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On the Entropy of Quotients

Vincent A Ssembatya

ABSTRACT. Bowen [**2**] proved a theorem on the entropy of the quotient of spaces, giving a useful estimate in computations about compact spaces. It would be desirable to have this extend to locally compact situations. We construct a locally compact (but non compact) space on which Bowen's Theorem on the Entropy of quotient of spaces does not apply.

1. Introduction

It is often desirable to compute the topological entropy for quotients of spaces and maps. In the case when all spaces are compact, Bowen's theorem provides a good upper bound for the entropy of the lower maps. It sometimes happens however, as in the case of Tori and their covers (or in the more complicated situations involving solenoids), that the spaces upstairs are only locally compact. It is natural to attempt to use Bowen's theorem in these situations. It is the purpose of this paper to show that Bowen's theorem does not cover all situations in which spaces upstairs are only locally compact.

1.1. Topological Entropy. Let (X, d) be a metric space and $T : X \to X$ be a uniformly continuous map. For subsets $F, K \subset X$, say that $F(n, \epsilon)$-spans K (with respect to T) provided that for each x in K, there is a y in F for which

$$d(T^j(x), T^j(y)) \leq \epsilon$$

for all $0 \leq j \leq n$.

For a compact set $K \subset X$ let $r_n(\epsilon, K)$ be the smallest cardinality of any set F which (n, ϵ)-spans K (with respect to T). One can interpret the meaning of that quantity as the minimal number of initial conditions whose behavior up to time n approximates the behavior of any initial condition up to ϵ. Let

$$r_T(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\epsilon, K)$$

Thus $r_T(\epsilon, K)$ measures the exponential growth rate for the quantity $r_n(\epsilon, K)$. The proof of the following lemma is in Bowen [Bo].

LEMMA 1. *Let (X; d) be a metric space. Let $T : X \to X$ be a uniformly continuous map. Let $\epsilon > 0$. For each compact set $K \subset X$ and for each positive integer n, the quantity $r_n(\epsilon, K) < \infty$.*

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For a uniformly continuous map T on X and $K \subset X$ compact, set $h_d(T,K) = lim_{\epsilon \to 0} r_T(\epsilon, K)$ and

$$h_d(T) = \sup_{K\ compact} h_d(T,K)$$

Let (X,d), (Y,e) be metric spaces and $T : X \to X$, $S : Y \to Y$, $\pi : X \to Y$ (surjective), be continuous maps with $\pi \circ T = S \circ \pi$.

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \big\downarrow & & \pi \big\downarrow \\ Y & \xrightarrow{S} & Y \end{array}$$

Bowen [**2**]showed that if X and Y are compact, then

$$h_d(T) \leq h_e(S) + \sup_{y \in Y} h_d(T, \pi^{-1}(y)).$$

By means of a counter example, we show that the compactness requirement cannot be dropped from the hypothesis of this theorem. In this example, we have a uniformly continuous function T with positive entropy yet

$$h_e(S) = \sup_{y \in Y} h_d(T, \pi^{-1}(y)) = 0.$$

2. An Example to which Bowen's theorem does not Extend

Let $\Sigma_2 = \prod_{n=-\infty}^{\infty} \{0,1\}$, be the space of binfinite sequences of 0's and 1's. A point $s \in \Sigma_2$ is represented as $s = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots\}$ where $s_i \in \{0,1\}$ for $i = 0, 1, \ldots$. We shall write $s = \{\ldots s_{-n}, \ldots s_{-1}.s_0 s_1 \ldots s_n \ldots\}$ to the effect of separating the symbol sequence into two parts with both parts being infinite.

For our example let X be the union of the spaces X_1 and X_2 described below.

$$X_1 = A \times \Sigma_2.$$

where $A = \mathbb{N} \times \{2^s | s \in \mathbb{Z}\}$ and

$$X_2 = B \times \{\{\ldots 0 \ldots 0.00 \ldots 0 \ldots\}\}$$

where $B = \mathbb{N} \times \{0\} = \bar{A} \backslash A$ with the operation of closure taken in $\mathbb{R}^2$. Generally, an element $x \in X$ will be of the form (x_1, x_2, s) where $x_1, x_2 \in \mathbb{R}$ and $s \in \Sigma_2$.

2.1. The Metric on X.

For $x = (x_1, x_2, s)$, $x' = (x_1', x_2', s') \in X$ define the distance between them to be

$$d(x, x') = \sup\{|x_1 - x_1'|, |x_2 - x_2'|, \sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0, t_1 + t_2\}} s_i - 2^{\min\left\{0, t_1' + t_2'\right\}} s_i'\right|}{2^{|i|}}\}$$

where t, and t' are such that $x_2 = 2^t$ and $x_2' = 2^{t'}$. Roughly (x_1, x_2, s) and (x_1', x_2', s') are "close" together if their first and second coordinates are respectively "close" together in the euclidean metric and the Σ_2 coordinates agree on a long central block.

Claim 1.

Theorem 1. *With the above, X is a locally compact metric space.*

Proof. Clearly $d(x,x') \geq 0$ for all $x,x' \in X$ with equality if and only if $x = x'$. Also $d(x,x') = d(x',x)$ for all $x,x' \in X$. Now suppose $x = (x_1,x_2,s)$, $x' = (x'_1,x'_2,s')$, $x = (x''_2,x''_2,s'') \in X$, then

$$d(x,x') = d((x_1,x_2,s),(x'_1,x'_2,s'))$$

$$= \sup\left\{|x_1-x'_1|,|x_2-x'_2|,\sum_{i=-\infty}^{\infty}\frac{\left|2^{\min\{0,t_1+t_2\}}s_i-2^{\min\{0,t'_1+t'_2\}}s'_i\right|}{2^{|i|}}\right\}$$

$$\leq \sup\{|x_1-x''_1|+|x''_1-x'_1|,|x_2-x''_2|+|x''_2-x'_2|,$$

$$\sum_{i=-\infty}^{\infty}\frac{\left|2^{\min\{0,t_1+t_2\}}s_i-2^{\min\{0,t''_1+t''_2\}}s''_i+2^{\min\{0,t''_1+t''_2\}}s''_i-2^{\min\{0,t'_1+t'_2\}}s'_i\right|}{2^{|i|}}\}$$

$$\leq \sup\{|x_1-x''_1|+|x''_1-x'_1|,|x_2-x''_2|+|x''_2-x'_2|,$$

$$\sum_{i=-\infty}^{\infty}\frac{\left|2^{\min\{0,t_1+t_2\}}s_i-2^{\min\{0,t''_1+t''_2\}}s''_i\right|+\left|2^{\min\{0,t''_1+t''_2\}}s''_i-2^{\min\{0,t'_1+t'_2\}}s'_i\right|}{2^{|i|}}\}$$

$$\leq \sup\{|x_1-x''_1|+|x''_1-x'_1|,|x_2-x''_2|+|x''_2-x'_2|,$$

$$\sum_{i=-\infty}^{\infty}\frac{\left|2^{\min\{0,t_1+t_2\}}s_i-2^{\min\{0,t''_1+t''_2\}}s''_i\right|}{2^{|i|}}+\sum_{i=-\infty}^{\infty}\frac{\left|2^{\min\{0,t''_1+t''_2\}}s''_i-2^{\min\{0,t'_1+t'_2\}}s'_i\right|}{2^{|i|}}\}$$

$$= d(x,x'')+d(x'',x').$$

Next we prove that (X,d) is locally compact. Let $\epsilon > 0$ be given (we may assume $0 < \epsilon < \frac{1}{2}$). Consider an $\epsilon-$neighborhood U of $(0,0,\bar{0})$, where $\bar{0} = \{\ldots 0\ldots 0.00\ldots 0\ldots\}$. Let N be the smallest integer such that $2^N \geq \epsilon$ .Then the subspace $V = \left\{(0,2^j,s)|j<N, s\in\Sigma_2\right\} \subset U$ and $\bar{V}$ is compact. Observe that it's enough to check for such a neighbourhood. ☐

Let $Y = \bar{A}$ be equiped with the euclidean metric from $\mathbb{R}^2$.
Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map defined by

$\sigma(\{\ldots,s_{-n},\ldots,s_{-1},s_0,s_1,\ldots,s_n,\ldots\}) = \{\ldots s_{-n}\ldots s_{-1}s_0.s_1\ldots s_n\ldots\}$

and let $\tau : \Sigma_2 \to \Sigma_2$ be the identity map on Σ_2. Define $T : X \to X$ by

$$T(x_1,x_2=2^t,s) = (x_1+1,x_2,\sigma(s)) \text{ if } x_1 \leq t$$
$$T(x_1,x_2=2^t,s) = (x_1+1,x_2,\tau(s)) \text{ if } x_1 > t$$

and $T(x,0,\bar{0}) = (x+1,0,\bar{0})$

Claim 2. *T is uniformly continuous on X*

Proof. This follows from the fact that $T\left(B_{\frac{\epsilon}{2}}(x)\right) \subset B_\epsilon(T(x))$, where $B_a(z)$ stands for the ball of radius a around the point z. ☐

THEOREM 2. *T has positive topological entropy.*

PROOF. We need to find a compact set K such that $h_d(T, K) > 0$. Let K be the closure of $\left\{(x_1, x_2, s) \in X | x_1 = 0, x_2 \leq \frac{1}{2}\right\} \subseteq cl_X(B_{\frac{1}{2}}(0))$. For any given $\epsilon > 0$ let m be the largest integer such that $0 < \epsilon \leq \frac{1}{2^m}$.

Then

$$\begin{aligned}
r_0(\epsilon, K) &\geq 2^m + 2^{m-1} + \cdots + 2 + 1 + 1 \\
r_1(\epsilon, K) &\geq 2^m + 2^m + 2^{m-1} + \cdots + 2^2 + 2 + 2 \ (m{+}2 \text{ terms}) \\
r_2(\epsilon, K) &\geq 2^m + 2^m + 2^m + 2^{m-1} + \cdots + 2^2 + 2 + 2 \\
&\cdots \\
r_2(\epsilon, K) &\geq (m+1)2^m + 2^n
\end{aligned}$$

So that $\sup \frac{1}{n} \log r_n(\epsilon, K) \geq \sup \frac{1}{n} \log(m2^m + 2^n) \geq \sup \frac{1}{n} \log 2^n = \log 2$.
Therefore $h_d(T, K) = \lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) \geq \log 2$. $\square$

Let $Y = \bar{X}_1$ and π be the corresponding quotient map from X to Y. $S : Y \to Y$ be the map $S(y_1, y_2) = (y_1 + 1, y_2)$.

CLAIM 3.

THEOREM 3. *S has topological entropy zero.*

PROOF. For any compact subset K of X and any given $\epsilon > 0$, $r_n(\epsilon, K)$ is constant for all n. It follows that $\lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) = 0$ for every $\epsilon > 0$ and therefore the topological entropy of S is 0. $\square$

THEOREM 4. *$h_d(T, \pi^{-1}(y)) = 0$ for every $y \in Y$.*

PROOF. Let $y = (y_1, y_2) \in Y$, $y_1 = t_1, y_2 = 2^{t_2}$, $t_1, t_2 \in \mathbb{Z}$. So for any x, x' $\in \pi^{-1}(y)$,

$$d(x, x') = 2^{\min\{0, t_1+t_2\}} \sum_{i=-\infty}^{\infty} \frac{|s_i - s_i'|}{2^{|i|}}$$

where $x = (t_1, 2^{t_2}, s)$, $x' = (t_1, 2^{t_2}, s')$.

Since $T(x_1, 2^t, s) = (x_1 + 1, 2^t, s)$ if $x_1 > t$ and $T(x_1, 2^t, s) = (x_1 + 1, 2^t, \sigma(s))$ if $x_1 \leq t$, there exists an integer m such that $T^j((x_1, 2^t, s)) = (x_1 + j, 2^t, \sigma^m(s))$ for all $j \geq m$.

It follows that if K is a compact subset of $\pi^{-1}(y)$, for a given $\epsilon > 0$, $r_n(\epsilon, K)$ attains a maximum (as n increases) of $r_m(\epsilon, K)$. So $\lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) = \lim_{n\to\infty} \sup \frac{1}{n} \log r_m(\epsilon, K) = 0$ and the result follows. $\square$

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