ISSN: 1401-5617

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Daniela Ferrarello
Ralf Fröberg

Research Reports in Mathematics
Number 5, 2004

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2004/5

Date of publication: April 19, 2004
2000 Mathematics Subject Classification: Primary 13O25.
Keywords: Clique complex, Hilbert series, matching complex, independent set complex.

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

THE HILBERT SERIES OF THE CLIQUE COMPLEX

DANIELA FERRARELLO

Department of Mathematics and Computer Science

University of Catania

Viale Andrea Doria 6, 95125 Catania, Italy

e-mail: ferrarello@dmi.unict.it

RALF FRÖBERG

Department of Mathematics

Stockholm University

SE-106 91 Stockholm, Sweden

e-mail: ralff@math.su.se

Abstract

For a graph G, we show a theorem that establishes a correspondence between the fine Hilbert series of the Stanley-Reisner ring of the clique complex for the complementary graph of G and the fine subgraph polynomial of G. We obtain from this theorem some corollaries regarding the independent set complex and the matching complex.

1 Introduction

There are several simplicial complexes we can associate to a graph.
Recall that a simplicial complex is a collection of subsets of a set V, called faces, including all the singletons and all the subsets of an element in the simplicial complex.
It is thus natural to associate to a graph simplicial complexes for properties which are "closed by subsets". Here we consider the *clique complex*, whose faces are the cliques of the graph, i.e. its complete subgraphs, the *independent set complex*, whose faces are the independent sets of the graph, i.e. the sets of vertices having no edges in common and the *matching complex*, whose faces are the matchings of the graph, i.e. the sets of edges sharing no vertices.
Precise definitions will be given later in the paper.

To a simplicial complex one can associate its *Stanley-Reisner ring*, that is the quotient ring of the polynomial ring over a field in the n variables (where n is the number of vertices) modulo the ideal generated by those monomials representing non-faces of the simplicial complex. This ring is also called the *face ring.*

Paul Renteln proved in 2002 that the Hilbert series of the face ring of a clique complex (in a few words the generating function of the dimension of the graded parts of the face ring) is related with a polynomial, called the *subgraph polynomial*; this last is a polynomial in two variables, say x and y, where the coefficient of $x^i y^j$ is the number of subgraphs of the graph with i vertices and j edges.

The theorem by Renteln is proved via homological algebra. Here we will give a more precise result than the one given by Renteln and moreover we will obtain the proof by pure combinatorics, without using any concepts of homological algebra.

2 Basic concepts

2.1 The Hilbert Series

DEFINITION 2.1. *Let $R = K[x_1, \dots, x_n]/I$ be a homogeneous algebra. Then $R = \bigoplus_{i \geq 0} R_i$, where R_i is the vector space of the homogeneous elements of R of degree i. The* Hilbert function *of R is*

$H(R,i) = dim_K R_i,$

and the Hilbert series *of* R *is*

$H_R(t) = \sum_{i\geq 0} H(R,i)t^i.$

If I is generated by monomials, then R is $\mathbb{N}^n$-graded, $R = \bigoplus_{\alpha\in\mathbb{N}^n} R_\alpha$, and we can define the *fine Hilbert series* of R in the following way

DEFINITION 2.2. *If* $R = K[x_1,\dots,x_n]/I$ *and* I *is generated by monomials, then the* fine Hilbert series *is*

$\mathbf{H}_R(t) = \sum_{\alpha\in\mathbb{N}^n} dim_K R_\alpha \mathbf{t}^\alpha$

where $\mathbf{t}^\alpha = t_1^{\alpha_1},\dots,t_n^{\alpha_n}$ *if* $\alpha = (\alpha_1,\dots,\alpha_n)$.

If $M = \bigoplus_{i\geq 0} M_i$ is a finitely generated module over a homogeneous algebra R (or $M = \bigoplus_{\alpha\in\mathbb{N}^n} M_\alpha$ a finitely generated module over an $\mathbb{N}^n$-graded algebra, respectively), we can extend the definitions of Hilbert series above. We set $H_M(t) = \sum_{n\geq 0} dim_K M_n t^n$ and $\mathbf{H}_M(t_1,\dots,t_n) = \sum_{\alpha\in\mathbb{N}} dim_K M_\alpha \mathbf{t}^\alpha$, respectively.

It is clear that $\mathbf{H}_M(t,\dots,t) = H_M(t)$.

If $R = K[x_1,\dots,x_n]$ then $\mathbf{H}_R(t_1,\dots,t_n) = 1/\prod_{i=1}^n(1-t_i)$. If $R = K[x_1,\dots,x_n]/I$ where I is generated by monomials, then $\mathbf{H}_R(t_1,\dots,t_n) = \mathbf{H}_{K[x_1,\dots,x_n]}(t_1,\dots,t_n) - \mathbf{H}_I(t_1,\dots,t_n)$.

If I is generated by the monomial $x_1^{i_1}\cdots x_n^{i_n}$, then

$\mathbf{H}_I(t_1,\dots,t_n) = t_1^{i_1}\cdots t_n^{i_n}\mathbf{H}_{K[x_1,\dots,x_n]}(t_1,\dots,t_n).$

EXAMPLE 2.1. *If* $R = K[x_1, x_2]/(x_1x_2)$*, then*

$\mathbf{H}_R(t_1, t_2) = 1/((1 - t_1) \cdot (1 - t_2)) - (t_1t_2)/((1 - t_1) \cdot (1 - t_2)) =$

$= (1 - t_1t_2)/((1 - t_1) \cdot (1 - t_2))$*, and thus*

$H_R(t) = (1 - t^2)/(1 - t)^2 = (1 + t)/(1 - t)$*.*

2.2 Graphs and Simplicial Complexes

Here we define the topics about graphs and simplicial complexes we will use in the paper.

DEFINITION 2.3. *The* complementary graph *of* $G = (V, E)$ *is the graph* $\bar{G}$ *with the vertices of* V *and edges all the couples* $\{v_i, v_j\}$ *such that* $i \neq j$ *and* $\{v_i, v_j\} \notin E$*.*

DEFINITION 2.4. *The* line graph *of* $G = (V, E)$ *is the graph* $L(G)$ *whose vertices are the edges of* G *and two vertices of* $L(G)$ *are joined is they share a vertex in common as edges of* G*.*

DEFINITION 2.5. *A* clique *of a graph* G *is a complete subgraph of* G*.*

DEFINITION 2.6. *An* independent set S *of a graph* $G = (V, E)$ *is a subset of* V *such that no vertices in* S *share an edge in* E*.*

DEFINITION 2.7. *A* matching M *of a graph* $G = (V, E)$ *is a subset of* E *such that no edges in* M *share a vertex in* V*.*

DEFINITION 2.8. *A* simplicial complex $\triangle$ *on the set* $V = \{v_1, \dots, v_n\}$ *is a set of subsets of* V*, called* faces *or* simplices *such that* $\{v_i\} \in \triangle, \forall i$ *and every subset of a face is itself a face.*

If we have a graph G we can associate several simplicial complexes to it. Among these, the clique complex, the independent set complex and the matching one.

DEFINITION 2.9. *The* clique complex $\triangle(G)$ *is the simplicial complex whose faces are the cliques of* G*.*

DEFINITION 2.10. *The* independent set complex $\bar{\triangle}(G)$ *is the simplicial complex whose faces are the independent sets of* G*.*

Since a clique of G is clearly an independent set of $\bar{G}$, it is easy to note that $\bar{\triangle}(G) = \triangle(\bar{G})$.

DEFINITION 2.11. *The* matching complex $\triangle_M(G)$ *is the simplicial complex whose faces are all the subsets of the edges which are matchings of* G*.*

By definition, the matchings of G are all the independent sets of the line graph $L(G)$. So, we can easily notice that $\triangle_M(G) = \bar{\triangle}(L(G))$.

2.3 The Subgraph Polynomial

The definition and some properties of the subgraph polynomial can be found in [Ren].

DEFINITION 2.12. *If G is a graph on the vertex set V and with edges in E, the* Subgraph Polynomial *is the following polynomial:*

$S_G(x, y) = \sum_{ij} b_{ij} x^i y^j$

where b_{ij} is the number of subgraphs of G with i vertices and j edges.

We will also define here a finer invariant:

DEFINITION 2.13. *If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, the* fine Subgraph Polynomial *of G is $\mathbf{S_G}(x_1, \ldots, x_n, y) = \sum_{T \subset E} x_{i_1} \cdots x_{i_t} y^{|T|}$, where $v_{i_1} \ldots v_{i_t}$ are the vertices involved in the edges in T.*

Clearly, $\mathbf{S}_G(x, \ldots, x, y) = S_G(x, y)$.

3 The Hilbert series of the Stanley-Reisner ring of $\triangle(G)$

The theorem we want to show is the following:

THEOREM 3.1. *Let G be a simple graph on the vertex set $V = \{v_1, \dots, v_n\}$ and let $\triangle(\bar{G})$ be the clique complex of the complement of G.*

Then, if R is the Stanley-Reisner ring of $\triangle(\bar{G})$, we have

$\mathbf{H}_R(v_1, \dots, v_n) = \mathbf{S}_G(v_1, \dots, v_n, -1)/\prod_{i=1}^{n}(1 - v_i)$.

Proof

The Stanley-Reisner ring of $\triangle(\bar{G})$ is the quotient $R = K[v_1, \dots, v_n]$ / $I_{\triangle(\bar{G})}$, where $I_{\triangle(\bar{G})}$=($\{v_{i_1} \cdots v_{i_r}$ s.t. $i_1 < \dots < i_r$, $\{v_{i_1}, \dots, v_{i_r}\} \notin \triangle(\bar{G})\}$).

Now, if $\{v_{i_1}, \dots, v_{i_r}\} \notin \triangle(\bar{G})$ it means that the subgraph induced by the set of vertices $\{v_{i_1}, \dots, v_{i_r}\}$ is not complete and so $\exists$ at least one edge, call it $\{v_{i_1}, v_{i_2}\}$, that does not belong to the edges of $\bar{G}$, i.e. $\{v_{i_1}, v_{i_2}\} \in E(G)$.

So, we can write $I_{\triangle(\bar{G})}$=($\{v_{i_1} \cdot v_{i_2}$ s.t. $i_1 < i_2$, $\{v_{i_1}, v_{i_2}\} \in G\}$).

The fine Hilbert series of a polynomial ring modulo an ideal generated by monomials is the sum of all the nonzero monomials, so in general, to compute the Hilbert series of the quotient $K[v_1, \dots, v_n]/$ $(m_1, \dots, m_k)$, where the $m_i's$ are the monomials generating the ideal, one can use the following formula coming from the principle of inclusion-exclusion:

$\mathbf{H}(K[v_1, \dots, v_n]/(m_1, \dots, m_k))$=$\mathbf{H}(K[v_1, \dots, v_n])$-$\sum_{i=1}^{k} \mathbf{H}(m_i)$+$\sum_{i<j} \mathbf{H}(m_i \cap m_j)$ -$\sum_{i<j<h} \mathbf{H}(m_i \cap m_j \cap m_h)$+....

It is easy to see that $(m_{i_1}) \cap \dots \cap (m_{i_h}) = l.c.m.(m_{i_1}, \dots, m_{i_h})$.

In our case $\mathbf{H}(R)$=$\mathbf{H}(K[v_1,\ldots,v_n]/(v_{1_1}\cdot v_{1_2},\ldots,v_{k_1}\cdot v_{k_2}))=$
$=\frac{1}{\prod_{i=1}^{k}(1-v_i)}\cdot\ [(-1)^1\sum_{i=1}^{k}v_{i_1}\cdot v_{i_2}+(-1)^2\sum_{i<j}l.c.m.(v_{i_1}\cdot v_{i_2},v_{j_1}\cdot v_{j_2})+$
$+(-1)^3\sum_{i<j<h}l.c.m.(v_{i_1}\cdot v_{i_2},v_{j_1}\cdot v_{j_2},v_{h_1}\cdot v_{h_2})+\ldots].$

Notice that the coefficient of the sums of subgraphs with an odd number of edges is -1 and the one of subgraphs with an even number of edges is 1.

And the fine Subgraph Polynomial computed for $y=-1$ is
$\mathbf{S_G}(v_1,\ldots,v_n,-1)=1+(\sum_{i=1}^{|E|}v_{i_1}\cdot v_{i_2})(-1)^1+\sum_{i<j}l.c.m.(v_{i_1}\cdot v_{i_2},v_{j_1}\cdot v_{j_2})(-1)^2$
$+\sum_{i<j<h}l.c.m.(v_{i_1}\cdot v_{i_2},v_{j_1}\cdot v_{j_2},v_{h_1}\cdot v_{h_2})(-1)^3+\ldots].$

So, the thesis holds. □

By the last theorem we can easily obtain three corollaries, two of these last are the results given by Renteln.

COROLLARY 3.1. *Let G be a graph on n vertices and let $\triangle(\bar{G})$ be the clique complex of the complementary graph and $\bar{\triangle}(G)$ be the independent sets complex of G. Then the Hilbert series of $K[\triangle(\bar{G})]$ is given by*
$H_{K[\bar{\triangle}(G)]}(v)=H_{K[\triangle(\bar{G})]}(v)=S_G(v,-1)/(1-v)^n.$

COROLLARY 3.2. *Let G be a graph on n vertices and let $\triangle(G)$ be its clique complex. Then the Hilbert series of $K[\triangle(G)]$ is given by*
$H_{K[\triangle(G)]}(v)=S_{\bar{G}}(v,-1)/(1-v)^n.$

COROLLARY 3.3. *Let G be a graph on n vertices and let $\triangle_M(G)$ be its match-*

ing complex. Then the Hilbert series of $K[\triangle_M(G)]$ *is given by* $H_{K[\triangle_M(G)]}(v) = H_{K[\bar{\triangle}(L(G))]} = S_{L(G)}(v, -1)/(1-v)^n$.

References

[Ren] Renteln P. *"The Hilbert series of the Face Ring of a Flag Complex" Graphs and Combinatorics* **18** *(2002) no.3 605–619*