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ON CONVERGENCE OF FORMAL SOLUTIONS TO A SYSTEM OF PDE'S

S.P. CHULKOV

1. Introduction

The aim of this paper is to study a variant of the classical problem of the convergence of formal solutions to a system of PDE's. The theory of ODE's studies differential equations written in the explicit form with respect to highest order derivative. For analytic ODE's the Cauchy problem is solvable and any formal solution is analytic. A possible generalization of this on the case of many-dimensions is the Cauchy-Kovalevskaya theorem. It describes, in particular, all formal solutions to the Cauchy-Kovalevskaya type PDE's and gives convergency conditions for them. For equations of this type, as for ODE's, a certain partial derivative is marked as leading and the equations are written in the explicit form with respect to this derivative.

Riquier in [1] has developed a theory generalizing the Cauchy-Kovalevskaya theorem. In the multidimensional case there is no natural way to select the leading partial derivative. Riquier considered which paritial derivative of unknown function should be selected as leading and how to state initial conditions for which the theorem of existence and uniqueness of formal and analytic solutions holds. In his great work Riquier introduces a total ordering on the set of parital derivatives of a function in several variables and by using it significantly advances in the mentioned problem. In the case of linear PDE's with constant coefficients the Riquier method, in substance, consists of what is actually called the Groebner bases and significantly advanced in the computing aspects of the commutative algebra.

The matter of study of Riquier's paper are stating of "correct" initial conditions and constructing corresponding formal solutions. Besides, Riquier studies convergence of the formal solutions. However his work is not complete in some sense because of the great generality of the stated problem: the Riquier method could be applied to a system written in the explicit form with respect to the leading derivatives only if certain strong and difficult to control conditions on the considered system are satisfied.

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One can find an exposition of [1] in Russian in the book of S.P. Finikov [2].

The most complete theorems of existence and uniqueness of formal and analytic solutions have been proved in the case of systems of linear PDEs with analytic coefficients. This case has been studied by V.P. Palamodov in [3]. V.P. Palamodov follows the idea of Riquier. Using a special algebraic technique, Palamodov finds new formal solutions which Riquier has not considered (see [4]). The theorem of existence and uniqueness proven in [3] contains the statement concerning convergence of formal solutions which statement Riquier's theorem does not cover. Palamodov's proof of convergence of formal solutions is based on the special algebraic technique developed by himself for the case of linear differential operators and can not be generalized to the case of systems of non-linear PDE's.

The problem of constructing formal solutions is mostly algebraic (see [4, 5]). We do not consider this. The aim of our work is studying the convergency of a given formal solution (found by any possible way). One can apply our theorem to any system of PDE's which is written in the explicit form with respect to leading derivatives and, most important, to any system of PDE's written in the "almost explicit form with respect to leading derivatives", (see below). Our main theorem states that a formal power series which is a formal solution to a system converges iff some special partial sum of this series converges. One can see that our theorem generalizes the Riquier convergence theorem. In the proof we use some ideas of his work [1]. But our generalization can be applied to formal solutions of "Palamodov‘s type", (such solutions characterize the system written in the "almost explicit form with respect to leading derivatives"), Riquier did not know about these solutions and his method can not be applied to this case. Our theorem generalizes also some corollaries of Palamodov's work [3] concerning the convergence of formal solutions: in contrast to Palamodov's work it can be applied to non-linear systems.

One can see that our main result can be generalized to the case of systems with several unknown functions (see Section 5).

Besides the works of Palamodov and Riquier, there are different proofs and generalizations of the Cauchy–Kovalevskaya theorem. One can find them in the works of Ovsyannikov [6] , Pate [7], Treves [8], Nirenberg [9] and Nishida [10].

This paper is organized as follows. In Section 2 we give several definitions and formulate the main result. In Section 3 some properties of the ordered semigroup $\mathbb{Z}^n_{\geqslant 0}$ are considered. In Section 4 we prove the main theorem. Followed by examples and remarks in Section 5.

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2. Main result

Consider the semigroup $\mathbb{Z}^n_{\geqslant 0} = \{(\alpha_1, \ldots, \alpha_n) | \alpha_i \in \mathbb{Z}, \alpha_i \geqslant 0\}$. The modulus $|\alpha|$ of an element $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ is the non-negative integer $\sum \alpha_i$.

Fix a total ordering $\prec$ on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ such that:

i) for any elements α and β of the semigroup, the condition $|\alpha| < |\beta|$ implies $\alpha \prec \beta$;

ii) the ordering relation $\prec$ is compatible with the sum operation on $\mathbb{Z}^n_{\geqslant 0}$, i.e., for any elements α, β and γ, the inequality $\alpha \prec \beta$ implies that $\alpha + \gamma \prec \beta + \gamma$.

From the condition i) it follows obviously that $(\mathbb{Z}^n_{\geqslant 0}, \preceq)$ is a complete ordered set.

As an example of the total ordering $\prec$ one can consider the following total ordering. For any elements α and β of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ we compare theirs modulus, if theirs modulus are equal to each other, we compare these elements with respect to the lexicographic order.

Let us consider the following finite system of PDE's in a neighborhood of the 0 of the space $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$:

$$\begin{cases} \partial_{\gamma_1} z = F_1(x, \partial_\alpha z) + M_1(x, \partial_\alpha z) \\ \qquad \ldots \\ \partial_{\gamma_k} z = F_k(x, \partial_\alpha z) + M_k(x, \partial_\alpha z), \end{cases}$$

where $F_1, \ldots, F_k$ and $M_1, \ldots, M_k$ are holomorphic functions in variables $x_1, \ldots, x_n$ and derivatives $\partial_\alpha z$ of the function z. Here and below ∂_α, where $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, denotes the differentiation operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

Assume that the formal power series $z = \sum\limits_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} z_\alpha x^\alpha$, where z_α are complex numbers and x^α is the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, is a formal solution for the system.

Suppose also that the following conditions are satisfied

1) for any index i, the function F_i depends only on the variables $x_1, \ldots, x_n$ and derivatives $\partial_\alpha z$, where $\alpha \prec \gamma_i$;

2) for any index i, the function M_i is of the form

$$M_i(x, \partial_\alpha z) = \sum_{\beta, |\beta| = |\gamma_i|} M_i^\beta(x, \partial_\alpha z) \partial_\beta z, \tag{1}$$

where M_i^β are holomorphic function in variables $x_1, \ldots, x_n$ and derivatives $\partial_\alpha z$ such that $|\alpha| < |\gamma_i|$, and for any i and β, the following equality

$$M_i^\beta(0, \alpha! z_\alpha) = 0 \tag{2}$$

holds at the initial point. Here and below $\alpha!$ is the product of factorials $\alpha_1! \ldots \alpha_n!$

Saying informally, the conditions above mean that our system is written in the "almost explicit form with respect to the leading derivatives in the sense of our total ordering", but on the right side of every equation one can add a linear function with respect to derivatives of highest order (peculiar for each equation), and coefficients of this linear function are equal to zero at the initial point. In the each equation some partial derivatives added to the right side could be larger with respect to the total ordering $\prec$ than the derivative from the left side.

An octant $O^n(a)$ with the vertex at the point a of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ is the set $\{\alpha \in \mathbb{Z}^n_{\geqslant 0} \mid \exists \beta \in \mathbb{Z}^n_{\geqslant 0}$ such that $\alpha = a + \beta\}$.

Consider a subset $I = \cup_{i=1}^k O(\gamma_i)$ of the semigroup $\mathbb{Z}^n_{\geqslant 0}$. Provided the conditions above are fulfilled,the following theorem holds

Theorem 1. *Suppose that the power series* $\tilde{z}(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus I} z_\alpha x^\alpha$ *has a non-zero radius of convergence. Then the formal solution* $z(x)$ *has a non-zero radius of convergence.*

The proof is in Section 4.

Remark 1. Note that the condition (1) that the functions M_i are linear with respect to highest order derivatives does not restrict generality of considered systems very strong. Indeed, if we start with some function M_i and it is not linear with respect to dirivatives of the highest order, then we can derivate corresponding equation by any variable and get new equation with new function M_i which will be linear with respect to derivatives of the highest order. All other conditions are preserved.

3. Properties of the total ordered semigroup $\mathbb{Z}^n_{\geqslant 0}$

Recall that in the previous section we fixed the total ordering on the semigroup $\mathbb{Z}^n_{\geqslant 0}$. The next Lemma shows that the total ordering $\prec$ on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ on any finite subset of the semigroup could be defined by one linear function. More clearly,

Lemma 1. *Let A be a finite subset of the semigroup $\mathbb{Z}^n_{\geqslant 0}$. There exists a linear function*

$$\Pi_A : \mathbb{Z}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$$
$$\Pi_A(\alpha) = \sum_{i=1}^n \pi_i \alpha_i,$$

where π_i are some positive real numbers and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$, which function posseses the property:

for any $\alpha, \beta \in A$ such that $\alpha \prec \beta$ the $\Pi_A(\alpha) < \Pi_A(\beta)$ holds.

Proof. Consider the chain of natural inclusions $\mathbb{Z}^n_{\geqslant 0} \subset \mathbb{Z}^n \subset \mathbb{R}^n$. Denote by B the following finite subset of the semogroup $\mathbb{Z}^n$:

$$B = \{\delta \in \mathbb{Z}^n | \exists \alpha, \beta \in A : \alpha \prec \beta, \delta = \beta - \alpha\}.$$

Let $conv(B)$ be the convex hull of the set B in the space $\mathbb{R}^n$. As total ordering $\prec$ is compatible with the sum operation on the semigroup $\mathbb{Z}^n_{\geqslant 0}$, we claim that the set $conv(B)$ does not contain the origin. Indeed, assume the converse. Let the following equality holds

$$\sum_1^N p_i \delta^i = 0, \tag{3}$$

where $\delta^i \in B \subset \mathbb{Z}^n$ ($\delta^i = \beta^i - \alpha^i$, $\beta^i, \alpha^i \in A$) and $p_i \in \mathbb{R}$, $p_i > 0$. One can consider the equation (3) as a system of homogeneous linear equations with integer coefficients on the coordinates of the vector $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$. The existence of a nontrivial solutions of the system implies the existence of a nontrivial vector space of solutions. The coefficients of equations in the system (3) are integer numbers, hence rational vectors are everywhere dense in the space of solutions. Therefore, there exist rational and, hence, integer positive numbers $\tilde{p}_i$ such that

$$\sum_1^N \tilde{p}_i \delta^i = 0.$$

Then,

$$\sum_1^N \tilde{p}_i \beta^i = \sum_1^N \tilde{p}_i \alpha^i.$$

In the other hand, $\alpha_i \prec \beta_i$, hence $\sum_1^N \tilde{p}_i \alpha_i \prec \sum_1^N \tilde{p}_i \beta_i$. This is a contrudiction.

Because the closed bounded convex set $conv(B)$ does not contain the point 0 there exists a linear function

$$L : \mathbb{R}^n \to \mathbb{R}$$
$$L(x) = \sum_{i=1}^n l_i x_i,$$

such that for any point $x \in conv(B)$ the value of the function $L(x) > 0$. Now if we put $\pi_i = S + l_i$, where S is a big enough natural number, we get needed linear function Π_A. □

Remark 2. Using the argument above, it is easy to prove the following well-known statement (see, for example, [11] or [12])

Proposition 1. *On the semigroup $\mathbb{Z}^n_{\geqslant 0}$ consider some total ordering relation $\prec$ such that it is compatible with the sum operation. Then there exist a linear functions $\Pi^1, \ldots, \Pi^j$, $j \leqslant n$,*

$$\Pi^i : \mathbb{Z}^n_{\geqslant 0} \mapsto \mathbb{R}$$

such that the total ordering $\prec$ is the lexicographic ordering with respect to this set of linear functions, i.e. the statement $\alpha \prec \beta$ is equivalent to the the statement

$$\Pi^1(\alpha) = \Pi^1(\beta), \ldots, \Pi^i(\alpha) = \Pi^i(\beta), \Pi^{i+1}(\alpha) < \Pi^{i+1}(\beta),$$

for some $i \in \{0, \ldots, j-1\}$.

Denote by Π_k, for any natural number k, a linear function posseses the conditions of the previous Lemma for the set $A_k = \{\alpha \in \mathbb{Z}^n_{\geqslant 0} | |\alpha| \leqslant k\}$. Denote $\mu_k = \min_{\{\alpha, \beta \in A_k\}} |\Pi_k(\beta) - \Pi_k(\alpha)|$. Note that $\mu_k > 0$.

4. The proof of Theorem 1

Our proof is based on the majorant method. Let $\mathbb{C}[[y_1, \dots, y_l]]$ be the ring of formal power series of variables $y_1, \dots, y_l$. Let $A, B \in \mathbb{C}[[y_1, \dots, y_l]]$.

Definition 1. We say that a formal power series $A(y) = \sum_{\alpha \in \mathbb{Z}^l_{\geqslant 0}} a_\alpha y^\alpha$ *majorize (or is majorant)* the power series $B(y) = \sum_{\alpha \in \mathbb{Z}^l_{\geqslant 0}} b_\alpha y^\alpha$, if for any element α of the semigroup $\mathbb{Z}^l_{\geqslant 0}$ the following conditions hold

$$a_\alpha \in \mathbb{R}_{\geqslant 0} \text{ and } |b_\alpha| \leqslant a_\alpha.$$

The idea of the proof is to construct a majorant convergent power series for our formal solution. More presisely, the majorant power series is a solution to some equation which, rather say, is majorant of each partial differential equation of the our system. The majorant equation is an ordinary differential equation and the standard theorem of existence and uniqueness applies to proving the existence of the needed solution.

In Section 4.1 we prove some lemmas about the majority relation.

In Sections 4.2 - 4.5 we prove a special case of the theorem when the equations of the systems are linear with respect to derivative of the high order and and all leading derivatives have the same order. In Section 4.2 we state the conditions of the special case. In Section 4.3 by means of an appropriate coordinate change our systems is reduced to the form that is very useful for constructing needed majorant equation. In Section 4.4 we obtain the majorant differantional equation and prove the existence of needed anaylitic solution of this equation. In Section 4.5 using founded solution of the majorant equation we construct a convergent power series and using Lemma 2 of Section 4.1, we prove that this power series majorizes the formal solution of our system. And, finally, in the Section 4.6 we complete the proof reducing general case to the considered one.

4.1. **Some properties of the majority relation.** In the first of two Lemmas of this Section we state that the operation of composition preserves in some sense the majority relation.

Consider holomorphic functions f_1 and f_2 defined in the neighborhood of the origin in the space $\mathbb{C}^{n+m} = \{(x_1, \dots, x_n, \xi_1, \dots, \xi_m) | x_i, \xi_j \in \mathbb{C}\}$. Suppose that the series expansion about 0 of the function f_2 majorizes the series expansion of the f_1.

Fix an (m+1)-tuple $\alpha_1 \prec \dots \prec \alpha_m \prec \alpha_0$ of elements of the semigroup $\mathbb{Z}^n_{\geqslant 0}$.

Let $w = \sum w_\alpha x^\alpha \in \mathbb{R}_{\geqslant 0}[[x]]$ and $z = \sum z_\alpha x^\alpha \in \mathbb{C}[[x]]$ be some power series and suppose that the following two conditions are satisfied

$$\text{for any } \alpha \prec \alpha_0,\ |z_\alpha| \leqslant w_\alpha \text{ holds;}$$
$$\text{for any } i(1 \leqslant i \leqslant m),\ w_{\alpha_i} = z_{\alpha_i} = 0\text{holds.}$$

In this case the power series $W \in \mathbb{R}_{\geqslant 0}[[x]]$ wich is the result of substituting in the expansion of the function f_2 variables ξ_i by the series $\partial_{\alpha_i} w$ $((1 \leqslant i \leqslant m))$, and $Z \in \mathbb{C}[[x]]$ wich is the result of substituting in the expansion of the function f_1 variables ξ_i by the series $\partial_{\alpha_i} z$ $((1 \leqslant i \leqslant m))$, are well-defined.

Lemma 2. *The inequality $|\partial_\beta Z|_0| \leqslant \partial_\beta W|_0$ holds for any $\beta \prec \alpha_0$.*

Proof. Assume that $Z = \sum_\alpha Z_\alpha x^\alpha$ and $W = \sum_\alpha W_\alpha x^\alpha$. The series $\partial_\beta Z|_0 = \beta! Z_\beta$ and $\partial_\beta W|_0 = \beta! W_\beta$ are the sums of the following expressions over the same set of indices

$$\beta! f^1_{(\alpha,\delta)} \prod_{i=1}^{m} (\alpha_i!)^{\delta_i} z_{\theta_1+\alpha_i} \cdot \dots \cdot z_{\theta_{\delta_i}+\alpha_i} \tag{4}$$

and

$$\beta! f^2_{(\alpha,\delta)} \prod_{i=1}^{m} (\alpha_i!)^{\delta_i} w_{\theta_1+\alpha_i} \cdot \dots \cdot z_{\theta_{\delta_i}+\alpha_i} \tag{5}$$

respectively; in (4) and (5) above $\alpha, \theta_j \in \mathbb{Z}^n_{\geqslant 0}$, $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{Z}^m_{\geqslant 0}$, and $f^j_{(\alpha,\delta)}$, $j = 1, 2$, are the coefficients of the series expansions of the functions f^j. To prove the Lemma it remains to note that $\alpha + \sum \theta_i = \beta \prec \alpha_0$ in (4) and (5), hence, by the conditions above, each summand in (4) is greater or equal than the modulus of the corresponding summand in (5). □

The proof of the following simple lemma one can find in [2].

Lemma 3. *Suppose that the power series $A(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} a_\alpha x^\alpha$ converges absolutly at the point $x_1 = \dots = x_n = \rho > 0$ and let M be a positive number greater than absolute value of any term of the series $A(\rho)$. Then the power series expansions about the origin of the fuctions $F_1(x) = \frac{M}{(1-x_1/\rho)\dots(1-x_n/\rho)}$ and $F_2 = \frac{M}{(1-(x_1+\dots+x_n)/\rho)}$ majorize the power series $A(x)$.*

4.2. **The condition of the special case.** Without loss of generality, one can believe that the coefficients z_α for $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus I$ of the power series z are equal 0. Indeed, according to the conditions of the theorem, the power series naturally constructed by this defines some analytic function φ in the neighborhood of the origin. Considering new unknown function $\tilde{z} = z - \phi$ we get the statement.

Consider now the following special case. Suppose that every equation of our system is linear with respect to derivatives of highest order and all leading paritial derivatives (derivatives on the left side of the equations) have the same order. Clearly, the system is written in the following form

$$
\begin{cases}
\partial_{\gamma_1} z = \sum\limits_{|\alpha|=N, \alpha \prec \gamma_1} f^1_\alpha \partial_\alpha z + f^1 + \sum\limits_{|\alpha|=N} M_1^\alpha \partial_\alpha z \\
\dots \\
\partial_{\gamma_k} z = \sum\limits_{|\alpha|=N, \alpha \prec \gamma_k} f^k_\alpha \partial_\alpha z + f^k + \sum\limits_{|\alpha|=N} M_k^\alpha \partial_\alpha z,
\end{cases} \tag{6}
$$

where $|\gamma_1| = \cdots = |\gamma_k| = N > 0$ and the holomorphic functions $f^i_\alpha, f^i, M^\alpha_k$ depend on the variables $x_1, \dots, x_n$ and derivatives $\partial_\beta z$ such that $|\beta| < N$. Besides, as it was above, for any admissible i and α $M^\alpha_i(0)$=0 holds.

4.3. **The change of the coordinates.** In this section by means of an appropriate coordinates change we obtain that the leading derivatives (with respect to our total ordering) become "chief". The coefficient of the another derivatives multiplies by some small numbers.

In the given conditions, $f^i_\alpha, f^i, M^\alpha_i$ are the holomorphic funcions in the neighborhood of 0. Consider power series expansions about the origin of the functions f^i_α, f^i and M^α_i for all admissible values of the parameters i and α. Suppose that all these series converge absolutely in the point $x_1 = \cdots = x_n = \partial_\alpha z = \rho$, where $|\alpha| < N$. By Lemma 3 of Section 4.1, let us choose a positive real number C such that the series expansion of the function

$$
\frac{C}{(1 - (x_1 + \cdots + x_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho)}
$$

majorizes the corresponding expansions of the functions f^i_α, f^i, M^α_i for all admissible i and α.

Denote by Π the linear functional Π_N (see Section 3), and let $\mu = \mu_N$. Consider a positive real number $\theta < 1$ such that

$$
\theta^\mu < \varepsilon = \frac{1}{2} \frac{1}{\Delta_N C},
$$

where Δ_i $(i \in \mathbb{Z}_{\geqslant 0})$ is the number of elements α of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ such that $|\alpha| = i$.

Suppose $y_i = \theta^{-\pi_i} x_i$, $1 \leqslant i \leqslant n$. Then

$$
\frac{\partial^{|\alpha|} z}{\partial x^\alpha}(x) = \theta^{-\Pi(\alpha)} \frac{\partial^{|\alpha|} z}{\partial y^\alpha}(y(x)).
$$

The equations (6) have a form

$$
\partial_{\gamma_i} z = \sum_{|\alpha|=N} f^i_\alpha(y, \theta^{-\Pi(\gamma)} \partial_\gamma z(y)) \theta^{\Pi(\gamma_i) - \Pi(\alpha)} \partial_\alpha z + \tag{7}
$$

$$
+ f^i(y, \theta^{-\Pi(\beta)} \partial_\beta z(y)) \theta^{\Pi(\gamma_i)} + \sum_{|\alpha|=N} M^\alpha_i(y, \theta^{-\Pi(\gamma)} \partial_\gamma z(y)) \theta^{\Pi(\gamma_i) - \Pi(\alpha)} \partial_\alpha z,
$$

for every i, after the coordinates change and dividing by the coefficients of the leading derivatives. Here and below the operators of partial differentiation are considered in the new coordinate system. The formal power series $z(y) = z(x(y))$ satysfies (7) in the new coordinate system (with zero initial conditions). Suppose that

$$\tilde{f}^i_\alpha(y, \partial_\gamma z) = f^i_\alpha(y, \theta^{-\Pi(\gamma)}\partial_\beta z)\theta^{\Pi(\gamma_i)-\Pi(\alpha)};$$
$$\tilde{f}^i(y, \partial_\gamma z) = f^k(y, \theta^{-\Pi(\gamma)}\partial_\gamma z)\theta^{\Pi(\gamma_i)};$$
$$\tilde{M}^\alpha_i(y, \partial_\gamma z) = M^\alpha_i(y, \theta^{-\Pi(\gamma)}\partial_\gamma z)\theta^{\Pi(\gamma_i)-\Pi(\alpha)},$$

for any admissible α, i. As before, $\tilde{M}^\alpha_i(0) = 0$ for any admissible i and α. Let us prove the following lemma

Lemma 4. *For any admissible i and α, the power series expansion (at the point 0) of the function*

$$\frac{\varepsilon C}{1 - (y_1 + \cdots + y_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho_1},$$

for some $0 < \rho_1 \ll \rho$, majorizes corresponding expansions of the functions $\tilde{f}^i_\alpha$ and $\tilde{f}^i$.

Proof. Choose ρ_1 as follows

(8) $$\rho_1 = \rho \min_{|\alpha|\leqslant N} \theta^{\pi(\alpha)}.$$

Check Lemma's statement for the functions f^i_α. Indeed, in the coordinates $x_1, \dots, x_n$ the series expansion about the origin of the function

$$\frac{C}{1 - (x_1 + \cdots + x_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho}$$

majorizes the series expansion of each function f^i_α. By the formulas of the change of coordinates, we claim that the series expansion of the function (in the new coordinates)

$$\frac{C}{1 - (y_1 + \cdots + y_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho_1}$$

majorizes the expansion of each function $f^i_\alpha(y, \theta^{-\Pi(\gamma)}\partial_\beta z)$. But

$$\tilde{f}^i_\alpha(y, \partial_\gamma z) = f^i_\alpha(y, \theta^{-\Pi(\gamma)}\partial_\beta z)\theta^{\Pi(\gamma_i)-\Pi(\alpha)},$$

where the coefficient $\theta^{\Pi(\gamma_i)-\Pi(\alpha)} \leqslant \theta^\mu < \varepsilon$, as $\alpha \prec \gamma_i$ and $|\alpha| \leqslant |\gamma_i| = N$. This proves the statment.

For the functions $\tilde{f}^i$ we can check mentioned relations in the same way, if we take into account the inquality $\theta^{\gamma_i} \leqslant \theta^\mu$ which holds due to the choice of the constant μ for each i. □

Consider a positive real number K such that the function

(9) $$\frac{KC}{1 - (y_1 + \cdots + y_n + \sum_{|\alpha|<N} \partial_\alpha z)/\rho_1}$$

majorizes the series expansion about the point 0 of each function $\tilde{M}_i^{\alpha}$ for any possible i and α.

4.4. **Constructing of the majorant equation.** Consider an ODE

$$
(10)\quad Y^{(N)}(t) = \frac{\varepsilon C}{1-(t+\sum_{j=1}^{N-1}\Delta_j Y^{(j)}(t))/\rho_1}(\Delta_N Y^{(N)}(t)+1)+ \\
+\Big(\frac{KC}{1-(t+\sum_{j=1}^{N-1}\Delta_j Y^{(j)}(t))/\rho_1} - KC\Big)\Delta_N Y^{(N)}(t).
$$

Recall that Δ_j is the number of elements α of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ such that $|\alpha| = j$. Rewrite the equation (10) in the explicit form with respect to the derivative of highest order, we have

$$
(11)\qquad Y^{(N)}(t) = \frac{2\varepsilon C}{1-2(KC\Delta_N+1)(t+\sum_{j=1}^{N-1}\Delta_j Y^{(j)}(t))/\rho_1}.
$$

We take into account the relation $\varepsilon \Delta_N C = 1/2$. By the theorem of existence and uniqness of solutions of an ODE, we claim that there exists a unique solution $Z(t)$ to the equation (11)((10)) with the initial conditions $Z^{(0)} = \cdots = Z^{(N-1)} = 0$.

As the power series expansion about the point 0 of the function

$$
\frac{2\varepsilon C}{1-2(KC\Delta_N+1)(t+\sum_{j=1}^{N-1}\Delta_j Y^{(j)}(t))/\rho_1}
$$

of variables t and $Y, Y^{(1)}, \ldots, Y^{(N-1)}$ has positive coefficients, we claim that the power series expansion about the origin of the solution $Z(t)$ has positive coefficients.

Denote

$$
G(Y,t) = \frac{C}{1-(t+\sum_{j=1}^{N-1}\Delta_j Y^{(j)}(t))/\rho_1}.
$$

4.5. **Constructing of the majorant power series.** Let us to prove the following

Lemma 5. *The power series expansion about the point* 0 *of the function* $Z(\sum_{i=1}^{n} y_i)$ *majorizes the formal solution* $z(y)$*. Hence, the power series* $z(y)$ *converges in some neighborhood of* 0*.*

Proof. Assume $Z(\sum_i y_i) = \sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} Z_\alpha y^\alpha$. Show by induction on $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ that the following eniquality holds

$$
(12)\qquad |z_\alpha| \leqslant Z_\alpha
$$

for each α. Indeed, the condition (12) holds for $\alpha \in \mathbb{Z}^n_{\geqslant 0}\setminus I$, and, hence, for $\alpha = 0$. Suppose now that (12) holds for every $\alpha \prec \alpha_0$. Prove the eniquality for α_0. Let $\alpha_0 = \beta + \gamma_i$ for some i $(1 \leqslant i \leqslant k)$.

Then

$$\alpha_0!|z_{\alpha_0}| = |\partial_{\alpha_0} z|_{y=0}| =$$
$$= |(\partial_\beta(\sum_{|\alpha|=N, \alpha\prec\gamma_i} \tilde{f}^i_\alpha \partial_\alpha z + \tilde{f}^i + \sum_{|\alpha|=N} \tilde{M}^\alpha_i \partial_\alpha z))|_{y=0}|.$$

Futher more, by Lemma 2 and the equality

$$\partial_\alpha F(\sum y_i) = F^{(|\alpha|)}(\sum y_i), \tag{13}$$

where F is an arbitrary holomorphic function, we get

$$|(\partial_\beta \left[\sum_{|\alpha|=N, \alpha\prec\gamma_i} \tilde{f}^i_\alpha \partial_\alpha z + \tilde{f}^i + \sum_{|\alpha|=N} \tilde{M}^\alpha_i \partial_\alpha z \right])|_{y=0}| \leqslant$$
$$\leqslant (\partial_\beta \Big[\varepsilon G(Z, \sum\nolimits_i y_i)(\Delta_N Z^{(N)}(\sum\nolimits_i y_i) + 1) +$$
$$+ (KG(Z, \sum\nolimits_i y_i) - KC)\Delta_N Z^{(N)}(\sum\nolimits_i y_i)\Big])|_{y=0}$$

Indeed, it follows from Lemma 4 and the inductive assumption that each summand of the form

$$\tilde{f}^i_\alpha \partial_\alpha z \tag{14}$$

is majorized by the summand

$$\varepsilon G(Z, \sum\nolimits_i y_i) Z^{(N)}(\sum\nolimits_i y_i), \tag{15}$$

but the number of summand of the form (14) is not larger than Δ_N. Similarly, using Lemma 4, relations on the constant K (see (9)) and by inductive assuption, we get the same for other summand; then we apply Lemma 2. But by (10) and (13) we have

$$(\partial_\beta \Big[\varepsilon G(Z, \sum\nolimits_i y_i)(\Delta_N Z^{(N)}(\sum\nolimits_i y_i) + 1) +$$
$$+ (KG(Z, \sum\nolimits_i y_i) - KC)\Delta_N Z^{(N)}(\sum\nolimits_i y_i)\Big])|_{y=0} =$$
$$\partial_{\alpha_0} Z(\sum_i y_i)|_{y=0} = \alpha_0! Z_{\alpha_0}$$

This proves the Lemma. □

4.6. **The completing of the proof of the Theorem.** To complete the proof we have to note that the general case can be reduced to the case considered above. Indeed, instead of the initial system one can consider the finite tuple of differential corolaries of the equations of the system

$$\partial_\beta \partial_{\gamma_i} z = \partial_\beta f_i(x, \partial_\alpha z) + \partial_\beta M_i(x, \partial_\alpha z),$$

where $|\beta| + |\gamma_i| = N$, N is big enough natural number (for example, one can take $N = \max_i |\gamma_i| + 1$). This new system satisfies all the conditions of the considered special case. Formal power series $z(x)$ iz a formal solution to the new system. The set $\mathbb{Z}^n_{\geqslant 0} \setminus I$ changes for finite tuple of elements and it does not affect the convergence of the power series $\tilde{z}(x)$.

5. Remarks and examples

5.1. **Example. The case of one equation.** Consider the PDE

$$\partial_\gamma z = F(x, \partial_\alpha z) + M(x, \partial_\alpha z).$$

Suppose that all conditions of Theorem 1 are satisfied. In this case it is obvious that there exists a unique formal solution to this equation with the following initial conditions $\partial_\alpha z|_{x=0} = \alpha! z_\alpha$, where α lies in the set $\mathbb{Z}^n_{\geqslant 0} \setminus O(\gamma)$ and z_α is an arbitrary complex number, i.e., as an initial conditions we assign on an arbitrary way the coefficients of soughted solution by the monomials x^α with $\alpha \in \mathbb{Z}^n_{\geqslant 0} \setminus O(\gamma)$. The power series $\tilde{z}(x)$ coincides with the power series of the initial conditions. For this case Theorem 1 claims the convergence of formal solution, if the power series of the initial conditions converges.

5.2. **Example. On the necessity of conditions of the Theorem.** Consider the equation

$$(1+i)\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial y^2}.$$

For this equation the conditions of Theorem 1 does not hold for any total ordering sastisfying conditions i) and ii) of Section 2. In this case it is easy to construct a nontrivial formal solution $z = \sum z_\alpha x^\alpha$ such that $z_\alpha = 0$ for any $\alpha \in \mathbb{Z}^2_{\geqslant 0} \setminus O((1,1))$ but the formal power series $z = \sum z_\alpha x^\alpha$ does not converge at any point except the origin. Indeed, soughted formal solution is uniquely difined by the following conditions

$z_{(0,n)} = 0$ for any non-negative integer n;

$z_{(1,n)} = n!(1-i)$, if the remainder on dividing non-negative integer n by 4 is equal to 3, and $z_{(1,n)} = 0$ otherwise.

5.3. **The case of several unknown functions.** It is important to note that Theorem1 may be obviously generalized to the case of systems of several unkown functions $z_1, \dots, z_p$.

The set of partial derivatives of the tuple of unknown function $z_1, \dots, z_p$ is parametrize by points of the product $Z = \mathbb{Z}^n_{\geqslant 0} \times \{1, \dots, p\}$. Let $\prec_{\mathbb{Z}^n_{\geqslant 0}}$ be a total ordering on the semigroup $\mathbb{Z}^n_{\geqslant 0}$ satisfying conditions i) and ii) of Section 2. Consider the following total ordering $\prec$ on the set Z. For any two elements (α, i) and (β, j) of the set Z we compare first the element α and β of the semigroup $\mathbb{Z}^n_{\geqslant 0}$ with respect to the total ordering $\prec_{\mathbb{Z}^n_{\geqslant 0}}$, if the element coincides, we compare (as integer numbers) numbers i and j. Repeating of the argument of this work, it is simple to extend Theorem 1 to this case.

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