ISSN: 1401-5617

# Bound states for semilinear Schrödinger equations with sign-changing potential

Yanheng Ding
Andrzej Szulkin

Research Reports in Mathematics
Number 5, 2005

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2005/5

Date of publication: November 15, 2005

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

# Bound states for semilinear Schrödinger equations with sign-changing potential

Yanheng Ding
Institute of Mathematics, AMSS, Chinese Academy of Sciences
100080 Beijing, China

Andrzej Szulkin
Department of Mathematics, Stockholm University
106 91 Stockholm, Sweden

**Abstract**

We study the existence and the number of decaying solutions for the semilinear Schrödinger equations $-\varepsilon^2\Delta u+V(x)u=g(x,u)$, $\varepsilon>0$ small, and $-\Delta u+\lambda V(x)u=g(x,u)$, $\lambda>0$ large. The potential $V$ may change sign and $g$ is either asymptotically linear or superlinear (but subcritical) in $u$ as $|u|\to\infty$.

## 1 Introduction and statement of main results

In this paper we will be concerned with the existence and the number of nontrivial solutions for the following two problems:

$$(1.1)\qquad \begin{cases} -\varepsilon^2\Delta u+V(x)u=g(x,u), & x\in\mathbb{R}^N \\ u(x)\to 0 \quad \text{as } |x|\to\infty \end{cases}$$

and

$$(1.2)\qquad \begin{cases} -\Delta u+\lambda V(x)u=g(x,u), & x\in\mathbb{R}^N \\ u(x)\to 0 \quad \text{as } |x|\to\infty, \end{cases}$$

respectively when $\varepsilon>0$ is small and $\lambda>0$ is large. If $g(x,u)=|u|^{p-2}u$ and $\varepsilon^2=\lambda^{-1}$, then $u$ is a solution of (1.2) if and only if $v=\lambda^{-1/(p-2)}u$ is a solution of (1.1). Hence as far as the existence and the number of solutions are concerned, these problems are equivalent. However, for more general $g$ this is no longer true and as we shall see, if $g$ is asymptotically linear, then (1.1) and (1.2) are in fact quite different (cf. Remark 3.6).

The first to study (1.1) by modern methods of nonlinear functional analysis were Floer and Weinstein [11]. Using a reduction of Liapunov-Schmidt type they have shown that if $N=1$, $g(x,u)=u^3$ and $V>0$, then there exist solutions $u_\varepsilon>0$ which concentrate at a nondegenerate critical point of $V$ as $\varepsilon\to 0$. This result has been subsequently generalized by Oh [15, 16] to $N\ge 2$ and $g(x,u)=|u|^{p-2}u$, $2<p<2N/(N-2)$. Variational approach to (1.1) was initiated by Rabinowitz in [18], and since then several authors have studied (1.1) under different assumptions on $V$ and $g$. We mention here the work by del Pino and Felmer [8], Ambrosetti, Malchiodi and

Secchi [2], Byeon and Wang [6], Jeanjean and Tanaka [12], and Byeon and Jeanjean [5]. A more complete up-to-date list of references may be found e.g. in [5, 12]. A problem similar to (1.2) has been considered by Bartsch, Pankov and Wang [4], and Liu, van Heerden and Wang [14]. In these papers the potential was of the form $V_\lambda(x) = a_0(x) + \lambda a_1(x)$, $\lambda > 0$ large.

In all the work mentioned above it has been assumed that $V$ (or $V_\lambda$) is positive and bounded away from 0 ($V \geq 0$ in [6]). Also, much attention has been paid to the asymptotic shape of the solutions as $\varepsilon \to 0$, and in particular, to the occurrence of solutions possessing well localized peaks. In this paper we are mostly interested in sign-changing $V$ though in a few cases we need to have $V \geq 0$. We consider the existence and the number of nontrivial solutions to (1.1) and (1.2) but leave open the problem of existence of solutions exhibiting sharp peaks. The only earlier work on (1.1) and (1.2) we know of where $V$ was allowed to change sign is that by Felmer and Torres [10] and our recent paper [9].

Let $G(x,u) := \int_0^u g(x,s)\,ds$,

$$\widetilde{\mathcal{G}}(x,u) := \frac{1}{2}g(x,u)u - G(x,u)$$

and denote the spectrum of a densely defined operator $A$ in $L^2(\mathbb{R}^N)$ by $\sigma(A)$. We make the following assumptions on $V$ and $g$:

($V_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V$ is bounded below;

($V_2$) There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure;

($g_1$) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $G(x,u) \geq 0$ for all $(x,u)$ and $g(x,u) = o(u)$ uniformly in $x$ as $u \to 0$;

($g_2$) $g(x,u) = a_\infty(x)u + g_\infty(x,u)$, where $a_\infty \in L^\infty(\mathbb{R}^N)$, $g_\infty(x,u) = o(u)$ uniformly in $x$ as $|u| \to \infty$ and $\hat{a}_\infty := \inf a_\infty(\mathbb{R}^N) > 0$;

($g_3$) Either (i) $0 \notin \sigma(-\Delta + V - a_\infty)$, or (ii) $\widetilde{\mathcal{G}}(x,u) \geq 0$ for all $(x,u)$ and $\widetilde{\mathcal{G}}(x,u) \geq \delta_0$ for some $\delta_0 > 0$ and all $(x,u)$ with $|u|$ large enough;

($g_4$) $\gamma < b_{\max}$, where $\gamma := \sup_{u \neq 0,\, x \in \mathbb{R}^N} g(x,u)/u$ and $b_{\max} := \sup\big\{b \in \mathbb{R} :$ the measure of the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is finite$\big\}$;

($g_5$) $G(x,u)/u^2 \to \infty$ uniformly in $x$ as $|u| \to \infty$;

($g_6$) $\widetilde{\mathcal{G}}(x,u) > 0$ whenever $u \neq 0$;

($g_7$) $|g(x,u)|^\tau \leq a_1 \widetilde{\mathcal{G}}(x,u)|u|^\tau$ for some $a_1 > 0$, $\tau > \max\{1, N/2\}$ and all $(x,u)$ with $|u|$ large enough.

Clearly, ($g_1$) implies that (1.1) and (1.2) have the trivial solution $u = 0$. Solutions $u \neq 0$ will be called *nontrivial*.

We shall either require that $g$ satisfies ($g_1$)-($g_4$) (the asymptotically linear case) or ($g_1$), ($g_5$)-($g_7$) (the superlinear case). A simple example of asymptotically linear $g$ satisfying ($g_1$), ($g_2$) and (ii) of ($g_3$) is $g(x,u) = f(x)\alpha(|u|)u$, where $0 < \inf f \leq \sup f < \infty$, $\alpha(0) = 0$ and $\alpha$ is increasing and bounded. If $(g_1)$ and $(g_7)$ hold, then $|g(x,u)|^\tau \le \frac{1}{2}a_1|g(x,u)||u|^{\tau+1}$ for large $|u|$, hence $g$ satisfies the growth restriction

$$|g(x,u)| \le a_2(|u| + |u|^{p-1}), \tag{1.3}$$

where $p = 2\tau/(\tau-1) \in (2, 2^*)$ ($2^* := 2N/(N-2)$ if $N \ge 3$, $2^* := \infty$ if $N = 1$ or $2$). On the other hand, if $g$ satisfies (1.3) with $p \in (2, 2^*)$ and the Ambrosetti-Rabinowitz superlinearity condition

$$0 < \mu G(x,u) \le g(x,u)u \text{ for some } \mu > 2 \text{ and all } (x,u) \text{ with } u \ne 0, \tag{1.4}$$

then it is easy to see that $(g_5)$ and $(g_6)$ hold, and it will be shown in Lemma 2.2 that so does $(g_7)$. We shall also show in this lemma that $(g_5)$-$(g_7)$ imply $\widetilde{\mathcal{G}}(x,u) \to \infty$ as $|u| \to \infty$. An example of $g$ satisfying $(g_1)$, $(g_5)$-$(g_7)$ but not (1.4) is $g(x,u) = f(x)u\ln(1+|u|)$, $0 < \inf f \le \sup f < \infty$.

Now we are ready to state our main results.

**Theorem 1.1** *Suppose $(V_1)$, $(V_2)$ and $(g_1)$-$(g_4)$ are satisfied. If $V(x) < a_\infty(x)$ for some $x$, then there exists $\varepsilon_0 > 0$ such that (1.1) has at least 1 nontrivial solution whenever $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $g$ is odd in $u$, then for each $k \ge 1$ there exists $\varepsilon_k > 0$ such that (1.1) has at least $k$ pairs of nontrivial solutions whenever $\varepsilon \in (0, \varepsilon_k)$.*

**Theorem 1.2** *Suppose $(V_1)$, $(V_2)$ and $(g_1)$-$(g_3)$ are satisfied. If $V(x) < 0$ for some $x$, then there exists a sequence $\lambda_k \to \infty$ such that (1.2) has a nontrivial solution for each $\lambda = \lambda_k$.*

Denote the interior of $V^{-1}(0)$ by $\Omega$. If $\Omega \ne \emptyset$, let $0 < \mu_1(\Omega) < \mu_2(\Omega) \le \dots$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$.

**Theorem 1.3** *Suppose $V \ge 0$, $(V_1)$, $(V_2)$ and $(g_1)$-$(g_3)$ are satisfied, $V^{-1}(0)$ has nonempty interior $\Omega$ and $\mu_k(\Omega) < \hat{a}_\infty$ for some $k \ge 1$. Then there exists $\Lambda_0 > 0$ such that (1.2) has at least 1 nontrivial solution whenever $\lambda > \Lambda_0$. Moreover, if $g$ is odd in $u$, then (1.2) has at least $k$ pairs of nontrivial solutions whenever $\lambda > \Lambda_0$.*

**Theorem 1.4** *The conclusions of Theorem 1.1 remain valid if $(V_2)$ and $(g_4)$ are replaced by*

$(V_2')$ There exists $b_\infty > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) - a_\infty(x) < b_\infty\}$ is nonempty and has finite measure;

$(g_4')$ $\gamma_\infty < b_{\infty,\max}$, where $\gamma_\infty := \sup_{u \ne 0,\, x \in \mathbb{R}^N} g_\infty(x,u)/u$ and $b_{\infty,\max} := \sup\{b_\infty \in \mathbb{R} :$ the measure of the set $\{x \in \mathbb{R}^N : V(x) - a_\infty(x) < b_\infty\}$ is finite$\}$.

**Theorem 1.5** *Suppose $(V_1)$, $(V_2)$, $(g_1)$, $(g_5)$-$(g_7)$ are satisfied and $V^{-1}(0)$ has nonempty interior $\Omega$.*
*(i) If $G(x,u) \ge a_0|u|^\delta$ for some $a_0 > 0$, $\delta \in (2, 2^*)$ and all $|u|$ small enough, then there exists $\varepsilon_0 > 0$ such that (1.1) has at least 1 nontrivial solution whenever $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $g$ is odd in $u$, then for each $k \ge 1$ there exists $\varepsilon_k > 0$ such that (1.1) has at least $k$ pairs of nontrivial solutions whenever $\varepsilon \in (0, \varepsilon_k)$.*
*(ii) There exists $\Lambda_0 > 0$ such that (1.2) has at least 1 nontrivial solution whenever $\lambda > \Lambda_0$. Moreover, if $g$ is odd in $u$, then for each $k \ge 1$ there exists $\Lambda_k > 0$ such that (1.2) has at least $k$ pairs of nontrivial solutions whenever $\lambda > \Lambda_k$.*

**Theorem 1.6** *Suppose* $(V_1)$, $(V_2)$, $(g_1)$ *and* $(g_5)$-$(g_7)$ *are satisfied.*
*(i) If* $V(x) < 0$ *for some* $x$ *and* $G(x,u) \geq a_0|u|^\delta$ *for some* $a_0 > 0$, $\delta \in (2, 2^*)$ *and all* $|u|$ *small enough, then there exists a sequence* $\varepsilon_k \to 0$ *such that (1.1) has a nontrivial solution for each* $\varepsilon = \varepsilon_k$.
*(ii) If* $V(x) < 0$ *for some* $x$, *then there exists a sequence* $\lambda_k \to \infty$ *such that (1.2) has a nontrivial solution for each* $\lambda = \lambda_k$.

The paper is organized as follows. In Section 2 we introduce a variational setting, discuss a linear eigenvalue problem and state some results which will be needed later. Section 3 is concerned with the asymptotically linear and Section 4 with the superlinear case. In Section 5 concentration of solutions to (1.2) on the set $V^{-1}(0)$ as $\lambda \to \infty$ is briefly discussed.

**Notation.** "$\rightharpoonup$" denotes weak convergence. $|M|$ is the Lebesgue measure of the set $M$. $B_\rho(x)$ and $S_\rho(x)$ are respectively the ball and the sphere of radius $\rho$ and center $x$, $B_\rho := B_\rho(0)$ and $S_\rho := S_\rho(0)$. $\|\,.\,\|_p$ is the usual norm in $L^p(\mathbb{R}^N)$ and $(\,.\,,.)_2$ the usual inner product in $L^2(\mathbb{R}^N)$. Similarly, $\|\,.\,\|_{p,\Omega}$ is the norm in $L^p(\Omega)$ and $(\,.\,,.)_{2,\Omega}$ the inner product in $L^2(\Omega)$.

## 2 Variational setting and preliminaries

In addition to (1.1) and (1.2) we shall consider the problem

$$(2.1) \qquad \begin{cases} -\Delta u + V(x)u = g(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

with $V$ and $g$ satisfying $(V_1)$, $(V_2)$, $(g_1)$ and either $(g_2)$-$(g_4)$ or $(g_5)$-$(g_7)$. Let

$$E := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x)u^2\,dx < \infty\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V^+(x)uv)\,dx, \quad \|u\| := \langle u, u\rangle^{1/2}.$$

We shall also need the inner product

$$\langle u, v\rangle_\lambda := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda V^+(x)uv)\,dx, \quad \lambda > 0.$$

The corresponding norm will be denoted by $\|\,.\,\|_\lambda$ (so $\|\,.\,\| \equiv \|\,.\,\|_1$), and we set $E_\lambda := (E, \|\,.\,\|_\lambda)$. Clearly, $\|u\| \leq \|u\|_\lambda$ if $\lambda \geq 1$ and it follows from $(V_1)$, $(V_2)$ and the Poincaré inequality that the embedding $E \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Let

$$(2.2) \qquad \Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

It is well known (see e.g. [20]) that $\Phi \in C^1(E, \mathbb{R})$ and $\Phi'(u) = 0$ if and only if $u \in E$ is a solution of the equation in (2.1); moreover, $u(x) \to 0$ as $|x| \to \infty$ (see e.g. [7] where this is shown for a much more general class of Schrödinger equations).

Let

$$F := \{u \in E : \operatorname{supp} u \subset V^{-1}([0,\infty))\}$$

and denote the orthogonal complement of $F$ in $E$ by $F^\perp$. If $V \geq 0$, then $E = F$, otherwise $F^\perp \neq \{0\}$. Let

$$A := -\Delta + V;$$

then $A$ is formally self-adjoint in $L^2(\mathbb{R}^N)$ and the associated bilinear form

$$a(u,v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

is continuous in $E$. Consider the eigenvalue problem

$$-\Delta u + V^+(x)u = \mu V^-(x)u, \quad u \in F^\perp. \tag{2.3}$$

Since $\operatorname{supp} V^-$ is of finite measure, the quadratic form $u \mapsto \int_{\mathbb{R}^N} V^-(x)u^2\, dx$ is weakly continuous, hence there exists a sequence of positive eigenvalues $(\mu_j)$ which may be characterized by

$$\mu_j = \inf_{\substack{\dim M \geq j \\ M \subset F^\perp}} \sup \left\{ \|u\|^2 : u \in M,\ \int_{\mathbb{R}^N} V^-(x)u^2\, dx = 1 \right\}, \quad j = 1, 2, \ldots.$$

Moreover, $\mu_j \to \infty$ and the corresponding eigenfunctions $e_j$, which may be chosen so that $\langle e_i, e_j\rangle = \delta_{ij}$, are a basis for $F^\perp$ (see e.g. [19], Theorems 4.45, 4.46 and note that $\int_{\mathbb{R}^N} V^-(x)uv\, dx = 0$ if $u \in F^\perp$, $v \in F$). Let

$$\hat{E} := \operatorname{span}\{e_j : \mu_j \leq 1\} \quad \text{and} \quad E^+ := \operatorname{span}\{e_j : \mu_j > 1\}.$$

Then $E = \hat{E} \oplus E^+ \oplus F$ is an orthogonal decomposition, $\dim \hat{E} < \infty$, the quadratic form $a$ is negative semidefinite on $\hat{E}$, positive definite on $E^+ \oplus F$ and it is easy to see that $a(u,v) = 0$ if $u$, $v$ are in different subspaces of the above decomposition of $E$.

In a similar manner we set

$$A_\lambda := -\Delta + \lambda V, \quad a_\lambda(u,v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + \lambda V(x)uv)\, dx, \tag{2.4}$$

and to the problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \quad u \in F_\lambda^\perp,$$

where $\lambda > 0$ is fixed and $F_\lambda^\perp$ is the orthogonal complement of $F$ in $E_\lambda$, there corresponds a sequence

$$\begin{aligned} \mu_j(\lambda) &:= \inf_{\substack{\dim M \geq j \\ M \subset F_\lambda^\perp}} \sup \left\{ \lambda^{-1}\|u\|_\lambda^2 : u \in M,\ \int_{\mathbb{R}^N} V^-(x)u^2\, dx = 1 \right\} \\ &= \inf_{\substack{\dim M \geq j \\ M \cap F = \{0\}}} \sup \left\{ \lambda^{-1}\|u\|_\lambda^2 : u \in M,\ \int_{\mathbb{R}^N} V^-(x)u^2\, dx = 1 \right\}, \quad j = 1, 2, \ldots \end{aligned} \tag{2.5}$$

(the equality follows immediately from the fact that $\int_{\mathbb{R}^N} V^-(x)u^2\, dx = 0$ if $u \in F$). Then, in an obvious notation, $E_\lambda = \hat{E}_\lambda \oplus E_\lambda^+ \oplus F$ is an orthogonal decomposition and as before, $\dim \hat{E}_\lambda < \infty$, the quadratic form $a_\lambda$ is negative semidefinite on $\hat{E}_\lambda$, positive definite on $E_\lambda^+ \oplus F$ and $a_\lambda(u,v) = 0$ if $u$, $v$ are in different subspaces of the decomposition above. Below we collect some properties of $\mu_j(\lambda)$.

**Lemma 2.1** *Suppose $V^- \not\equiv 0$. Then, for each fixed $j$,*
*(i) $\mu_j(\lambda) \to 0$ as $\lambda \to \infty$ (so in particular, $\dim \hat{E}_\lambda \to \infty$ as $\lambda \to \infty$);*
*(ii) $\mu_j(\lambda)$ is a non-increasing continuous function of $\lambda$.*

**Proof** (i) Let $\varphi_i \in C_0^\infty(\mathbb{R}^N)$, $1 \le i \le j$, be functions such that $\operatorname{supp} \varphi_i \subset \operatorname{supp} V^-$, $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_m = \emptyset$ if $i \ne m$ and let $M := \operatorname{span}\{\varphi_1, \ldots, \varphi_j\}$. Then

$$\mu_j(\lambda) \le \sup_{u \in M \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} \lambda V^-(x) u^2 \, dx} \to 0 \text{ as } \lambda \to \infty.$$

(ii) Let $u \in M$, $M \cap F = \{0\}$ and $\int_{\mathbb{R}^N} V^-(x) u^2 \, dx = 1$. Then

$$\lambda_1^{-1} \|u\|_{\lambda_1}^2 - \lambda_2^{-1} \|u\|_{\lambda_2}^2 = (\lambda_1^{-1} - \lambda_2^{-1}) \|\nabla u\|_2^2, \tag{2.6}$$

hence $\mu_j$ is non-increasing in $\lambda$. To show continuity, let $\lambda_1, \lambda_2 \in (\lambda_0, \tilde{\lambda})$, where $\lambda_0 > 0$. It suffices to consider subspaces $M$ for which the supremum in the second line of (2.5) is $\le C := \mu_j(\lambda_0) + 1$. Then $\|\nabla u\|_2^2 \le C\tilde{\lambda}$ and it follows from (2.6) that $\mu_j(\lambda_2) \to \mu_j(\lambda_1)$ whenever $\lambda_2 \to \lambda_1$. □

**Lemma 2.2** *(i) If $g$ satisfies (1.3) and (1.4) for some $a_2 > 0$, $p \in (2, 2^*)$ and $\mu > 2$, then $(g_7)$ holds with $\tau \in (N/2, p/(p-2))$, $\tau > 1$.*
*(ii) If $(g_5)$-$(g_7)$ are satisfied, then $\widetilde{\mathcal{G}}(x, u) \to \infty$ uniformly in $x$ as $|u| \to \infty$.*

**Proof** (i) First we note that $p/(p-2) > \max\{1, N/2\}$ because $p \in (2, 2^*)$. Fix $\tau \in (N/2, p/(p-2))$, $\tau > 1$. If $|u| \ge 1$, then $|g(x, u)| \le a_3 |u|^{p-1}$ for some $a_3 > 0$. Choose $r \ge 1$ so large that

$$\frac{1}{\mu} \le \frac{1}{2} - \frac{a_3^{\tau-1}}{|u|^{p-(p-2)\tau}} \text{ whenever } |u| \ge r.$$

Then, for such $|u|$,

$$G(x, u) \le \frac{1}{\mu} g(x, u) u \le \left( \frac{1}{2} - \frac{a_3^{\tau-1}}{|u|^{p-(p-2)\tau}} \right) g(x, u) u \le \left( \frac{1}{2} - \frac{|g(x, u)|^{\tau-1}}{|u|^{\tau-1} u^2} \right) g(x, u) u,$$

and it follows that

$$\frac{|g(x, u)|^\tau}{|u|^\tau} \le \frac{1}{2} g(x, u) u - G(x, u) = \widetilde{\mathcal{G}}(x, u).$$

(ii) Using $(g_5)$-$(g_7)$, it follows that for $|u|$ large enough,

$$a_1 \widetilde{\mathcal{G}}(x, u) \ge \left( \frac{g(x, u)}{u} \right)^\tau \ge \left( \frac{2G(x, u)}{u^2} \right)^\tau \to \infty$$

uniformly in $x$ as $|u| \to \infty$. □

Recall that $(u_m)$ is called a Cerami sequence for $\Phi$ if $\Phi(u_m)$ is bounded and $(1 + \|u_m\|)\Phi'(u_m) \to 0$ as $m \to \infty$, and $\Phi$ satisfies the Cerami condition if each such sequence has a convergent subsequence. A Cerami sequence with $\Phi(u_m) \to c$ will be called a $(C)_c$-sequence, and we shall say that $\Phi$ satisfies the $(C)_c$-condition if each $(C)_c$-sequence has a convergent subsequence.

We shall make use of the following two propositions. The first one is Rabinowitz's linking theorem and may be found e.g. in [17, 20], and the second is a result by Bartolo, Benci and Fortunato [3, Theorem 2.4]. In the linking theorem it is usually assumed that $\Phi$ satisfies the stronger Palais-Smale condition; however, the Cerami condition is sufficient for the deformation lemma, and hence for the linking theorem to hold [3].

**Proposition 2.3** *Suppose* $\Phi \in C^1(E, \mathbb{R})$*,* $E = E_1 \oplus E_2$*, where* $\dim E_2 < \infty$*, and there exist* $R > \rho > 0$*,* $\kappa > 0$ *and* $e_0 \in E_1 \setminus \{0\}$ *such that* $\inf \Phi(E_1 \cap S_\rho) \geq \kappa$ *and* $\sup \Phi(\partial Q) \leq 0$*, where* $Q = \{u = v + te_0 : v \in E_2,\ t \geq 0,\ \|u\| \leq R\}$*. If* $\Phi$ *satisfies the* $(C)_c$*-condition for all* $c \in [\kappa, \sup \Phi(Q)]$*, then* $\Phi$ *has a critical value in* $[\kappa, \sup \Phi(Q)]$*.*

**Proposition 2.4** *Suppose* $\Phi \in C^1(E, \mathbb{R})$ *is even,* $\Phi(0) = 0$ *and there exist closed subspaces* $E_1$*,* $E_2$ *such that* $\operatorname{codim} E_1 < \infty$*,* $\inf \Phi(E_1 \cap S_\rho) \geq \kappa$ *for some* $\kappa, \rho > 0$ *and* $\sup \Phi(E_2) < \infty$*. If* $\Phi$ *satisfies the* $(C)_c$*-condition for all* $c \in [\kappa, \sup \Phi(E_2)]$*, then* $\Phi$ *has at least* $\dim E_2 - \operatorname{codim} E_1$ *pairs of critical points with corresponding critical values in* $[\kappa, \sup \Phi(E_2)]$*.*

## 3 Proofs of Theorems 1.1-1.4

First we shall be concerned with (2.1).

**Lemma 3.1** *Suppose that* $V$ *and* $g$ *satisfy* $(V_1)$*,* $(V_2)$ *and* $(g_1)$*-*$(g_4)$*. Then the functional* $\Phi$ *given by (2.2) satisfies the Cerami condition.*

**Proof** We adapt an argument in [8], see also [12]. Since $\gamma < b_{\max}$, we may choose $b$ in $(V_2)$ so that $\gamma < b$. Let $R > 0$ be given and let $\varphi_R \in C^\infty(\mathbb{R}^N, [0, 1])$ be such that $\varphi_R(x) = 1$ for $|x| \geq R$, $\varphi_R(x) = 0$ for $|x| \leq R/2$ and $|\nabla\varphi_R(x)| \leq a_0/R$ for some $a_0 > 0$. If $(u_m)$ is a Cerami sequence, then

$$\Phi'(u_m)(\varphi_R u_m) = \int_{\mathbb{R}^N} (|\nabla u_m|^2 + V^+ u_m^2)\varphi_R\, dx + \int_{\mathbb{R}^N} u_m \nabla u_m \cdot \nabla\varphi_R\, dx - \int_{\mathbb{R}^N} V^- u_m^2 \varphi_R\, dx - \int_{\mathbb{R}^N} g(x, u_m) u_m \varphi_R\, dx \to 0. \tag{3.1}$$

We show that $(u_m)$ is bounded. Assuming the contrary, $\|u_m\| \to \infty$, $v_m := u_m/\|u_m\| \rightharpoonup v$ in $E$ and $v_m \to v$ a.e. after passing to a subsequence. Since $|g(x, u)| \leq \gamma|u|$ and $g(x, u_m(x))/u_m(x) \to a_\infty(x)$ if $v(x) \neq 0$, it is easy to see that

$$-\Delta v + (V(x) - a_\infty(x))v = 0. \tag{3.2}$$

Below we shall denote different constants by $c_1, c_2, \ldots$. It follows from (3.1) that

$$\int_{\mathbb{R}^N} (|\nabla v_m|^2 + V^+ v_m^2)\varphi_R\, dx - \int_{\mathbb{R}^N} \frac{g(x, u_m)}{u_m} v_m^2 \varphi_R\, dx \leq \int_{\mathbb{R}^N} V^- v_m^2 \varphi_R\, dx + \frac{c_1}{R} + \varepsilon_m, \tag{3.3}$$

where $\varepsilon_m \to 0$. Since $g(x, u)/u \leq \gamma < b$, $(1 - \gamma/b)V^+(x) \leq V^+(x) - g(x, u)/u$ whenever $V^+(x) \geq b$. Using this and the fact that the measure $|D_R| \to 0$ as $R \to \infty$, where

$$D_R := \{x \in \mathbb{R}^N : V^+(x) < b \text{ and } |x| \geq R/2\},$$

we see from (3.3) and the inequality

$$\int_{D_R} v_m^2\, dx \leq c_2 |D_R|^{(p-2)/p} \|v_m\|^2$$

$(2 < p < 2^*)$ that

$$\left(1 - \frac{\gamma}{b}\right) \int_{|x| \geq R} (|\nabla v_m|^2 + V^+ v_m^2)\, dx \leq \delta(R) + \varepsilon_m, \tag{3.4}$$

where $\delta(R) \to 0$ as $R \to \infty$. Since $v_m \to v$ in $L^p_{loc}(\mathbb{R}^N)$, it follows from the Sobolev embedding theorem and (3.4) that $v_m \to v$ in $L^p(\mathbb{R}^N)$, $2 \le p < 2^*$. As a consequence of (3.1) (with $\varphi_R$ replaced by 1) and (3.2),

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla(v_m - v)|^2 + V^+(v_m - v)^2)\,dx = \lim_{m\to\infty} \int_{\mathbb{R}^N} \frac{g_\infty(x,u_m)}{u_m} v_m^2\,dx + \lim_{m\to\infty} \int_{\mathbb{R}^N} (V^- + a_\infty)(v_m - v)^2\,dx.$$

The first limit on the right-hand side is 0 by the dominated convergence theorem, and so is the second limit because $V^-$ and $a_\infty$ are bounded. Hence $v_m \to v$ in $E$ and in particular, $v \ne 0$. This is a contradiction if (i) of $(g_3)$ is satisfied. If (ii) holds, then $\widetilde{\mathcal{G}}(x,u) \ge 0$ and there is $\beta > 0$ such that $\widetilde{\mathcal{G}}(x,u) \ge \delta_0$ whenever $|u| \ge \beta$. Therefore

$$\begin{aligned} c_3 \ge \Phi(u_m) - \frac{1}{2}\Phi'(u_m)u_m &= \int_{\mathbb{R}^N} \widetilde{\mathcal{G}}(x,u_m)\,dx \\ &\ge \int_{|u_m|\ge\beta} \delta_0\,dx = \delta_0 \left|\{x \in \mathbb{R}^N : |u_m(x)| \ge \beta\}\right|, \end{aligned}$$

hence

$$\left|\{x \in \mathbb{R}^N : |u_m(x)| \ge \beta\}\right| \le \frac{c_3}{\delta_0}.$$

Since $v(x) \ne 0$ on a set of infinite measure (by the unique continuation property), there exist $\varepsilon > 0$ and $\omega \subset \mathbb{R}^N$ such that $|v(x)| \ge 2\varepsilon$ in $\omega$ and $2c_3/\delta_0 \le |\omega| < \infty$. By Egoroff's theorem we can find a set $\omega' \subset \omega$ of measure larger than $c_3/\delta_0$ on which $v_m \to v$ uniformly. So for almost all $m$, $|v_m(x)| \ge \varepsilon$ and hence $|u_m(x)| \ge \beta$ in $\omega'$. Consequently,

$$\frac{c_3}{\delta_0} < |\omega'| \le \left|\{x \in \mathbb{R}^N : |u_m(x)| \ge \beta\}\right| \le \frac{c_3}{\delta_0},$$

a contradiction again. This completes the proof of boundedness of $(u_m)$.

Passing to a subsequence, $u_m \rightharpoonup u$ in $E$ and $u_m \to u$ a.e. in $\mathbb{R}^N$. Moreover, $u$ is a solution of (2.1). Using (3.1) we obtain

$$\int_{\mathbb{R}^N} (|\nabla u_m|^2 + V^+u_m^2)\varphi_R\,dx - \int_{\mathbb{R}^N} \frac{g(x,u_m)}{u_m} u_m^2\varphi_R\,dx \le \int_{\mathbb{R}^N} V^- u_m^2 \varphi_R\,dx + \frac{c_4}{R} + \varepsilon'_m$$

(cf. (3.3)) and hence (3.4) with $u_m$ replacing $v_m$. It follows that $u_m \to u$ in $L^p(\mathbb{R}^N)$ for $2 \le p < 2^*$ and

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla(u_m - u)|^2 + V^+(u_m - u)^2)\,dx = \lim_{m\to\infty} \int_{\mathbb{R}^N} (g(x,u_m)u_m - g(x,u)u)\,dx + \lim_{m\to\infty} \int_{\mathbb{R}^N} V^-(u_m - u)^2\,dx.$$

The first limit on the right-hand side is 0 by the continuity of the Nemytskii operator and the second limit is 0 as before. Hence $u_m \to u$ in $E$. □

**Remark 3.2** If $(V_2)$ and $(g_4)$ are replaced by $(V_2')$ and $(g_4')$, then $\Phi$ still satisfies the Cerami condition. The proof above needs to be slightly modified. In (3.3) $V^\pm$ should be replaced by

$(V-a_\infty)^\pm$ and $g$ by $g_\infty$. Then $g_\infty(x,u)/u \le \gamma_\infty < b_\infty$ and $(1-\gamma_\infty/b_\infty)(V-a_\infty)^+ \le (V-a_\infty)^+ - g_\infty(x,u)/u$ whenever $(V-a_\infty)^+ \ge b_\infty$. A more convenient (for this proof) equivalent inner product in $E$ is

$$\langle\langle u,v\rangle\rangle := \int_{\mathbb{R}^N} (\nabla u\cdot\nabla v + (V-a_\infty)^+(x)uv)\,dx.$$

Taking these changes into account, the argument of Lemma 3.1 can be repeated.

**Proposition 3.3** *Suppose that* $(V_1)$, $(V_2)$ *and* $(g_1)$-$(g_4)$ *are satisfied and* $W$ *is a nontrivial finite-dimensional subspace of* $E$ *such that* $\hat{E}\cap W=\{0\}$. *If*

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x)-a_\infty(x))u^2)\,dx < 0$$

*for all* $u\in(\hat{E}\oplus W)\setminus\{0\}$, *then (2.1) has at least 1 solution* $u\ne 0$. *Moreover, if* $g$ *is odd in* $u$, *then (2.1) has at least* $\dim W$ *pairs of solutions* $u\ne 0$.

Before proving this result we show that $\Phi$ has geometric properties required in Propositions 2.3 and 2.4.

**Lemma 3.4** *There exist* $\kappa,\rho>0$ *such that* $\inf\Phi\left((E^+\oplus F)\cap S_\rho\right)\ge\kappa$.

**Proof** Let $p\in(2,2^*)$. Since the quadratic form in (2.2) is positive definite on $E^+\oplus F$ and for each $\varepsilon>0$ there exists $C_\varepsilon$ such that $0\le G(x,u)\le\varepsilon|u|^2+C_\varepsilon|u|^p$, the conclusion follows by a standard argument. □

**Lemma 3.5** *There exists* $R$ *(depending on* $W$*) such that* $\Phi(u)\le 0$ *for all* $u\in(\hat{E}\oplus W)\setminus B_R$.

**Proof** We have

$$\Phi(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x)-a_\infty(x))u^2)\,dx - \int_{\mathbb{R}^N} G_\infty(x,u)\,dx,$$

where the quadratic form above is negative definite on the finite-dimensional space $\hat{E}\oplus W$. Since $G_\infty(x,u)/u^2\to 0$ uniformly in $x$ as $|u|\to\infty$, it is easy to see that $\int_{\mathbb{R}^N} G_\infty(x,u)\,dx = o(\|u\|^2)$ as $\|u\|\to\infty$. Hence the conclusion. □

**Proof of Proposition 3.3** By Lemma 3.1, $\Phi$ satisfies the Cerami condition. If $\hat{E}=\{0\}$, then $\Phi$ has the mountain pass geometry and the argument is simpler. Therefore we assume $\hat{E}\ne\{0\}$. Let $w_0\in W\setminus\{0\}$ and $Q:=\{u=v+tw_0 : v\in\hat{E},\ t\ge 0,\ \|u\|\le R\}$. Since $G\ge 0$, $\Phi\le 0$ on $\hat{E}$. Hence it follows from Lemmas 3.4 and 3.5 that the hypotheses of Proposition 2.3 are satisfied (with $E_1=E^+\oplus F$, $E_2=\hat{E}$ and $e_0=$ the projection of $w_0$ on $E^+\oplus F$) which gives the first conclusion.

Suppose now $g$ is odd in $u$. Then we can use Proposition 2.4 with $E_1=E^+\oplus F$ and $E_2=\hat{E}\oplus W$. It is clear that $\dim E_2-\operatorname{codim}E_1=\dim W$, hence the second conclusion. □

**Remark 3.6** Let $u\in E$ be a solution of (2.1). Multiplying by $u$ and integrating we obtain

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2+\left(V(x)-\frac{g(x,u)}{u}\right)u^2\right)dx=0.$$

It follows that if $V(x) \geq \gamma$ (cf. $(g_4)$) for all $x$, then the only solution $u \in E$ of (2.1) is $u = 0$. This has the following consequences regarding (1.1) and (1.2). If $V \geq \gamma$, then there are no nontrivial solutions $u \in E$ of (1.1). In particular, if $g(x,u) = a_\infty u + g_\infty(x,u)$, where $a_\infty$ is a positive constant and $g_\infty(x,u)u \leq 0$, then $a_\infty = \gamma$. Hence there are no solutions $u \in E \setminus \{0\}$ if $V \geq a_\infty$ while according to Theorem 1.1 there are such solutions if $V(x) < a_\infty(x)$ somewhere, and their number tends to infinity as $\varepsilon \to 0$ provided $g$ is odd in $u$. For (1.2) the situation is different. If $V \geq b_0$ for some $b_0 > 0$, then, when $\lambda$ is large, $\lambda V(x) \geq g(x,u)/u$ and there are no solutions $u \in E \setminus \{0\}$. On the other hand, according to Theorems 1.2 and 1.3, solutions $u \neq 0$ do exist if $V$ changes sign or $V^{-1}(0)$ has nonempty interior. However, in the sign-changing case we have only been able to show the existence of nontrivial solutions for a sequence $\lambda_m \to \infty$ and not for all large $\lambda$.

**Proof of Theorem 1.1** Let $\lambda = 1/\varepsilon^2$ and

$$\Psi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2)\,dx - \int_{\mathbb{R}^N} \lambda G(x,u)\,dx. \tag{3.5}$$

Then critical points of $\Psi_\lambda$ are solutions of (1.1). Replacing $V$ by $\lambda V$ and $g$ by $\lambda g$ in Lemma 3.1 we see that $\Psi_\lambda$ satisfies the Cerami condition if $\lambda > 0$. In view of Proposition 3.3 it suffices to show that for each $k \geq 1$ there exist $\Lambda_k$ and a $k$-dimensional subspace $W$ of $E_\lambda$ such that $\hat{E}_\lambda \cap W = \{0\}$ and

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda(V(x) - a_\infty(x))u^2)\,dx < 0 \text{ for all } u \in (\hat{E}_\lambda \oplus W) \setminus \{0\},\ \lambda > \Lambda_k. \tag{3.6}$$

If $V^- \equiv 0$, then $\hat{E}_\lambda = \{0\}$. Let $W = \text{span}\{\varphi_1, \dots, \varphi_k\}$, where $\varphi_j \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$ whenever $i \neq j$ and $\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^N : V(x) < a_\infty(x)\}$, $j = 1, \dots, k$. Since $W$ is finite-dimensional, it is clear that (3.6) holds for $\lambda$ large enough.

Suppose $V^- \not\equiv 0$. Then $\hat{E}_\lambda \neq \{0\}$ for large $\lambda$ (see Lemma 2.1). Let $W$ be as above and in addition,

$$\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^N : \hat{a}_\infty/20 < V(x) < \hat{a}_\infty/10\}, \quad j = 1, \dots, k. \tag{3.7}$$

Since $W \subset F$, $\hat{E}_\lambda \cap W = \{0\}$. Write $u = v + w \in \hat{E}_\lambda \oplus W$ and let

$$w = \hat{w} + \tilde{w}, \text{ where } \hat{w} \in \hat{E}_\lambda \text{ and } (v, \tilde{w})_2 = 0 \text{ for all } v \in \hat{E}_\lambda.$$

Then

$$\|u\|_2^2 = \|v + \hat{w}\|_2^2 + \|\tilde{w}\|_2^2 \tag{3.8}$$

and since $w \in C_0^\infty(\mathbb{R}^N) \cap F$, we obtain using the notation (2.4) that

$$(A_\lambda w, \hat{w})_2 = a_\lambda(w, \hat{w}) = \langle w, \hat{w}\rangle_\lambda = 0.$$

Hence by (3.7),

$$(A_\lambda w, \tilde{w})_2 = (A_\lambda w, w)_2 = a_\lambda(w,w) \geq \frac{\lambda \hat{a}_\infty}{20}\|w\|_2^2$$

and

$$\|A_\lambda w\|_2^2 = \int_{\mathbb{R}^N} (-\Delta w + \lambda V w)^2\,dx \leq 2\int_{\mathbb{R}^N} ((\Delta w)^2 + (\lambda V w)^2)\,dx \leq \left(b_1 + \frac{\lambda^2 \hat{a}_\infty^2}{50}\right)\|w\|_2^2$$

for some $b_1 > 0$. Combining these inequalities gives

$$\frac{\lambda \hat{a}_\infty}{20}\|w\|_2^2 \le (A_\lambda w, \tilde{w})_2 \le \|A_\lambda w\|_2 \|\tilde{w}\|_2 \le \left(b_1 + \frac{\lambda^2 \hat{a}_\infty^2}{50}\right)^{1/2} \|w\|_2 \|\tilde{w}\|_2.$$

Therefore

$$\|w\|_2 \le \frac{(b_1 + \lambda^2 \hat{a}_\infty^2/50)^{1/2}}{\lambda \hat{a}_\infty/20}\|\tilde{w}\|_2 \le 3\|\tilde{w}\|_2 \tag{3.9}$$

provided $\lambda$ is large enough. Let $b_2$ be a constant such that $\|\nabla w\|_2^2 \le b_2\|w\|_2^2$ for all $w \in W$. Employing (3.7)-(3.9) and the fact that $a_\lambda(v, v) \le 0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda(V - a_\infty)u^2)\, dx &\le a_\lambda(w, w) - \lambda \hat{a}_\infty \|\tilde{w}\|_2^2 \\ \le b_2\|w\|_2^2 + \frac{\lambda \hat{a}_\infty}{10}\|w\|_2^2 - \lambda \hat{a}_\infty \|\tilde{w}\|_2^2 &\le \left(9b_2 - \frac{\lambda \hat{a}_\infty}{10}\right)\|\tilde{w}\|_2^2 < 0\end{aligned}$$

whenever $\lambda$ is sufficiently large and $w \neq 0$. Hence (3.6) is satisfied for $\lambda > \Lambda_k$ and $w \neq 0$. If $v \neq 0$ and $w = 0$, then (3.6) obviously holds. □

**Proof of Theorem 1.2** Let

$$\Phi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2)\, dx - \int_{\mathbb{R}^N} G(x, u)\, dx \tag{3.10}$$

and choose $\Lambda_0$ so that $\gamma = \Lambda_0 b_{\max}$. Then $(g_4)$ holds (with $\lambda V$ replacing $V$) and hence the Cerami condition is satisfied for all $\lambda > \Lambda_0$ according to Lemma 3.1. We need to show that for each $\bar{\lambda} > \Lambda_0$ there exists $\lambda \ge \bar{\lambda}$ such that (1.2) has a nontrivial solution. Since $\mu_j(\lambda) \to 0$ as $\lambda \to \infty$, we may find $k$ such that $\mu_k(\bar{\lambda}) > 1$ and then use Lemma 2.1 in order to obtain $\lambda \ge \bar{\lambda}$ with $1 < \mu_k(\lambda) < 1 + (\lambda\|V^-\|_\infty)^{-1}\hat{a}_\infty$. Let $W = \mathbb{R}e_k(\lambda)$ and $u \in \hat{E}_\lambda \oplus W$. Since $\mu_j(\lambda) \le \mu_k(\lambda)$ for all $j \le k$ and $\langle e_i(\lambda), e_j(\lambda)\rangle_\lambda = \delta_{ij}$, a simple computation shows that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) - a_\infty)u^2)\, dx \le (\lambda(\mu_k(\lambda) - 1)\|V^-\|_\infty - \hat{a}_\infty)\|u\|_2^2 < 0$$

whenever $u \neq 0$. Hence the conclusion follows from Proposition 3.3 with $\hat{E}$ replaced by $\hat{E}_\lambda$. □

**Proof of Theorem 1.3** Let $\Phi_\lambda$ and $\Lambda_0$ be as in the preceding proof and let $W = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$, where $(\varphi_j)$ is an orthonormal set of eigenvalues corresponding to $\mu_j(\Omega)$, $j = 1, \ldots, k$. Since $V \ge 0$, $\hat{E}_\lambda = \{0\}$. Hence

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + (\lambda V - a_\infty)w^2)\, dx = \int_{\mathbb{R}^N} (|\nabla w|^2 - a_\infty w^2)\, dx \le (\mu_k(\Omega) - \hat{a}_\infty)\|w\|_2^2 < 0$$

for all $w \in W \setminus \{0\}$ and the conclusion follows from Proposition 3.3.

**Proof of Theorem 1.4** By Remark 3.2, $\Psi_\lambda$ satisfies the Cerami condition. Since $a_\infty > 0$, $\operatorname{supp} V^-$ has finite measure again and $E_\lambda = \hat{E}_\lambda \oplus F$ as before ($E_\lambda = F$ if $V \ge 0$). Hence the argument of Theorem 1.1 goes through unchanged. □

## 4 Proofs of Theorems 1.5 and 1.6

**Lemma 4.1** *Let $(V_1)$, $(V_2)$, $(g_1)$ and $(g_5)$-$(g_7)$ be satisfied. Then any Cerami sequence for $\Phi$ is bounded.*

**Proof** Let $(u_m) \subset E$ be a Cerami sequence. Then, for large $m$ and some $C > 0$,

$$C \geq \Phi(u_m) - \frac{1}{2}\Phi'(u_m)u_m = \int_{\mathbb{R}^N} \widetilde{\mathcal{G}}(x, u_m)\, dx. \tag{4.1}$$

Set

$$g(r) := \inf\left\{\widetilde{\mathcal{G}}(x,u) : x \in \mathbb{R}^N,\ |u| \geq r\right\}.$$

By $(g_6)$ and Lemma 2.2, $g(r) > 0$ for all $r > 0$ and $g(r) \to \infty$ as $r \to \infty$. For $0 < \alpha < \beta$, let

$$\Omega_m(\alpha,\beta) := \left\{x \in \mathbb{R}^N : \alpha \leq |u_m(x)| < \beta\right\}$$

and

$$c_\alpha^\beta := \inf\left\{\widetilde{\mathcal{G}}(x,u)/u^2 : x \in \mathbb{R}^N,\ \alpha \leq |u| < \beta\right\}.$$

One has

$$\widetilde{\mathcal{G}}(x, u_m(x)) \geq c_\alpha^\beta u_m(x)^2 \text{ for all } x \in \Omega_m(\alpha,\beta).$$

It follows from (4.1) that

$$\begin{aligned} C &\geq \int_{\Omega_m(0,\alpha)} \widetilde{\mathcal{G}}(x,u_m)\, dx + \int_{\Omega_m(\alpha,\beta)} \widetilde{\mathcal{G}}(x,u_m)\, dx + \int_{\Omega_m(\beta,\infty)} \widetilde{\mathcal{G}}(x,u_m)\, dx \\ &\geq \int_{\Omega_m(0,\alpha)} \widetilde{\mathcal{G}}(x,u_m)\, dx + c_\alpha^\beta \int_{\Omega_m(\alpha,\beta)} u_m^2\, dx + g(\beta)|\Omega_m(\beta,\infty)|. \end{aligned} \tag{4.2}$$

Arguing indirectly, assume $\|u_m\| \to \infty$. Set $v_m := u_m/\|u_m\|$. Then $\|v_m\| = 1$ and $\|v_m\|_s \leq C_s$ for all $s \in [2, 2^*)$. Using (4.2),

$$|\Omega_m(\beta,\infty)| \leq \frac{C}{g(\beta)} \to 0 \text{ uniformly in } m \text{ as } \beta \to \infty, \tag{4.3}$$

and, for any fixed $0 < \alpha < \beta$,

$$\int_{\Omega_m(\alpha,\beta)} v_m^2\, dx = \frac{1}{\|u_m\|^2}\int_{\Omega_m(\alpha,\beta)} u_m^2\, dx \leq \frac{C}{c_\alpha^\beta \|u_m\|^2} \to 0 \text{ as } m \to \infty. \tag{4.4}$$

It follows from (4.3) and the Hölder inequality that for any $s \in [2, 2^*)$, $p \in (s, 2^*)$ and a suitable constant $c_1$,

$$\int_{\Omega_m(\beta,\infty)} |v_m|^s\, dx \leq c_1 |\Omega_m(\beta,\infty)|^{(p-s)/p} \to 0 \text{ uniformly in } m \text{ as } \beta \to \infty. \tag{4.5}$$

Write $u = \hat{u} + u^+$, $\hat{u} \in \hat{E}$, $u^+ \in E^+ \oplus F$, and similarly for $v$. Then

$$\|v_m^+\|_s \leq C_s \|v_m^+\| \leq C_s \|v_m\| = C_s \tag{4.6}$$

and

$$\|u_m\|^{-2}\Phi'(u_m)u_m^+ = a(v_m^+, v_m^+) - \int_{\mathbb{R}^N} \frac{g(x,u_m)}{u_m} v_m v_m^+ \, dx \to 0. \tag{4.7}$$

Let $\varepsilon > 0$. By $(g_1)$ there is $a_\varepsilon > 0$ such that $|g(x,u)| \le \frac{\varepsilon}{C_2^2}|u|$ for all $|u| \le a_\varepsilon$. Consequently,

$$\int_{\Omega_m(0,a_\varepsilon)} \frac{g(x,u_m)}{u_m}|v_m||v_m^+|\, dx \le \int_{\Omega_m(0,a_\varepsilon)} \frac{\varepsilon}{C_2^2}|v_m||v_m^+|\, dx \le \frac{\varepsilon}{C_2^2}\|v_m\|_2\|v_m^+\|_2 \le \varepsilon \tag{4.8}$$

for all $m$. By $(g_7)$, (4.2), (4.5) and (4.6) with $s = 2\tau/(\tau-1) \in (2, 2^*)$, we can take $b_\varepsilon$ so large that

$$\begin{aligned} &\int_{\Omega_m(b_\varepsilon,\infty)} \frac{g(x,u_m)}{u_m}|v_m||v_m^+|\, dx \\ &\le \left(\int_{\Omega_m(b_\varepsilon,\infty)} \left|\frac{g(x,u_m)}{u_m}\right|^\tau dx\right)^{1/\tau} \left(\int_{\Omega_m(b_\varepsilon,\infty)} |v_m|^s\, dx\right)^{1/s} \left(\int_{\Omega_m(b_\varepsilon,\infty)} |v_m^+|^s\, dx\right)^{1/s} \\ &\le \left(\int_{\Omega_m(b_\varepsilon,\infty)} a_1\widetilde{\mathcal{G}}(x,u_m)\, dx\right)^{1/\tau} \left(\int_{\Omega_m(b_\varepsilon,\infty)} |v_m|^s\, dx\right)^{1/s} \left(\int_{\Omega_m(b_\varepsilon,\infty)} |v_m^+|^s\, dx\right)^{1/s} < \varepsilon \end{aligned} \tag{4.9}$$

for all $m$. By (4.4) and (4.6) there are constants $c_2$ and $m_0$ such that

$$\int_{\Omega_m(a_\varepsilon,b_\varepsilon)} \frac{g(x,u_m)}{u_m}|v_m||v_m^+|\, dx \le c_2\int_{\Omega_m(a_\varepsilon,b_\varepsilon)} |v_m|\,|v_m^+|\, dx \le c_2\|v_m\|_{2,\Omega_m(a_\varepsilon,b_\varepsilon)}\|v_m^+\|_2 < \varepsilon \tag{4.10}$$

for all $m \ge m_0$. Now a combination of (4.8), (4.9) and (4.10) implies that for $m \ge m_0$,

$$\int_{\mathbb{R}^N} \frac{g(x,u_m)}{u_m} v_m v_m^+\, dx < 3\varepsilon.$$

Since the quadratic form $a$ is positive definite on $E^+ \oplus F$ and $\varepsilon$ has been chosen arbitrarily, it follows from (4.7) that $v_m^+ \to 0$ in $E$. Hence, passing to a subsequence, $v_m \to v \ne 0$ in $E$ (recall $\dim \hat{E} < \infty$). By (4.1), $(g_6)$, Lemma 2.2 and Fatou's lemma,

$$C \ge \int_{\mathbb{R}^N} \widetilde{\mathcal{G}}(x,u_m)\, dx \ge \int_{v\ne 0} \widetilde{\mathcal{G}}(x,u_m)\, dx \to \infty,$$

a contradiction. □

**Lemma 4.2** *Suppose* $(V_1)$, $(V_2)$, *(1.3) are satisfied and let* $(u_m)$ *be a Palais-Smale sequence for* $\Phi$ *such that* $\Phi(u_m) \to c$ *and* $u_m \rightharpoonup u$. *Then, passing to a subsequence, there exists a sequence* $v_m \to u$ *such that*

$$\Phi(u_m - v_m) \to c - \Phi(u) \quad \textit{and} \quad \Phi'(u_m - v_m) \to 0. \tag{4.11}$$

This result is essentially due to Ackermann [1]. Since the setting in [1] is rather different from ours, for the reader's convenience we prove Lemma 4.2 (in fact in a slightly more general form) in Appendix. It is clear that the conclusion of this lemma remains valid also for the functionals $\Phi_\lambda$ and $\Psi_\lambda$ defined respectively in (3.10) and (3.5).

**Lemma 4.3** *Let* $(V_1)$, $(V_2)$, $(g_1)$ *and* $(g_5)$-$(g_7)$ *be satisfied. Then for any* $M > 0$ *there is* $\Lambda = \Lambda(M) > 0$ *such that* $\Phi_\lambda$ *satisfies the* $(C)_c$*-condition for all* $c \le M$ *and* $\lambda \ge \Lambda$, *and* $\Psi_\lambda$ *for all* $c \le M\lambda^{-\alpha}$ *and* $\lambda \ge \Lambda$, *where* $\alpha > 2/(2^* - 2)$ *if* $N \ge 3$ *and* $\alpha > 0$ *if* $N = 1$ *or* $2$.

**Proof** First we consider $\Phi_\lambda$. Let $(u_m)$ be a $(C)_c$-sequence with $c \le M$. By Lemma 4.1, $(u_m)$ is bounded in $E_\lambda$, hence passing to a subsequence we may assume that $u_m \rightharpoonup u$ and $\Phi_\lambda$ satisfies (4.11). Let $w_m := u_m - v_m$. Since $V(x) < b$ on a set of finite measure and $w_m \rightharpoonup 0$,

$$\|w_m\|_2^2 = \int_{V(x)\ge b} w_m^2\,dx + \int_{V(x)<b} w_m^2\,dx \le \frac{1}{\lambda b}\|w_m\|_\lambda^2 + o(1); \tag{4.12}$$

moreover, if $2 < s < p < 2^*$, then by (4.12) and the Hölder and Sobolev inequalities,

$$\|w_m\|_s^s \le \|w_m\|_2^{2(p-s)/(p-2)}\|w_m\|_p^{p(s-2)/(p-2)} \le d_1(\lambda b)^{-(p-s)/(p-2)}\|w_m\|_\lambda^s + o(1), \tag{4.13}$$

where the constant $d_1$ is independent of $w_m$.

It is clear that given $\varepsilon > 0$, there is $\delta > 0$ such that $|g(x,u)| \le \varepsilon|u|$ for all $x \in \mathbb{R}^N$ and $|u| \le \delta$, and $(g_7)$ is satisfied for $|u| \ge \delta$ (with the same $\tau$ but possibly larger $a_1$). It follows from (4.12) that

$$\int_{|w_m|\le\delta} g(x,w_m)w_m\,dx \le \varepsilon\int_{|w_m|\le\delta} w_m^2\,dx \le \frac{\varepsilon}{\lambda b}\|w_m\|_\lambda^2 + o(1). \tag{4.14}$$

By (4.11),

$$\Phi_\lambda(w_m) - \frac{1}{2}\Phi'_\lambda(w_m)w_m = \int_{\mathbb{R}^N} \widetilde{\mathcal{G}}(x,w_m)\,dx \to c - \Phi_\lambda(u). \tag{4.15}$$

Using $(g_7)$, (4.13) with $s = 2\tau/(\tau-1)$ and (4.15), we obtain (cf. (4.9))

$$\begin{aligned}\int_{|w_m|>\delta} g(x,w_m)w_m\,dx &\le \left(\int_{|w_m|>\delta} a_1\widetilde{\mathcal{G}}(x,w_m)\,dx\right)^{1/\tau}\|w_m\|_s^2 \\ &\le a_1^{1/\tau}\,(c-\Phi_\lambda(u))^{1/\tau}\,\|w_m\|_s^2 + o(1) \le d_2M^{1/\tau}(\lambda b)^{-\theta}\|w_m\|_\lambda^2 + o(1),\end{aligned} \tag{4.16}$$

where $\theta = \frac{2(p-s)}{s(p-2)} > 0$. Now by (4.14), (4.16), and since $\Phi'_\lambda(w_m)w_m \to 0$ and $w_m \to 0$ in $L^2(\{x \in \mathbb{R}^N : V(x) < 0\})$,

$$o(1) = \|w_m\|_\lambda^2 - \int_{\mathbb{R}^N} g(x,w_m)w_m\,dx \ge \left(1 - \frac{\varepsilon}{\lambda b} - \frac{d_2M^{1/\tau}}{(\lambda b)^\theta}\right)\|w_m\|_\lambda^2 + o(1).$$

Letting $\Lambda = \Lambda(M)$ be so large that the term in the brackets above is positive when $\lambda \ge \Lambda$, we get $w_m \to 0$ in $E_\lambda$. Since $w_m = u_m - v_m$ and $v_m \to u$, it follows that also $u_m \to u$.

For $\Psi_\lambda$ we still have (4.11) and (4.14) but since $g$ has been replaced by $\lambda g$, the integrand in (4.15) is $\lambda\widetilde{\mathcal{G}}$ and (4.16) becomes

$$\int_{|w_m|>\delta} g(x,w_m)w_m\,dx \le d_2(M\lambda^{-\alpha})^{1/\tau}\lambda^{-1/\tau}(\lambda b)^{-\theta}\|w_m\|_\lambda^2 + o(1).$$

Hence

$$o(1) = \|w_m\|_\lambda^2 - \int_{\mathbb{R}^N} \lambda g(x,w_m)w_m\,dx + o(1) \tag{4.17}$$
$$\geq \left(1 - \frac{\varepsilon}{b} - \frac{d_2\lambda^{1-1/\tau}(M\lambda^{-\alpha})^{1/\tau}}{(\lambda b)^\theta}\right)\|w_m\|_\lambda^2 + o(1) = \left(1 - \frac{\varepsilon}{b} - d_3\lambda^\beta\right)\|w_m\|_\lambda^2 + o(1),$$

where $\beta = \frac{2-\alpha(p-2)}{\tau(p-2)}$. If $N \geq 3$, then $\alpha > 2/(2^* - 2)$; therefore $p \in (s, 2^*)$ may be chosen so that $\alpha > 2/(p-2)$ and hence $\beta < 0$. If $N = 1$ or 2 and $p > s$ is sufficiently large, we have $\alpha > 2/(p-2)$ and $\beta < 0$ again. Consequently, (4.17) will be satisfied for all $\lambda$ large provided $\varepsilon < b$, and it follows that $w_m \to 0$ and $u_m \to u$. □

**Proof of Theorem 1.5** Let $W = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$, where $\varphi_j \in C_0^\infty(\Omega)$ have disjoint supports. Then $W \subset F$. By Lemma 3.4, $\Phi_\lambda((E_\lambda^+ \oplus F) \cap S_\rho) \geq \kappa$ and $\Psi_\lambda((E_\lambda^+ \oplus F) \cap S_\rho) \geq \kappa$ ($\rho$ and $\kappa$ may depend on $\lambda$). We consider $\Phi_\lambda$ first and then point out the differences for $\Psi_\lambda$. According to Propositions 2.3, 2.4, Lemma 4.3 and since $G \geq 0$, it suffices to show that $\sup \Phi_\lambda(\hat{E}_\lambda \oplus W)$ is bounded above by a constant independent of $\lambda$ and there exists $R > 0$ (possibly depending on $\lambda$) such that $\Phi_\lambda(u) \leq 0$ whenever $u \in \hat{E}_\lambda \oplus W$ and $\|u\|_\lambda \geq R$. By $(g_5)$, for each $\eta > 0$ there is $r_\eta > 0$ such that $G(x,u) \geq \frac{1}{2}\eta u^2$ if $|u| \geq r_\eta$. Let $u = v + w \in \hat{E}_\lambda \oplus W$. Then

$$\Phi_\lambda(u) \leq \frac{1}{2}a_\lambda(u,u) - \int_\Omega G(x,u)\,dx \leq \frac{1}{2}a_\lambda(w,w) - \frac{1}{2}\eta\|u\|_{2,\Omega}^2 + \int_\Omega \left(\frac{1}{2}\eta u^2 - G(x,u)\right)dx \tag{4.18}$$
$$= \frac{1}{2}\|\nabla w\|_2^2 - \frac{1}{2}\eta\|u\|_{2,\Omega}^2 + \int_\Omega \left(\frac{1}{2}\eta u^2 - G(x,u)\right)dx \leq \frac{1}{2}\|\nabla w\|_2^2 - \frac{1}{2}\eta\|u\|_{2,\Omega}^2 + C_\eta,$$

where $C_\eta$ depends on $\eta$ but not $\lambda$. Furthermore, since $w \in C_0^\infty(\Omega)$ and $a_\lambda(v,w) = 0$,

$$\|\nabla w\|_2^2 = a_\lambda(w,u) = \int_\Omega (-\Delta w)u\,dx \leq \|\Delta w\|_2\|u\|_{2,\Omega} \leq b_0\|\nabla w\|_2\|u\|_{2,\Omega} \leq \frac{b_0^2}{2\eta}\|\nabla w\|_2^2 + \frac{\eta}{2}\|u\|_{2,\Omega}^2$$

($b_0$ is a constant depending on $W$). Choosing $\eta \geq b_0^2$, we obtain $\|\nabla w\|_2^2 \leq \eta\|u\|_{2,\Omega}^2$ and it follows from (4.18) that $\Phi_\lambda(u) \leq C_\eta$. That $\Phi_\lambda(u) \leq 0$ for $\|u\|_\lambda$ large enough is an easy consequence of $(g_5)$ and the finite-dimensionality of $\hat{E}_\lambda \oplus W$.

Now we turn to $\Psi_\lambda$. We still have (4.18) with $\lambda G$ replacing $G$, hence $\Psi_\lambda(u) \leq C_{\eta,\lambda}$, where $C_{\eta,\lambda}$ is an upper bound for $\int_\Omega \left(\frac{1}{2}\eta u^2 - \lambda G(x,u)\right)dx$. So we need to show that $C_{\eta,\lambda} \leq M\lambda^{-\alpha}$ ($\alpha$ is as in Lemma 4.3). Since $G \geq 0$,

$$\int_{\Omega\cap\{|u|\leq\lambda^{-\alpha/2}\}} \left(\frac{1}{2}\eta u^2 - \lambda G(x,u)\right)dx \leq c_1\lambda^{-\alpha} \tag{4.19}$$

for some constant $c_1$ ($\eta \geq b_0^2$ fixed). Moreover, $(g_6)$ and the condition on $G$ for $u$ close to 0 imply $G(x,u) \geq c_2|u|^\delta$ whenever $|u| \leq 1$ ($c_2 > 0$). Hence if $\lambda^{-\alpha/2} \leq |u| \leq 1$, then

$$\frac{1}{2}\eta u^2 - \lambda G(x,u) \leq \frac{1}{2}\eta u^2 - \lambda c_2|u|^\delta \leq u^2\left(\frac{1}{2}\eta - c_2\lambda^{1-\alpha(\delta-2)/2}\right).$$

Since $\delta < 2^*$, we may choose $\alpha > 2/(2^* - 2)$ so that $1 - \alpha(\delta-2)/2 > 0$. Hence the right-hand side above is $\leq 0$ for large $\lambda$. Obviously, $\frac{1}{2}\eta u^2 - \lambda G(x,u) \leq 0$ if $|u| \geq 1$ and $\lambda$ is large enough. It follows that $\Psi_\lambda(u) \leq c_1\lambda^{-\alpha}$, where $c_1$ is as in (4.19). □

**Proof of Theorem 1.6** We combine the arguments of Theorems 1.2 and 1.5. Given $\bar{\lambda} > 0$ there exists $\lambda \geq \bar{\lambda}$ such that $1 < \mu_k(\lambda) < 1 + 1/\lambda$ for some $k$. Let $W = \mathbb{R}e_k(\lambda)$, $u = v + w \in \hat{E}_\lambda \oplus W$ and recall $\int_{\mathbb{R}^N} V^- u^2\,dx = \int_{\mathbb{R}^N} V^- v^2\,dx + \int_{\mathbb{R}^N} V^- w^2\,dx$ by the orthogonality of $\hat{E}_\lambda$ and $W$. Hence (cf. (4.18))

$$\begin{aligned}\Phi_\lambda(u) &\leq \frac{1}{2}a_\lambda(w,w) - \int_{\operatorname{supp} V^-} G(x,u)\,dx = \frac{1}{2}\lambda(\mu_k(\lambda)-1)\int_{\mathbb{R}^N} V^- w^2\,dx - \frac{1}{2}\eta\|u\|^2_{2,\operatorname{supp} V^-} \\ &\quad + \int_{\operatorname{supp} V^-}\left(\frac{1}{2}\eta u^2 - G(x,u)\right)dx \leq \frac{1}{2}\int_{\mathbb{R}^N} V^- w^2\,dx - \frac{\eta}{2\|V^-\|_\infty}\int_{\mathbb{R}^N} V^- u^2\,dx + C_\eta \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} V^- w^2\,dx - \frac{\eta}{2\|V^-\|_\infty}\int_{\mathbb{R}^N} V^- w^2\,dx + C_\eta,\end{aligned}$$

where $C_\eta$ is independent of $\lambda$. Choosing $\eta \geq 1/\|V^-\|_\infty$, we obtain $\Phi_\lambda(u) \leq C_\eta$. If $\bar{\lambda}$ is large enough, then $\Phi_\lambda$ satisfies $(C)_c$ for all $c \leq C_\eta$. As in the proof of Theorem 1.5, $\Phi_\lambda \leq 0$ outside a large ball. Hence $\Phi_\lambda$ has a critical point $u \neq 0$ according to Proposition 2.3.

For $\Psi_\lambda$ we show as in the preceding proof that $C_{\eta,\lambda} \leq c_1\lambda^{-\alpha}$ whenever $\lambda \geq \bar{\lambda}$ and $\bar{\lambda}$ is large enough. □

## 5 Remarks on concentration of solutions

In this section we consider (1.2) with $\lambda = \lambda_m \to \infty$. As before, we denote the interior of $V^{-1}(0)$ by $\Omega$ (we do not exclude the case $\Omega = \emptyset$).

**Theorem 5.1** *Suppose* $(V_1)$, $(V_2)$ *and (1.3) with* $p \in (2, 2^*)$ *are satisfied, and let* $u_m \in E$ *be a solution of (1.2) with* $\lambda = \lambda_m$. *If* $\lambda_m \to \infty$ *and* $\|u_m\|_{\lambda_m} \leq C$ *for some* $C > 0$ *and all* $m$, *then, passing to a subsequence,* $u_m \to \bar{u}$ *in* $L^q(\mathbb{R}^N)$ *for all* $q \in (2, 2^*)$, $\bar{u}$ *is a weak solution of the equation*

$$-\Delta u = g(x,u), \quad x \in \Omega$$

*and* $\bar{u} = 0$ *a.e. in* $\mathbb{R}^N \setminus V^{-1}(0)$. *If in addition* $V \geq 0$ *and* $(g_1)$ *is satisfied, then* $u_m \to \bar{u}$ *in* $E$.

Note that if $\Omega = \emptyset$, then the conclusion of the theorem is that $u_m \to 0$ in $L^q(\mathbb{R}^N)$ (and in $E$ provided $V \geq 0$). Note also that if $V^{-1}(0) = \bar{\Omega}$ and $\partial\Omega$ is locally Lipschitz continuous, then $\bar{u} \in H_0^1(\Omega)$ (cf. [4]).

**Proof** We modify an argument in [4]. Since $\|u_m\| \leq \|u_m\|_{\lambda_m} \leq C$, $u_m \rightharpoonup \bar{u}$ in $E$ and $u_m \to \bar{u}$ in $L^r_{loc}(\mathbb{R}^N)$ for $2 \leq r < 2^*$ after passing to a subsequence. Since $\Phi'_{\lambda_m}(u_m)\varphi = 0$ and (1.3) holds, it follows easily that $\int_{\mathbb{R}^N} V(x)u_m\varphi\,dx \to 0$ and hence $\int_{\mathbb{R}^N} V(x)\bar{u}\varphi\,dx = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore $\bar{u} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$ and $-\Delta\bar{u} = g(x,\bar{u})$ in $\Omega$.

Next we show that $u_m \to \bar{u}$ in $L^q(\mathbb{R}^N)$. Assuming the contrary, it follows from P.L. Lions' vanishing lemma [13, Lemma I.1], [20, Lemma 1.21] that

$$\int_{B_\rho(x_m)} (u_m - \bar{u})^2\,dx \geq \gamma$$

for some $(x_m) \subset \mathbb{R}^N$, $\rho, \gamma > 0$ and almost all $m$. Moreover, $|x_m| \to \infty$ (for otherwise the left-hand side above tends to 0 as $m \to \infty$). Hence $\left|B_\rho(x_m) \cap \{x \in \mathbb{R}^N : V(x) < b\}\right| \to 0$ and $\int_{B_\rho(x_m)} \bar{u}^2\, dx \to 0$. Consequently,

$$\|u_m\|_{\lambda_m}^2 \geq \lambda_m b \int_{B_\rho(x_m)\cap\{V\geq b\}} u_m^2\, dx = \lambda_m b \left( \int_{B_\rho(x_m)} (u_m - \bar{u})^2\, dx + o(1) \right) \to \infty,$$

a contradiction.

Suppose $V \geq 0$ and $(g_1)$ holds. Since $\Phi'_{\lambda_m}(u_m)u_m = \Phi'_{\lambda_m}(u_m)\bar{u} = 0$,

$$\|u_m\|^2 \leq \|u_m\|_{\lambda_m}^2 = \int_{\mathbb{R}^N} g(x, u_m)u_m\, dx \tag{5.1}$$

and

$$\langle u_m, \bar{u}\rangle_{\lambda_m} = \int_{\mathbb{R}^N} g(x, u_m)\bar{u}\, dx. \tag{5.2}$$

Letting $m \to \infty$ in (5.2) and recalling that $\bar{u}(x) = 0$ if $V(x) > 0$, we obtain

$$\|\bar{u}\|^2 = \int_{\mathbb{R}^N} g(x, \bar{u})\bar{u}\, dx. \tag{5.3}$$

We claim that

$$\int_{\mathbb{R}^N} g(x, u_m)u_m\, dx \to \int_{\mathbb{R}^N} g(x, \bar{u})\bar{u}\, dx \tag{5.4}$$

(which is not obvious because we do not know whether $u_m \to \bar{u}$ in $L^2(\mathbb{R}^N)$). Assuming this, it follows from (5.1) and (5.3) that $\limsup_{m\to\infty} \|u_m\| \leq \|\bar{u}\|^2$; hence $u_m \to \bar{u}$ in $E$.

It remains to prove (5.4). We have

$$\int_{\mathbb{R}^N} |g(x, u_m)u_m - g(x, \bar{u})\bar{u}|\, dx \leq \int_{\mathbb{R}^N} |g(x, u_m)|\, |u_m - \bar{u}|\, dx + \int_{\mathbb{R}^N} |g(x, u_m) - g(x, \bar{u})|\, |\bar{u}|\, dx,$$

and since $u_m \to \bar{u}$ in $L^r_{loc}(\mathbb{R}^N)$, $2 \leq r < 2^*$, it is easy to see that the second integral on the right-hand side above tends to 0. Since for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|g(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1}$,

$$\int_{\mathbb{R}^N} |g(x, u_m)|\, |u_m - \bar{u}|\, dx \leq \varepsilon \int_{\mathbb{R}^N} |u_m|\, |u_m - \bar{u}|\, dx + C_\varepsilon \int_{\mathbb{R}^N} |u_m|^{p-1}\, |u_m - \bar{u}|\, dx,$$

and the conclusion follows from the Hölder inequality because $u_m \to \bar{u}$ in $L^p(\mathbb{R}^N)$, $(u_m)$ is bounded in $L^2(\mathbb{R}^N)$ and $\varepsilon$ has been chosen arbitrarily. □

Although the result above is not very satisfactory (it says nothing about the strong convergence in $E$ unless $V \geq 0$ and gives no conditions for $\bar{u}$ to be different from 0), it has some interesting consequences which we would like to point out. Under the assumptions of Theorem 5.1 (with possibly sign-changing $V$) we have $\Phi'_{\lambda_m}(u_m)u_m = 0$, hence if $(g_1)$ holds,

$$|a_{\lambda_m}(u_m, u_m)| \leq \int_{\mathbb{R}^N} |g(x, u_m)u_m|\, dx \leq \varepsilon\|u_m\|_2^2 + C_\varepsilon\|u_m\|_p^p.$$

If $\bar{u} = 0$, then $u_m \to 0$ in $L^p(\mathbb{R}^N)$ and therefore $a_{\lambda_m}(u_m, u_m) \to 0$ (recall $\|u_m\|_2$ is bounded and $\varepsilon$ arbitrary). Now it follows easily that $\Phi_{\lambda_m}(u_m) \to 0$. Hence if $\Phi_{\lambda_m}(u_m)$ is bounded away from 0, we either have $\|u_m\|_{\lambda_m}$ unbounded or $\bar{u} \neq 0$. Moreover, the second possibility cannot occur when $\Omega = \emptyset$.

Suppose now $V \geq 0$ and $g(x, u) = |u|^{p-2}u$, $2 < p < 2^*$. Then

$$\Phi_{\lambda_m}(u_m) = \frac{1}{2}\|u_m\|^2_{\lambda_m} - \frac{1}{p}\|u_m\|^p_p$$

and

$$\Phi_{\lambda_m}(u_m) = \Phi_{\lambda_m}(u_m) - \frac{1}{2}\Phi'_{\lambda_m}(u_m)u_m = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_m\|^p_p.$$

Hence if $\Phi_{\lambda_m}(u_m)$ is bounded, so is $\|u_m\|_p$, and therefore also $\|u_m\|_{\lambda_m}$. Passing to a subsequence, $u_m \to \bar{u}$ in $E$; moreover, $\bar{u} \neq 0$ whenever $\Phi_{\lambda_m}(u_m)$ is bounded away from 0. In particular, $\Omega \neq \emptyset$ in this case. It is easy to see from the proof of Theorem 1.5 that for a fixed $k$, $\sup \Phi_\lambda(W) \leq C$ and $\inf \Phi_\lambda(S_\rho) \geq \kappa$, where $C, \kappa, \rho$ are positive constants independent of $\lambda$. So if $(u_m)$ is a sequence of solutions of (1.2) (with $\lambda = \lambda_m \to \infty$) obtained with the aid of Theorem 1.5, then, up to a subsequence, $u_m \to \bar{u} \neq 0$ in $E$ and $-\Delta\bar{u} = |\bar{u}|^{p-2}\bar{u}$ in $\Omega$ (this conclusion may also be found in [9]). On the other hand, if $\Omega = \emptyset$, there can be no sequence of solutions $(u_m)$ with $\Phi_{\lambda_m}(u_m) \in [\kappa, C]$ while in view of Theorem 1.6 such sequence necessarily exists provided $V$ changes sign. This does not exclude the possibility of having nontrivial solutions to (1.2) also when $V \geq 0$, regardless of whether $\Omega$ is empty or not. In fact it has been shown in [9] that for such $V$ and each $\lambda > 0$ (1.2) has a solution $u_\lambda \neq 0$ such that $\Phi_\lambda(u_\lambda) \to \infty$. The proof uses a compactness argument which heavily relies on the fact that $g(x, u) = |u|^{p-2}u$ and it is not clear whether it can be made applicable to the larger class of nonlinearities considered in this paper.

# A Appendix

In this appendix we prove a convergence result for the Nemytskii operator. As an easy consequence of it we obtain Lemma 4.2 in a somewhat more general form (see Corollary A.3 and Remark A.4 below).

**Proposition A.1** *Let $\Omega$ be an open set in $\mathbb{R}^N$ and $f \in C(\Omega\times\mathbb{R}, \mathbb{R})$ a function such that $|f(x, u)| \leq a|u|^s$ for some $a > 0$ and $s \in [1, \infty)$. Suppose $s \leq p < \infty$, $p > 1$, $(u_m)$ is a bounded sequence in $L^p(\Omega)$, $u_m \to u$ a.e. in $\Omega$ and in $L^p(\Omega \cap B_R)$ for all $R > 0$. Then, passing to a subsequence, there exists a sequence $v_m \to u$ in $L^p(\Omega)$ such that*

$$f(x, u_m) - f(x, u_m - v_m) - f(x, u) \to 0 \text{ in } L^{p/s}(\Omega). \tag{A.1}$$

Under somewhat more restrictive assumptions on $f$ it can be shown that $v_m$ in (A.1) can be replaced by $u$. We do not know whether this can be done under the hypotheses of Proposition A.1.

**Proof** We adapt an argument in [1]. Since $(u_m)$ is bounded in $L^p(\Omega)$ and $u_m \to u$ a.e., $u_m \rightharpoonup u$ in $L^p(\Omega)$ [19, Theorem 10.36]. We claim that there is a subsequence $(u_{m_j})$ of $(u_m)$ and a sequence $R_{m_j} \to \infty$ such that for each $\varepsilon > 0$, each $R \geq R(\varepsilon)$ and $j \geq j(\varepsilon)$,

$$\int_{\Omega\cap B_{R_{m_j}}\setminus B_R} |u_{m_j}|^p\, dx \leq \varepsilon. \tag{A.2}$$

Since $u_m \to u$ in $L^p(\Omega \cap B_R)$, it follows that for each fixed $j \geq 1$ and almost all $m$,

(A.3) $$\int_{\Omega \cap B_j} (|u_m|^p - |u|^p)\, dx \leq \frac{1}{j}.$$

Hence there exists $m_j \geq j$ such that the above inequality holds, and we may assume $m_j < m_{j+1}$ for all $j$. Choose $R$ so that

(A.4) $$\int_{\Omega \setminus B_R} |u|^p\, dx \leq \frac{\varepsilon}{2}.$$

Then, writing $R_{m_j} = j$,

$$\int_{\Omega \cap B_{R_{m_j}} \setminus B_R} |u_{m_j}|^p\, dx = \int_{\Omega \cap B_{R_{m_j}}} (|u_{m_j}|^p - |u|^p)\, dx$$
$$+ \int_{\Omega \cap B_{R_{m_j}} \setminus B_R} |u|^p\, dx + \int_{\Omega \cap B_R} (|u|^p - (|u_{m_j}|^p)\, dx$$

whenever $R_{m_j} > R$, and (A.2) is a consequence of (A.3), (A.4) and the fact that the last term on the right-hand side above tends to 0 as $j \to \infty$.

Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be a function such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$ and set $v_{m_j}(x) := \chi(2|x|/R_{m_j})u(x)$. Clearly, $v_{m_j} \to u$ in $L^p(\Omega)$. By the continuity of the Nemytskii operator,

(A.5) $$f(x, u_{m_j}) - f(x, u_{m_j} - v_{m_j}) - f(x, u) \to 0 \text{ in } L^{p/s}(\Omega \cap B_R).$$

Furthermore,

$$\|f(x, u_{m_j}) - f(x, u_{m_j} - v_{m_j}) - f(x, u)\|_{L^{p/s}(\Omega \setminus B_R)}$$
$$\leq \|f(x, u_{m_j}) - f(x, u_{m_j} - v_{m_j}) - f(x, v_{m_j})\|_{L^{p/s}(\Omega \setminus B_R)} + \|f(x, v_{m_j}) - f(x, u)\|_{L^{p/s}(\Omega \setminus B_R)}.$$

Since $v_{m_j} \to u$ in $L^p(\Omega)$, the second term on the right-hand side above tends to 0 by the continuity of the Nemytskii operator. Since $|v_{m_j}| \leq |u|$ and $\operatorname{supp} v_{m_j} \subset \bar{B}_{R_{m_j}}$,

$$|f(x, u_{m_j}) - f(x, u_{m_j} - v_{m_j}) - f(x, v_{m_j})|^{p/s} \leq a^{p/s}(|u_{m_j}|^s + |u_{m_j} - v_{m_j}|^s + |v_{m_j}|^s)^{p/s}$$
$$\leq b(|u_{m_j}|^p + |u|^p)$$

for some constant $b > 0$ and the left-hand side above is 0 for $x \in \Omega \setminus B_{R_{m_j}}$. Hence the conclusion follows from (A.2), (A.4) and (A.5). □

Recall that $L^p(\Omega) + L^q(\Omega)$ is the space of functions $u$ such that $u = u_1 + u_2$, $u_1 \in L^p(\Omega)$, $u_2 \in L^q(\Omega)$, normed by $\|u\|_{p \vee q} = \inf\{\|u_1\|_p + \|u_2\|_q\}$, where the infimum is taken with respect to all decompositions $u = u_1 + u_2$ as above.

**Theorem A.2** *Let $\Omega$ be an open set in $\mathbb{R}^N$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ a function such that $|f(x,u)| \leq a(|u|^r + |u|^s)$ for some $1 \leq r < s < \infty$. Suppose $s \leq p < \infty$, $r \leq q < \infty$, $q > 1$, $(u_m)$ is a bounded sequence in $L^p(\Omega) \cap L^q(\Omega)$, $u_m \to u$ a.e. in $\Omega$ and in $L^p(\Omega \cap B_R) \cap L^q(\Omega \cap B_R)$ for all $R > 0$. Then, passing to a subsequence, there exists a sequence $v_m \to u$ in $L^p(\Omega) \cap L^q(\Omega)$ such that*

$$f(x, u_m) - f(x, u_m - v_m) - f(x, u) \to 0 \text{ in } L^{q/r}(\Omega) + L^{p/s}(\Omega).$$

**Proof** We follow an argument in [20, Theorem A.4]. Let $\eta \in C(\mathbb{R}, [0,1])$ be such that $\eta(t) = 1$ for $|t| \le 1$ and $\eta(t) = 0$ for $|t| \ge 2$. Set

$$f_1(x,u) := \eta(|u|)f(x,u), \quad f_2(x,u) := (1-\eta(|u|))f(x,u).$$

Then $|f_1(x,u)| \le a_1|u|^r$ and $|f_2(x,u)| \le a_2|u|^s$ for appropriate constants $a_1$, $a_2$, so by Theorem A.1, $f_1(x,u_m) - f_1(x,u_m-v_m) - f_1(x,u) \to 0$ in $L^{q/r}(\Omega)$ and $f_2(x,u_m) - f_2(x,u_m-v_m) - f_2(x,u) \to 0$ in $L^{p/s}(\Omega)$ after passing to a subsequence. Now the conclusion follows if we note that the same subsequence $(u_{m_j})$ can be chosen both for $f_1$ and $f_2$. □

Let $E$ be the Sobolev space defined in Section 2, suppose $L : E \to E$ is a bounded linear selfadjoint operator and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

$$|g(x,u)| \le a(|u| + |u|^{p-1}),$$

where $2 < p < 2^*$. Set $\Phi(u) := \frac{1}{2}\langle Lu, u\rangle - \int_{\mathbb{R}^N} G(x,u)\,dx$.

**Corollary A.3** *Let $u_m \rightharpoonup u$ in $E$. Then, passing to a subsequence, there exists a sequence $v_m \to u$ in $E$ such that*

$$\Phi(u_m) = \Phi(u_m - v_m) + \Phi(u) + o(1)$$

*and*

$$\Phi'(u_m) = \Phi'(u_m - v_m) + \Phi'(u) + o(1)$$

*as $m \to \infty$. In particular, if $(u_m)$ is a $(PS)_c$-sequence, then $\Phi(u_m - v_m) \to c - \Phi(u)$ and $\Phi'(u_m - v_m) \to 0$ after passing to a subsequence.*

**Proof** Passing to a subsequence, $u_m \to u$ a.e. in $\mathbb{R}^N$, $u_m \rightharpoonup u$ in $L^t(\mathbb{R}^N)$ and $u_m \to u$ in $L^t_{loc}(\mathbb{R}^N)$, $2 \le t < 2^*$. Let $f(x,u) = G(x,u)$ in Theorem A.2. Then $r = 2$, $s = p$ and if we set $q = 2$, we obtain

$$\int_{\mathbb{R}^N} G(x,u_m)\,dx = \int_{\mathbb{R}^N} G(x,u_m - v_m)\,dx + \int_{\mathbb{R}^N} G(x,u)\,dx + o(1).$$

Next we let $f(x,u) = g(x,u)$. Then $r = 1$, $s = p-1$ and we may set $q = 2$ again. By Theorem A.2, $g(x,u_m) - g(x,u_m - v_m) - g(x,u) \to 0$ in $L^2(\Omega) + L^{p/(p-1)}(\Omega)$, and therefore

$$\sup_{\|\varphi\|\le 1} \int_{\mathbb{R}^N} g(x,u_m)\varphi\,dx = \sup_{\|\varphi\|\le 1} \int_{\mathbb{R}^N} (g(x,u_m - v_m) + g(x,u))\,\varphi\,dx + o(1).$$

The sequence $(v_m)$ constructed in the proof of Proposition A.1 has the property that $v_m \to u$ in $E$; hence $Lu_m = L(u_m - v_m) + Lu + o(1)$ and the conclusion follows. □

**Remark A.4** (i) The conclusions of Corollary A.3 remain valid if $E \hookrightarrow H^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain with regular boundary (in the sense that the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^t(\Omega)$, $2 \le t < 2^*$, are continuous) and $v_m \to u$ in $E$, where $(v_m)$ is the sequence appearing in the proof of Proposition A.1.
(ii) Since the embedding $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N}) \hookrightarrow L^t(\mathbb{R}, \mathbb{R}^{2N})$ is continuous if $2 \le t < \infty$, the conclusions of Corollary A.3 remain valid for $E = H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$. This is useful when looking for homoclinic solutions of the Hamiltonian systems $\dot{z} = JH_z(z,t)$ in $\mathbb{R}^{2N}$.

## References


[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. 248 (2004), 423–443.

[2] A. Ambrosetti, A. Malchiodi and S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rat. Mech. Anal. 159 (2001), 253–271.

[3] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlin. Anal. 7 (1983), 981–1012.

[4] T. Bartsch, A. Pankov and Z.Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Comm. Contemp. Math. 3 (2001), 549–569.

[5] J. Byeon and L. Jeanjean, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Preprint.

[6] J. Byeon, Z.Q. Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations, II*, Calc. Var. PDE 18 (2003), 207–219.

[7] J. Chabrowski and A. Szulkin, *On the Schrödinger equation involving a critical Sobolev exponent and magnetic field*, Top. Meth. Nonl. Anal. 25 (2005), 3–21.

[8] M. del Pino and P. Felmer, *Local mountain passes for semilinear problems in unbounded domains*, Calc. Var. PDE 4 (1996), 121–137.

[9] Y.H. Ding and A. Szulkin, *Existence and number of solutions for a class of semilinear Schrödinger equations.* In: Nonlinear Differential Equations. A tribute to D.G. de Figueiredo, T. Cazenave et al. eds., Birkhäuser, to appear.

[10] P. Felmer and J.J. Torres, *Semi-classical limit for the one dimensional nonlinear Schrödinger equation*, Comm. Contemp. Math. 4 (2002), 481–512.

[11] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Func. Anal. 69 (1986), 397–408.

[12] L. Jeanjean and K. Tanaka, *Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities*, Calc. Var. PDE 21 (2004), 287–318.

[13] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. Part I*, Ann. IHP, Analyse Non Linéaire 1 (1984), 109–145.

[14] Z.L. Liu, F.A. van Heerden and Z.Q. Wang, *Nodal type bound states of Schrödinger equations via invariant set and minimax methods*, J. Diff. Eq. 214 (2005), 358–390.

[15] Y.G. Oh, *Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class* $(V)_a$, Comm. PDE 13 (1988), 1499–1519.

[16] Y.G. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. 131 (1990), 223–253.

[17] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS 65, Amer. Math. Soc., Providence, R.I., 1986.

[18] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. 43 (1992), 270–291.

[19] M. Willem, *Analyse Harmonique Réelle*, Hermann, Paris, 1995.

[20] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.