Lemma 4. *Let $(\Lambda V, d_V)$ be a $\mathbb{N}^n$-graded dg-algebra with $\mathrm{H}_0(\Lambda V, d_V) = R$, and assume that*

$$\mathrm{H}_i(\Lambda V, d_V)_\tau = 0$$

for all $i > 0$. Then $(\Lambda V, d_V)$ can be embedded into a model $(\Lambda W, d_W)$ of R such that $(\Lambda V, d_V)_\tau = (\Lambda W, d_W)_\tau$. Furthermore, if $(\Lambda V, d)$ is minimal, then $(\Lambda W, d)$ may be chosen minimal.

Proof. A minimal model is constructed inductively, by successively adjoining basis elements to V in order to kill homology, see [4] Propositions 2.1.10 and 7.2.4 for details. Since ΛV is $\mathbb{N}^n$-graded, we can do the inductive step one multidegree at a time. Adding a basis element of multidegree α will not affect the part of the algebra below α. Since $\mathrm{H}_i(\Lambda V, d_V)_\tau = 0$ for all $i > 0$, we do not need to add variables in the multidegrees $\{0,1\}^n$ in order to kill homology. Applying this technique, we get a minimal model $(\Lambda W, d_W)$ of R, where W is a vector space obtained from V by adjoining basis elements of degrees outside $\{0,1\}^n$. In particular $(\Lambda W, d_W)_\tau = (\Lambda V, d_V)_\tau$. □

3.1. **Homotopy Lie algebras of dg Γ-algebras.** A *dg Γ-algebra* is a commutative dg-algebra (F, d) together with a system of divided power operations. This means that for each element $x \in F$ of even positive degree and each $i \geq 0$ there is an element $x^{(i)}$ of degree $i|x|$ subject to certain conditions, cf. [23] Definition 1.7.1. We do not reproduce the definition and elementary properties of dg Γ-algebras here since they will not be used in our further arguments. However, two special cases should be mentioned. If $F = F_0$ is a commutative ring, then $(F, 0)$ is trivially a dg Γ-algebra. If k has characteristic zero, then every dg-algebra over k has a unique structure of dg Γ-algebra. It is given by $x^{(i)} = x^i/i!$.

The functors $\mathrm{Ext}_R(k, k)$ and $\mathrm{Tor}^R(k, k)$ for augmented k-algebras R extend to the category of augmented dg-algebras. For instance, to an augmented dg-algebra (A, d) one associates the 'differential Ext-algebra' $\mathrm{Ext}_{(A,d)}(k, k)$, which can be defined using *semi-free* resolutions of k. We refer to [17] for the yoga of differential homological algebra.

For an augmented dg Γ-algebra $(F, d) \to k$ such that $\mathrm{H}_0(F, d)$ is a noetherian ring and $\mathrm{H}_i(F, d)$ is a noetherian $\mathrm{H}_0(F, d)$-module for each i, the Yoneda algebra $\mathrm{Ext}_{(F,d)}(k, k)$ is the universal enveloping algebra of a uniquely determined graded Lie algebra $\pi(F) = \bigoplus_{n\geq 1} \pi^n(F)$, called the *homotopy Lie algebra* of F, cf. [3] Theorems 1.1 and 1.2. A sketch of the argument goes as follows. One proves that $\mathrm{Tor}^{(F,d)}(k, k)$ admits the structure of a graded commutative Hopf Γ-algebra. Then a structure theorem, due to Milnor and Moore [29] in characteristic $p = 0$, André [1] in $p > 2$ and Sjödin [33] in $p = 2$, says that the dual Hopf algebra, in this case $\mathrm{Ext}_{(F,d)}(k, k)$, is the universal enveloping algebra of a uniquely determined graded Lie algebra $\pi(F)$. This may seem like an awkward definition of $\pi(F)$. The question arises of how to compute $\pi(F)$ before knowing the structure of $\mathrm{Ext}_{(F,d)}(k, k)$ as a graded Hopf algebra.

A commutative k-algebra R is a dg Γ-algebra concentrated in degree 0 with trivial differential. Thus, if R is noetherian and augmented with a morphism of algebras $R \to k$, then it makes sense to talk about the homotopy Lie algebra $\pi(R)$ of R. In this case, $\pi(R)$ can be computed as the homology of the dg Lie algebra of Γ-derivations on an 'acyclic closure' of k over R, cf. [4] section 10.2.

Another method of computing homotopy Lie algebras is by the use of minimal models. The following theorem will be our main tool.

Theorem 1 ([3], Theorem 4.2)**.** *Let F be an augmented dg Γ-algebra such that $\mathrm{H}_0(F) = k$ and $\mathrm{H}_i(F)$ is a finite dimensional vector space over k for each i, and let $(\Lambda V, d)$ be a minimal model of F. Then there is an isomorphism of graded Lie algebras*

$$\mathfrak{L}(\Lambda V, d) \cong \pi(F).$$

4. Minimizing free commutative dg-algebras

The following is a counterpart of Lemma 3.2.1 in [23], but taking the $\mathbb{N}^n$-grading into account. It tells us how to 'minimize' a given dg-algebra. Here the assumption that the augmentation ideal (V) of ΛV is concentrated in positive weight becomes essential.

Proposition 1. *Let $(\Lambda V, d)$ be a free commutative dg-algebra and let $H = \mathrm{H}(V, d_0)$. There is a differential d_H on ΛH making it a dg-algebra, and a surjective map of dg-algebras*

$$(\Lambda V, d) \xrightarrow{\psi} (\Lambda H, d_H)$$

such that

- *$(\Lambda H, d_H)$ is minimal, i.e., $d_H(H) \subseteq \Lambda^{\geq 2} H$.*
- *$\mathrm{H}_0(\Lambda V, d) \cong \mathrm{H}_0(\Lambda H, d_H)$.*

- *The induced map of Lie algebras*
$$\mathfrak{L}_\psi \colon \mathfrak{L}(\Lambda(H_{\geq 1}), \bar{d}_H) \to \mathfrak{L}(\Lambda(V_{\geq 1}), \bar{d})$$
is an isomorphism.
- *The squarefree truncation of ψ is a quasi-isomorphism, i.e.,*
$$(\psi_\tau)_* \colon \mathrm{H}(\Lambda V, d)_\tau \xrightarrow{\cong} \mathrm{H}(\Lambda H, d_H)_\tau$$
.
- *If k has characteristic 0, then ψ is a quasi-isomorphism.*

Proof. Let W be a graded subspace of V such that $V = \operatorname{Ker} d_0 \oplus W$ and similarly split $\operatorname{Ker} d_0$ as $H \oplus \operatorname{Im} d_0$ (hence $H \cong \mathrm{H}(V, d_0)$). Note that since $dV_1 \subseteq \mathfrak{m}^2$, $W_0 = W_1 = 0$. As V is concentrated in positive weight, so is W. The map d_0 induces an isomorphism $W \to \operatorname{Im} d_0$, so we may write
$$V = H \oplus W \oplus d_0(W).$$
Consider the graded subspace $U = H \oplus W \oplus dW$ of ΛV. The induced homomorphism of graded algebras $f \colon \Lambda U \to \Lambda V$ is an isomorphism by Lemma 1, because f_0 is the identity on ΛH_0 and the linear part of f is the map $1_H \oplus 1_W \oplus g$, where $g \colon dW \to d_0(W)$ is the isomorphism taking an element to its linear part (isomorphism precisely because $\operatorname{Ker} d_0 \cap W = 0$). Thus we may identify ΛU and ΛV via f. In particular $f^{-1} df$ is a differential on ΛU, which we also will denote by d, and $(\Lambda U, d)$ is a dg-algebra in which $\Lambda(W \oplus dW)$ is a dg-subalgebra. The projection $U \to H$ induces an epimorphism of graded algebras $\phi \colon \Lambda U \to \Lambda H$ with kernel $(W \oplus dW)\Lambda U$, the ideal generated by $W \oplus dW$ in ΛU. Define a differential d_H on ΛH by
$$d_H(h) = \phi d\iota(h),$$
where ι is induced by the inclusion $H \subseteq U$. With this definition it is evident that $(\Lambda H, d_H)$ is minimal, and ϕ becomes a morphism of dg-algebras. Let
$$\psi = \phi f^{-1} \colon \Lambda V \to \Lambda H.$$
The linear part ψ_0 of ψ is the projection of V onto H given by the above splitting, and it induces an isomorphism in homology $\mathrm{H}(V, d_0) \cong H$. The map of Lie algebras $\mathfrak{L}(\Lambda(H_{\geq 1}), \bar{d}_H) = (s(H_{\geq 1}))^* \to \mathrm{H}^*((s(V_{\geq 1}))^*) = \mathfrak{L}(\Lambda(V_{\geq 1}), \bar{d})$ is the map induced in cohomology by the dual of the suspension of ψ_0 restricted to positive degrees. Since we work over a field, this is an isomorphism by the universal coefficient theorem.

Consider the increasing filtration
$$F_p = (\Lambda H)_{\leq p} \cdot \Lambda(W \oplus dW).$$
Obviously $\cup F_p = \Lambda U$, and $dF_p \subseteq F_p$ since d preserves $\Lambda(W \oplus dW)$. The associated first quadrant spectral sequence is convergent, with
$$E^2_{p,q} = \mathrm{H}_p(\Lambda H, d_H) \otimes_k \mathrm{H}_q(\Lambda(W \oplus dW), d) \Longrightarrow \mathrm{H}_{p+q}(\Lambda U, d).$$
Since $W_0 = W_1 = 0$, we have $\mathrm{H}_0(\Lambda(W \oplus dW), d) = k$, and therefore $\mathrm{H}_0(\Lambda H, d_H) = E^2_{0,0} = E^3_{0,0} = \ldots = E^\infty_{0,0} = \mathrm{H}_0(\Lambda U, d) = \mathrm{H}_0(\Lambda V, d)$. If the field k has characteristic zero, then $(\Lambda(W \oplus dW), d)$ is acyclic, so in this case the spectral sequence degenerates, showing that $\mathrm{H}(\Lambda H, d_H) \cong \mathrm{H}(\Lambda V, d)$. However, $\Lambda(W \oplus dW)$ need not be acyclic in positive characteristic p — if $x \in W$ is of even degree, then x^{np} and $x^{np-1}dx$ represent non-trivial homology classes for all $n \geq 1$. Recall however that

we are working with $\mathbb{N}^n$-graded objects and maps. Since W is concentrated in positive weight, the truncation $\Lambda(W \oplus dW)_\tau$ is acyclic, simply because no elements of the form $x^n a$, for $x \in (W \oplus dW), a \in \Lambda(W \oplus dW), n > 1$, are there. In particular the dissidents x^{np} and $x^{np-1}dx$ live in non-squarefree degrees. Hence the truncated spectral sequence collapses, regardless of characteristic, and so

$$\mathrm{H}_i(\Lambda H, d_H)_\tau \cong \mathrm{H}_i(\Lambda V, d)_\tau,$$

for all i. □

Remark 1. A fact that should be clear from the proof, but which we would like to emphasize, is that the map ψ need not be a quasi-isomorphism in positive characteristics. Suppose k has characteristic $p > 0$. Let $V = \langle x, y\rangle_k$, where $|x| = 2$, $|y| = 1$, $\deg(x) = \deg(y) = 1$ $(n = 1)$. Let $dx = y$ and $dy = 0$. Then $H = \mathrm{H}(V, d_0) = 0$ and hence $\Lambda H = k$. The map $\psi \colon \Lambda V \to k$ is not a quasi-isomorphism, because for instance x^p represents a non-trivial homology class in $\mathrm{H}_{2p}(\Lambda V, d)$. On the other hand $(\Lambda V, d)_\tau$ is the algebra $\Lambda V/(x^2, xy)$. It has basis $x, y, 1$ and obviously the map $\psi_\tau \colon (\Lambda V, d)_\tau \to k$ is a quasi-isomorphism, as asserted by the proposition.

Algebras with monomial relations

Let k be any field. Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$ minimally generated by a set M of monomials of degree at least 2, and let $R = S/I$. The Yoneda algebra $\mathrm{Ext}_R(k, k)$ is the universal enveloping algebra of the graded Lie algebra $\pi(R)$. Let $\pi^{\geq 2}(R) = \bigoplus_{i\geq 2} \pi^i(R)$. It is an ideal and in particular a sub Lie algebra of $\pi(R)$. The *multigraded Poincaré series* of R is the formal power series

$$\mathrm{P}_R(\mathbf{x}, z) = \sum_{i\geq 0, \alpha\in\mathbb{N}^n} \dim_k \mathrm{Ext}^i_R(k, k)_\alpha x^\alpha z^i.$$

We begin by citing the theorems which were the starting point for our work.

Theorem 2 ([7])**.** *The multigraded Poincaré series of R is a rational of the form*

$$\mathrm{P}_R(\mathbf{x}, z) = \frac{\prod_{i=1}^n (1 + x_i z)}{b_R(\mathbf{x}, z)},$$

for a polynomial $b_R(\mathbf{x}, z) \in \mathbb{Z}[x_1, \ldots, x_n, z]$.

Theorem 3 ([5], Theorem 1)**.** *Let I and J be ideals generated by monomials of degree at least 2 in the polynomial rings $k[\mathbf{x}]$, $k[\mathbf{y}]$ respectively, where $\mathbf{x}$ and $\mathbf{y}$ are finite sets of variables. Let $Q = k[\mathbf{x}]/I$ and $R = k[\mathbf{y}]/J$. If L_I and L_J are isomorphic as partially ordered graphs, then there is an isomorphism of graded Lie algebras*

$$\pi^{\geq 2}(Q) \cong \pi^{\geq 2}(R).$$

We are aiming at a description of the homotopy Lie algebra and the Poincaré series of R in combinatorial terms. It turns out that the machinery of minimal models is very well suited for this task. To a monomial set M we will associate two objects

- A finite lattice K_M, called the 'lattice of saturated subsets of M'.
- A finite dimensional L_∞-algebra $\mathfrak{L}_\infty(M)$.

These objects are accompanied by their respective theorems. Theorem 5 says that the denominator b_R of the Poincaré series of R with set of minimal relations M is given by

$$b_R = 1 + \sum_{S\in \hat{K}_M} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset, S); k)(z).$$

Here $(\emptyset, S)$ is the open interval from $\emptyset$ to S in the lattice K_M.

We define a functor F on the category of multigraded Lie algebras, whose restriction to the subcategory of truncated Lie algebras is left adjoint to the truncation functor, and Theorem 6 says that

$$\pi^{\geq 2}(R) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(M))$$

as multigraded Lie algebras.

Thus the study of P_R and $\pi^{\geq 2}(R)$ is reduced to combinatorics via these two objects. In proving the results we will reduce to the case when M consists of squarefree monomials. This is done by a procedure called 'polarization'.

4.1. **Polarization.** We invoke a construction of Fröberg, [19] pp. 30, which is often referred to as *polarization.* Let I be any monomial ideal in $S = k[x_1, \dots, x_n]$, and let $M = \mathrm{Gen}(I)$. Let $d_i = \max_{m\in M} \deg_{x_i}(m)$. To each $m \in M$ we associate a squarefree monomial m° in $Q = k[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d_i]$ as follows: If $m = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ then

$$m^\circ = \prod_{i=1}^{n}\prod_{j=1}^{\alpha_i} x_{i,j}.$$

The set $M^\circ = \{m^\circ \mid m \in M\}$ minimally generates an ideal in Q, which we denote by I°. The map $M^\circ \to M$, $m^\circ \mapsto m$, extends to a map $f\colon L_{I^\circ} \to L_I$ characterized by the property that $x_{i,j}$ divides $m \in L_{I^\circ}$ if and only if x_i^j divides $f(m)$. From this defining property it is easily seen that f is an isomorphism of pographs. Hence by Theorem 3, with $R = S/I$ and $R^\circ = Q/I^\circ$, we have

$$\pi^{\geq 2}(R) \cong \pi^{\geq 2}(R^\circ).$$

It is also easy to see that

$$b_R(x_1, \dots, x_n, z) = b_{R^\circ}(x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_n, z).$$

Therefore, questions about P_R or $\pi^{\geq 2}(R)$ may always be reduced to the squarefree case, if necessary.

5. Lattices and simplicial complexes associated to monomial sets

Elementary definitions and facts about simplicial complexes are found in Appendix A.

Definition/Lemma 1. If M is a monomial set, then the set

$$\{S \subseteq M \mid m_S \neq m_M \text{ or } S \text{ disconnected}\}$$

is the set of faces of a simplicial complex Δ'_M with vertex set M.

If $M = M_1 \cup \ldots \cup M_r$ is the decomposition of M into its connected components, then let Δ_M be the simplicial complex $\Delta_M = \Delta'_{M_1} \cdot \ldots \cdot \Delta'_{M_r}$ (cf. Appendix A). Thus Δ_M has vertices M and faces

$$\{S \subseteq M \mid m_S \neq m_M \text{ or } M_i \cap S \text{ disconnected for some } i\}.$$

Proof. To see that Δ'_M is indeed a simplicial complex, suppose $T \subseteq S \in \Delta'_M$. If $m_S \neq m_M$, then clearly $m_T \neq m_M$. If S is disconnected, decompose S as $S = S_1 \cup S_2$ where S_1 and S_2 are separated and non-empty. Then both m_{S_1}, m_{S_2} strictly divide m_M. Since T is a connected subset of S, we have that $T \subseteq S_i$ for some i, and therefore m_T strictly divides m_M. □

Definition 2. Let S be a subset of a monomial set M, and let $S = S_1 \cup \ldots \cup S_r$ be its decomposition into connected components. The *saturation* of S in M is the set $\overline{S} = \overline{S_1} \cup \ldots \cup \overline{S_r}$, where $\overline{S_i} = M_{m_{S_i}}$ for connected S_i. Clearly $S \subseteq \overline{S}$, and S is called *saturated in* M if equality holds. Equivalently, S is saturated in M if for all $m \in M$, $m \mid m_T$ implies $m \in S$ if T is a connected subset of S.

Define K_M to be the set of saturated subsets of M. It is a lattice with intersection as meet and the saturation of the union of saturated subsets as join. Set $\hat{K}_M = K_M - \{\emptyset\}$ and $\bar{K}_M = \hat{K}_M - \{M\}$.

It is easily checked that if $T \subseteq S \subseteq M$ and S is saturated in M, then T is saturated in S if and only if T is saturated in M. Therefore, if $S \in K_M$, then K_S is equal to the sublattice $(K_M)_{\subseteq S} = \{T \in K_M \mid T \subseteq S\}$ of K_M.

As usual, a partially ordered set P is considered to be a topological space by passage to the simplicial complex of chains in P. If L is a lattice, with top and bottom element $\hat{1}$ and $\hat{0}$, then its *proper part* is the poset $\bar{L} = L - \{\hat{0}, \hat{1}\}$. If L is atomic, with atoms A, the *crosscut complex* $\Gamma(\bar{L}, A)$ is the simplicial complex with vertices A and faces all subsets S of A such that $\vee S \neq \hat{1}$. The 'Crosscut Theorem', cf. [12] Theorem 10.8, asserts that $\bar{L}$ is homotopy equivalent to $\Gamma(\bar{L}, A)$. Note that K_M is an atomic lattice with atoms $A_M = \{\{m\} \mid m \in M\}$.

Proposition 2. *Δ_M is isomorphic to the crosscut complex $\Gamma(\bar{K}_M, A_M)$. In particular, for each $S \in K_M$, the open interval $(\emptyset, S)$ in K_M is homotopy equivalent to Δ_S.*

Proof. Identify $\Gamma(K_M, A_M)$ with the complex $\big\{S \subseteq M \mid \overline{S} \neq M\big\}$. Decompose M into connected components, $M = M_1 \cup \ldots \cup M_p$, and let $m_i = m_{M_i}$. We need to show that $\overline{S} = M$ is equivalent to $m_S = m_M$ and $S \cap M_i$ connected for each i. Let $S_1, \ldots, S_r$ be the components of S. Suppose $\overline{S} = M$. Then $m_S = m_{\overline{S}} = m_M$ and $\overline{S}_1, \ldots, \overline{S}_r$ are the components of M. Note that $S \cap \overline{S}_i = S_i$, which is connected.

Conversely, suppose that $m_S = m_M$ and that each $S \cap M_i$ is connected. If $m \in M$, then $m \in M_i$ for some i. Since $m_S = m_M = m_1 \ldots m_p$ and $\gcd(m_i, m_j) = 1$ when $i \neq j$, it follows that $m_i = m_{S \cap M_i}$. Since $S \cap M_i$ is connected and $m \mid m_{S \cap M_i}$ it follows that $m \in \overline{S}$.

The second assertion follows because the open interval $(\emptyset, S)$ in K_M equals $\bar{K}_S$. □

Two monomial ideals I, J are called *equivalent* if there is an isomorphism of posets $f\colon L_I \to L_J$ which is also an isomorphism of graphs, that is, f is an isomorphism of *pographs.* In particular, one checks that polarization yields an isomorphism of pographs $L_I \to L_{I^\circ}$, so I and I° are equivalent in this sense.

Let $\epsilon_I\colon K_I \to L_I$ be the map of join-semilattices taking S to m_S.

Proposition 3. *Let $f\colon L_I \to L_J$ be a bijective morphism of pographs. Then there is a surjective morphism of join-semilattices $\overline{f}\colon K_I \to K_J$ such that the following* *diagram of join-semilattices commutes:*

$$\begin{tikzcd} K_I \arrow[r, two heads, "\bar{f}"] \arrow[d, "\epsilon_I"] & K_J \arrow[d, "\epsilon_J"] \\ L_I \arrow[r, "f"] & L_J \end{tikzcd}$$

If in addition f^{-1} is a morphism of graphs, then $\bar{f}$ is an isomorphism of lattices and $c(S) = c(\bar{f}(S))$ for all $S \in K_I$.

Proof. Let M and N be the sets of minimal generators for I and J respectively. M and N are the atoms of L_I and L_J respectively, so f restricts to a bijective morphism of graphs $M \to N$. This means that if $S \subseteq M$ is connected, then so is $f(S)$. Define $\bar{f}\colon K_I \to K_J$ by $\bar{f}(S) = \overline{f(S)}$. To show that $\bar{f}$ is a morphism of join-semilattices we need to show that $\overline{f(S)} = \overline{f(\overline{S})}$ for all $S \subseteq M$. Indeed, assume $m \in \overline{f(\overline{S})}$. Then $m \mid m_T$ for some connected $T \subseteq f(\overline{S})$. If $n \in T$, then $f^{-1}(n) \in \overline{S}$, so $f^{-1}(n) \mid m_U$ for some connected $U \subseteq S$. Since f is an isomorphism of lattices, $n \mid m_{f(U)}$, and by the above $f(U)$ is a connected subset of $f(S)$. Therefore $T \subseteq \overline{f(S)}$ and then $m \mid m_T$ and T connected implies $m \in \overline{f(S)}$. The reverse inclusion is obvious.

If $S \in K_J$ then $f^{-1}(S) \in K_I$, because $m \mid m_T$ and $T \subseteq f^{-1}(S)$ connected implies $f(m) \mid m_{f(T)}$ and $f(T) \subseteq S$ connected, whence $f(m) \in S$, i.e, $m \in f^{-1}(S)$. Moreover, $\bar{f}(f^{-1}(S)) = S$. This shows that $\bar{f}$ is surjective.

If f is an isomorphism of graphs, then it restricts to an isomorphism of graphs $M \to N$. In particular $c(S) = c(f(S))$ for all $S \in K_I$. Also $\bar{f}(S) = f(S)$, so in this case $\bar{f}$ is bijective. It is trivial to verify that the diagram is commutative. □

Corollary 1. *An isomorphism of pographs $f\colon L_I \to L_J$ induces an isomorphism of lattices $K_I \to K_J$ such that $c(S) = c(f(S))$ for all $S \in K_I$.*

6. Minimal model of a monomial ring

Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$ minimally generated by a set M of monomials of degree at least 2. Fix a total order on M. Let $R = S/I$. We will construct explicitly the squarefree part of a multigraded minimal model for R. This will enable us to read off the homotopy Lie algebra and to compute the Poincaré series of R. The construction is modelled on the Taylor complex, whose definition we now will recall.

6.1. The Taylor complex.

The Taylor complex associated to I is a finite S-free resolution of S/I. It was originally introduced by Taylor in [34].

Let E be the vector space with basis $\{e_S \mid \emptyset \neq S \subseteq M\}$, with gradings defined by $|e_S| = |S|$ and $\deg(e_S) = \deg(m_S)$. Let X be the span of $x_1, \ldots, x_n$, with $|x_i| = 0$ and the standard $\mathbb{N}^n$-grading. Let

$$T_I = \Lambda(X \oplus E)/J,$$

where J is the ideal generated by the elements $e_S e_T - \operatorname{sgn}(S,T)\gcd(m_S, m_T)e_{S\cup T}$ for all pairs of subsets $S, T \subseteq M$. Recall that $\operatorname{sgn}(S,T) = 0$ if S and T are not disjoint, so for such a pair we get the generator $e_S e_T$. The generators of J are homogeneous with respect to all gradings, so T_I is an $\mathbb{N} \times \mathbb{N}^n$-graded algebra.

Clearly, $(T_I)_i$ is a free ΛX-module with basis e_S for $|S| = i$. Set $e_\emptyset = 1$. Define a map $d\colon E \to \Lambda(X \oplus E)$ by

$$de_S = \sum_{s \in S} \operatorname{sgn}(s, S - \{s\}) \frac{m_S}{m_{S-\{s\}}} e_{S-\{s\}},$$

and extend it to $\Lambda(X \oplus E) = \Lambda X \otimes_k \Lambda E$ as a ΛX-linear derivation. One checks that $d^2 = 0$ and that d preserves J, so we get an induced differential δ on the quotient T_I. We have a canonical map of algebras $(T, \delta) \to \mathrm{H}_0(T, \delta) = \Lambda X/I = R$.

Definition 3. The *Taylor complex* on the monomial ideal I is the dg-algebra (T_I, δ).

Proposition 4. *The canonical map $(T_I, \delta) \to R$ is a quasi-isomorphism. In particular, (T_I, δ) is a resolution of R by free S-modules.*

Proof. See for instance [18] for a short proof. □

Note that the underlying algebra of (T_I, δ) is not free, so it is not a model of R. It will become apparent later that the Taylor algebra is actually the truncation of a model of R.

The following is proved in [5].

Proposition 5. *(T_I, δ) is a dg Γ-algebra and*

$$\pi^{\geq 2}(R) \cong \pi(T_I \otimes_S k)$$

as graded Lie algebras.

Note that it makes sense to talk about $\pi(T_I \otimes_S k)$ only because we know that T_I, and hence $T_I \otimes_S k$, is a dg Γ-algebra.

6.2. **Minimal model of a monomial ring.** Let $V = X \oplus Y$, where $X = V_0$ is the linear span over k of $x_1, \ldots, x_n$ with $|x_i| = 0$ and the standard $\mathbb{N}^n$-grading. The space $Y = V_{\geq 1}$ is defined by

$$Y = \langle y_S \mid S \text{ non-empty connected subset of } M \rangle_k,$$

where gradings are given by $|y_S| = |S|$ and $\deg(y_S) = \deg(m_S)$. We extend the definition of the symbol y_S to arbitrary subsets of M as follows. If S is any, not necessarily connected, subset of M and $S = S_1 \cup \ldots \cup S_r$ is its decomposition into connected components, then define y_S to be the element

$$y_S = \operatorname{sgn}(S_1, \ldots, S_r) y_{S_1} \cdot \ldots \cdot y_{S_r} \in \Lambda V.$$

Set $y_\emptyset = 1$. With these definitions it is clear that $|y_S| = |S|$ and $\deg(y_S) = \deg(m_S)$ for any $S \subseteq M$.

A differential d on ΛV is defined on the basis as follows. We set $dx_i = 0$ for all i and if S is a connected subset of M, then

$$dy_S = \sum_{s \in S} \operatorname{sgn}(s, S - \{s\}) \frac{m_S}{m_{S-\{s\}}} y_{S-\{s\}}, \tag{2}$$

The differential is extended to all of ΛV as an ΛX-linear derivation. Obviously, this definition is modelled on the differential of the Taylor complex. Note that it may happen that $y_{S-\{s\}}$ becomes decomposable as a product in the sum above. One verifies easily that the formula (2) remains valid for disconnected S. By definition, d is of homological degree -1 and is homogeneous with respect to the $\mathbb{N}^n$-grading. The verification of $d^2 = 0$ is routine.

The algebra ΛY is isomorphic to $\Lambda V \otimes_{\Lambda X} k$, and therefore inherits a differential $\bar{d}$ from ΛV.

Proposition 6. *Let I be a an ideal in S generated by squarefree monomials of degree at least 2, and let $R = S/I$. There is a multigraded minimal model $(\Lambda W, d_W)$ of R such that $W_\tau = \mathrm{H}(V, d_0)$ and $\mathfrak{L}(\Lambda(W_{\geq 1}), \bar{d}_W)_\tau \cong \mathfrak{L}(\Lambda Y, \bar{d})$.*

Proof. By definition, $\mathrm{H}_0(\Lambda V, d) = R$. Furthermore, we have an isomorphism of dg-algebras $\Lambda(V)_\tau \cong T_\tau$, where T is the Taylor complex of the ideal I. In particular, since the Taylor complex is a multigraded resolution of R over S, we have $\mathrm{H}_i(\Lambda V, d)_\tau = 0$ for $i > 0$. Let $H = \mathrm{H}(V, d_0)$. From Proposition 1 we get a differential d_H on ΛH making it a minimal dg-algebra such that $\mathrm{H}_0(\Lambda H, d_H) = R$, $\mathrm{H}_i(\Lambda H, d_H)_\tau = 0$ for $i > 0$ and $\mathfrak{L}(\Lambda(H_{\geq 1}), \bar{d}_H) \cong \mathfrak{L}(\Lambda Y, \bar{d})$. Applying Lemma 4 to the minimal dg-algebra $(\Lambda H, d_H)$, we get a vector space W and minimal model $(\Lambda W, d_W)$ of R, such that $(\Lambda W, d_W)_\tau = (\Lambda H, d_H)_\tau$. Then it follows immediately from the definition of the functor $\mathfrak{L}$ that $\mathfrak{L}(\Lambda(W_{\geq 1}), \bar{d}_W)_\tau = \mathfrak{L}(\Lambda(H_{\geq 1}), \bar{d}_H)_\tau = \mathfrak{L}(\Lambda(H_{\geq 1}), \bar{d}_H) \cong \mathfrak{L}(\Lambda Y, \bar{d})$ and that $W_\tau = H$. □

Proposition 7. *If I is generated by squarefree monomials, then $\pi^{\geq 2}(R)_\tau \cong \mathfrak{L}_Y$.*

Proof. Let $(\Lambda W, d_W)$ be the minimal model of R constructed in Proposition 6. The Taylor complex T_I is a dg Γ-algebra resolution of R by free S-modules. Therefore, by Lemma 3, $(\Lambda W \otimes_S k, \bar{d}_W) = (\Lambda(W_{\geq 1}), \bar{d}_W)$ is a minimal model of the dg Γ-algebra $T_I \otimes_S k$. Hence, by Theorem 1 and Proposition 5,

$$\mathfrak{L}_{W_{\geq 1}} = \pi(T_I \otimes_S k) = \pi^{\geq 2}(R).$$

By Proposition 6, we conclude that

$$\mathfrak{L}_Y \cong (\mathfrak{L}_{W_{\geq 1}})_\tau = \pi^{\geq 2}(R)_\tau.$$

□

7. $\mathbb{N}^n$-graded deviations

The next observation is the extension of Remark 7.1.1 of [4] to the $\mathbb{N}^n$-graded situation, and it is proved in a similar manner, cf. [15] Proposition 3.1.

Let $P = 1 + \sum_{i \geq 1, \alpha \in \mathbb{N}^n} b_{i,\alpha} x^\alpha z^i$ be a formal power series with integer coefficients $b_{i,\alpha}$ such that for i fixed, $b_{i,\alpha} = 0$ when $|\alpha| \gg 0$. Then there are uniquely determined integers $e_{i,\alpha}$ such that

$$P = \prod_{i \geq 1, \alpha \in \mathbb{N}^n} \frac{(1 + x^\alpha z^{2i-1})^{e_{2i-1,\alpha}}}{(1 - x^\alpha z^{2i})^{e_{2i,\alpha}}},$$

the product converging in the (z)-adic topology. Furthermore, for a fixed i, we have $e_{i,\alpha} = 0$ when $|\alpha| \gg 0$.

This observation applies to the Poincaré series $\mathrm{P}_R(\mathbf{x}, z)$ of a monomial ring R: since $\mathrm{Ext}^i_R(k,k)$ is a finite dimensional vector space for each i, it can be non-zero in only finitely many multidegrees. Thus we have a product decomposition

$$\mathrm{P}_R(\mathbf{x}, z) = \prod_{i \geq 1, \alpha \in \mathbb{N}^n} \frac{(1 + x^\alpha z^{2i-1})^{\epsilon_{2i-1,\alpha}}}{(1 - x^\alpha z^{2i})^{\epsilon_{2i,\alpha}}}. \tag{3}$$

The numbers $\epsilon_{i,\alpha}(R) = \epsilon_{i,\alpha}$ are called the $\mathbb{N}^n$*-graded deviations* of R. These refine the ordinary deviations ϵ_i of R (cf. [4], Section 7.1):

$$\epsilon_i = \sum_{\alpha \in \mathbb{N}^n} \epsilon_{i,\alpha}.$$

Multigraded deviations have been introduced also in [15].

It is a fundamental result that the ordinary deviations $\epsilon_i(R)$ can be computed from a minimal model of R, cf. [4] Theorem 7.2.6. The same is true in the multigraded setting. Recall that $\operatorname{Ext}_R(k,k) = U\pi(R)$ as multigraded algebras. By the Poincaré-Birkhoff-Witt theorem, there is an isomorphism of graded vector spaces $U\pi(R) \cong \Lambda\pi(R)$. In view of the vector space structure of $\Lambda\pi(R)$ there results a product decomposition of the Poincaré series

$$\mathrm{P}_R(\mathbf{x}, z) \quad = \quad \prod_{i \geq 1, \alpha \in \mathbb{N}^n} \frac{(1 + x^\alpha z^{2i-1})^{p_{2i-1,\alpha}}}{(1 - x^\alpha z^{2i})^{p_{2i,\alpha}}},$$

where $p_{i,\alpha} = \dim_k \pi^i(R)_\alpha$. It follows from the remark that the numbers $\dim_k \pi^i(R)_\alpha$ equal the deviations $\epsilon_{i,\alpha}(R)$ of R.

The space $\pi^1(R) \cong \operatorname{Ext}^1_R(k,k)$ can be identified with the dual of the vector space $s\langle x_0, \dots, x_n \rangle_k$ of minimal algebra generators for R. By Lemma 3, if $(\Lambda W, d)$ is a minimal model for R, then $(\Lambda(W_{\geq 1}), \bar{d})$ is a minimal model for $T_I \otimes_S k$, whence by Proposition 5 and Theorem 1, $\pi^{\geq 2}(R) = \mathfrak{L}_{W_{\geq 1}} = (s(W_{\geq 1}))^*$. Therefore $\epsilon_{i,\alpha}(R) = \dim_k \pi^i(R)_\alpha = \dim_k W_{i-1,\alpha}$ for $i \geq 2$. Furthermore, $W_0 = \langle x_0, \dots, x_n \rangle_k$, so it is also true that $\pi^1(R) \cong (sW_0)^*$. We state this as a lemma for future reference.

Lemma 5. *Let $(\Lambda W, d)$ be an $\mathbb{N}^n$-graded minimal model of a monomial ring R. Then the $\mathbb{N}^n$-graded deviations $\epsilon_{i,\alpha}$ of R are given by*

$$\epsilon_{i,\alpha} = \dim_k W_{i-1,\alpha},$$

for $i \geq 1$ and $\alpha \in \mathbb{N}^n$. □

7.1. **Squarefree deviations.** In the squarefree case, there is a nice interpretation of the squarefree deviations in terms of simplicial homology. Recall the definition of Δ'_M found in Section 5. As usual, M_α denotes the set of monomials in M dividing x^α.

Theorem 4. *Assume that I is minimally generated by a set M of squarefree monomials of degree at least 2. Let $\alpha \in \{0,1\}^n$ and let $i \geq 2$. If $x^\alpha \notin L_I$, then $\epsilon_{i,\alpha} = 0$, and if $x^\alpha \in L_I$ then*

$$\epsilon_{i,\alpha} = \dim_k \widetilde{\mathrm{H}}_{i-3}(\Delta'_{M_\alpha}; k).$$

Proof. By Proposition 6 there is a minimal model $(\Lambda W, d_W)$ of R such that $W_\tau = \mathrm{H}(V, d_0)$. By Lemma 5 we get that

$$\epsilon_{i,\alpha} = \dim_k W_{i-1,\alpha} = \dim_k H_{i-1,\alpha} = \dim_k \mathrm{H}_{i-1,\alpha}(V, d_0), \tag{4}$$

for $\alpha \in \{0,1\}^n$. We will now proceed to give a combinatorial description of the complex $V = (V, d_0)$. As a complex, V splits into its $\mathbb{N}^n$-graded components

$$V = \bigoplus_{\alpha \in \mathbb{N}^n} V_\alpha.$$

V_{e_i} is one-dimensional and concentrated in degree 0 for $i = 1, \ldots, n$. This accounts for the first deviations $\epsilon_{1,e_i} = 1$. If $|\alpha| > 1$, then V_α has basis y_S for S in the set

$$C_\alpha = \{S \subseteq M \mid m_S = x^\alpha,\ S \text{ connected}\}.$$

In particular $V_\alpha = 0$ if $x^\alpha \notin L_I$. The differential of V_α is given by

$$dy_S = \sum_{\substack{s\in S\\ S-\{s\}\in C_\alpha}} \operatorname{sgn}(s, S-\{s\}) y_{S-\{s\}}. \tag{5}$$

Let Σ_α be the simplicial complex whose faces are all subsets of the set $M_\alpha = \{m \in M \mid m \mid x^\alpha\}$, with orientation induced from the orientation $\{m_1, \ldots, m_n\}$ of M. Define a map from the chain complex $\widetilde{C}(\Sigma_\alpha; k)$ to the desuspended complex $s^{-1}V_\alpha$ by sending a face $S \subseteq M_\alpha$ to $s^{-1}y_S$ if $S \in C_\alpha$ and to 0 otherwise. In view of (5), this defines a morphism of complexes, which clearly is surjective. The kernel of this morphism is the chain complex associated to Δ'_{M_α}, so we get a short exact sequence of complexes

$$0 \to \widetilde{C}(\Delta'_{M_\alpha}; k) \to \widetilde{C}(\Sigma_\alpha; k) \to s^{-1}V_\alpha \to 0$$

Since Σ_α is acyclic, the long exact sequence in homology derived from the above sequence shows that $\mathrm{H}_i(V_\alpha) \cong \widetilde{\mathrm{H}}_{i-2}(\Delta'_{M_\alpha}; k)$. The theorem now follows from (4). □

8. Poincaré series

This section is devoted to the deduction of the following theorem which gives a formula for the Poincaré series of a monomial ring in terms of simplicial homology.

Theorem 5. *Let k be any field. Let I be an ideal in $S = k[x_1, \ldots, x_n]$ generated by monomials of degree at least 2, and let M be its minimal set of generators. The denominator of the Poincaré series of $R = S/I$ is given by*

$$b_R(\mathbf{x}, z) = 1 + \sum_{S\in \hat{K}_M} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset, S); k)(z), \tag{6}$$

Some intermediate results will be needed before we can give the proof. Retain the notations of Theorem 5 throughout this section. We will frequently suppress the variables and write $b_R = b_R(\mathbf{x}, z)$ and $\mathrm{P}_R = \mathrm{P}_R(\mathbf{x}, z)$.

Assume that the ideal I is minimally generated by squarefree monomials $M = \{m_1, \ldots, m_g\}$ of degree at least 2. By Backelin [7], the Poincaré series of R is rational of the form

$$\mathrm{P}_R(\mathbf{x}, z) = \frac{\prod_{i=1}^n (1 + x_i z)}{b_R(\mathbf{x}, z)},$$

where $b_R(\mathbf{x}, z)$ is a polynomial with integer coefficients and x_i-degree at most 1 for each i. We start with the following observation made while scrutinizing Backelin's proof.

Lemma 6. *If I is generated by squarefree monomials, then the polynomial b_R is squarefree with respect to the x_i-variables. Moreover b_R depends only on the deviations $\epsilon_{i,\alpha}$ for $\alpha \in \{0,1\}^n$. In fact, there is a congruence modulo $(x_1^2, \ldots, x_n^2)$:*

$$b_R \equiv \prod_{\alpha\in\{0,1\}^n} (1 - x^\alpha p_\alpha(z)),$$

where $p_\alpha(z)$ is the polynomial $p_\alpha(z) = \sum_{i=1}^{|\alpha|} \epsilon_{i,\alpha} z^i$.

Proof. Note that $\epsilon_{1,e_i} = 1$ and $\epsilon_{1,\alpha} = 0$ for $\alpha \neq e_i = \deg(x_i)$ $(i = 1, \dots, n)$. Hence using the product representation (3) and reducing modulo $(x_1^2, \dots, x_n^2)$ we get (note that $(1 + mp(z))^n \equiv 1 + nmp(z)$ for any integer n and any squarefree monomial m):

$$\begin{aligned} b_R &= \frac{\prod_{i\geq 1,\alpha}(1 - x^\alpha z^{2i})^{\epsilon_{2i,\alpha}}}{\prod_{i\geq 2,\alpha}(1 + x^\alpha z^{2i-1})^{\epsilon_{2i-1,\alpha}}} \\ &\equiv \prod (1 - x^\alpha(\epsilon_{2i-1,\alpha} z^{2i-1} + \epsilon_{2i,\alpha} z^{2i})) \\ &\equiv \prod (1 - x^\alpha p_\alpha(z)), \end{aligned} \tag{7}$$

product taken over all $\alpha \in \{0,1\}^n$, where $p_\alpha(z) \in \mathbb{Z}[z]$ is the polynomial $p_\alpha(z) = \sum_{i=1}^{|\alpha|} \epsilon_{i,\alpha} z^i$. $\square$

This gives a formula for b_R in terms of finitely many deviations $\epsilon_{i,\alpha}$. In terms of the polynomials $p_\alpha(z)$, Theorem 4 may be stated as

$$p_\alpha(z) = z^3 \widetilde{\mathrm{H}}(\Delta'_{M_\alpha}; k)(z), \tag{8}$$

for $x^\alpha \in L_I$.

Proof of Theorem 5. Square-free case. By Theorem 4, $p_\alpha(z) = 0$ unless $x^\alpha \in L_I$, in which case $p_\alpha(z) = z^3 \widetilde{\mathrm{H}}(\Delta'_{M_\alpha}; k)(z)$. But Δ'_{M_α} is a simplex and hence contractible if M_α is disconnected, so $p_\alpha(z) = 0$ unless $x^\alpha \in cL_I$, where cL_I denotes the subset of L_I consisting of elements $m \neq 1$ such that M_m is connected. Hence by Lemma 6

$$b_R \equiv \prod_{x^\alpha \in cL_I} (1 - x^\alpha p_\alpha(z)) \qquad \bmod (x_1^2, \dots, x_n^2).$$

If we carry out the multiplication in the above formula and use that b_R is squarefree with respect to the x_i-variables (by Lemma 6) we get the equality

$$b_R = 1 + \sum_{N\in D(cL_I)} \prod_{x^\alpha \in N} (-x^\alpha p_\alpha(z)) = 1 + \sum_{N \in D(cL_I)} m_N (-1)^{|N|} \prod_{x^\alpha\in N} p_\alpha(z)$$

(the identity $\prod_{x^\alpha \in N} x^\alpha = m_N$ holds because N is independent). Using (8) the formula takes the form

$$b_R = 1 + \sum_{N\in D(cL_I)} m_N (-1)^{|N|} \prod_{x^\alpha\in N} z^3 \widetilde{\mathrm{H}}(\Delta'_{M_\alpha}; k)(z).$$

By (13) this may be written

$$b_R = 1 + \sum_{N\in D(cL_I)} m_N (-1)^{|N|} z^{|N|+2} \widetilde{\mathrm{H}}(\Gamma; k)(z),$$

where $\Gamma = \Delta'_{M_{\alpha_1}} \cdot \ldots \cdot \Delta'_{M_{\alpha_r}}$, if $N = \{x^{\alpha_1}, \dots, x^{\alpha_r}\}$. The point here is that $M_N = M_{\alpha_1} \cup \ldots \cup M_{\alpha_r}$ is the decomposition of M_N into its connected components: every M_{α_i} is connected because $x^{\alpha_i} \in cL_I$, and since N is independent, there are no edges between M_{α_i} and M_{α_j} if $i \neq j$. Therefore

$$\Delta_{M_N} = \Delta'_{M_{\alpha_1}} \cdot \ldots \cdot \Delta'_{M_{\alpha_r}} = \Gamma.$$

For any $N \in D(cL_I)$, the set M_N is obviously saturated in M. Conversely, for any saturated subset S of M, let $S = S_1 \cup \ldots \cup S_r$ be the decomposition of S into connected components. Then $N = \{m_{S_1}, \dots, m_{S_r}\} \in D(L_I)$ and since $S_i = M_{m_{S_i}}$, as S is saturated, it follows that $M_{m_{S_i}}$ is connected for each i, so that $N \in D(cL_I)$.

This sets up a one-to-one correspondence between $\hat{K}_M$ and $D(cL_I)$. Furthermore, under this correspondence $m_S = m_N$ and $c(S) = |N|$, so it translates our formula to:

$$b_R = 1 + \sum_{S \in \hat{K}_M} m_S(-z)^{c(S)+2} \widetilde{\mathrm{H}}(\Delta_S; k)(z).$$

To finish the proof, we use Proposition 2, which says that for any $S \in K_M$, the simplicial complex Δ_S is homotopy equivalent to the order complex of the open interval $(\emptyset, S)$ in K_M. $\square$

Introduce the auxiliary notation

$$F(M) = 1 + \sum_{S \in \hat{K}_M} m_S(-z)^{c(S)+2} \widetilde{\mathrm{H}}(\Delta_S; k)(z),$$

when M is a set of monomials of degree at least 2. If I is a monomial ideal in some polynomial ring Q over k, then set $F(I) = F(\mathrm{Gen}(I))$. So far we have proved that $b_{Q/I} = F(I)$ whenever I is generated by squarefree monomials. The claim of Theorem 5 is that $b_{Q/I} = F(I)$ for all monomial ideals I.

Lemma 7. *Let I and J be equivalent monomial ideals, and let $f\colon L_I \to L_J$ be an isomorphism of pographs. Then $f(F(I)) = F(J)$, where $f(F(I))$ denotes the result of applying f to the coefficients m_S of $F(I)$, regarding it as a polynomial in z with coefficients in L_I.*

Proof. Let $M = \mathrm{Gen}(I)$ and $N = \mathrm{Gen}(J)$. By Proposition 3, f induces an isomorphism of lattices $K_M \to K_N$ which maps $S \in K_M$ to $f(S) \in K_M$. In particular the open intervals $(\emptyset, S)_{K_M}$ and $(\emptyset, f(S))_{K_N}$ are isomorphic. Furthermore, $c(S) = c(f(S))$ and $f(m_S) = m_{f(S)}$ for all $S \subseteq M$. The result follows. $\square$

Proof of Theorem 5. General case. We use polarization. Let $I^\circ \subseteq Q$ be the squarefree monomial ideal associated to I as in section 4.1, and let $R = S/I$, $R^\circ = Q/I^\circ$. The map $f\colon L_{I^\circ} \to L_I$ is an isomorphism of pographs. Therefore we get

$$b_R = f(b_{R^\circ}) = f(F(I^\circ)) = F(I),$$

where the first equality follows from the construction in [19], the second from the squarefree case of Theorem 5, and the third from Lemma 7. This proves Theorem 5 in general. $\square$

8.1. **Applications and remarks.** We will here give the proofs of some corollaries to Theorem 5 and make some additional remarks.

Corollary 1. *With notations as in Theorem 5*

$$\deg b_R(z) \leq g + d,$$

where $b_R(z) = b_R(1, \ldots, 1, z)$, $g = |M|$ is the number of minimal generators of I and d is the independence number of M, i.e., the largest size of an independent subset of M. In particular

$$\deg b_R(z) \leq 2g,$$

with equality if and only if R is a complete intersection.

Proof. If Δ is a simplicial complex with v vertices, then $\deg \widetilde{\mathrm{H}}(\Delta; k)(z) \leq v - 2$, because either $\dim \Delta = v-1$, in which case Δ is the $(v-1)$-simplex and $\widetilde{\mathrm{H}}(\Delta; k) = 0$, or else $\dim \Delta \leq v - 2$, in which case $\widetilde{\mathrm{H}}_i(\Delta; k) = 0$ for $i > v - 2$. The simplicial complex Δ_S has $|S|$ vertices. Thus the z-degree of a general summand in the formula (6) for $b_R(\mathbf{x}, z)$ is bounded above by $c(S)+2+|S|-2 \leq d+g$, because the number of components of S can not exceed the independence number of M. Since $d \leq g$ we get in particular that

$$\deg b_R(z) \leq 2g,$$

with equality if and only if M is independent itself, which happens if and only if R is a complete intersection. $\square$

Now that we know that Q and R below satisfy $b_Q = F(I)$ and $b_R = F(J)$, the next corollary is merely a restatement of Lemma 7.

Corollary 2. *Let I and J be ideals generated by monomials of degree at least 2 in the rings $k[\mathbf{x}]$ and $k[\mathbf{y}]$ respectively, where $\mathbf{x}$ and $\mathbf{y}$ are finite sets of variables. Let $Q = k[\mathbf{x}]/I$ and $R = k[\mathbf{y}]/J$. If $f : L_I \to L_J$ is an equivalence, then*

$$b_R(\mathbf{y}, z) = f(b_Q(\mathbf{x}, z)),$$

where $f(b_Q(\mathbf{x}, z))$ denotes the result of applying f to the coefficients of $b_Q(\mathbf{x}, z)$, regarding it as a polynomial in z. $\square$

Remark 2. Given formula (6), it is easy to reproduce the result, implicit in [18] and explicit in [16], that

$$b_R(\mathbf{x}, z) = \sum_{S \subseteq M} (-1)^{c(S)} z^{|S|+c(S)} m_S,$$

when the Taylor complex on M is minimal. The Taylor complex is minimal precisely when $m_T = m_S$ implies $S = T$, for $S, T \subseteq M$, i.e., when L_I is isomorphic to the boolean lattice of subsets of M. In this case every non-empty subset S of M is saturated, because $m \mid m_T$ implies $m \in T$ for any $T \subseteq M$, and Δ_S is a triangulation of the $(|S|-2)$-sphere, because $m_S = m_M$ only if $S = M$.

9. (Strongly) homotopy Lie algebras

Fix a field k and let $S = k[x_1, \ldots, x_n]$. Let $I \subseteq S$ be an ideal generated by a set M of squarefree monomials of degree at least 2, and set $R = S/I$. Recall the definition of the dg-algebra $(\Lambda Y, \bar{d})$ constructed in Section 6.2. By Proposition 7, $\pi^{\geq 2}(R) \cong \mathfrak{L}_Y$, and this is by definition the cohomology of the L_∞-algebra obtained from dualizing the dg-algebra $(\Lambda Y, \bar{d})$. We will now describe this L_∞-algebra combinatorially.

Definition 4. Let M be any set of monomials. Then $\mathfrak{L}_\infty(M)$ is a multigraded L_∞-algebra with

$$\mathfrak{L}_\infty(M)^i_\alpha = \langle \xi_S \mid S \text{ connected subset of } M;\ |S| = i-1,\ m_S = x^\alpha \rangle.$$

Thus ξ_S has cohomological degree $|S|+1$ and multidegree m_S. For each $r \geq 1$, we have an r-ary bracket $\mathfrak{L}_\infty(M)^{\otimes r} \to \mathfrak{L}_\infty(M)$ of cohomological degree $2-r$ which is homogeneous with respect to the $\mathbb{N}^n$-grading. Whenever $S_1, \ldots, S_r$ are connected subsets of M satisfying $\gcd(m_{S_i}, m_{S_j}) = 1$ when $i \neq j$, then this bracket is given by

$$[\xi_{S_1}, \ldots, \xi_{S_r}] = (-1)^{\epsilon} \sum \operatorname{sgn}(m, S_r, \ldots, S_1)\xi_{S\cup m}. \tag{9}$$

Here $S = S_1 \cup \ldots \cup S_r$ and the summation is over all $m \in M - S$ such that $S \cup m$ is connected and $m \mid m_S$. The number ϵ is defined by

$$\epsilon = \begin{cases} 1 + |\xi_{S_1}| + |\xi_{S_3}| + \ldots |\xi_{S_r}|, & \text{if } r \text{ is odd} \\ 1 + |\xi_{S_2}| + |\xi_{S_4}| + \ldots |\xi_{S_r}|, & \text{if } r \text{ is even} \end{cases}$$

The bracket is anti-symmetric in the sense that

$$[\xi_{S_1}, \ldots, \xi_{S_i}, \xi_{S_{i+1}}, \ldots, \xi_{S_r}] = -(-1)^{|\xi_{S_i}||\xi_{S_{i+1}}|}[\xi_{S_1}, \ldots, \xi_{S_{i+1}}, \xi_{S_i}, \ldots, \xi_{S_r}].$$

The bracket is zero whenever $\gcd(m_{S_i}, m_{S_j}) \neq 1$ for some $i \neq j$. All operations coming from higher divided powers, such as the squaring operations $x \mapsto x^{[2]}$, are zero in $\mathfrak{L}_\infty(M)$.

If I is a monomial ideal, then $\mathfrak{L}_\infty(I)$ is defined to be $\mathfrak{L}_\infty(M)$, where M is the minimal set of generators for I.

Proposition 8. *Let $I \subseteq S$ be an ideal generated by squarefree monomials and let $R = S/I$. Then we have an isomorphism of multigraded Lie algebras*

$$\pi^{\geq 2}(R)_\tau \cong \mathrm{H}^*(\mathfrak{L}_\infty(I)).$$

Proof. If I is generated by squarefree monomials, then $\mathfrak{L}_Y = \mathrm{H}^*(\mathfrak{L}_\infty(I))$, by definition of $\mathfrak{L}_\infty(I)$. Therefore the proposition follows from Proposition 7. □

Remark 3. Since the definition of $\mathfrak{L}_\infty(M)$ only uses properties of M which can be extracted from the pograph L_M, it is clear that, up to isomorphism of L_∞-algebras, $\mathfrak{L}_\infty(M)$ depends only on the equivalence class of the monomial set M.

A sort of converse to this is true in the sense that the lattice L_M can be recovered from the graph structure of M and the L_∞-algebra $\mathfrak{L}_\infty(M)$ with the given base indexed by connected non-empty subsets of M. Indeed, the lattice L_M is determined by what relations $m \mid m_S$ hold for $m \in M$ and subets S of M. The relation $m \mid m_S$ holds if and only if there is a subset T of S, with connected components $T_1, \ldots, T_r$ say, such that $T \cup m$ is connected and $\xi_{T\cup m}$ occurs with a non-zero coefficient in the bracket $[\xi_{T_1}, \ldots, \xi_{T_r}]$.

Next we wish to prove that $\pi^{\geq 2}(R)$ is obtained from $\pi^{\geq 2}(R)_\tau$ by 'extending it freely in higher multidegrees'. Before doing so, we need to make this last sentence precise. Recall that for homogeneous elements x, y of a vector space, $x \perp y$ means that the multidegrees of x and y have disjoint supports.

Definition 5. If L is a Lie algebra, then let

$$FL = \frac{\mathbb{L}(L)}{\langle [\![x,y]\!] - [x,y] \mid x \perp y,\ x, y \in L\rangle},$$

where $[\![x,y]\!]$ denotes the bracket in the free Lie algebra $\mathbb{L}(L)$ on the vector space L and $[x,y]$ the bracket in L. This defines a functor $F\colon \mathfrak{Lie} \to \mathfrak{Lie}$.

One can check that the restriction of F to the full subcategory of truncated Lie algebras is left adjoint to the truncation functor $L \mapsto L_\tau$ from Lie algebras to truncated Lie algebras. It is also easy to check that if L is a truncated Lie algebra with a presentation $\mathbb{L}(V)_\tau/\langle W\rangle$, where V is a truncated vector space and $\langle W\rangle$ is the ideal in $\mathbb{L}(V)_\tau$ generated by a subspace $W \subseteq \mathbb{L}(V)_\tau$, then $FL = \mathbb{L}(V)/\langle W\rangle$, where $\langle W\rangle$ is the ideal generated in $\mathbb{L}(V)$ by W. In this sense FL is obtained from the truncated Lie algebra L by extending it freely in multidegrees outside $\{0,1\}^n$, taking only into account relations in degrees $\{0,1\}^n$. Note also that $(FL)_\tau = L$, so one could say that FL is the largest Lie algebra whose truncation is L.

If L is a multigraded Lie algebra, then its universal enveloping algebra UL is a multigraded associative algebra and the cohomology vector spaces $\mathrm{H}^*(L,k) = \mathrm{Ext}^*_{UL}(k,k)$ are multigraded. $\mathrm{Ext}^1_{UL}(k,k)$ can be seen as the vector space of minimal generators for L and similarly $\mathrm{Ext}^2_{UL}(k,k)$ is the space of minimal relations among the minimal generators of L. Therefore the next statement should be clear.

The following are equivalent for a multigraded Lie algebra L:

- $\mathrm{Ext}^i_{UL}(k,k)$ is a truncated vector space for $i = 1,2$.
- L has a free presentation $L \cong \mathbb{L}(V)/\langle W\rangle$, where V and W are truncated vector spaces.
- $L = F(L_\tau)$.

In this case we say that L is presented in squarefree multidegrees.

Proposition 9. *The $\mathbb{N}^n$-graded Lie algebra $\pi^{\geq 2}(R)$ is presented in squarefree multidegrees. In other words*
$$\pi^{\geq 2}(R) = F(\pi^{\geq 2}(R)_\tau).$$

Proof. $\mathrm{Ext}_R(k,k)$ is the universal enveloping algebra of the Lie algebra $\pi(R)$.

Let
$$H(z_1,z_2,\mathbf{x}) = \sum_{i,j,\alpha} \dim_k \mathrm{Ext}^i_{\mathrm{Ext}^j_R(k,k)}(k,k)_\alpha z_1^i z_2^j x^\alpha$$
be the Hilbert series of the $\mathbb{N}^{n+2}$-graded algebra $\mathrm{Ext}_{\mathrm{Ext}_R(k,k)}(k,k)$ According to [8] Theorem 5$'$, this algebra is generated by elements in squarefree degrees, so $\mathrm{Ext}^1_{\mathrm{Ext}_R(k,k)}(k,k)_\alpha = 0$ unless $\alpha \in \{0,1\}^n$. Furthermore the Hilbert series is of the form
$$H(z_1,z_2,\mathbf{x}) = \frac{p(z_1,z_2,\mathbf{x})}{\prod_{i=1}^n (1 - z_1 z_2 x_i)},$$
where $p(z_1,z_2,\mathbf{x})$ is reduced modulo $(x_1^2,\dots,x_n^2)$. By looking at the z_1^2-coefficient of this series, one sees that $\mathrm{Ext}^2_{\mathrm{Ext}_R(k,k)}(k,k)$ is concentrated in multidegrees of the form $x_i \cdot m$, where m is a squarefree monomial in $x_1,\dots,x_n$. Therefore, $\pi(R)$ is generated by elements of squarefree degrees, and the minimal relations between these generators are situated in degrees of the form $x_i m$, where m is squarefree.

Consider the ring $R' = k[x_1,\dots,x_n,y_1,\dots,y_n]/J$, where J is generated by the monomials $m' = m(x_1y_1,\dots,x_ny_n)$, for $m \in M$. Clearly the monomials m' are squarefree and the correspondence $m \leftrightarrow m'$ gives an equivalence $M \sim M'$. According to Theorem 3, we have an isomorphism $\pi^{\geq 2}(R) \cong \pi^{\geq 2}(R')$ of Lie algebras, and in particular a set of generators for the Lie algebra $\pi^{\geq 2}(R)$ is transferred to one for $\pi^{\geq 2}(R)$. Since $\pi(R')$ is generated as a Lie algebra by elements of squarefree multidegrees it follows that $\pi^{\geq 2}(R')$ as a $\pi(R')$-module is generated by squarefree elements. But we have constructed R' so that the action of $\pi^1(R')$ on $\pi^{\geq 2}(R')$ is trivial. Indeed, it follows from Lemma 5 and the fact that the generators are in squarefree multidegrees that $\pi^{\geq 2}(R')_\alpha = 0$ unless $x^\alpha \in [L_{M'}]$. Here $[L_{M'}]$ denotes the sub-semigroup of $[x_1 \dots, x_n, y_1, \dots, y_n]$ generated by $L_{M'}$. But by construction any such α has the property that $x_i \mid x^\alpha$ if and only if $y_i \mid x^\alpha$. Therefore, as $\pi^1(R')$ is concentrated in the multidegrees $x_1,\dots,x_n,y_1,\dots,y_n$ and $\pi^{\geq 2}(R')$ is concentrated in the multidegrees $[L_{M'}]$, the product $[\pi^1(R'), \pi^{\geq 2}(R')]$ is zero for multidegree reasons. So $\pi^{\geq 2}(R')$ is generated by elements of squarefree degrees. By the above, the minimal relations among the generators for $\pi^{\geq 2}(R')$ are situated in multidegrees of the form $x_i m$ or $y_i m$, where m is squarefree. It follows that the minimal relations are concentrated in squarefree multidegrees as it is impossible to reach a multidegree of the form $x_i^2 n$ or $y_i^2 n$, where n is squarefree, from the generators of $\pi^{\geq 2}(R')$.

Thus the Lie algebra $\pi^{\geq 2}(R')$ has a squarefree presentation. By [15], squarefree multidegrees are mapped to squarefree ones by the isomorphism $\pi^{\geq 2}(R) \cong \pi^{\geq 2}(R')$, so the same is true for $\pi^{\geq 2}(R)$. □

Remark 4. Replacing M by an equivalent monomial set M' does not change the isomorphism class of $\pi^{\geq 2}(R)$, but the action of $\pi^1(R)$ is altered. The key point in the proof is that any R is equivalent to some R' where $\pi^1(R')$ has a trivial action on $\pi(R')$.

We are now in position to give the main theorem of this section in which we also remove the squarefree hypothesis.

Theorem 6. *Let $I \subseteq S$ be any monomial ideal, and let $R = S/I$. There is an isomorphism of multigraded Lie algebras*

$$\pi^{\geq 2}(R) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(I)).$$

Proof. Proposition 9 together with Proposition 8 yield that $\pi^{\geq 2}(R) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(I))$ when I is generated by squarefree monomials. If I and J are equivalent ideals, then $\mathfrak{L}_\infty(I) \cong \mathfrak{L}_\infty(J)$ and this isomorphism respects the relation $\perp$, whence also $F\,\mathrm{H}^*(\mathfrak{L}_\infty(I)) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(J))$. Applying this to the equivalence of I and its polarization I° and using Theorem 3, we get

$$\pi^{\geq 2}(R) \cong \pi^{\geq 2}(R^\circ) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(I^\circ)) \cong F\,\mathrm{H}^*(\mathfrak{L}_\infty(I)).$$

□

The relevant data for computing $\pi^{\geq 2}(R)$ is the differential and the binary bracket of $\mathfrak{L}_\infty(I)$. As a favour to the reader, we write these out. The differential d of $\mathfrak{L}_\infty(I)$ has degree 1 and is given on basis elements by

$$d\xi_S = \sum_{\substack{m \in M - S \\ m | m_S}} \mathrm{sgn}(S, m)\xi_{S \cup m}.$$

The bracket of L, which is a Lie bracket only up to homotopy, is given by

$$[\xi_S, \xi_T] = \sum_{\substack{m \in M - (S \cup T) \\ m | m_{S \cup T} \\ S \cup T \cup m \text{ connected}}} \mathrm{sgn}(T, m, S)\xi_{S \cup T \cup m},$$

if $\gcd(m_S, m_T) = 1$ and zero otherwise. Note the order T, m, S in the sign. The reduced square is $\xi_S^{[2]} = 0$ for all $\xi_S \in L$.

We note two special cases when the structure of $\pi^{\geq 2}(R)$ is simple.

- If the lcm-lattice L_M is boolean, then there are no cover relations $m \mid m_S$, and therefore all operations in $\mathfrak{L}_\infty(M)$ are trivial. Therefore the homotopy Lie algebra $\pi^{\geq 2}(R)$ is the free graded Lie algebra generated by ξ_S for non-empty connected subsets S of M divided by the relations $[\xi_S, \xi_T] = 0$ for all pairs $S, T \subseteq M$ such that $\gcd(m_S, m_T) = 1$.

- If M is a complete graph, i.e., if $\gcd(m,n) \neq 1$ for all $m, n \in M$, then $\pi^{\geq 2}(R)$ is the free graded Lie algebra on the vector space $\mathrm{H}^*(\mathfrak{L}_\infty(M))$. In this case, this vector space is isomorphic to $\mathrm{Ext}_S^{\geq 1}(R,k)$.

Example 1. Forgetting the multigrading, Theorem 5 shows that the graded vector space $\pi^{\geq 2}(R)$ is determined by the combinatorial data (K_M, c). Indulging ourselves in a comparison, this is reminiscent of the fact that the homotopy type of the complement of an affine subspace arrangement is determined by its intersection lattice and its dimension function, cf. [11]. Despite this analogy, the datum (K_M, c) is not sufficient for determining the Lie bracket on $\pi^{\geq 2}(R)$, as is shown by the following example. Consider the monomial rings R, Q defined by the sets $M = \{x^2, xy, y^2\}$ and $N = \{x^2, xyz, y^2\}$ respectively. These monomial sets are isomorphic as graphs and the pairs (K_M, c_M), (K_N, c_N) are isomorphic. Therefore $\pi^{\geq 2}(R) \cong \pi^{\geq 2}(Q)$ as graded vector spaces. They are not isomorphic as graded Lie algebras. $\pi^{\geq 2}(R)$ is the free graded Lie algebra generated by $\xi_1, \xi_2, \xi_3, \xi_{1,3}, \xi_{2,3}$, whereas $\pi^{\geq 2}(Q)$ has generators $\zeta_1, \zeta_2, \zeta_3, \zeta_{1,3}, \zeta_{2,3}, \zeta_{1,2,3}$ and the single relation $[\zeta_1, \zeta_2] = 0$.

Example 2. In spite of what the previous example might suggest, it is possible to have $\pi^{\geq 2}(R) \cong \pi^{\geq 2}(Q)$ without M and N being equivalent. The monomial sets $M = \{x^2, y^2, z^2, xyz\}$ and $N = \{x^2, y^2, z^2, xyzw\}$ are not equivalent because their lcm-lattices are not isomorphic, but their homotopy Lie algebras are isomorphic. Both have the presentation

$$\mathbb{L}(\xi_1, \xi_2, \xi_3, \xi_{1,4}, \xi_{2,4}, \xi_{3,4}, \xi_{1,2,4}, \xi_{1,3,4}, \xi_{2,3,4}, \xi_{1,2,3,4})/([\xi_1, \xi_2], [\xi_1, \xi_3], [\xi_2, \xi_3]).$$

10. Goldness

As a conclusion, we note how our formula gives a combinatorial criterion for when a monomial ring is Golod. Interesting sufficient combinatorial conditions have been found earlier, see for instance [25], but the author is not aware of any necessary condition which is formulated in terms of the combinatorics of the monomial generators.

Our formula for the Poincaré series denominator could be compared with the result of [21] that the Betti numbers $\dim_k \mathrm{Tor}_{i,\alpha}^S(R,k)$ of a monomial ring $R = k[x_1, \dots, x_n]/I$ can be computed from the homology of the lower intervals of the lcm-lattice, L_I, of I. Specifically, Theorem 2.1 of [21] can be stated as

$$\mathrm{P}_R^S(\mathbf{x}, z) = 1 + \sum_{1 \neq m \in L_I} m z^2 \widetilde{\mathrm{H}}((1,m)_{L_I}; k)(z). \tag{10}$$

Here $\mathrm{P}_R^S(\mathbf{x}, z)$ is the polynomial

$$\mathrm{P}_R^S(\mathbf{x}, z) = \sum_{i \geq 0, \alpha \in \mathbb{N}^n} \dim_k \mathrm{Tor}_{i,\alpha}^S(R,k) x^\alpha z^i.$$

Recall that R is called a *Golod ring* if there is an equality of formal power series

$$\mathrm{P}_R(\mathbf{x}, z) = \frac{\prod_{i=1}^n (1 + x_i z)}{1 - z(\mathrm{P}_R^S(\mathbf{x}, z) - 1)}.$$

In terms of the denominator polynomial the condition reads

$$b_R(\mathbf{x}, z) = 1 - z(\mathrm{P}_R^S(\mathbf{x}, z) - 1). \tag{11}$$

It is easily seen that S is saturated in M if and only if S is saturated in M_{m_S}. Note also that $(1,m)_{L_M} = L_{M_m} - \{1, m\} =: \bar{L}_{M_m}$. Therefore, after plugging the formulas (6) and (10) into (11) and equating the coefficients of each $m \in L_I$, we get a criterion for R to be a Golod ring as follows:

Definition 6. A monomial set N is called *pre-Golod over* k if

$$\widetilde{\mathrm{H}}(\bar{L}_N; k)(z) = \sum_{\substack{S \in \hat{K}_N \\ m_S = m_N}} (-z)^{c(S)-1} \widetilde{\mathrm{H}}((\emptyset, S)_{K_N}; k)(z).$$

Theorem 7. *Let k be a field and let I be a monomial ideal in $k[x_1, \ldots, x_n]$ with minimal set of generators M. Then the monomial ring $R = k[x_1, \ldots, x_n]/I$ is Golod if and only if every non-empty subset of M of the form M_m is pre-Golod over k.* □

Combinatorics

11. Realizations of lattices

Recall that the lcm-lattice of a monomial set M is the set $L_M = \{m_S \mid S \subseteq M\}$ of least common multiples of subsets of M partially ordered by divisibility. It is natural to ask which finite lattices occur as lcm-lattices. An isomorphism of a lattice L with the lcm-lattice of some set of monomials M such that M maps to the irreducible elements of L is called a *realization* of L. Sometimes we will abuse language and call M a realization of L. Construction 2.3 of [30] provides a realization of any geometric lattice. We will show that actually any finite lattice is an lcm-lattice.

A realization of a lattice induces a graph structure on it, by viewing sets of monomials, and in particular lcm-lattices, as graphs with edges going between monomials having non-trivial common factors. The next lemma says that every graph structure induced on L via some realization contains the edges

$$\{(x, y) \mid x, y \not\leq c, \text{ for some coirreducible } c \in L\}.$$

Proposition 10. *Let M be a monomial set. If m and n are elements of L_M satisfying $\gcd(m, n) = 1$, then for all coirreducible elements $c \in L_M$ either $m \mid c$ or $n \mid c$ holds.*

Proof. Let X be the variables used in M. For each $x \in X$ and each $n \geq 1$, consider the function $\alpha_{x^n} : L_M \to \underline{2}$ defined by

$$\alpha_{x^n}(w) = \begin{cases} 0 & x^n \nmid w \\ 1 & x^n \mid w \end{cases}$$

α_{x^n} is an element of $(L_M)^*$. We claim that the set $\{\alpha_{x^n} \mid x \in X, n \geq 1\}$ generates $(L_M)^*$ as a join-semilattice. To see this, note that if $f \in (L_M)^*$, then with $v = \vee f^{-1}(0)$ we have $f = f_v$, where $f_v(w) = 0$ if and only if $w \leq v$. Hence, as is easy to check,

$$f = \bigvee_{x^n \nmid v} \alpha_{x^n}.$$

Now let c be a coirreducible element of L_M. Then f_c is irreducible in $(L_M)^*$. Therefore $f_c = \alpha_{x^n}$ for some $x \in X$ and some $n \geq 1$, that is, $c \mid w$ if and only if $x \nmid w$ for $w \in L_M$. If $\gcd(m, n) = 1$, then either $x^n \nmid m$ or $x^n \nmid n$, i.e., $m \mid c$ or $n \mid c$. □

Definition 7. Let L be a finite lattice and let I (C) be its set of irreducible (coirreducible) elements. The *minimal realization* of L is the monomial set $M = \{m_a \mid a \in I\}$, where for each $z \in L$, m_z is the squarefree monomial in the variables $\{x_c\}_{c\in C}$ defined by

$$m_z = \prod_{\substack{c \in C \\ z \nleq c}} x_c.$$

If one starts with a geometric lattice and takes its minimal realization then one obtains the same monomial set as that constructed by Peeva, cf. [30] Construction 2.3. The next proposition justifies the name 'minimal realization'.

Proposition 11. *Let L be a finite lattice and let M be its minimal realization. The map $L \to L_M$, $z \mapsto m_z$ is an isomorphism of lattices. Furthermore the graph structure induced on L via this isomorphism is the minimal possible, i.e., $x, y \in L$ are connected by an edge if and only if $x, y \nleq c$ for some coirreducible $c \in L$.*

Proof. That we have an isomorphism of lattices follows from the fact that $x \leq y$ in L if and only if $C_y \subseteq C_x$, where C_x denotes the set of coirreducible elements above x. Also, the graph structure on L is the minimal allowed by Proposition 10 — $\gcd(m_a, m_b) = 1$ if and only if for all $c \in C$ either $a \leq c$ or $b \leq c$, or both. $\square$

Remark 5. Proposition 11 shows that any finite lattice is the lcm-lattice of some set of monomials. Restricting attention to antichains of monomials, which is the same thing as minimal generators for monomial ideals, we see that a finite lattice is the lcm-lattice of some monomial ideal if and only if it is atomic.

12. Complete monomial sets and geometric lattices

As suggested by Theorem 7, the Golod property of a monomial ring $S/(M)$ is a property of the morphism of semilattices $K_M \to L_M$. This part contains a closer investigation of this morphism. We introduce a new class of finite lattices, called *complete* lattices, which is closed under direct products and contains all geometric lattices. The main feature of this class is that monomial sets whose lcm-lattices are complete define Golod rings if and only if their corresponding graphs are complete. This generalizes the previously known result that this holds if the lcm-lattice is boolean. It should be noted that the arguments in this part do not formally depend on Theorem 7 or any other result established in preceding parts. It is rather the case that this theorem pointed toward which structure to examine more carefully.

We will investigate the two lattices L_M and K_M associated to a monomial set M. Notions and definitions concerning semilattices are collected in Appendix B.

Recall that M_m denotes the set of all $n \in M$ that divide m, if m is a monomial and M is a set of monomials. L_M embeds into K_M as a meet-semilattice by mapping $x \in L_M$ to $\overline{\{x\}} = M_x$. The map $K_M \to L_M$ sending S to m_S is a map of join-semilattices and a retraction onto L_M, because $m_{M_x} = x$. Thus K_M is isomorphic to L_M if and only if the equality $M_{m_S} = S$ holds for every saturated subset S of M.

Definition 8. A monomial set M is called *complete* if K_M is isomorphic to L_M, i.e., if $M_{m_S} = S$ holds for all $S \in K_M$.

For instance, it is easily seen that if the graph underlying M is complete, i.e., if every two monomials in M have a non-trivial common factor, then M is a complete monomial set.

Proposition 12. *M is complete if and only if for all $x, y \in L_M$ with $\gcd(x, y) = 1$ and for all $m \in M$, $m \mid xy$ implies $m \mid x$ or $m \mid y$.*

Proof. Assume M complete and suppose $x, y \in L_M$ and $\gcd(x, y) = 1$. Let $S = M_x \cup M_y$. S is saturated in M because the saturated sets M_x and M_y are the connected components of S. Note that $m_S = xy$, so by completeness $M_x \cup M_y = M_{xy}$, which is exactly what is required.

Conversely, if $M_{xy} = M_x \cup M_y$ whenever $\gcd(x, y) = 1$ and $x, y \in L_M$, then for $S \in K_M$, decompose S into connected components as $S = S_1 \cup \ldots \cup S_r$. Since $S_i = M_{m_{S_i}}$, it follows that $S = M_{m_{S_1}} \cup \ldots \cup M_{m_{S_r}} = M_{m_{S_1} \ldots m_{S_r}} = M_{m_S}$. $\square$

Let M be the minimal realization of a lattice L and let N be any realization of L. Then by Propositions 10 and 11, the induced lattice isomorphism $f \colon L_M \to L_N$ is morphism of graphs, i.e., the graph L_M is obtained from the graph L_N by removing some edges. Then as in Proposition 3 there is a commutative diagram of join-semilattices

$$\begin{tikzcd} K_M \arrow[r, two heads, "\bar{f}"] \arrow[d, "m_-"] & K_N \arrow[d, "m_-"] \\ L_M \arrow[r, "\cong", "f"'] & L_N \end{tikzcd}$$

If M is complete, i.e., if $K_M \to L_M$ is an isomorphism, then so is N. In other words, if the minimal realization of a lattice L is complete, then all realizations of L are complete. In view of this fact we call the lattice L complete if its minimal realization is a complete monomial set, and we have the following characterization.

Proposition 13. *The following are equivalent for a lattice L:*

- *L is complete.*
- *The minimal realization of L is complete.*
- *Every realization of L is complete.*
- *For any $x, y \in L$ such that $L_{\geq x} \cup L_{\geq y}$ contains all coirreducible elements of L, if $a \in L$ is irreducible, then $a \leq x \vee y$ only if $a \leq x$ or $a \leq y$.*

If M and N are sets of monomials in the variables X and Y, respectively, then $M \bigoplus N$ is the monomial set $M \cup N$ in the variables $X \sqcup Y$. The graph underlying $M \bigoplus N$ is the disjoint union of the graphs of M and N. Clearly, $L_{M \bigoplus N} \cong L_M \times L_N$ and $K_{M \bigoplus N} \cong K_M \times K_N$. Therefore $M \bigoplus N$ is complete if M and N are complete. One can also verify that if M and N are the minimal realizations of the lattices L and K, then $M \bigoplus N$ is the minimal realization of $L \times K$. Consequently, direct products of complete lattices are complete.

Recall that a finite lattice L is called *geometric* if it is atomic and if it is *semimodular*, meaning that for all $x, y \in L$, if x and y both cover $x \wedge y$, then both x and y are covered by $x \vee y$. Geometric lattices abound in combinatorics and other areas of mathematics. Among the many results concerning geometric lattices, we cite here a structure theorem which will be useful to us. A lattice is called *indecomposable* if it is not isomorphic to a direct product of smaller lattices.

Theorem 8 ([22], Theorems IV.3.5 and IV.3.6). *Every geometric lattice is isomorphic to a direct product of indecomposable geometric lattices. A geometric lattice L is indecomposable if and only if for any two atoms $a, b \in L$, there is a coatom $c \in L$ such that $a \not\leq c$ and $b \not\leq c$.*

The next result was discovered by J.Blasiak and P.Hersh. Their original proof uses matroid theory. We present here an alternative proof using the above structure theorem.

Corollary 2. *The graph underlying the minimal realization of a geometric lattice L is a disjoint union of complete graphs, the components being in one-to-one correspondence with the factors of the decomposition of L as a direct product of indecomposable geometric lattices.*

Proof. A geometric lattice is coatomic, so the coirreducible elements of L are precisely the coatoms. Thus, in the minimal realization $f\colon L \to L_M$, two monomials $m, n \in M$ have a common factor if and only if there is a coatom c of L such that $f^{-1}(m), f^{-1}(n) \not\leq c$. □

By Proposition 13 we conclude

Corollary 3. *Geometric lattices are complete.*

Remark 6. Not all complete lattices are geometric. The complete monomial set $M = \{x^2y, xz, yz\}$ is a minimal realization of its lcm-lattice, but this lattice is not ranked, let alone geometric.

It is well known that geometric lattices are shellable (that is, the order complex of the proper part of a geometric lattice is a shellable simplicial complex). However, not all shellable lattices are complete; the lcm-lattice of the monomial set $M = \{x^2, xy, y^2\}$ is shellable, but not complete.

12.1. **The Golod property and L_∞-algebra of complete monomial sets.** Our interest in complete monomial sets stems from the fact that it is trivial to determine whether or not such a set is Golod. It is easily seen that if the graph underlying a monomial set M is complete, then M is Golod over any field k. In [16] it is proved that if L_M is boolean (i.e., isomorphic to the lattice of subsets of M) then the converse holds, i.e., M is Golod if and only if its underlying graph is complete. Boolean lattices are geometric and hence complete, and we have the following generalization of the quoted result.

Proposition 14. *If M is a complete antichain of monomials, then M is Golod if and only if the graph underlying M is complete.*

Proof. We need only the following two properties of the class of Golod sets:

- A Golod set is connected.
- If M is Golod, then M_m is Golod for any monomial m.

Suppose M is Golod and let $x, y \in M$. If $\gcd(x, y) = 1$, then M_{xy} must be Golod, and hence connected. On the other hand $M_{xy} = M_x \cup M_y$ by completeness, but this set is not connected. Therefore $\gcd(x, y) \neq 1$ for all $x, y \in M$. □

Corollary 4. *If the lcm-lattice of M is geometric, then M is Golod if and only if its underlying graph is complete.*

The L_∞-algebra $\mathfrak{L}_\infty(M)$ is particularly simple when M is a complete monomial set. In fact, completeness of M has the following algebraic characterization.

Proposition 15. *A monomial set M is complete if and only if all higher operations of the L_∞-algebra $\mathfrak{L}_\infty(M)$ are trivial.*

Proof. Assume that M is complete. Let $S_1, \ldots, S_r$ be pairwise separated connected subsets of M, where $r \geq 2$, and let S be their union. If $m \mid m_{S_1} \ldots m_{S_r}$, then $m \mid m_{S_i}$ for some i, by completeness of M. Therefore $S \cup \{m\}$ cannot be connected. From the definition (9) of the brackets, we see that $[\xi_{S_1}, \ldots, \xi_{S_r}] = 0$.

Conversely, assume that all r-ary operations of $\mathfrak{L}_\infty(M)$ are zero, for all $r \geq 2$. In view of Proposition 12, it is clear that M is complete if and only if for all $m \in M$ and all $S \subseteq M$, if $m \mid m_S$ then $m \mid m_{S_i}$ for some connected component S_i of S. Thus let $m \in M$, and let S be a subset of M with connected components $S_1, \ldots, S_r$. Suppose $m \mid m_S$. Since $[\xi_{S_1}, \ldots, \xi_{S_r}] = 0$, we conclude that $S \cup \{m\}$ is not connected — otherwise $\xi_{S\cup\{m\}}$ would have occured with a non-zero coefficient in the bracket. Therefore $\gcd(m, m_{S_i}) = 1$ for some i, and by renumbering we may assume that $i = r$. Therefore $m \mid m_T$, where $T = S_1 \cup \ldots \cup S_{r-1}$. Continuing in this way we end up with $m \mid m_{S_j}$ for some j. This proves the claim. □

Remark 7. It follows from the previous proposition that $\mathrm{H}^*(\mathfrak{L}_\infty(M))$ is an abelian Lie algebra if M is complete. We stress that this does not imply that $\pi^{\geq 2}(R)$ is abelian. What it does imply is that the product of two elements in $\pi^{\geq 2}(R)$ of relatively prime multidegrees is zero.

13. Connections to real subspace arrangements

We first recall the notion of a subspace arrangement. Let $\mathbb{F}$ be a field, often $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *subspace arrangement* is a finite collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ of affine subspaces of $\mathbb{F}^n$. The complement of the arrangement is the space

$$\mathcal{M}_\mathcal{A} = \mathbb{F}^n - \bigcup_{i=1}^{n} A_i.$$

An arrangement is called *central* if all A_i are linear subspaces.

The *intersection lattice* of an arrangement $\mathcal{A}$ is the set of intersections of subspaces,

$$L_\mathcal{A} = \{A_{i_1} \cap \ldots \cap A_{i_r} \mid r \geq 0\},$$

ordered by *reverse* inclusion. Here an empty intersection is interpreted as $\mathbb{F}^n$.

The cohomology of the complement of a subspace arrangement can be understood in terms of the combinatorics of intersection lattices. The Goresky-MacPherson formula states that for a real arrangement $\mathcal{A}$ the integral cohomology groups of the complement are given by (cf. [11])

$$\widetilde{\mathrm{H}}^*(\mathcal{M}_\mathcal{A}) \cong \bigoplus_{x \in L_\mathcal{A}, x \neq \hat{0}} \widetilde{\mathrm{H}}_{\mathrm{codim}(x)-2-i}(\hat{0}, x).$$

Relations between certain types of complex arrangements and monomial rings have been investigated. In some cases the cohomology algebra of the complement can be described purely algebraically. Specifically, to a simplicial complex Δ with vertex set $\{1, 2, \ldots, n\}$ one may associate a complex arrangement $\mathcal{A}(\Delta)$. It consists of all subspaces $W_F = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in F\}$, where F ranges over the set of minimal non-faces of Δ. Let $U(\Delta)$ denote the complement of the arrangement $\mathcal{A}(\Delta)$. Theorem 8.13 of [14] states that there is an isomorphism of graded algebras

$$\mathrm{H}^*(U(\Delta)) \cong \mathrm{Tor}^S_*(k[\Delta], k).$$

For a related result in the real case see [20].

13.1. **Real diagonal subspace arrangements.** Let m be a squarefree monomial in the variables $x_1, \dots, x_n$. To m we associate the *diagonal subspace*

$$U_m = \{u \in \mathbb{R}^n \mid u_i = u_j, \text{ if } x_i x_j \text{ divides } m\},$$

which is a linear subspace of $\mathbb{R}^n$. If M is a set of squarefree monomials in the variables $x_1, \dots, x_n$, then let $\mathcal{A}_M$ be the arrangement $\{U_m \mid m \in M\}$. The *intersection lattice* of $\mathcal{A}_M$ is the set

$$L_{\mathcal{A}_M} = \{U_{m_1} \cap \dots \cap U_{m_r} \mid r \geq 0, m_i \in M\}.$$

partially ordered by *reverse* inclusion. Here an empty intersection is interpreted as $\mathbb{R}^n$.

Proposition 16. *Let M be a set of squarefree monomials. The lattice of saturated subsets of M, K_M, is isomorphic to the intersection lattice $L_{\mathcal{A}_M}$ of the diagonal arrangement $\mathcal{A}_M$ associated to M. Furthermore for $S \in K_M$*

$$\mathrm{codim}(S) + c(S) = \deg m_S,$$

where $\mathrm{codim}(S)$ *is the codimension of the image of S in the intersection lattice.*

Proof. For the diagonal subspaces, we have that $U_m \cap U_n = U_{\mathrm{lcm}(m,n)}$ if $\gcd(m, n) \neq 1$. Hence any intersection may be brought to the form

$$U_{m_1} \cap \dots \cap U_{m_r},$$

where m_i are pairwise relatively prime and $m_i \in cL_M$, that is, $\{m_1, \dots, m_r\}$ is an element of $D(cL_M)$. Recall that cL_M denotes the subset of L_M consisting of monomials $m \in L_M$ such that M_m is connected. Conversely, if $m \in cL_M$, then

$$U_m = \bigcap_{n \in M_m} U_n.$$

This establishes an isomorphism of partial orders $L_{\mathcal{A}_M} \cong D(cL_M)$.

An isomorphism $D(cL_M) \cong K_M$ is given by mapping

$$\{m_1, \dots, m_r\} \mapsto M_{m_1} \cup \dots \cup M_{m_r}.$$

The inverse is given by

$$S \mapsto \{m_{S_1}, \dots, m_{S_r}\},$$

where $S_1, \dots, S_r$ are the connected components of $S \in K_M$.

Clearly, $\mathrm{codim}(U_m) = \deg(m) - 1$. The element in the intersection lattice corresponding to $S \in K_M$ is

$$U_{m_1} \cap \dots \cap U_{m_r},$$

where $m_i = m_{S_i}$ and $S_1, \dots, S_r$ are the connected components of S. Since the monomials m_i are pairwise relatively prime

$$\begin{aligned} \mathrm{codim}(U_{m_1} \cap \dots \cap U_{m_r}) &= \mathrm{codim}(U_{m_1}) + \dots + \mathrm{codim}(U_{m_r}) \\ &= (\deg(m_1) - 1) + \dots + (\deg(m_r) - 1) \\ &= \deg(m_S) - c(S). \end{aligned}$$

$\square$

The next proposition could be given a direct proof, but instead we give a neat proof using the formula for the Poincaré series.

Proposition 17. *Let M be a monomial set, and let $\alpha \in \mathbb{N}^n$. Then we have an isomorphism of graded vector spaces*

$$\mathrm{Tor}_*^{S/(M)}(k,k)_\alpha \cong \mathrm{Tor}_*^{S/(M_\alpha)}(k,k)_\alpha.$$

Proof. $\mathrm{Tor}_*^{S/(M)}(k,k)_\alpha(z)$ is the coefficient of x^α in the formal power series $\mathrm{P}_{S/(M)}$. Therefore the result follows if we can show the congruence

$$\mathrm{P}_{S/(M)} \equiv \mathrm{P}_{S/(M_\alpha)} \mod (x_1^{\alpha_1+1}, \ldots, x_n^{\alpha_n+1}).$$

But this follows since

$$\begin{aligned} b_{S/(M)} &= 1 + \sum_{S \in \hat{K}_M} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset, S); k)(z) \\ &\equiv 1 + \sum_{S \in \hat{K}_M, S \subseteq M_\alpha} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset, S); k)(z) \\ &= b_{S/(M_\alpha)}. \end{aligned}$$

The last equality holds because $\{S \in K_M \mid S \subseteq M_\alpha\} = K_{M_\alpha}$. □

Let $\mathcal{M}_{\mathcal{A}_M} = \mathbb{R}^n - \cup\mathcal{A}_M$ be the complement of the union of all subspaces in $\mathcal{A}_M$. The Goresky-Macpherson formula [11] expresses the cohomology of $\mathcal{M}_{\mathcal{A}_M}$ in terms of the homology groups of the lower intervals in the intersection lattice. We state here a generating functions version of the formula using coefficients from a field k:

$$\widetilde{\mathrm{H}}^*(\mathcal{M}_{\mathcal{A}_M}; k)(z) = \sum_{x \in L_{\mathcal{A}_M}, x \neq \hat{0}} z^{\operatorname{codim} x - 2}\widetilde{\mathrm{H}}((\hat{0}, x); k)(z^{-1}).$$

Using the Bar resolution to resolve k over R, Peeva, Reiner and Welker relates the cohomology of real diagonal subspace arrangements, with coefficients in a field k, to $\mathrm{Tor}_*^R(k,k)$ for monomial rings R. We are able give a new proof of their result using our formula for the deviations $\epsilon_{i,\alpha}$ and the Goresky-MacPherson formula. Let $\tau = (1, \ldots, 1) \in \mathbb{N}^n$.

Theorem 9 ([31], Theorem 1.3)**.** *For a monomial ring $R = k[x_1, \ldots, x_n]/I$ there is an isomorphism of vector spaces*

$$\mathrm{H}^i(\mathcal{M}_{\mathcal{A}_M}; k) \cong \mathrm{Tor}_{n-i}^R(k,k)_\tau,$$

where M is the set of squarefree monomials in the minimal set of generators for I.

Proof. Since we work over a field, the required isomorphism is equivalent to the equality

$$\mathrm{H}^*(\mathcal{M}_{\mathcal{A}_M}; k)(z) = z^n \,\mathrm{Tor}_*^R(k,k)_\tau(z^{-1}).$$

According to the Goresky-MacPherson formula

$$\widetilde{\mathrm{H}}^*(\mathcal{M}_{\mathcal{A}_M}; k)(z) = \sum_{x \in L_{\mathcal{A}_M}, x \neq \hat{0}} z^{\operatorname{codim}(x)-2}\widetilde{\mathrm{H}}((\hat{0}, x); k)(z^{-1}). \tag{12}$$

On the other hand consider the $x_1 \ldots x_n$-part of the Poincaré series P_R. According to Proposition 17, this is the same as the $x_1 \ldots x_n$ part of $\mathrm{P}_{S/(M)}$. We use the product decomposition of $\mathrm{P}_{S/(M)}$ and reduce modulo $(x_1^2, \ldots, x_n^2)$:

$$P_{S/(M)} = \prod_{i \geq 1, \alpha \in \mathbb{N}^n} (1 - x^\alpha(-z)^i)^{(-1)^{i-1}\epsilon_{i,\alpha}} \equiv \prod_{\alpha \in \{0,1\}^n} (1 + x^\alpha p_\alpha(z)),$$

where

$$p_\alpha(z) = \sum_{i \geq 1} \epsilon_{i,\alpha} z^i.$$

Since M is squarefree, we infer from Theorem 4 that $p_{e_i}(z) = z$, for e_i a standard basis vector, and

$$p_\alpha(z) = z^3 \widetilde{\mathrm{H}}(\Delta_{M_\alpha}; k)(z),$$

for $x^\alpha \in L_M$. Also $p_\alpha(z) = 0$ unless $x^\alpha \in cL_M$. Therefore

$$P_{S/(M)} \equiv \prod_{i=1}^{n} (1 + x_i z) \prod_{x^\alpha \in cL_M} (1 + x^\alpha p_\alpha(z)).$$

Note that the right factor is the same as the expression (7) for $b_{S/(M)}$, except for a sign. Carrying out the same manipulations as for $b_{S/(M)}$ we get

$$P_{S/(M)} \equiv \prod_{i=1}^{n} (1 + x_i z)(1 + \sum_{S \in \hat{K}_M} m_S z^{c(S)+2} \widetilde{\mathrm{H}}((\emptyset, S); k)(z)).$$

The $x_1 \dots x_n$-part of P_R is therefore given by

$$\mathrm{Tor}^R(k,k)_\tau(z) = z^n + \sum_{S \in \hat{K}_M} z^{n + c(S) - \deg(m_S) + 2} \widetilde{\mathrm{H}}((\emptyset, S); k)(z).$$

Proposition 16 now tells us that $K_M \cong L_{\mathcal{A}_M}$ and $\deg(m_S) - c(S) = \mathrm{codim}(S)$. Thus we see that

$$\begin{aligned} z^n \,\mathrm{Tor}^R(k,k)_\tau(z^{-1}) &= 1 + \sum_{S \in \hat{K}_M} z^{\deg(m_S) - c(S) - 2} \widetilde{\mathrm{H}}((\emptyset, S); k)(z^{-1}) \\ &= 1 + \sum_{x \in L_{\mathcal{A}_M}} z^{\mathrm{codim}(x) - 2} \widetilde{\mathrm{H}}((\hat{0}, x); k)(z^{-1}) \\ &= \mathrm{H}^*(\mathcal{M}_{\mathcal{A}_M}; k)(z), \end{aligned}$$

by (12). This is what we wanted to prove. □

Appendix

Appendix A. Simplicial complexes

A simplicial complex on a set V is a set Δ of subsets of V such that $F \subseteq G \in \Delta$ implies $F \in \Delta$. The set V is called the *vertex set* of Δ. The *i-faces* or *i-simplices* of Δ are the elements in Δ of cardinality $i+1$. We do not require that $\{v\} \in \Delta$ for all $v \in V$, but if a simplicial complex Δ is given without reference to a vertex set V, then it is assumed that $V = \cup\Delta$.

If Δ is a simplicial complex then $\widetilde{C}(\Delta)$ will denote the augmented chain complex associated to Δ. Thus $\widetilde{C}_i(\Delta)$ is the free abelian group on the i-faces of Δ, the empty set being considered as the unique (-1)-face, and $\widetilde{C}(\Delta)$ is equipped with the standard differential of degree -1. Therefore

$$\mathrm{H}_i(\widetilde{C}(\Delta)) = \widetilde{\mathrm{H}}_i(\Delta).$$

As usual, if G is an abelian group, then $\widetilde{C}(\Delta; G) = \widetilde{C}(\Delta) \otimes G$ and $\widetilde{\mathrm{H}}_i(\Delta; G) = \mathrm{H}_i(\widetilde{C}(\Delta; G))$. Also $\widetilde{\mathrm{H}}(\Delta; G) = \bigoplus_{i \in \mathbb{Z}} \widetilde{\mathrm{H}}_i(\Delta; G)$.

The Alexander dual of a simplicial complex Δ with vertices V is the complex

$$\Delta^\vee = \{F \subseteq V \mid V - F \notin \Delta\}.$$

The join of two complexes Δ_1, Δ_2 with disjoint vertex sets V_1, V_2 is the complex with vertex set $V_1 \cup V_2$ and faces

$$\Delta_1 * \Delta_2 = \{F_1 \cup F_2 \mid F_1 \in \Delta_1, F_2 \in \Delta_2\}.$$

With Δ_1 and Δ_2 as above, we define what could be called the *dual join* of them:

$$\Delta_1 \cdot \Delta_2 = (\Delta_1^\vee * \Delta_2^\vee)^\vee.$$

Thus $\Delta_1 \cdot \Delta_2$ is the simplicial complex with vertex set $V_1 \cup V_2$ and simplices

$$\{F \subseteq V_1 \cup V_2 \mid F \cap V_1 \in \Delta_1 \text{ or } F \cap V_2 \in \Delta_2\}.$$

We will now briefly review the effects of these operations on the homology groups when the coefficients come from a field k.

If $|V| = n$, then ([13] Lemma 5.5.3)

$$\widetilde{\mathrm{H}}_i(\Delta; k) \cong \widetilde{\mathrm{H}}_{n-i-3}(\Delta^\vee; k).$$

If H is a graded vector space, then sH denotes the graded vector space with $(sH)_i = H_{i-1}$. Because of the convention that a set with d elements has dimension $d-1$ considered as a simplex there is a shift in the following formula:

$$\widetilde{\mathrm{H}}(\Delta_1 * \Delta_2; k) \cong s(\widetilde{\mathrm{H}}(\Delta_1; k) \otimes_k \widetilde{\mathrm{H}}(\Delta_2; k)).$$

If $H = \bigoplus_{i\in\mathbb{Z}} H_i$ is a graded vector space such that each H_i is of finite dimension, then let $H(z) = \sum_{i\in\mathbb{Z}} \dim H_i z^i$ be the *generating function* of H. The above isomorphisms of graded vector spaces can be interpreted in terms of generating functions. If Δ has n vertices, then

$$z^n \widetilde{\mathrm{H}}(\Delta^\vee; k)(z^{-1}) = z^3 \widetilde{\mathrm{H}}(\Delta; k)(z),$$

and if $\Delta = \Delta_1 * \Delta_2$, then

$$\widetilde{\mathrm{H}}(\Delta; k)(z) = z\widetilde{\mathrm{H}}(\Delta_1; k)(z) \cdot \widetilde{\mathrm{H}}(\Delta_2; k)(z).$$

From these two identities and an induction one can work out the following formula. If $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_r$, then

$$\widetilde{\mathrm{H}}(\Delta; k)(z) = z^{2r-2} \widetilde{\mathrm{H}}(\Delta_1; k)(z) \cdot \ldots \cdot \widetilde{\mathrm{H}}(\Delta_r; k)(z). \tag{13}$$

Appendix B. Semilattices

By a *semilattice* we mean a commutative monoid $(X, \vee, 0)$ such that $x \vee x = x$ holds for all $x \in X$. The element $x \vee y$ is called the *join* of x and y. We will only consider finite semilattices. Morphisms of semilattices are required to preserve 0.

A semilattice is partially ordered by the relation $a \leq b \Leftrightarrow a \vee b = b$. Any finite semilattice L has a *meet* operation $x \wedge y$ such that $(L, \vee, \wedge, 0, 1)$ becomes a lattice, where $1 = \vee L$. If X and Y are finite semilattices, then the set $\mathrm{Hom}_\vee(X, Y)$ consisting of all morphisms of semilattices from X to Y is a semilattice with operations defined pointwise. The set $\underline{2} = \{0, 1\}$ becomes a semilattice by setting $i \vee j = \max(i, j)$. As partial orders, $X^* := \mathrm{Hom}_\vee(X, \underline{2})$ is anti-isomorphic to X by associating to $x \in X$ the morphism $f_x \colon X \to \underline{2}$ defined by $f_x(y) = 0 \Leftrightarrow y \leq x$. To $f \in X^*$ we associate $x = \vee f^{-1}(0)$. These functions are readily seen to be order-reversing bijections $X \leftrightarrows X^*$. In particular, if $x \in X$ is coirreducible, meaning that it cannot be written as the meet of strictly greater elements, then f_x is irreducible, meaning that it cannot be written as the join of strictly lower elements.

An element x of a poset is said to *cover* another element y if $x > y$ and $x \geq z \geq y$ implies $z = x$ or $z = y$. An *atom* of a lattice is an element covering $\hat{0}$, and a *coatom* is an element covered by $\hat{1}$. A lattice is called *atomic* if every element is the join of all atoms below it.

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