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A Hardy inequality in the Half-space

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Abstract

Here we prove a Hardy-type inequality in the upper half-space which generalize an inequality originally proved by V. G. Maz'ya (see [10], p. 99). Here we present a different proof, which enable us to improve the constant in front of the remainder term. We will also generalize the inequality to the L^p case.

1 Introduction

There are many Hardy inequalities that can be called classical, see for example [4],[5],[9], or [10]. They are all, in some way, generalizations of the original inequality by G. Hardy [6].
Hardy inequalities are, in general, not sharp in the sense that there exist no extremal functions. Remainder terms might therefore be added to improve the inequalities. Those terms can be of many different forms.
The problem of improving the remainder terms has received much attention during the last decade and many articles have been written on the subject.

Some results concerning a type of Hardy inequality involving the distance to the boundary were obtained in [1], [3], [7] and [11]. These inequalities are all improvements and generalizations of the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|u|^2}{\delta(x)^2}dx \quad u \in C_0^\infty(\Omega),$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$ and Ω is a convex domain in $\mathbb{R}^n$.
Here we shall obtain an estimate for a remainder term for a special case suggested by a result of V. G. Maz'ya.

In the well known book 'Sobolev Spaces' by V. G. Maz'ya [10], the following inequality is derived

$$\int_{\mathbb{R}^n} |x_n|^{p-1}|\nabla u(x)|^p dx \geq \frac{1}{(2p)^p}\int_{\mathbb{R}^n} \frac{|u(x)|^p}{(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}}dx, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{1.1}$$

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If $p = 2$ and one substitutes $u(x) = |x_n|^{-1/2}v(x)$ into (1.1) one gets

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^n} \frac{v^2 dx}{x_n^2} + \frac{1}{16}\int_{\mathbb{R}^n} \frac{v^2 dx}{(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}|x_n|}, \tag{1.2}$$

valid for all $v \in C_0^\infty(\mathbb{R}^n)$ that vanishes for $x_n = 0$.
Here we will derive (1.2) (but with the constant 1/16 replaced by 1/8) using a different method, which is partly based on techniques from [2]. Notice that, in the case $p = 2$, this will improve inequality (1.1) as well.

It is an open problem, formulated by V. G. Maz'ya, whether the following generalization of the above inequality holds or not :

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx + \alpha(p,\tau)\int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} dx,$$

where $p > 1$, $\tau > 0$, $\alpha(p,\tau)$ is a positive constant and $u \in C_0^\infty(\mathbb{R}^n_+)$.
Here we will answer the question affirmative.

The paper will be organized in the following way.
First, we prove the original L^2 inequality, with the improved constant.
Then we prove the theorem for $p > 1$ and $\tau = 1$. The proof will be slightly different from the general case and gives us better constants in this particular case.
At last we present a proof of the general τ-version.

2 Auxiliary results

Let

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

Our starting point will be the following lemma.

Lemma 2.1. *Let $u \in C_0^\infty(\Omega)$, $d \in (-\infty, mp-1)$ where $m \in \mathbb{N}_+$ and let $F = (F_1, \ldots, F_n)$ be a vector field in $\mathbb{R}^n$ with components in $C^1(\Omega)$. Furthermore, let $w(x) \in C^1(\Omega)$ be a nonnegative weight function and*

$$h_{p,m,d} = \left(\frac{mp-d-1}{p}\right)^p,$$

then

$$\int_\Omega \frac{|\nabla u|^p \cdot w}{\delta^{(m-1)p-d}} dx \geq h_{p,m,d}\left(\int_\Omega \frac{|u|^p \cdot w}{\delta^{mp-d}} - \frac{p \cdot |u|^p \Delta\delta \cdot w}{(mp-d-1)\delta^{mp-d-1}} dx\right)$$

$$+ \quad h_{p,m,d}\int_\Omega \left(\frac{p \cdot \mathrm{div} F}{mp-d-1} + \frac{(p-1)}{\delta^{mp-d}}\left(1 - \left|\nabla\delta - \delta^{mp-d-1}F\right|^{\frac{p}{p-1}}\right)\right)|u|^p \cdot w dx$$

$$+ \left(\frac{np-d-1}{p}\right)^{p-1} \int_\Omega \nabla w \cdot \left(F - \frac{\nabla\delta}{\delta^{np-d-1}}\right) dx. \tag{2.1}$$

Proof. Hölders inequality and partial integration gives

$$p^p \int_\Omega \frac{|\nabla u|^p \cdot w}{\delta^{(m-1)p-d}} dx \quad \cdot \quad \left(\int_\Omega \left| \frac{\nabla \delta}{\delta^{m(p-1)+\frac{d}{p}-d}} - \delta^{m-1-\frac{d}{p}} F \right|^{\frac{p}{p-1}} |u|^p \cdot w dx \right)^{p-1}$$

$$\geq \quad p^p \left| \int_\Omega \left(\frac{\nabla \delta \cdot w}{\delta^{mp-d-1}} - F \cdot w \right) (\text{sign } u) |u|^{p-1} \cdot \nabla u dx \right|^p$$

$$= \left| \int_\Omega \left(\left(\frac{mp-d-1}{\delta^{mp-d}} - \frac{\Delta \delta}{\delta^{mp-d-1}} + \text{div} F \right) w + \nabla w \left(F - \frac{\nabla \delta}{\delta^{np-d-1}} \right) \right) |u|^p dx \right|^p .$$

By factoring out the constant $(mp-d-1)^p$ from the R.H.S. and applying the ineq. $\frac{|A|^p}{B^{p-1}} \geq pA - (p-1)B \quad (B > 0)$ to the resulting inequality, we get (2.1) and the proof is complete. □

Corollary 2.1. *Let the notation be as in the above lemma, then*

$$\int_\Omega \frac{|\nabla u|^p \cdot w}{\delta^{(m-1)p-d}} dx \geq h_{p,m,d} \int_\Omega \frac{|u|^p \cdot w}{\delta^{mp-d}} dx$$

$$- \left(\frac{np-d-1}{p} \right)^{p-1} \int_\Omega \left(\frac{\Delta \delta \cdot w}{\delta^{np-d-1}} + \frac{\nabla w \cdot \nabla \delta}{\delta^{np-d-1}} \right) |u|^p dx$$

Proof. Simply put $F \equiv 0$ in the previous lemma. □

Notice that if we also require that Ω should be convex and $\nabla w \cdot \nabla \delta \leq 0$, then both terms occuring in the last integral on the R.H.S. in the corollary become nonpositive.

The corollary is, in general, a very rough estimate and can be improved by choosing the vector field F differently. The most suitable F, of course, depend on the domain in question and what type of remainder terms we wish to obtain.

It should be noted however, that the constant, $h_{p,m,d}$ in front of the main term cannot be improved (see [5]), at least if we don't want it to depend on the dimension n.

We will also need some simple lemmas :

Lemma 2.2. *Let $-1 \leq x \leq 0$ and $\alpha \geq 2$, then*

$$(1+x)^\alpha \leq 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2.$$

Proof. Taylor expansion of $(1+x)^\alpha$ around 0 gives

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \alpha(\alpha-1)(\alpha-2)(1+\xi x)^{\alpha-3} \frac{x^3}{6},$$

where $0 \leq \xi \leq 1$. The last term is non positive if $-1 \leq x \leq 0$. □

Lemma 2.3. *Let $-1 \leq x \leq \infty$ and $\alpha \leq 2$, then*

$$(1+x)^\alpha \leq 1 + \alpha x + \frac{\alpha}{2} x^2.$$

Proof.

$$(1+x)^\alpha = (1+2x+x^2)^{\frac{\alpha}{2}} \leq 1 + \frac{\alpha}{2}(2x+x^2) = 1 + \alpha x + \frac{\alpha}{2} x^2,$$

according to a variant of Bernoulli's inequality. □

3 Improvement of the constant in inequality (1.2)

We will, in the next section, generalize the proof here, to prove a L^p variant of the inequality. The L^2 case will then follow from the general L^p ineqality, but since the proof is easier and shorter in the L^2 case, I present it here before I proceed to the more general case.

Lemma 3.1. *Let $u \in C_0^\infty(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = \{x_1, \dots, x_n : x_n > 0\}$, then we have*

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^2} dx + \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_{n-1}^2 + x_n^2} dx$$

Proof. Apply lemma 2.1 to the case where $\Omega = \mathbb{R}_+^n, p = 2, m = 0, w \equiv 1$ and

$$F = \left(0, \dots, 0, \frac{x_{n-1}}{x_{n-1}^2 + x_n^2}, \frac{x_n}{x_{n-1}^2 + x_n^2}\right).$$

□

Lemma 3.2. *Let $u \in C_0^\infty(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = \{x_1, \dots, x_n : x_n > 0\}$ and $a \geq 0$, then*

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^2} dx + \frac{a}{2} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n (x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} dx \quad (3.1)$$

$$- \frac{(a^2 + 2a)}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_{n-1}^2 + x_n^2} dx \quad (3.2)$$

Proof. Now apply lemma 2.1 to the case where $\Omega = \mathbb{R}_+^n, p = 2, m = 0, w \equiv 1$ and

$$F = \left(0, \dots, 0, \frac{a}{(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}}\right)$$

and use the simple estimate

$$\text{div} F = \frac{-a x_n}{(x_{n-1}^2 + x_n^2)^{\frac{3}{2}}} \geq \frac{-a}{x_{n-1}^2 + x_n^2}$$

to get inequality (3.2). □

Corollary 3.1. *Let u be as in the above lemma. Then the following inequality holds*

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^2} dx + \frac{1}{8} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n (x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} dx.$$

Proof. Multiply the inequality in lemma (3.1) by $a^2 + 2a$ and add the result to the inequality in lemma (3.2) to get

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n^2} dx + \frac{a}{2(a^2 + 2a + 1)} \int_{\mathbb{R}_+^n} \frac{|u|^2}{x_n (x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} dx.$$

The maximum value of the expression in front of the second integral on the right side is obtained when $a = 1$ and equals $\frac{1}{8}$. □

4 L^p inequalities

We will now prove an analogous inequality for L^p.

The special case of lemma 2.1 when $m = 1, d = 0$ and $w \equiv 1$ gives

$$\int_{\Omega} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{\delta^p} dx \quad + \quad \left(\frac{p-1}{p}\right)^p \cdot$$

$$\cdot \int_{\mathbb{R}^n_+} \left(\frac{p}{p-1} \cdot \mathrm{div} F - \frac{p \cdot \Delta\delta}{(p-1)\delta} + \frac{(p-1)}{\delta^p} \left(1 - \left| \nabla\delta - \delta^{p-1} F \right|^{\frac{p-1}{p}} \right) \right) |u|^p dx. \tag{4.1}$$

We will apply this inequality to two different vector fields F and then add the corresponding inequalities to get our desired result. As before, let $\Omega = \mathbb{R}^n_+, \delta = x_n$. Substituting this into (4.1) gives

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \quad + \quad \left(\frac{p-1}{p}\right)^p \cdot$$

$$\cdot \int_{\mathbb{R}^n_+} \left(\frac{p}{p-1} \cdot \mathrm{div} F + \frac{(p-1)}{x_n^p} \left(1 - \left| (0, \ldots, 0, 1) - x_n^{p-1} F \right|^{\frac{p-1}{p}} \right) \right) |u|^p dx. \tag{4.2}$$

This inequality will be our starting point when we prove the following theorem.

Theorem 4.1. *Let $\Omega = \mathbb{R}^n_+$ and $u \in C_0^\infty(\Omega)$, then*

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^p dx \quad &\geq \quad \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \\ &+ \quad D(p) \left(\frac{p-1}{p}\right)^{p-1} \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-1}(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} dx, \end{aligned}$$

where

$$D(p) = \begin{cases} \frac{2}{2+3p} & \text{if} \quad 1 < p < 2 \\ \frac{1}{4(p-1)} & \text{if} \quad p \geq 2 \end{cases}.$$

Proof. We will split the proof into two parts, corresponding to different inequalities. Each part, will in turn, be divided into two cases depending on whether $1 < p < 2$ or $p \geq 2$. The inequalities in the two parts will then be combined to prove the theorem.
Let

$$m_p = \left(\frac{p-1}{p}\right)^{p-1}.$$

Part 1 :
Apply inequality (4.1) to the vector field

$$F = \left(0, \ldots, 0, \frac{a_p}{x_n^{p-2}(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}}\right), \quad 0 \leq a_p \leq 1$$

and note that

$$\begin{aligned}\frac{p\cdot \mathrm{div}F}{p-1} &= \frac{a_p p\cdot(2-p)}{(p-1)x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}} - \frac{a_p p\cdot x_n^{3-p}}{(p-1)(x_{n-1}^2+x_n^2)^{\frac{3}{2}}}\\ &\geq \frac{a_p p\cdot(2-p)}{(p-1)x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}} - \frac{a_p\cdot p}{(p-1)x_n^{p-2}(x_{n-1}^2+x_n^2)}.\end{aligned}$$

Now we must estimate the expression

$$\frac{p-1}{\delta^p}\left(1-\left|(0,\ldots,0,1)-\delta^{p-1}F\right|^{\frac{p}{p-1}}\right)$$

$$= \frac{p-1}{x_n^p}\left(1-\left(1-\frac{a_p\cdot x_n}{(x_{n-1}^2+x_n^2)^{\frac{1}{2}}}\right)^{\frac{p}{p-1}}\right). \qquad (4.3)$$

Case 1. $1<p<2$
Lemma 2.2 enables us to estimate (4.3) from below by

$$\frac{a_p\cdot p}{x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}} - \frac{1}{2}\frac{p}{p-1}\frac{a_p^2}{x_n^{p-2}(x_{n-1}^2+x_n^2)}.$$

Altogether, (4.2) and the above estimates leads to the inequality

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla u|^p dx &\geq \left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx + a_p\cdot m_p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}}dx\\ &- \left(a_p+\frac{a_p^2}{2}\right)m_p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-2}(x_{n-1}^2+x_n^2)}. \qquad (4.4)\end{aligned}$$

Case 2. $p\geq 2$
By lemma 2.3 one can now show that (4.3) is bounded from below by

$$\frac{a_p\cdot p}{x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}} - \frac{p}{2}\cdot\frac{a_p^2}{x_n^{p-2}(x_{n-1}^2+x_n^2)}.$$

In this case our lower estimates of the R.H.S. of (4.2) gives us

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla u|^p dx &\geq \left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx + a_p\cdot m_p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}}dx\\ &- \left(a_p+(p-1)\frac{a_p^2}{2}\right)m_p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-2}(x_{n-1}^2+x_n^2)}. \qquad (4.5)\end{aligned}$$

In the above inequalities, there is a negative term in the R.H.S. which must be taken care of. This will be done by adding yet another inequality to the above ones.

Part 2 :

To begin with, consider the vector field

$$F=\left(0,\ldots,0,\frac{c_p\cdot x_{n-1}x_n^{2-p}}{x_{n-1}^2+x_n^2},\frac{c_p\cdot x_n^{3-p}}{x_{n-1}^2+x_n^2}\right),\quad 0\leq c_p\leq 1.$$

Direct calculations shows that

$$\frac{p}{p-1}\cdot \mathrm{div}F = \frac{c_p\cdot(2-p)p}{p-1}\frac{1}{x_n^{p-2}(x_{n-1}^2+x_n^2)}$$

and

$$\frac{p-1}{\delta^p}\left(1-\left|\nabla\delta-\delta^{p-1}F\right|^{\frac{p}{p-1}}\right)=\frac{p-1}{x_n^p}\left(1-\left(1+\frac{(c_p^2-2c_p)x_n^2}{x_{n-1}^2+x_n^2}\right)^{\frac{p}{2(p-1)}}\right). \tag{4.6}$$

Again, the estimates of the expressions above will be different depending on whether $1<p<2$ or $p\geq 2$.

Case 1. $1<p<2$
If we assume $-1\leq c_p^2-2c_p\leq 0$, then (4.6) is not greater than

$$\frac{(p-1)(2c_p-c_p^2)}{x_n^{p-2}(x_{n-1}^2+x_n^2)}.$$

If we apply the above estimates to (4.2) we can conclude that the inequality

$$\int_{\mathbb{R}^n_+}|\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx+\alpha(p,c_p)\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-2}(x_{n-1}^2+x_n^2)}dx \tag{4.7}$$

holds, where

$$\alpha(p,c_p)=\left((p^2-2p+2)c_p-(p-1)^2c_p^2\right)\frac{1}{p-1}\left(\frac{p-1}{p}\right)^p.$$

One may put $c_p=1$ to get

$$\alpha(p,1)=\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}.$$

By multiplying this inequality by a suitable (positive) constant and adding the result to (4.4) we get, after maximizing with respect to the parameter a_p

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla u|^p dx \quad &\geq \quad \left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx \\ &+ \quad \left(\frac{p-1}{p}\right)^{p-1}\frac{2}{2+3p}\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-1}(x_{n-1}^2+x_n^2)^{\frac{1}{2}}}dx.\end{aligned}$$

Case 2. $p\geq 2$
If $x\geq -1$ and $0\leq\alpha\leq 1$ then

$$(1+x)^\alpha\leq 1+\alpha x$$

according to a variant of Bernoulli's inequality. So, if one assume, as in the case $1<p<2$, that $-1\leq c_p^2-2c_p\leq 0$, then (4.6) is bounded from below by

$$\frac{(2c_p-c_p^2)p}{2x_n^{p-2}(x_{n-1}^2+x_n^2)}.$$

In this case, our estimates applied to (4.2) gives us

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx + \beta(p, c_p) \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-2}(x_{n-1}^2 + x_n^2)} dx, \quad (4.8)$$

where

$$\beta(p, c_p) = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \left((1-p)c_p^2 + 2c_p\right).$$

The choice $c_p = \frac{1}{p-1}$ maximizes $\beta(p, c_p)$ and

$$\beta\left(p, \frac{1}{p-1}\right) = \frac{1}{2(p-1)}\left(\frac{p-1}{p}\right)^{p-1}.$$

This inequality may be multiplied by a (positive) constant which, when added to (4.5), gives (after maximizing with respect to the parameter a_p)

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^p dx &\geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \\ &+ \left(\frac{p-1}{p}\right)^{p-1} \frac{1}{4(p-1)} \in t_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-1}(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} dx. \end{aligned}$$

□

Remark 1. *Note that putting $p = 2$ gives us back our previous inequality with $\frac{1}{8}$ as the constant in front of the integral in the remainder term.*

Remark 2. *It should be noted that the above estimates are very rough and can probably be improved.*

5 A generalized L^p inequality

We will now generalize the above inequalities by introducing a parameter τ. The proof of this result will be a slightly modified variant of the proof above.

Theorem 5.1. *Let $0 < \tau \leq 1$ and let $\Omega = \mathbb{R}^n_+$ and u be as in the above, then the following inequality holds*

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx + A(p, \tau) m_p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} dx,$$

where

$$A(p, \tau) = \begin{cases} \frac{\tau^2}{2(1+p\tau^2)} & \text{if} \quad 1 < p < 2 \\ \frac{\tau^2}{2(p-1)(1+2\tau^2)} & \text{if} \quad p \geq 2. \end{cases}$$

Proof. Let

$$m_p = \left(\frac{p-1}{p}\right)^{p-1}.$$

Recall inequality (4.1), which gives a remainder term of the form

$$\left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \left(\frac{p}{p-1}\cdot \mathrm{div}F + \frac{(p-1)}{x_n^p}\left(1-\left|\nabla x_n - x_n^{p-1}F\right|^{\frac{p-1}{p}}\right)\right)|u|^p dx \tag{5.1}$$

in the ordinardy Hardy inequality.
Put

$$F = \left(0,\ldots,0,\frac{cx_n^{\tau-p+1}}{(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}}\right), \quad c=c(\tau,p) \quad ,0\le c\le 1.$$

This implies

$$\begin{aligned}\frac{p\cdot \mathrm{div}F}{p-1} &= \frac{cp(\tau-p+1)}{p-1}\frac{1}{x_n^{p-\tau}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}} - \frac{\tau cp}{p-1}\frac{x_n^\tau}{x_n^{p-2}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}+1}}\\ &\ge \frac{cp(\tau-p+1)}{p-1}\frac{1}{x_n^{p-\tau}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}} - \frac{\tau cp}{p-1}\frac{1}{x_n^{p-2}(x_{n-1}^2+x_n^2)}.\end{aligned}$$

Furthermore

$$\frac{(p-1)}{x_n^p}\left(1-\left|\nabla x_n - x_n^{p-1}F\right|^{\frac{p-1}{p}}\right) = \frac{(p-1)}{x_n^p}\left(1-\left(1-\frac{cx_n^\tau}{(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}}\right)^{\frac{p-1}{p}}\right) \tag{5.2}$$

Again, the estimates of this will depend on whether $1<p<2$ or $p\ge 2$.

Case 1. $1<p<2$
According to lemma 2.2, (5.2) is not greater than

$$\begin{aligned}&\frac{cp}{x_n^{p-\tau}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}} - \frac{c^2p}{2(p-1)}\frac{1}{x_n^{p-2\tau}(x_{n-1}^2+x_n^2)^\tau} \ge\\ &\ge \left(cp-\frac{c^2p}{2(p-1)}\right)\frac{1}{x_n^{p-\tau}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}}.\end{aligned}$$

In this case, from (5.1), we get

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla u|^p dx &\ge \left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx\\ &+ m_p\frac{\tau^2}{2}\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2+x_n^2)^{\frac{\tau}{2}}}dx\\ &- m_p\tau^2\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-2}(x_{n-1}^2+x_n^2)}dx\end{aligned}$$

after choosing $c=\tau$ (which maximizes the expression in front of the second integral in the R.H.S). By adding the inequality

$$\begin{aligned}p\tau^2\int_{\mathbb{R}^n_+}|\nabla u|^p dx &\ge p\tau^2\left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^p}dx\\ &+ m_p\tau^2\int_{\mathbb{R}^n_+}\frac{|u|^p}{x_n^{p-2}(x_{n-1}^2+x_n^2)}dx\end{aligned}$$

(which was proven in the above) to this one, one gets

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx + B(p,\tau) \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} dx,$$

where

$$B(p,\tau) = m_p \frac{\tau^2}{2(1+p\tau^2)}.$$

Case 2. $p \geq 2$
By lemma 2.3, (5.2) is greater or equal to

$$\frac{cp}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} - \frac{cp^2}{2} \frac{1}{x_n^{p-2\tau}(x_{n-1}^2 + x_n^2)^{\tau}} \geq \frac{cp - \frac{pc^2}{2}}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}}.$$

This leads to the inequality

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^p dx &\geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \\ &+ m_p \frac{\tau^2}{2(p-1)} \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} dx \\ &- m_p \frac{\tau^2}{p-1} \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-2}(x_{n-1}^2 + x_n^2)} dx, \end{aligned}$$

which, when added to

$$\begin{aligned} 2\tau^2 \int_{\mathbb{R}^n_+} |\nabla u|^p dx &\geq 2\tau^2 \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \\ &+ m_p \frac{\tau^2}{p-1} \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-2}(x_{n-1}^2 + x_n^2)} dx \end{aligned}$$

(which also was proven in the above) gives

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx + C(p,\tau) \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} dx,$$

where

$$C(p,\tau) = m_p \cdot \frac{\tau^2}{2(p-1)(1+2\tau^2)}.$$

□

It should be noted that the constants $A(p,\tau)$ are, in general, not optimal, since for example $A(2,1) = \frac{1}{12}$ and we know that it can be improved to $\frac{1}{8}$.

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