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HILBERT POLYNOMIAL FOR A SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

A. G. KHOVANSKII, S. P. CHULKOV

Abstract. Systems of linear partial differential equations with analytic coefficients are considered. Suppose the coefficients of a system are defined in a domain $U \in \mathbb{C}^n$. We study the spaces of germs of formal and analytic solutions of the system at a point u of the domain U . We discuss the following questions. 1) How to pose "proper" initial conditions for formal and analytic solutions for the system. 2) How depend dimensions of the spaces of k-jets of germs of formal and analytic solutions of the system at a point u in the domain U on the positive integer k and the point u.

1. Introduction

Consider a system of linear partial differential equations with analytic coefficients on a one unknown function z in a domain U of the space $\mathbb{C}^n$. We study the spaces of germs of formal and analytic solutions of the system at a point u of the domain U. We discuss the following questions:

1) How to pose "proper" initial conditions for formal and analytic solutions for the system. More precise, how to fix some set of derivatives of unknown function z at the point u such that for any values of fixed derivatives there exists unique formal (analytic) solution with such initial conditions.

2) How depend dimensions of the spaces of k-jets of germs of formal and analytic solutions of the system at a point u in on the positive integer k and the point u.

There are obtained the following results. We show that exists a "bad" hypersurface Σ such that in the complement $U \setminus \Sigma$ the spaces of germs at a point u of formal and analytic solutions of the system do not depend on point u in some sense. More precise, there exists a set of partial derivatives (which set does not depend on u) such that for any values of fixed derivatives there exits unique formal solution with such initial conditions (see Theorem 4.1). A formal solution will be analytic if and only if partial sum of Taylor series consisted only fixed derivatives converges (see Theorem 5.3).

For any point u in $U \setminus \Sigma$ and any positive integer k denote by $A_u(k)$ and $F_u(k)$ the spaces of k-jets at the point u of germs of formal and analytic solutions of the system respectively. We prove that for any nonnegative integer k dimensions of the spaces $A_u(k)$ $F_u(k)$ are coincide and do not depend on the point u (see Corollary 5.4). More over, the function $H(k) = \dim A_u(k) = \dim F_u(k)$ is a polynomial for sufficiently large positive integers k (see Corollary 4.3). The function H has the following geometrical interpretation. To the system corresponds a family of algebraic varieties analytically parameterized by a point u of the domain U. For the values of parameters u from the complement $U \setminus \Sigma$ to the hypersurface Σ the Hilbert functions of these varieties are coincide to each other and the function H (see Paragraph 6.4).

Studied problems are classical and described results are more or less known. The most important works in this area are Riquier's book [1] and V.P. Palamodov's paper [2]. Riquier [1] considered the problem how to pose "proper" initial condition for formal and analytic solutions of a system of partial differential equations. In his distinguished paper, Riquier for the first time introduced a well-ordering on the set of partial derivatives of a function of several variables and use it to make substantial progress in the problem. In the case of linear partial differential equations with constant coefficients, Riquier's method essentially contains what has later been called Gröbner bases and has made a breakthrough in computational aspects of commutative algebra. However, owing to the great generality of the problem stated, Riquier's work does not give definitive answers.

In this paper we apply Riquier's method (now it is called differential Gröbner bases) with some modifications to systems of linear partial differential equations only. For this situation it gives quite good results. We especially interested in corollaries of the existence and uniqueness theorem such as existence of Hilbert polynomial of the system.

The most comprehensive existence and uniqueness theorems for formal and analytic solutions for the case a system of linear partial differential equations were proved by Palamodov [2]. Palamodov proves the theorems of existence and uniqueness for formal and analytic solutions which are strictly stronger than Riquier's method gives: Palamodov proves that "bad" set is smaller than "bad" hypersurface Σ which one can get by Requiter's method. Palamodov's paper [2] is complicated and based on a special technique developed by him for linear differential operators.

Note that Palamodov's results concerning the existence and uniqueness for formal and analytic solutions of a system of linear differential equations one can prove on the following way. First, one can prove the theorem of existence and uniqueness for formal solution of the system. The proof is purely algebraic and not complicated. Second, for studying analytic solutions one use the convergence theorem from [3].

2. The semigroup $\mathbb{Z}^n_{\geqslant 0}$

Consider the semigroup $\mathbb{Z}^n_{\geqslant 0} = \{\alpha = (\alpha_1, \ldots, \alpha_n) | \alpha_i \in \mathbb{Z}, \alpha_i \geqslant 0\}$.

The set $O^n(a) = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} | \exists \beta \in \mathbb{Z}^n_{\geqslant 0}$ such that $\alpha = a + \beta\}$ will be called the *octant with vertex* a in $\mathbb{Z}^n_{\geqslant 0}$.

An *ideal* in the semigroup $\mathbb{Z}^n_{\geqslant 0}$ ($\mathbb{Z}^n_{\geqslant 0}$-ideal) is a subset of the semigroup satisfying the following condition if a point a lies in the ideal, then the octant $O^n(a)$ is a subset of the ideal. It is clear that any octant is an ideal of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

The following well-known statements (see, for example, [4]) fully describes structure of ideals of the semigroup$\mathbb{Z}^n_{\geqslant 0}$:

Proposition 2.1. *(Noetherianness of the semigroup $\mathbb{Z}^n_{\geqslant 0}$) Any $\mathbb{Z}^n_{\geqslant 0}$-ideal can be written as a union of a finite number of octant (in other words, if we take a union of infinite number of octant we can ever take several octant such that the union of taken octant contains all others).*

The semigroup $\mathbb{Z}^n_{\geqslant 0}$ contains 2^n special subsemigroups: for each subset I of the set $\{1, \dots, n\}$ is defined the semigroup $\mathbb{Z}_{\geqslant 0}(I)$ consisting of the points $a = (a_1, \dots, a_n)$ such that $a_i = 0, i \in I$ and $a_i \geq 0, i \notin I$. Note that, $\mathbb{Z}_{\geqslant 0}(\{1, \dots, n\}) = 0$ and $\mathbb{Z}_{\geqslant 0}(\emptyset) = \mathbb{Z}^n_{\geqslant 0}$.

A subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ is called *moved special subsemigroup*, if it has a form $a + \mathbb{Z}_{\geqslant 0}(I)$ for $a \in \mathbb{Z}^n_{\geqslant 0}$.

Proposition 2.2. *The complement $\mathbb{Z}^n_{\geqslant 0} \setminus \mathcal{I}$ to an ideal $\mathcal{I}$ of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ can be decomposed in a disjoint union of a finite number of moved special subsemigroups.*

The nonnegative integer $|\alpha| = \sum \alpha_i$ will be called the *modulus* of an element α of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

We take linear ordering $\prec$ on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ such that

(a) $\alpha \prec \beta$ for any elements α and β such that $|\alpha| < |\beta|$;

(b) the ordering $\prec$ is compatible with addition on the semigroup $\mathbb{Z}^n_{\geqslant 0}$: $\alpha + \gamma \prec \beta + \gamma$ for any elements α, β and γ such that $\alpha \prec \beta$.

It is obviously follows from condition (a) that $(\mathbb{Z}^n_{\geqslant 0}, \prec)$ is a well-ordered set.

3. Gröbner map and bases of a differential ideal

In this section we define Gröbner map for the ring of differentials operators and by means of it we study ideals of this ring.

Consider an arbitrary domain U in the space $\mathbb{C}^n = \{x_1, \dots, x_n | x_i \in \mathbb{C}\}$. Let B be a subring of the ring $\mathcal{O}(U)$ of holomorphic functions in the domain U such that B contains 1 and is closed with respect to differentiation.

Denote by Dif_B the ring of linear differential operators with coefficient in B. If $d \in Dif_B$ then $d = \sum\limits_{\alpha \in \operatorname{supp} d} b_\alpha \partial_\alpha$, where $b_\alpha (\not\equiv 0) \in B$ and $\operatorname{supp} d$ is a finite subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$; here and in what follows ∂_α denotes the operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. The finite subset $\operatorname{supp} d$ of the semigroup is called *support* of the operator d. Note that the ring Dif_B has the evident natural structure of a left B-module.

Definition 1. The following map is called Gröbner map

$$Grb : Dif_B \setminus \{0\} \to \mathbb{Z}^n_{\geqslant 0}$$
$$Grb(d) = \max_{\alpha \in \operatorname{supp} d} \alpha.$$

Maximal element in the right-hand side of the last expression is taken with respect to linear ordering $\prec$.

The following simple lemma describes important property of the Gröbner map.

Lemma 3.1. *For any two nonzero elements D and d of the ring Dif_B holds*

$$Grb(D \circ d) = Grb(d \circ D) = Grb(D) + Grb(d). \tag{1}$$

More over, denote by D_α and d_β the coefficients by the leading derivatives of operators D and d respectively. Then the coefficients by the leading derivatives of operators $D \circ d$ and $d \circ D$ are coincide and equal to $D_\alpha \cdot d_\beta$.

Proof. Assume $\sum_{|\alpha|=N(D)} D_\alpha \partial_\alpha$ and $\sum_{|\alpha|=N(d)} d_\alpha \partial_\alpha$, where $D_\alpha, d_\alpha \in A$, are highest homogeneous parts of operator D and d respectively. It is clear that highest homogeneous parts of operators $D \circ d$ and $d \circ D$ are coincide and equal to

$$\sum_{|\alpha|=N(D)+N(d)} \sum_{\{|\beta|=N(D), |\gamma|=N(d), \beta+\gamma=\alpha\}} D_\beta d_\gamma \partial_\alpha. \tag{2}$$

But by the condition () on the order relation $\prec$ (see Sec.2)) the image by the Gröbner map of an operator and the image of its highest homogeneous part are the same. This completes the proof. $\square$

Corollary 3.1. *The image by Gröbner map of an ideal of the ring Dif_B is an ideal of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.*

Let $\mathcal{I}$ be a left ideal of the ring Dif_B. By Corollary 3.1 the image $Grb(\mathcal{I})$ is an ideal of the semigroup $\mathbb{Z}^n_{\geqslant 0}$, hence by Proposition 2.1, it can be written as a union of finite number of octants. Assume

$$Grb(\mathcal{I}) = \cup_{i=1}^N O(\gamma_i). \tag{3}$$

Consider some elements $l_1, \ldots, l_k$ of the ideal $\mathcal{I}$ such that $Grb(l_i) = \gamma_i$. For each index i ($1 \leqslant i \leqslant N$) denote by $a_{\gamma_i} \in B$ the coefficient by the derivative ∂_{γ_i} in the operator l_i.

Let $\mathcal{M}$ be an multiplicatively closed subset of B, generated by the functions $a_{\gamma_1}, \ldots, a_{\gamma_N}$. I.e. $\mathcal{M}$ is the minimal closed with respect to multiplication subset of the ring B containing the elements $1, a_{\gamma_1}, \ldots, a_{\gamma_N}$. Consider the ring of fractions $\mathcal{M}^{-1}B$. It consists of all formal fractions $\frac{b}{m}$, where $b \in B$ and $m \in \mathcal{M}$, and $\frac{b_1}{m_1} = \frac{b_2}{m_2}$ if and only if equality holds $b_1 m_2 = b_2 m_1$ in the ring B.

It is clear that there exists the natural monomorphism of the ring of fractions $\mathcal{M}^{-1}B$ to the ring $\mathcal{O}(U \setminus M)$ of holomorphic functions in the domain $U \setminus M$, $M = \{a_{\gamma_1} \cdot \cdots \cdot a_{\gamma_N} = 0\}$. Moreover, the ring $\mathcal{M}^{-1}B$ is closed with respect to differentiation.

Consider the ring $Dif_{\mathcal{M}^{-1}B}$. Let us consider elements of the ring Dif_B as elements of the ring $Dif_{\mathcal{M}^{-1}B}$ by means of natural monomorphism

$$\pi : Dif_B \mapsto Dif_{\mathcal{M}^{-1}B} \tag{4}$$

$$\sum b_\alpha \partial_\alpha \mapsto \sum \frac{b_\alpha}{1} \partial_\alpha.$$

Proposition 3.3. *The elements $l_1, \dots, l_N$ generates the ideal $\mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B}$ of the ring $Dif_{\mathcal{M}^{-1}B}$.*

By $\mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B}$ we denote the minimal left ideal of the $Dif_{\mathcal{M}^{-1}B}$ containing the image by the monomorphism (4) of the ideal $\mathcal{I}$.

Proof. For the rings $Dif_{\mathcal{M}^{-1}B}$ and Dif_B the Gröbner maps Grb are defined. Moreover, for any element d of the ring Dif_B holds $Grb(\pi(d)) = Grb(d)$. Hence, the following equality holds $Grb(\mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B}) = Grb(\mathcal{I})$.

Consider arbitrary nonzero element u of the ideal $\mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B}$. Suppose $u = f_u \partial_{Grb(u)} + \{\text{lower terms}\}$. There exists a number $k \in 1, \dots, N$ such that the image $Grb(u)$ belongs to the octant $O^n(\gamma_k)$. Hence, $Grb(u) = \gamma_k + \alpha$, $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. Assume $u_1 = u - \frac{f_u}{a_{\gamma_k}} \partial_\alpha \circ l_k$. There are only two possibilities or $u_1 = 0$, or by Lemma 1 holds inequality $Grb(u_1) \prec Grb(u)$. If the element u_1 of the ideal $\mathcal{I} \cdot \mathcal{M}^{-1} \cdot Dif_{\mathcal{M}^{-1}B}$ is nonzero, we can repeat described the construction and get the elements u_2 and so on. But infinite chain of elements $u, u_1, u_2, \dots \in \mathcal{I} \cdot \mathcal{M}^{-1} \cdot Dif_{\mathcal{M}^{-1}B}$ such that $Grb(u) \succ Grb(u_1) \succ Grb(u_2) \succ \dots$ can not exist. Indeed, condition () on the linear ordering $\prec$ immediately implies that exists only finite numbers of elements of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ smaller than a given one. Hence, there exists a positive integer l such that $u_l = 0$. Substituting in the last equality the element u_l by it expression from u and l_i we complete the proof. $\square$

Remark 1. The expansion of the element $u = \sum_{i=1}^k p_i \circ l_i$ constructed in the proof of the proposition satisfies the following property for any i holds $Grb(p_i \circ l_i) \preceq Grb(u)$.

A system of generators of the ideal given by the proposition above is called a *Gröbner basis* of the ideal.

Consider the submodule $MI \subset Dif_{\mathcal{M}^{-1}B}$ generated over $\mathcal{M}^{-1}B$ by the elements $\{\partial_\alpha\}$, $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$.

Proposition 3.4. *The following decomposition in a direct sum of vector spaces holds*

$$Dif_{\mathcal{M}^{-1}B} = MI \oplus \mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B} \tag{5}$$

Proof. We have to prove that an arbitrary element d of the ring $Dif_{\mathcal{M}^{-1}B}$ can be only decomposed to a sum

$$d = m(d) + i(d), \tag{6}$$

where $i(d) \in \mathcal{I} \cdot Dif_{\mathcal{M}^{-1}B}$ and $m(d) \in MI$ (i.e., $\operatorname{supp} m(d) \subset \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$). Let us prove the uniqueness. Assume the inverse. Consider two different decompositions:

$$d = i_1 + m_1 = i_2 + m_2,$$

where $i_1, i_2 \in \mathcal{I}$ and $m_1, m_2 \in MI$. Then $0 \neq i_1 - i_2 = m_2 - m_1$, hence $Grb(i_1 - i_2) = Grb(m_2 - m_1)$. But $(i_1 - i_2) \in \mathcal{I}$, hence $Grb(i_1 - i_2) \in Grb(\mathcal{I})$. On the other side, $\operatorname{supp}(m_2 - m_1) \subset \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$. Therefore, we have $Grm(m_2 - m_1) \notin Grb(\mathcal{I})$. The contradiction proves the statement.

Let us prove the existence of the decomposition. The decomposition could be constructed by the algorithm. Suppose $d \in Dif_{\mathcal{M}^{-1}B}$. If $\operatorname{supp} d \cap Grb(\mathcal{I}) = \emptyset$, assume $m(d) = d$ and $i(d) = 0$. Otherwise, consider the element $\mu(d) = \max_{\alpha \in \operatorname{supp} d \cap Grb(\mathcal{I})} \alpha$ of the semigroup. Let $\mu(d) = \gamma_k + \alpha$ for some $1 \leqslant k \leqslant N$ and $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. Consider the difference $d - a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})$, where the coefficient $a \in B$ is taken such that

or $\operatorname{supp}(d - a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})) \cap Grb(\mathcal{I}) = \emptyset$, then $m(d) = d - a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})$ and $i(d) = a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})$;

or $\operatorname{supp}(d - a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})) \cap Grb(\mathcal{I}) \neq \emptyset$ and the repeat the process.

Note that on each step holds $\mu(d - a\partial_\alpha(\frac{l_k}{a_{\alpha_k}})) \prec \mu(d)$. The pair $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a well-ordered set, hence after a finite number of steps we will get desired decomposition. □

4. Formal solutions of a system of linear partial differential equations

Let us introduce necessary notions. Fix a domain U of the space of independent variables $\mathbb{C}^n$. Consider a subring A of the ring $\mathcal{O}(U)$ of holomorphic functions in the domain U containing 1 and closed with respect to differentiation.

Consider a system of linear partial differential equations in the domain U

$$\begin{cases} D_1 z = 0 \\ \quad \dots \\ D_k z = 0 \\ \quad \dots \end{cases}, \tag{S}$$

where $D_i \in Dif_A$ $(i = 1, 2, \dots)$.

The system (S) may contain infinite number of equations. In this Section we describe the space of germs of formal and analytic solutions for the system (S) for each point u of some open everywhere dense subset of the set U.

4.1. **Formal solution and linear function on the ring of differential operators.** Let u be a point of the domain U. Consider a subring B of the ring $\mathcal{O}_u$ of germs of holomorphic functions at the point u.

Definition 2. A map $\varphi : M \mapsto \mathbb{C}$ of B-module M is called u-linear map if

$$\varphi(\sum_{j=1}^{N} f_j L_j) = \sum_{j=1}^{N} f_j(u)\varphi(L_j), \tag{7}$$

for arbitrary $L_i \in M$ and $f_i \in B$.

By $L_u(M)$ we denote the space of the u-linear maps of the B-module M.

It is obvious that the ring Dif_A has a natural structure of an A-module. The following lemma describes the space $L_u(Dif_A)$, $u \in U$.

Lemma 4.2. *For any point u in U there exists the natural isomorphism of the vector spaces*

$$L_u(Dif_A) \cong \mathbb{C}[[x-u]]. \tag{8}$$

Proof. It is clear that the following map gives us desired isomorphism

$$\mathbb{C}[[x-u]] \mapsto L_u(Dif_A) \tag{9}$$

$$f(d) = d(f)|_{x=u}, \tag{10}$$

where $d \in Dif_A$, $f, d(f) \in \mathbb{C}[[x-u]]$, and by $d(f)|_{x=u}$ we denote the constant term of the power series $d(f)$. □

Denote by $\mathcal{I}(S)$ a left ideal of the ring Dif_A generated by the operators from the left-hand side of the equations of system (S).

Proposition 4.5. *A formal power series $f \in \mathbb{C}[[x-u]]$ defines u-linear map of the ring Dif_A vanishing on the ideal $\mathcal{I}(S)$ if and only if for any element $d \in \mathcal{I}(S)$ the formal power series $d(f)$ equals 0 (in other words, the formal power series f is a formal solution of system (S)).*

Proof. Indeed, the statement that for any operator $d \in Dif_A$ holds $d(f) = 0$ is equivalent to $\partial_\alpha(d(f))|_{x=u} = (\partial_\alpha \circ d)(f)|_{x=u} = 0$ for any $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. The proof is complete. □

Let us denote by $F_u(S)$ the space of formal solutions at a point u of the system (S).

Corollary 4.2. *There is the natural isomorphism*

$$F_u(S) = L_u(Dif_A/\mathcal{I}(S)). \tag{11}$$

We will describe spaces of u-linear maps using the following lemma. Let $\mathcal{M}$ be a multiplicatively closed subset in A. The ring of fractions $\mathcal{M}^{-1}A$ is evidently closed with respect to differentiation. It is clear that the natural monomorphism (dividing by 1) $\pi : A \mapsto \mathcal{M}^{-1}A$ induce the ring morphism

$$\pi_* : Dif_A \mapsto Dif_{\mathcal{M}^{-1}A} \tag{12}$$

Consider a left ideal $\mathcal{I}$ of the ring Dif_A. Denote by $\mathcal{M}^{-1}\mathcal{I}$ the minimal left ideal of the ring $Dif_{\mathcal{M}^{-1}A}$ containing the image $\pi_*(\mathcal{I})$ of the ideal $\mathcal{I}$.

Lemma 4.3. *Consider a point u of the domain U. If for any function $f \in \mathcal{M}$ holds $f(u) \neq 0$ then there exists the natural isomorphism between the vector space of the u-linear maps $L_u(Dif_A/\mathcal{I})$ and $L_u(Dif_{M^{-1}A}/M^{-1}\mathcal{I})$.*

Proof. Inclusion π induces the map

$$\pi^* : L_u(Dif_{\mathcal{M}^{-1}A}/\mathcal{M}^{-1}\mathcal{I}) \mapsto L_u(Dif_A/\mathcal{I}) \tag{13}$$

The following computation shows that π^* is an isomorphism:

$$(\pi^*)^{-1}(l)([\sum_{j=1}^{N} \frac{d_\alpha}{x_\alpha}\partial^\alpha]) = \sum_{j=1}^{N} \frac{d_\alpha(u)}{x_\alpha(u)} l([\partial^\alpha]), \tag{14}$$

$d_{\alpha_j} \in A$, $x_{\alpha_j} \in \mathcal{M}$. By $[d] \in Dif_{\mathcal{M}^{-1}A}/\mathcal{M}^{-1}\mathcal{I}$ here we denote the equivalence class of the element $d \in Dif_A$. □

4.2. **Existence of the formal solutions.** Consider the $\mathbb{Z}^n_{\geqslant 0}$-ideal $Grb(\mathcal{I}(S))$. By Proposition 2.1 we can find a finite number of octants such that

$$Grb(\mathcal{I}(S)) = \cup_{i=1}^{l} O(\gamma_i). \tag{15}$$

Choose elements $s_1, \ldots, s_l$ of the ideal $\mathcal{I}$ such that $Grb(s_i) = \gamma_i$ holds for each i. Denote by Γ a multiplicatively closed subset in the ring A generated by the leading coefficients of the operators s_i (i.e. the coefficients s_{γ_i} by the derivatives ∂_{γ_i}). Besides, by Σ we denote an analytic hypersurface defined by the equation $s_{\gamma_1} \cdot \ldots \cdot s_{\gamma_l} = 0$.

Proposition 3.3 immediately implies the following proposition.

Proposition 4.6. *The elements $s_1, \ldots, s_l$ generate the ideal $\mathcal{I} \cdot Dif_{\Gamma^{-1}A}$. In other words, in the domain $U \setminus \Sigma$ considered system (S) is equivalent to the system consisting of equations $s_i z = 0$, where index i ranges over the set $\{1, \ldots, l\}$.*

The *support* $\operatorname{supp} f$ of an formal (convergence) power series $f = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} f_\alpha (x-u)^\alpha$ is the following subset of the semigroup

$$\operatorname{supp} f = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} | f_\alpha \neq 0\}. \tag{16}$$

Theorem 4.1. *Consider a point $u \in U \setminus \Sigma$. Then*

$$F_u(S) \cong \{f \in \mathbb{C}[[x-u]] \mid \operatorname{supp} f \subset \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})\}. \tag{17}$$

Proof. Consider a free $\Gamma^{-1}A$-module $MI(S)$ with basis $\{\partial_\alpha\}$, $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$. It is clear that

$$L_u(MI) \cong \{f \in \mathbb{C}[[x-u]] \mid \operatorname{supp} f \subset \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})\}. \tag{18}$$

Then by Proposition 3.4 there is an isomorphism of $\Gamma^{-1}A$-modules

$$MI \cong Dif_{\Gamma^{-1}A}/\Gamma^{-1}\mathcal{I}. \tag{19}$$

Therefore, since for any point $u \in U \setminus \Sigma$ and any function $f \in \Gamma \subset A$ holds $f(u) \neq 0$, we have (using Lemma 4.3)

$$F_u(S) \cong L_u(Dif_A/\mathcal{I}) \cong L_u(Dif_{\Gamma^{-1}A}/\Gamma^{-1}\mathcal{I}). \tag{20}$$

To complete the proof we just have to combine the statements (18), (19) and (20). □

For any nonnegative integer i we put by definition

$$F_{u,i}(S) = F_u(S)/(f \sim g \overset{def}{\Leftrightarrow} f - g = o((x-u)^i)). \tag{21}$$

$F_{u,i}(S)$ is the space of i-jets of formal solutions at a point $u \in U$ of the system.

Definition 3. The function $H(u,i) = \dim F_{u,i}(S)$ of nonnegative integer argument i is called the Hilbert function at a point u of system (S).

Corollary 4.3. *The function $H(u,i)$ does not depend on $u \in U \setminus \Sigma$ and coincide with a polynomial for sufficiently large i.*

Proof. By (17) holds

$$\dim F_{u,i}(S) = |\{\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I}) | |\alpha| \leqslant i\}|, \tag{22}$$

where for a finite subset A we denote by $|A|$ the number of elements of the set A. This proves the first statement of the theorem.

By Proposition 2.2, we can decompose the set $\mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$ to the disjoint union of moved special subsemigroups

$$\mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I}) = \cup_{k=1}^l \{a_k + \mathbb{Z}_{\geqslant 0}(I_k)\}. \tag{23}$$

But for any moved special subsemigroup $a_k + \mathbb{Z}_{\geqslant 0}(I_k)$ the function

$$H_{(a_k,I_k)}(i) = |\{\alpha \in a_k + \mathbb{Z}_{\geqslant 0}(I_k) | |\alpha| \leqslant i\}|,$$

coincides with a polynomial for $i \geqslant |a_k|$ (it is easy to compute that for $i \geqslant |a_k|$ holds $H_{(a_k,I_k)}(i) = (\binom{n-|I|+i-a_k}{i-a_k}))$. Hence, the Hilbert function $H(u,i) = \sum_{j=1}^{l} H_{(a_j,I_j)}(i)$ coincides with a polynomial for $i \geqslant \max_k |a_k|$. The proof is complete. $\square$

5. Convergence of formal solutions

Suppose that a formal power series $z(x) = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} a_\alpha x^\alpha$ satisfies the following finite system of differential relations

$$\begin{cases} \partial_{\gamma_1} z = F_1(x, \partial_\alpha z), & \alpha \prec \gamma_1 \\ \quad\quad\ldots & \\ \partial_{\gamma_1} z = F_k(x, \partial_\alpha z), & \alpha \prec \gamma_k \end{cases}$$

Here $F_1,\ldots,F_k$ are holomorphic functions of the variables $x_1,\ldots,x_n$ and the derivatives $\partial_\alpha z$ such that α satisfy inequalities of the right-hand column above. Consider the subset $I = \cup_{i=1}^{k} O(\gamma_i)$ of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

Theorem 5.2. *Suppose that the truncated power series* $\tilde{z}(x) = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}\setminus I} a_\alpha x^\alpha$ *has a nonzero radius of convergence. Then so does the formal solution* $z(x)$.

The Theorem 5.2 is a particular case of the convergence theorem form [3].

Denote by $A_u(S)$ the space of germs of analytic solutions at a point u of the system (S). Combining the Theorem 5.2 and 4.1 we have

Theorem 5.3. *For any point* $u \in U \setminus \Sigma$ *holds*

$$A_u \cong \{f \in \mathbb{C}\{(x-u)\} \mid \operatorname{supp} f \subset \mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})\}. \tag{24}$$

Proof. Consider a system of partial differential equations consisting of equations $s_1 z = 0, \ldots, s_l z = 0$, where s_i are taken above elements of ideal $\mathcal{I}$ such that $Grb(s_i) = \gamma_i$ holds for each i. By Proposition 4.6 in the domain $U \setminus \Sigma$ this system is equivalent to system (S). Let us solve each equation $s_i z = 0$, where i ranges over the set $\{1,\ldots,l\}$, for the derivative γ_i which is the highest-order derivative with respect to the ordering $\prec$ (we can do this by choice of Σ). Now we can apply mentioned above convergence theorem 5.2. $\square$

Let us consider a decomposition of the set $\mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I})$ on moved special subsemigroups

$$\mathbb{Z}^n_{\geqslant 0} \setminus Grb(\mathcal{I}) = \cup_{k=1}^{l} \{a_k + \mathbb{Z}_{\geqslant 0}(I_k)\}. \tag{25}$$

The following theorem is a corollary of Theorem 5.3

Theorem 5.4. *For any point* $u \in U \setminus \Sigma$ *exists unique analytic in some neighborhood of the point* u *solution* $z(x)$ *for the system* (S) *satisfying the following initial conditions: for any* l $(1 \leqslant k \leqslant l)$

$\partial_{a_k} z(x)|_{\{x_i=u_i, i\in I_k\}} = \psi_1(x_{i_1},\ldots,x_{i_m})$, *where* $\{i_1,\ldots,i_k\} = \{1,\ldots,n\}\setminus I_k$

and ψ_i are an arbitrary holomorphic functions in some neighborhood of the point u (if $I_k = \{1, \dots, n\}$ for some k the ψ_k is just a complex number).

Proof. To deduce this theorem from the previous one it is sufficient to note that a convergence power series consisting of monomials x^α, where α belongs to a special subsemigroup $\mathbb{Z}_{\geqslant 0}(I)$ and $I = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, is a gem of holomorphic function depending on variables $x_{i_1}, \dots, x_{i_k}$. □

Denote by $A_{u,i}$ the space of i-jets at a point u of analytic solutions of the system.

Corollary 5.4. *For any point $u \in U \setminus \Sigma$ and each nonnegative integer i the dimensions of the following spaces are coincide*

$$\dim F_{u,i}(S) = \dim A_{u,i}(S). \tag{26}$$

6. Remarks and examples

6.1. **The condition (a) on linear ordering $\prec$ and convergence of formal solutions.** The following weaker condition (a') on the linear ordering $\prec$ is enough to develop the theory of existence and uniqueness of formal solutions

(') for any element α of the semigroup holds $0 \prec \alpha$.

The constructions of the Gröbner map and all needed statement holds for this case. In particular, the condition (a') implies that $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a well-ordered set. Besides, the statement and the proof of the Theorem 4.1 for this case does not change.

But the following example due to S. Kowalevsky(see [5, 6]) shows that the condition (') are not enough to prove convergence of formal solutions (Theorem 5.3 and 5.2). Consider the equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}.$$

It easy to construct a linear ordering $\prec$ satisfying conditions (a') (b) such that $\frac{\partial^2}{\partial x^2} \prec \frac{\partial}{\partial y}$. We claim that the formal solutions $z(x)$ with initial data

$$z = \phi(x) \text{ for } y = y^0,$$

where $\phi(x)$ is a holomorphic function, could has the zero radius of convergence.

6.2. **The case of several unknown functions.** It is important to note that all presented results can readily generalized to the case of systems of several unknown functions $z_1, \dots, z_p$.

The set of partial derivatives of the tuple of unknown function $z_1, \dots, z_p$ is parameterize by points of the product $Z = \mathbb{Z}^n_{\geqslant 0} \times \{1, \dots, p\}$. Let $\prec_{\mathbb{Z}^n_{\geqslant 0}}$ be a linear ordering on the semigroup

$\mathbb{Z}^n_{\geqslant 0}$ satisfying conditions (a) and (b) of Section 2. Consider the following well-ordering $\prec$ on the set Z. For any two elements (α, i) and (β, j) of the set Z we first compare the element α and β of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ with respect to the linear ordering $\prec_{\mathbb{Z}^n_{\geqslant 0}}$, if the element coincides, we compare numbers i and j (as integers). Reproducing the argument in this paper almost word for word, one can readily extend Theorems 4.1, 5.2, 5.3 and 5.4 to this case.

6.3. **On germs of solutions at points of "bad" hypersurface Σ.** Above we described the space of germs of formal and analytic solutions of the system at points belonging to the complement of some analytic hypersurface Σ only. The following example shows that at some points belonging to Σ the space of germs of formal and analytic solutions of the system may be different form described above. Consider an equation of the form

$$\sum a_i x_i \frac{\partial}{\partial x_i} z = 0,$$

where a_i are integer points. "Bad" hypersurface for this case is one of the hyperplanes $x_i = 0$ (it depends on choice of the linear ordering). It is clear that for appropriate a_i all theorem proved above (Theorem 4.1, 5.2, 5.3 and 5.4) are fails for the point $0 \in \Sigma$. In particular, the function $H(0, i)$ does not coincide with a polynomial for sufficiently large positive integers.

6.4. **Algebraic sense of Hilbert function of a system.** Recall that *highest homogeneous symbol* of the system is a family of algebraic varieties (more precise, a family of ideals of the polynomial ring $\mathbb{C}[\xi_1, \dots, \xi_n]$) $M(u)$ depending on a point u of the domain U. The family $M(u)$ is defined on the following way. For equation of the system we take an operator

$$D = \sum_{\alpha \in \operatorname{supp} D} d_\alpha \partial_\alpha, \tag{27}$$

from the left-hand side of the equation. We corresponds to the operator a family of homogeneous polynomials

$$\widetilde{D}(u, \xi) = \sum_{\alpha \in \operatorname{supp} D_i, |\alpha| = r(D)} d_\alpha(u) \xi^\alpha, \tag{28}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and $r(D)$ is the order of the operator D. $M(u)$ is a family of ideals polynomials of the ring $\mathbb{C}[\xi_1, \dots, \xi_n]$ generated by polynomials $\widetilde{D}$.

The Gröbner bases construction immediately implies

Proposition 6.7. *For any point $u \in U \setminus \Sigma$ and nonnegative integer i holds*

$$H_{M(u)}(i) = H(u, i), \tag{29}$$

where by $H_{M(u)}$ we denote the Hilbert function of algebraic variety $M(u)$ (i.e., the Hilbert function of the corresponding ideal).

References

[1] *C. Riquier* Les systèmes d'èquations aux derivées partielles // Paris: Gauthier-Villars.-1910.

[2] *V.P. Palamodov* Differential operators in the class of convergent power series and the Weierstrass preparation theorem // Funkts. Anal. Prilozhen., 1968, vol. 2, No. 3, pp. 58-69.

[3] *S.P. Chulkov* On the convergence of formal solutions of a system of partial differential equations // Funkts. Anal. Prilozhen., 2005, vol. 39, No. 3, pp. 64-75.

[4] *A.G. Khovanskii* Sums of finite sets, orbits of finite semigroups and Hilbert functions // Funkts. Anal. Prilozhen., 1995, vol. 29, No. 2, pp. 36-50.

[5] *S. Kowalevsky* Zür Theorie der partiellen Differentialgleichungen // J. für Math.-1875.-V. 20-P. 1

[6] *S.P. Finikov* Cartan's Method of Exterior Forms in Differential Geometry. The Theory of Consistency of Systems of Total and Partial Differential Equations [in Russian]// M.-L.: Gostekhizdat, 1948.