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p-Laplacian Problems where the Nonlinearity Crosses an Eigenvalue

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Abstract

Using linking arguments and a cohomological index theory we obtain nontrivial solutions of p-Laplacian problems with nonlinearities that interact with the spectrum.

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1 Introduction

Consider the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta_p\, u = f(x,u) \text{ in } \Omega, \\ \qquad u = 0 \qquad \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 1$, $\Delta_p\, u = \operatorname{div}(|\nabla u|^{p-2}\, \nabla u)$ is the p-Laplacian, $1 < p < \infty$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$\frac{f(x,t)}{|t|^{p-2}\, t} \to \begin{cases} \lambda_0 & \text{as } t \to 0, \\ \lambda_\infty & \text{as } |t| \to \infty \end{cases} \quad \text{uniformly in } x \tag{1.2}$$

with $\lambda_0, \lambda_\infty \notin \sigma(-\Delta_p)$, the Dirichlet spectrum of $-\Delta_p$ on Ω. In the semilinear case $p = 2$, a well-known theorem of Amann and Zehnder [1] states that this problem has a nontrivial solution if there is an eigenvalue λ_l of $-\Delta$ between λ_0 and λ_∞. In this paper we extend their result to the quasilinear case $p \ne 2$.

The quasilinear problem is far more difficult as a complete description of the spectrum is not available and there are no eigenspaces to work with. Although there is a sequence of variational eigenvalues $\lambda_l \nearrow \infty$ defined by a standard minimax scheme involving the Krasnoselskii genus it is not known whether this is a complete list when $n > 1$. Using the cohomological index of Fadell and Rabinowitz [9] we will construct an unbounded sequence of minimax eigenvalues $\mu_l \ge \lambda_l$ for which the following theorem holds.

Theorem 1.1. *Assume that $\mu_{l-1} < \mu_l$. Then for each $\varepsilon_0 \in (0, \mu_l - \mu_{l-1})$, there is an eigenvalue $\widetilde{\mu}_l \ge \mu_l$ such that problem* (1.1) *has a nontrivial solution if*

$$F(x,t) := \int_0^t f(x,s)\, ds \ge \frac{1}{p}\, (\mu_{l-1} + \varepsilon_0)\, |t|^p \quad \forall (x,t) \tag{1.3}$$

and

(i) $\lambda_0 < \mu_l \le \widetilde{\mu}_l < \lambda_\infty$, *or*

(ii) $\lambda_\infty < \mu_l \le \widetilde{\mu}_l < \lambda_0$.

In the ODE case $n = 1$, $\widetilde{\mu}_l = \mu_l = \lambda_l$.

It follows from (1.2), (1.3) and l'Hospital's rule that $\lambda_0, \lambda_\infty > \mu_{l-1} + \varepsilon_0$. It will be easily seen from our construction of μ_l that μ_1 is the smallest eigenvalue of $-\Delta_p$, and it is well known that $\mu_1 < \mu_2$ (see, e.g., Drábek, Kufner, and Nicolosi [8], Theorem 3.1 and Lemma 3.9). Hence if $\lambda_0 > \mu_1$ (respectively $\lambda_\infty > \mu_1$) is given, there always exists $l \geq 2$ such that $\mu_{l-1} < \lambda_0 < \mu_l$ (or $\mu_{l-1} < \lambda_\infty < \mu_l$). If $l = 1$ or 2, the conclusions above are known. Moreover, in these cases $\widetilde{\mu}_l = \mu_l$ and hypothesis (1.3) is unnecessary (see Dancer and Perera [5]). We suspect that (1.3) is unnecessary and one can take $\widetilde{\mu}_l = \mu_l$ also if $l \geq 3$.

When f is odd in t we will also prove the following multiplicity result.

Theorem 1.2. *Assume that f is odd in t for all x. Then problem* (1.1) *has $m - l$ pairs of nontrivial solutions if*

(i) $\mu_{l-1} < \lambda_0 < \mu_l \leq \mu_{m-1} < \lambda_\infty < \mu_m$, *or*

(ii) $\mu_{l-1} < \lambda_\infty < \mu_l \leq \mu_{m-1} < \lambda_0 < \mu_m$.

Case (i) of Theorem 1.2 generalizes a recent result of Li and Zhou [10] where it was assumed that $\lambda_0 = 0$ and a different sequence of numbers $\geq \mu_l$ (not necessarily eigenvalues) was used.

2 Cohomological Index

Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of W. Fadell and Rabinowitz constructed an index theory $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ with the following properties ([9], Sections 5 and 6, see also Bartsch [2], Example 4.4 and Remark 4.6):

(i) Definiteness: $i(A) = 0 \iff A = \emptyset$.

(ii) Monotonicity: If there is an odd map $A \to A'$, then

$$i(A) \leq i(A'). \tag{2.1}$$

In particular, equality holds if A and A' are homeomorphic.

(iii) Subadditivity: $i(A \cup A') \leq i(A) + i(A')$.

(iv) Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that

$$i(N) = i(A). \tag{2.2}$$

(v) Neighborhood of zero: If U is a bounded symmetric neighborhood of 0 in W, then

$$i(\partial U) = \dim W. \tag{2.3}$$

(vi) Stability: If A is closed and $A * \mathbb{Z}_2$ is the join of A with $\mathbb{Z}_2$, realized in $W \oplus \mathbb{R}$, then

$$i(A * \mathbb{Z}_2) = i(A) + 1. \tag{2.4}$$

(vii) Piercing property: If A, A_0, A_1 are closed and $\varphi : A \times [0,1] \to A_0 \cup A_1$ is an odd map such that $\varphi(A \times [0,1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$, $\varphi(A \times \{1\}) \subset A_1$, then

$$i(\varphi(A \times [0,1]) \cap A_0 \cap A_1) \geq i(A). \tag{2.5}$$

For a definition of join $A * B$ we refer, e.g., to Bartsch [2]. Here we only recall that if $\mathbb{Z}_2 = \{1, -1\} \subset \mathbb{R}$, then $A * \mathbb{Z}_2$ is the union of all line segments in $W \oplus \mathbb{R}$, joining $\{1\}$ and $\{-1\}$ to points of A. Hence $A * \mathbb{Z}_2$ is the suspension of A.

Note that $i(A) \leq \gamma(A)$, where γ denotes the Krasnoselskii genus. Indeed, if $\gamma(A) = k < \infty$, then there exists an odd map $A \to S^{k-1}$, hence by (ii) and (v), $i(A) \leq i(S^{k-1}) = k$. We also note that for compact A the Fadell-Rabinowitz index is equivalent to that of Yang [9].

3 Variational Eigenvalues

Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space, normed by

$$\|u\| := \left(\int_\Omega |\nabla u|^p \right)^{1/p}. \tag{3.1}$$

We see from the Lagrange multiplier rule that the Dirichlet eigenvalues of the p-Laplacian are the critical values of the functional

$$I(u) = \frac{1}{\displaystyle\int_\Omega |u|^p}, \quad u \in S := \Big\{u \in W = W_0^{1,p}(\Omega) : \|u\| = 1\Big\}. \tag{3.2}$$

We use the customary notation

$$I^c := \Big\{u \in S : I(u) \le c\Big\}, \qquad I_c := \Big\{u \in S : I(u) \ge c\Big\}. \tag{3.3}$$

Note for further reference that if $u \neq 0$, then

$$\frac{u}{\|u\|} \in I^c \quad \Longleftrightarrow \quad \int_\Omega |\nabla u|^p \le c \int_\Omega |u|^p \tag{3.4}$$

and

$$\frac{u}{\|u\|} \in I_c \quad \Longleftrightarrow \quad \int_\Omega |\nabla u|^p \ge c \int_\Omega |u|^p. \tag{3.5}$$

Lemma 3.1. *I satisfies the Palais-Smale compactness condition* (PS)*, i.e., every sequence $\{u_j\}$ such that $\{I(u_j)\}$ is bounded and $I'(u_j) \to 0$, called a Palais-Smale sequence, has a convergent subsequence.*

Proof. Since $\|u_j\| = 1$ for all j, for a subsequence, u_j converges to some u weakly in W and strongly in $L^p(\Omega)$. Moreover, $u \neq 0$ as $\{I(u_j)\}$ is bounded. Let

$$J(u) := \|u\|^p \quad \text{and} \quad \widetilde{I}(u) := \frac{1}{\displaystyle\int_\Omega |u|^p}, \; u \in W \setminus \{0\}. \tag{3.6}$$

Then there exists a sequence $\{\nu_j\} \subset \mathbb{R}$ such that

$$I'(u_j) = \widetilde{I}'(u_j) - \nu_j J'(u_j) \to 0 \tag{3.7}$$

(cf. Willem [13], Proposition 5.12). Since

$$\langle \widetilde{I}'(u_j), u_j \rangle = -p\, \widetilde{I}(u_j) \quad \text{and} \quad \langle J'(u_j), u_j \rangle = p\, \|u_j\|^p = p, \tag{3.8}$$

$\nu_j = -\widetilde{I}(u_j) \to -\widetilde{I}(u) \neq 0$. Moreover, J' has a continuous inverse (see, e.g., Drábek, Kufner, and Nicolosi [8], Lemma 3.3), so it follows from follows from (3.7) that $u_j \to u$. □

Denote by $\mathcal{A}$ the class of compact symmetric subsets of S, let

$$\mathcal{F}_l := \Big\{A \in \mathcal{A} : i(A) \geq l\Big\}, \tag{3.9}$$

and set

$$\mu_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} I(u). \tag{3.10}$$

Proposition 3.2. *μ_l is an eigenvalue of $-\Delta_p$ and $\mu_l \nearrow \infty$.*

Proof. Note first that critical values of I coincide with eigenvalues of $-\Delta_p$. If μ_l is not a critical value of I, then there is an $\varepsilon > 0$ and an odd homeomorphism η of S such that $\eta(I^{\mu_l+\varepsilon}) \subset I^{\mu_l-\varepsilon}$ by the first deformation lemma. Let us remark here that since S is of class C^1 but not $C^{1,1}$ if $1 < p < 2$, the standard deformation lemma cannot be used. However, a more general version of it, see, e.g., Corvellec, Degiovanni, and Marzocchi [4] does apply in our situation. Taking $A \in \mathcal{F}_l$ with $\max I(A) \leq \mu_l + \varepsilon$, we have $A' = \eta(A) \in \mathcal{F}_l$, but $\max I(A') \leq \mu_l - \varepsilon$, contradicting (3.10).

Clearly, $\mu_{l+1} \geq \mu_l$. To see that $\mu_l \to \infty$, recall that this holds for the Ljusternik-Schnirelmann eigenvalues λ_l defined using the genus γ (see, e.g., Struwe [12]). But $i(A) \leq \gamma(A)$, so $\mu_l \geq \lambda_l$. □

If $\mu_{l-1} < \mu_l$ and $\varepsilon_0 \in (0, \mu_l - \mu_{l-1})$, take $A_0 \in \mathcal{F}_{l-1}$ with $A_0 \subset I^{\mu_{l-1}+\varepsilon_0/2}$, let

$$\mathcal{G} := \Big\{g \in C(CA_0, S) : g|_{A_0} = \mathrm{id}\Big\}, \tag{3.11}$$

where $CA_0 = (A_0 \times [0,1])/(A_0 \times \{1\})$ is the cone over A_0, and set

$$\widetilde{\mu}_l := \inf_{g \in \mathcal{G}} \max_{u \in g(CA_0)} I(u). \tag{3.12}$$

Proposition 3.3. *$\widetilde{\mu}_l \geq \mu_l$ is an eigenvalue of $-\Delta_p$.*

Proof. Let $g \in \mathcal{G}$. Regarding $A_0 * \mathbb{Z}_2$ as the suspension of A_0, g can be extended to an odd map $\widetilde{g} \in C(A_0 * \mathbb{Z}_2, S)$. Then $\widetilde{g}(A_0 * \mathbb{Z}_2) \in \mathcal{A}$ and

$$i(\widetilde{g}(A_0 * \mathbb{Z}_2)) \geq i(A_0 * \mathbb{Z}_2) = i(A_0) + 1 \geq l, \tag{3.13}$$

so

$$\max I(g(CA_0)) = \max I(\widetilde{g}(A_0 * \mathbb{Z}_2)) \geq \mu_l. \tag{3.14}$$

It follows that $\widetilde{\mu}_l \geq \mu_l$.

If $\widetilde{\mu}_l$ is not a critical value of I, then there is an $\varepsilon \in (0, \widetilde{\mu}_l - \mu_{l-1} - \varepsilon_0/2)$ and an odd homeomorphism η of S such that $\eta|_{A_0} = \text{id}$ and $\eta(I^{\widetilde{\mu}_l+\varepsilon}) \subset I^{\widetilde{\mu}_l-\varepsilon}$. Taking $g \in \mathcal{G}$ with $\max I(g(CA_0)) \le \widetilde{\mu}_l + \varepsilon$, we have $g' = \eta \circ g \in \mathcal{G}$, but $\max I(g'(CA_0)) \le \widetilde{\mu}_l - \varepsilon$, contradicting (3.12). □

4 Variational Setting

Solutions of (1.1) are the critical points of

$$\Phi(u) = \int_\Omega |\nabla u|^p - p\, F(x,u), \quad u \in W. \tag{4.1}$$

Lemma 4.1. *Φ satisfies* (PS).

Proof. First we show that every Palais-Smale sequence $\{u_j\}$ is bounded. Suppose that $\rho_j = \|u_j\| \to \infty$ for a subsequence. Setting $v_j = u_j/\rho_j$ and passing to a further subsequence, v_j converges to some v weakly in W and strongly in $L^p(\Omega)$. We have

$$\frac{1}{\rho_j^{p-1}} \langle \Phi'(u_j), w\rangle = \langle J'(v_j), w\rangle - p \int_\Omega \frac{f(x,u_j)}{|u_j|^{p-2} u_j}\, |v_j|^{p-2}\, v_j\, w \to 0. \tag{4.2}$$

If $v_j \rightharpoonup 0$, then it follows from (4.2) with $w = v_j$ that $p = p\,\|v_j\|^p = \langle J'(v_j), v_j\rangle \to 0$. Hence $v \ne 0$. For each $w \in W$, passing to the limit in (4.2) gives

$$\int_\Omega |\nabla v|^{p-2}\, \nabla v \cdot \nabla w - \lambda_\infty\, |v|^{p-2}\, v\, w = 0, \tag{4.3}$$

so λ_∞ is an eigenvalue of $-\Delta_p$, contrary to our assumption.

Since $\{u_j\}$ is bounded, for a subsequence, u_j converges to some u weakly in W and strongly in $L^p(\Omega)$. We have

$$\langle \Phi'(u_j), w\rangle = \langle J'(u_j), w\rangle - p \int_\Omega f(x,u_j)\, w \to 0, \tag{4.4}$$

so $u_j \to u$ (recall J' has a continuous inverse). □

Let

$$\Phi_0(u) := \int_\Omega |\nabla u|^p - \lambda_0\, |u|^p, \qquad \Phi_\infty(u) := \int_\Omega |\nabla u|^p - \lambda_\infty\, |u|^p. \tag{4.5}$$

In the proofs of Theorems 1.1 and 1.2 it will be convenient to replace Φ by the functional $\widetilde{\Phi}$ defined below.

Proposition 4.2. *For all sufficiently small $\rho > 0$ and sufficiently large $R > 4\rho$, there is a functional $\widetilde{\Phi} \in C^1(W, \mathbb{R})$ such that*

(i) $\widetilde{\Phi}(u) = \begin{cases} \Phi_0(u), & \|u\| \le \rho, \\ \Phi(u), & 2\rho \le \|u\| \le R/2, \\ \Phi_\infty(u), & \|u\| \ge R, \end{cases}$

(ii) $u = 0$ *is the only critical point of* Φ *and* $\widetilde{\Phi}$ *with* $\|u\| \le 2\rho$ *or* $\|u\| \ge R/2$, *in particular, critical points of* $\widetilde{\Phi}$ *are the solutions of* (1.1),

(iii) $\widetilde{\Phi}$ *satisfies* (PS),

(iv) $\widetilde{\Phi}(u) \le \int_\Omega |\nabla u|^p - (\mu_{l-1} + \varepsilon_0)\, |u|^p$ *for all* u *if* (1.3) *holds,*

(v) $\widetilde{\Phi}$ *is even if* f *is odd in* t *for all* x.

Proof. Since $\lambda_0, \lambda_\infty \notin \sigma(-\Delta_p)$, Φ_0 and Φ_∞ satisfy (PS) and have no critical points with $\|u\| = 1$, so

$$\delta_0 := \inf_{\|u\|=1} \|\Phi_0'(u)\| > 0, \qquad \delta_\infty := \inf_{\|u\|=1} \|\Phi_\infty'(u)\| > 0, \tag{4.6}$$

and

$$\inf_{\|u\|=\rho} \|\Phi_0'(u)\| = \rho^{p-1}\, \delta_0, \qquad \inf_{\|u\|=R} \|\Phi_\infty'(u)\| = R^{p-1}\, \delta_\infty \tag{4.7}$$

by homogeneity. Let

$$\Psi_0(u) = -\int_\Omega p\, F(x,u) - \lambda_0\, |u|^p, \quad \Psi_\infty(u) = -\int_\Omega p\, F(x,u) - \lambda_\infty\, |u|^p. \tag{4.8}$$

By (1.2),

$$\sup_{\|u\|=\rho} |\Psi_0(u)| = o(\rho^p), \qquad \sup_{\|u\|=R} |\Psi_\infty(u)| = o(R^p) \tag{4.9}$$

and

$$\sup_{\|u\|=\rho} \|\Psi_0'(u)\| = o(\rho^{p-1}), \qquad \sup_{\|u\|=R} \|\Psi_\infty'(u)\| = o(R^{p-1}) \tag{4.10}$$

as $\rho \to 0$ and $R \to \infty$. Since $\Phi = \Phi_0 + \Psi_0 = \Phi_\infty + \Psi_\infty$, it follows from (4.7) and (4.10) that

$$\inf_{\|u\|=\rho} \|\Phi'(u)\| = \rho^{p-1}(\delta_0 + o(1)), \qquad \inf_{\|u\|=R} \|\Phi'(u)\| = R^{p-1}(\delta_\infty + o(1)). \tag{4.11}$$

Take smooth functions $\varphi_0, \varphi_\infty : [0, \infty) \to [0, 1]$ such that

$$\varphi_0(t) = \begin{cases} 1, & t \le 1, \\ 0, & t \ge 2, \end{cases} \qquad \varphi_\infty(t) = \begin{cases} 0, & t \le 1/2, \\ 1, & t \ge 1 \end{cases} \tag{4.12}$$

and set

$$\widetilde{\Phi}(u) = \Phi(u) - \varphi_0(\|u\|/\rho)\, \Psi_0(u) - \varphi_\infty(\|u\|/R)\, \Psi_\infty(u). \tag{4.13}$$

Since

$$\|d(\varphi_0(\|u\|/\rho))\| = O(\rho^{-1}), \qquad \|d(\varphi_\infty(\|u\|/R))\| = O(R^{-1}), \tag{4.14}$$

(4.11) holds with Φ replaced by $\widetilde{\Phi}$ also, and (i) and (ii) follow.

By construction, $\|\widetilde{\Phi}'\|$ is bounded away from 0 for $\rho \le \|u\| \le 2\rho$ and $\|u\| \ge R/2$, so every Palais-Smale sequence for $\widetilde{\Phi}$ has a subsequence in $\|u\| < \rho$ or $2\rho < \|u\| < R/2$, which is then a Palais-Smale sequence for Φ_0 or Φ, respectively.

To see (iv), note that

$$\widetilde{\Phi}(u) = \int_\Omega |\nabla u|^p - \Big(\lambda_0\, \varphi_0(\|u\|/\rho) + \lambda_\infty\, \varphi_\infty(\|u\|/R)\Big)|u|^p \\ - p\, \Big(1 - \varphi_0(\|u\|/\rho) - \varphi_\infty(\|u\|/R)\Big) F(x, u), \tag{4.15}$$

$1 - \varphi_0(\|u\|/\rho) - \varphi_\infty(\|u\|/R) \ge 0$ for all u, and $\lambda_0, \lambda_\infty \ge \mu_{l-1} + \varepsilon_0$ if (1.3) holds. (v) is clear. □

5 Proof of Theorem 1.1

Let A be a closed subset of a metric space K, B a closed subset of W, $A \ne \emptyset \ne B$, and let $f \in C(A, W)$ be a map such that $f(A) \cap B = \emptyset$. We shall say that (A, f) links B with respect to K if $\gamma(K) \cap B \neq \emptyset$ for every map $\gamma \in C(K, W)$, $\gamma|_A = f$. If $A \subset K \subset W$ and f is the identity map on A, then we say A links B.

Suppose (A, f) links B with respect to K and $\sup \widetilde{\Phi}(f(A)) < \inf \widetilde{\Phi}(B)$, then

$$c := \inf_{\substack{\gamma \in C(K,W) \\ \gamma|_A = f}} \sup_{z \in K} \widetilde{\Phi}(\gamma(z)) \geq \inf \widetilde{\Phi}(B) \tag{5.1}$$

is a critical value of $\widetilde{\Phi}$ according to a general minimax principle (see, e.g., Willem [13]).

5.1 Case (i)

Take $g \in \mathcal{G}$, $g(CA_0) \subset I^{\lambda_\infty}$. Then, employing (iv) of Proposition 4.2 and (3.4),

$$\widetilde{\Phi}(u) \leq \int_\Omega |\nabla u|^p - (\mu_{l-1} + \varepsilon_0)\, |u|^p \leq 0, \quad \frac{u}{\|u\|} \in A_0 \tag{5.2}$$

(recall $A_0 \in \mathcal{F}_{l-1}$, $A_0 \subset I^{\mu_{l-1}+\varepsilon_0/2}$) and, since $g(CA_0) \subset I^{\lambda_\infty}$,

$$\widetilde{\Phi}(u) = \Phi_\infty(u) \leq 0, \quad \|u\| = R,\ \frac{u}{R} \in g(CA_0) \tag{5.3}$$

by (3.4) again (here ρ and R are as in Proposition 4.2). We may regard W as a subspace of $W \oplus \mathbb{R}$ and we may assume CA_0 is a (geometric) cone over A_0 in $W \oplus \mathbb{R}$, with vertex at some point $\notin W$. Let

$$A_1 = \Big\{ tu : u \in A_0,\ t \in [0, 1] \Big\}, \quad A = A_1 \cup CA_0 \tag{5.4}$$

and $f(z) = Rz$ for $z \in A_1$, $f(z) = Rg(z)$ for $z \in CA_0$. Since $g|_{A_0} = \mathrm{id}$, f is well defined. By (5.2) and (5.3), $\widetilde{\Phi}(f(z)) \leq 0$ whenever $z \in A$. On the other hand, by (3.5),

$$\widetilde{\Phi}(u) = \Phi_0(u) \geq \left(1 - \frac{\lambda_0}{\mu_l}\right) \rho^p > 0 \tag{5.5}$$

on

$$B = \left\{ u \in S_\rho : \frac{u}{\rho} \in I_{\mu_l} \right\}, \tag{5.6}$$

where $S_\rho = \Big\{u \in W : \|u\| = \rho\Big\}$. We will complete the proof by showing that (A, f) links B with respect to

$$K = \Big\{tz : z \in A,\, t \in [0, 1]\Big\} \tag{5.7}$$

and hence $\widetilde{\Phi}$ has a positive critical value c.

Any $\gamma \in C(K, W)$ such that $\gamma|_A = f$ can be extended to an odd map $\widetilde{\gamma}$ on

$$\widetilde{K} = \Big\{tz : z \in \widetilde{A},\, t \in [0, 1]\Big\}, \tag{5.8}$$

where $\widetilde{A} := A_1 \cup CA_0 \cup (-CA_0) = A_1 \cup (A_0 * \mathbb{Z}_2)$ and it suffices to show that

$$\widetilde{\gamma}(\widetilde{K}) \cap B \neq \emptyset. \tag{5.9}$$

We note that $\widetilde{\gamma}(0) = 0$ (by oddness), $\widetilde{\gamma}|_{A_0 * \mathbb{Z}_2} = R\widetilde{g}$, where $\widetilde{g}$ is as in the proof of Proposition 3.3, and $\widetilde{K} = \Big\{tz : z \in A_0 * \mathbb{Z}_2,\, t \in [0, 1]\Big\}$. Applying the piercing property to

$$C = A_0 * \mathbb{Z}_2, \quad C_0 = \overline{B}_\rho, \quad C_1 = W \setminus B_\rho, \tag{5.10}$$

where $B_\rho = \Big\{u \in W : \|u\| < \rho\Big\}$, and

$$\varphi : C \times [0, 1] \to C_0 \cup C_1, \quad (z, t) \mapsto \widetilde{\gamma}(tz) \tag{5.11}$$

gives

$$i(\widetilde{\gamma}(\widetilde{K}) \cap S_\rho) = i(\varphi(C \times [0, 1]) \cap C_0 \cap C_1) \geq i(C) = i(A_0 * \mathbb{Z}_2) \geq l \tag{5.12}$$

by (3.13), so

$$\max_{u \in \widetilde{\gamma}(\widetilde{K}) \cap S_\rho} I\left(\frac{u}{\rho}\right) \geq \mu_l \tag{5.13}$$

and (5.9) follows.

5.2 Case (ii)

We have

$$\widetilde{\Phi}(u) = \Phi_\infty(u) \ge \left(1 - \frac{\lambda_\infty}{\mu_l}\right) R^p, \quad \|u\| \ge R,\ \frac{u}{\|u\|} \in I_{\mu_l} \tag{5.14}$$

(by (3.5)) and $\widetilde{\Phi}$ is bounded on bounded sets, so $\widetilde{\Phi}$ is bounded below on

$$B = \left\{ tu : u \in I_{\mu_l},\ t \ge 0 \right\}. \tag{5.15}$$

On the other hand,

$$\widetilde{\Phi}(u) = \Phi_\infty(u) \le -\left(\frac{\lambda_\infty}{\mu_{l-1} + \varepsilon_0/2} - 1\right) \|u\|^p, \quad \|u\| \ge R,\ \frac{u}{\|u\|} \in A_0 \tag{5.16}$$

and the coefficient of $\|u\|^p$ is negative since $\lambda_\infty \ge \mu_{l-1} + \varepsilon_0$, so taking

$$A = \left\{ u \in S_{R'} : \frac{u}{R'} \in A_0 \right\} \tag{5.17}$$

with $R' \ge R$ sufficiently large, $\max \widetilde{\Phi}(A) < \inf \widetilde{\Phi}(B)$. We will complete the proof by showing that A links B with respect to

$$K = \left\{ tu : u \in A,\ t \in [0,1] \right\} \tag{5.18}$$

and that the critical value c defined by (5.1) is negative.

Let $\gamma \in C(K, W)$, $\gamma|_A = \text{id}$. We are done if $0 \in \gamma(K)$, so suppose not. Then the map

$$g(u,t) = \frac{\gamma(R'(1-t)\, u)}{\|\gamma(R'(1-t)\, u)\|}, \quad (u,t) \in CA_0 \tag{5.19}$$

is in $\mathcal{G}$, and it suffices to show that

$$g(CA_0) \cap B \ne \emptyset. \tag{5.20}$$

But $\max I(g(CA_0)) \ge \mu_l$ by (3.14), so (5.20) follows.

To see that $c < 0$, take $\varepsilon \in (0, \lambda_0 - \widetilde{\mu}_l)$ and $g \in \mathcal{G}$, $g(CA_0) \subset I^{\widetilde{\mu}_l + \varepsilon}$. Then

$$\widetilde{\Phi}(u) \le -\frac{\varepsilon_0/2}{\mu_{l-1} + \varepsilon_0/2}\, \rho^p < 0, \quad \frac{u}{\|u\|} \in A_0, \ \|u\| \ge \rho \tag{5.21}$$

by (iv) of Proposition 4.2 and (3.4), and

$$\widetilde{\Phi}(u) = \Phi_0(u) \le -\left(\frac{\lambda_0}{\widetilde{\mu}_l + \varepsilon} - 1\right)\rho^p < 0, \quad \|u\| = \rho, \ \frac{u}{\rho} \in g(CA_0), \tag{5.22}$$

so $\max \widetilde{\Phi}(\gamma(K)) < 0$ for

$$\gamma(tu) = \begin{cases} \rho\, g(u/\|u\|, 1 - R't/\rho), & 0 \le t \le \rho/R', \\ tu, & \rho/R' \le t \le 1. \end{cases} \tag{5.23}$$

5.3 ODE Case

Let $\Omega = (0, 1)$. The spectrum in this case consists of a sequence of simple eigenvalues $\lambda_l \nearrow \infty$ given by the usual minimax scheme involving the genus, and the eigenfunction φ_l of λ_l has exactly l nodal domains (see, e.g., Drábek [7], Theorem 11.3, or del Pino, Elgueta, and Manásevich [6]).

As we noted in the proof of Proposition 3.2, $\mu_l \ge \lambda_l$. Let $\xi_j = \varphi_l$ on the j-th nodal domain of φ_l and 0 everywhere else in $(0, 1)$. Then $I = \lambda_l$ on the $(l-1)$-sphere $S^{l-1} = S \cap \operatorname{span}\{\xi_1, \dots, \xi_l\} \in \mathcal{F}_l$, so $\mu_l = \lambda_l$.

To see that $\widetilde{\mu}_l = \lambda_l$, let $\varepsilon_0 \in (0, \lambda_l - \lambda_{l-1})$ and let S^{l-1}_+ be the hemisphere of S^{l-1} that contains φ_l and has boundary $S^{l-2} = S^{l-1} \cap \operatorname{span}\{\xi_1, \dots, \xi_{l-1}\}$. Since $\pm\varphi_l \notin S^{l-2}$ and I has no critical values in $[\lambda_{l-1} + \varepsilon_0/2, \lambda_l)$, there is an odd homeomorphism η of S such that $A_0 = \eta(S^{l-2}) \subset I^{\lambda_{l-1} + \varepsilon_0/2}$ and $\eta(S^{l-1}_+) \subset I^{\lambda_l}$ by a repeated application of the first deformation lemma. Then the map

$$g(u, t) = \eta\left(\frac{(1-t)\,\eta^{-1}(u) + t\varphi_l}{\|(1-t)\,\eta^{-1}(u) + t\varphi_l\|}\right), \quad (u, t) \in CA_0 \tag{5.24}$$

is in $\mathcal{G}$ and $I \le \lambda_l$ on $g(CA_0) = \eta(S^{l-1}_+)$.

6 Proof of Theorem 1.2

6.1 Case (i)

Denote by $\mathcal{A}$ the class of compact symmetric subsets of W and by Γ the group of odd homeomorphisms γ of W such that $\gamma|_{\widetilde{\Phi}^0} = \mathrm{id}$, let

$$i^*(A) := \min_{\gamma \in \Gamma} i(\gamma(A) \cap S_\rho), \quad A \in \mathcal{A}, \tag{6.1}$$

where ρ is as in Proposition 4.2, be the pseudo-index of Benci [3] related to i, S_ρ, and Γ, and set

$$c_j := \inf_{\substack{A \in \mathcal{A} \\ i^*(A) \geq j}} \max_{u \in A} \widetilde{\Phi}(u), \quad j = l, \dots, m-1. \tag{6.2}$$

We will show that $0 < c_l \leq \dots \leq c_{m-1} < +\infty$ and hence $\widetilde{\Phi}$ has $m - l$ pairs of nontrivial critical points (see Benci [3]).

If $i^*(A) \geq l$, then $i(A \cap S_\rho) \geq l$, so

$$\max_{u \in A \cap S_\rho} I\left(\frac{u}{\rho}\right) \geq \mu_l \tag{6.3}$$

and hence

$$\max_{u \in A} \widetilde{\Phi}(u) \geq \max_{u \in A \cap S_\rho} \Phi_0(u) \geq \left(1 - \frac{\lambda_0}{\mu_l}\right)\rho^p > 0. \tag{6.4}$$

It follows that $c_l > 0$.

To show that c_{m-1} is well defined and finite, we construct a set $A \in \mathcal{A}$ with $i^*(A) \geq m - 1$. Take $A_0 \in \mathcal{F}_{m-1}$, $A_0 \subset I^{\lambda_\infty}$ and let

$$A = \left\{tu : \|u\| = R,\ \frac{u}{R} \in A_0,\ t \in [0,1]\right\}. \tag{6.5}$$

Then $\widetilde{\Phi} = \Phi_\infty \leq 0$ on

$$\partial A = \left\{u : \|u\| = R,\ \frac{u}{R} \in A_0\right\}, \tag{6.6}$$

so for any $\gamma \in \Gamma$, $\gamma|_{\partial A} = \mathrm{id}$ and hence applying the piercing property to

$$C = \partial A, \quad C_0 = \overline{B}_\rho, \quad C_1 = W \setminus B_\rho \tag{6.7}$$

and

$$\varphi : C \times [0,1] \to C_0 \cup C_1, \quad (u,t) \mapsto \gamma(tu) \tag{6.8}$$

gives

$$i(\gamma(A) \cap S_\rho) = i(\varphi(C \times [0,1]) \cap C_0 \cap C_1) \geq i(C) = i(A_0) \geq m-1. \tag{6.9}$$

6.2 Case (ii)

Set

$$c_j := \inf_{\substack{A \in \mathcal{A} \\ i(A) \geq j}} \max_{u \in A} \widetilde{\Phi}(u), \quad j = l, \dots, m-1. \tag{6.10}$$

We will show that $-\infty < c_l \leq \dots \leq c_{m-1} < 0$ and hence $\widetilde{\Phi}$ has $m-l$ pairs of nontrivial critical points (see, e.g., Rabinowitz [11]).

Take $\varepsilon \in (0, \lambda_0 - \mu_{m-1})$ and $A_0 \in \mathcal{F}_{m-1}$, $A_0 \subset I^{\mu_{m-1}+\varepsilon}$ and let

$$A = \left\{ u \in S_\rho : \frac{u}{\rho} \in A_0 \right\}. \tag{6.11}$$

Then $i(A) \geq m-1$ and

$$\widetilde{\Phi}(u) = \Phi_0(u) \leq -\left(\frac{\lambda_0}{\mu_{m-1}+\varepsilon} - 1 \right) \rho^p < 0 \tag{6.12}$$

on A, so $c_{m-1} < 0$.

We claim that $c_l \geq \inf \widetilde{\Phi}(\overline{B}_R)$. If not, take $A \in \mathcal{A}$, $i(A) \geq l$ with $\max \widetilde{\Phi}(A) < \inf \widetilde{\Phi}(\overline{B}_R) < 0$. Then $A \subset W \setminus \overline{B}_R$, so

$$\Phi_\infty(u) = \widetilde{\Phi}(u) < 0, \quad u \in A \tag{6.13}$$

and hence $I < \lambda_\infty < \mu_l$ on

$$A_0 = \left\{ \frac{u}{\|u\|} : u \in A \right\} \in \mathcal{F}_l, \tag{6.14}$$

contradicting (3.10).

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