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Formalized Limits and Colimits of Setoids

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Abstract

We construct arbitrary limits and colimits of setoids in Type Theory and prove the universal properties, hence proving constructively that the category of setoids is complete and cocomplete. In particular, it has products and disjoint unions of arbitrary setoid-indexed families of setoids.

Essential use is made of the Curry–Howard correspondence, exemplifying how it can relate different well-known concepts in a very elegant way.

The reader is supposed to be familiar with Martin-Löf's type theory [7, 8], the theory in which the following formalization will take place. The work we will present is straightforward, but might be of value anyway, since some of the concepts treated are central and have not, to the authors knowledge, been treated in type theory in full generality before.[1] It is also a very good example of the Curry–Howard correspondence at its best: the work shows very well that 'proof objects' should *not* be thought of as 'proofs' in the sense of *convincing arguments*. Indeed, we will define preorders where a proof of $i \leq j$ is an arrow in a category, rather than an 'argument' in any reasonable sense.

1 Graphs, Preorders, Deductive systems, and Categories

A graph is a class I of objects and for every i, j in I a set $\mathrm{Hom}(i, j)$ of arrows from i to j. We write $i \leq j$ for the proposition which is true precisely when there is an arrow from i to j. By the Curry–Howard correspondence, we may simply define $i \leq j \stackrel{\text{def}}{=} \mathrm{Hom}(i, j)$.

A graph (I, Hom) is *small* if I is a set.

Lambek and Scott [6] defines a *deductive system* to be a category but without certain axioms; or rather, a category as a deductive system satisfying certain axioms. Thus, a deductive system is a graph (I, Hom), with a certain arrow $1_i : \mathrm{Hom}(i, i)$ for every i in I and a composition $f \circ g : \mathrm{Hom}(i, k)$ whenever g :

[1] Categories have been treated in e.g. [5, 1]. A. Saïbi [9] defines limits in a way similar to ours. Our only advantage at this point is that we define limits of arbitrary diagrams, while Saïbi, using a general cone transformation, defines limits of *functors*. Therefore, our construction turns out to be intensionally simpler in some cases. Colimits of setoids seem not to have been treated before in type theory.

$\mathrm{Hom}(i,j)$ and $f : \mathrm{Hom}(j,k)$. A category[2] is a deductive system with equivalence relations on the Hom-sets, satisfying

$$1_j \circ f = f \tag{1}$$
$$f \circ 1_i = f \tag{2}$$
$$(f \circ g) \circ h = f \circ (g \circ h) \tag{3}$$

whenever the compositions make sense, and the replacement rule

$$\frac{p = p' \qquad q = q'}{p \circ q = p' \circ q'} \,. \tag{4}$$

A set with an equivalence relation is called a *setoid* and maps between setoids are required to preserve the equivalence relations.

By the Curry–Howard correspondence, a deductive system is nothing but a preorder:

$$\mathrm{Hom}(i,j) \leftrightsquigarrow i \leq j \tag{5}$$
$$1_i \leftrightsquigarrow \mathrm{refl}_i \tag{6}$$
$$f \circ g \leftrightsquigarrow \mathrm{tr}(g,f)\,. \tag{7}$$

Thus every category is a preorder with the order relation

$$i \leq j \stackrel{\mathrm{def}}{=} \mathrm{Hom}(i,j)\,.$$

Conversely, given a preorder we can give it a category structure by defining a suitable equivalence relation on the proofs. There are two extreme ways of doing that. The easiest one is to identify all proofs

$$p = q \stackrel{\mathrm{def}}{=} \top$$

which will give a category with 'unique' arrows between the objects (hence precisely what usually is called 'preorder' by category theorists, see e.g. [6, p. 6, example C3']. The dual approach is the finest possible equivalence relation, inductively generated[3] by the axioms (1), (2), (3), (4) and the equivalence relation axioms.

We will consider, for a given preorder $(I, \leq)$, the two extreme categories

$$I^* = (I, \leq) \text{ with maximal equivalence relations,} \tag{8}$$
$$I_* = (I, \leq) \text{ with minimal equivalence relations,} \tag{9}$$

the former being the one usually considered as a preorder.

A morphism of a graph into the underlying graph of a category is called a *diagram* in the category. A functor between categories is a morphism of the corresponding deductive systems (preorders), preserving also the equivalence relations on the Hom-sets. Any morphism from a preorder $(I, \leq)$ to a category

[2]We restrict ourselves to *locally small* categories in the sense that we require a *set* of arrows between every pair of objects.

[3]To formalize inductive definitions, one can either extend the type theory or use well-orderings or tree-types, see e.g. [2]. We will make no formal use of these equivalence relations, so we omit the formalizations.

C is automatically a functor $I_* \to C$, because the equations that hold in I_* are those which have to hold in any category.

Hence every functor is a morphism of preorders and every morphism of preorders is a functor.

It will turn out that the category structures of index categories are not used at all in the constructions of limits and colimits of functors. All that is needed is the underlying graph structure. Therefore, it is more natural to speak of limits and colimits of *diagrams*. It shows also that the generality introduced by going to general limits and colimits from projective and inductive limits is somewhat illusory: the limit of an arbitrary functor is the projective limit obtained by considering the index category as a preorder, and the colimit is the corresponding inductive limit.

2 Projective and Inductive Systems of Setoids

A projective system over $(I, \leq)$ is a contravariant functor $I_* \to \mathbf{Setoid}$. In elementary terms, this means that we have the following:

First of all, $(I, \leq)$ is a preorder:

$$I : \mathsf{set} \tag{10}$$

$$i \leq j : \mathsf{prop}\ (i, j : I) \tag{11}$$

$$\mathrm{refl}_i : i \leq i\ (i : I) \tag{12}$$

$$\mathrm{tr}_{ijk}(p, q) : i \leq k\ (i, j, k : I,\ p : i \leq j,\ q : j \leq k)\,. \tag{13}$$

Secondly, we have a family of setoids over I:

$$A_i : \mathsf{set}\ (i : I) \tag{14}$$

$$a =_i b : \mathsf{prop}\ (i : I,\ a, b : A_i) \tag{15}$$

$$\mathrm{refl}_{A_i a} : a =_i a\ (i : I,\ a : A_i) \tag{16}$$

$$\mathrm{sym}_{A_i ab}(p) : b =_i a\ (i : I,\ a, b : A_i,\ p : a =_i b) \tag{17}$$

$$\mathrm{tr}_{A_i abc}(p, q) : a =_i c\ (i : I,\ a, b, c : A_i,\ p : a =_i b,\ q : b =_i c) \tag{18}$$

and a family of functions

$$f_{ij}(p, x) : A_i\ (i, j : I,\ p : i \leq j,\ x : A_j)$$

which are extensional,

$$f_{ij}(p, x) =_i f_{ij}(p, y)\,\mathrm{true}\ (i, j : I,\ p : i \leq j,\ x, y : A_j,\ x =_j y\,\mathrm{true})\,, \tag{19}$$

and assumed to preserve the relation in inverse order:

$$f_{ii}(\mathrm{refl}_i, x) =_i x\,\mathrm{true}\ (i : I,\ x : A_i) \tag{20}$$

$$f_{ik}(\mathrm{tr}_{ijk}(p, q), x) =_i f_{ij}(p, f_{jk}(q, x))\,\mathrm{true}\ (i, j, k : I,\ p : i \leq j,\ q : j \leq k,\ x : A_k)\,. \tag{21}$$

Because I^* is what usually is considered as a preorder in category theory, one might instead want to have a projective system being a contravariant functor $I^* \to \mathbf{Setoid}$. That is obtained by adding the requirement of proof-irrelevance

$$f_{ij}(p, x) =_i f_{ij}(q, x)\,\mathrm{true}\ (i, j : I,\ p, q : i \leq j,\ x : A_j)\,. \tag{22}$$

Category theorists would probably argue that this principle is what distinguishes projective systems from contravariant functors in general. We shall, however, make no use of (22) in the following.[4]

The formalization of inductive systems is precisely analogous, but in this case we have covariant functors instead, so we reverse all arrows f_{ij}. Our family of functions is now

$$f_{ij}(p,x) : A_j \ (i,j : I,\, p : i \leq j,\, x : A_i)\,.$$

They are extensional,

$$f_{ij}(p,x) =_j f_{ij}(p,y) \,\mathrm{true}\ (i,j : I,\, p : i \leq j,\, x,y : A_i,\, x =_i y \,\mathrm{true}) \tag{23}$$

and assumed to preserve the relation:

$$f_{ii}(\mathrm{refl}_i, x) =_i x \,\mathrm{true}\ (i : I,\, x : A_i) \tag{24}$$

$$f_{ik}(\mathrm{tr}_{ijk}(p,q), x) =_k f_{jk}(q, f_{ij}(p,x)) \,\mathrm{true}\ (i,j,k : I,\, p : i \leq j,\, q : j \leq k,\, x : A_i)\,. \tag{25}$$

Again, this is a functor from I_*. If we want functoriality from I^*, we need to require also a principle of proof-irrelevance

$$f_{ij}(p,x) =_j f_{ij}(q,x) \,\mathrm{true}\ (i,j : I,\, p,q : i \leq j,\, x : A_i)\,. \tag{26}$$

3 Limits

In the sequel, we will assume that I is a set.

The projective limit of the projective system defined above is the limit of the functor $(I_*)^{\mathrm{op}} \to \mathbf{Setoid}$ or $(I^*)^{\mathrm{op}} \to \mathbf{Setoid}$, depending on our taste regarding the principle (22) of proof-irrelevance. It is interesting to notice that in both cases we get the same limit. More precisely, the limit does not depend at all on the equivalence relations on the Hom-sets. In this section we treat limits of arbitrary functors to **Setoid**, but because we never use the equivalence relations on Hom-sets, we will use the notation for preorders rather than categories, letting $i \leq j \stackrel{\mathrm{def}}{=} \mathrm{Hom}(i,j)$ etc. We will not even assume reflexivity and transitivity, so we actually use only the graph structure of $(I, \leq)$.

Thus, let $(I, \leq)$ be a graph (think of it as the underlying graph of an arbitrary small category) and consider an arbitrary diagram $D : (I, \leq) \to \mathbf{Setoid}$ (in particular, D can be a functor). It is formalized precisely as our formalization of inductive systems, but without requiring reflexivity (12) and transitivity (13) of $(I, \leq)$, and consequently without assuming that this structure is preserved (24), (25), (26). We cover in this way inductive systems, but also projective systems by thinking of $(I, \leq)$ as $(I, \geq)$. We will define the limit of D.

The universal property for limits says that given a cone of functions from a setoid $A/{=_A}$ to D, it should factor in a unique way through the limit. Any element of A defines a function of type $(i : I)A_i$, so the limit should consist of such functions. Not all functions, however, because being a cone means that

[4] Thanks to Per Martin-Löf, who communicated his suspicion that (22) might not be needed.

all triangles commute. Thus, we pick out the subsetoid consisting of acceptable functions:

$$(\Sigma f : \Pi(I, A))(\forall i, j : I,\, p : i \leq j)(f_{ij}(p, \mathsf{app}(f, i)) =_j \mathsf{app}(f, j))\,. \tag{27}$$

Now, we define $\varprojlim$ as this set with the equivalence relation being extensional equality:

$$f \stackrel{\text{ext}}{=} g \ \stackrel{\text{def}}{=} (\forall i : I)(\pi_i(f) =_i \pi_i(g))\,,$$

where the π_i are the projections

$$\pi_i(f) \stackrel{\text{def}}{=} \mathsf{app}(\pi_\ell(f), i)\,,$$

and π_ℓ is the left projection $(z)\,\mathsf{split}(z, (x, y)x)$. The equality is thus defined as the coarsest one making the projections π_i extensional.

The maps π_i are compatible with the functions f_{ij} precisely because we used a subsetoid of the setoid of functions:

$$f_{ij}(p, \pi_i(f)) =_j \pi_j(f)\,\text{true}\ (i, j : I,\, p : i \leq j,\, f : \varprojlim)\,. \tag{28}$$

This is easy to see if we replace f with a canonical element (f, q) and normalize:

$$\mathsf{app}(\mathsf{app}(\mathsf{app}(q, i), j, p) : f_{ij}(p, \mathsf{app}(f, i)) =_j \mathsf{app}(f, j)\ (\ \dots\ ) \tag{29}$$

with the dots replaced by the appropriate typing of i, j, p, f, q.

We have to verify the universal property. Assume there is a setoid $A/{=_A}$ with a cone of extensional functions

$$g_i(x) : A_i\ (i : I,\, x : A) \tag{30}$$

$$\mathrm{ext}_i(p, x, y) : g_i(x) =_i g_i(y)\ (i : I,\, x, y : A,\, p : x =_A y) \tag{31}$$

$$\mathrm{cone}_{ij}(p, x) : f_{ij}(p, g_i(x)) =_j g_j(x)\ (i, j : I,\, p : i \leq j,\, x : A)\,. \tag{32}$$

We should then have a function $g : A \to \varprojlim$ satisfying

$$\pi_i(g(x)) =_i g_i(x)\,\text{true}\ (i : I,\, x : A)\,. \tag{33}$$

We obtain such a function by letting

$$g_1(x) \stackrel{\text{def}}{=} \lambda i.g_i(x) \tag{34}$$

$$g_2(x) \stackrel{\text{def}}{=} \lambda i.\lambda j.\lambda p.\mathrm{cone}_{ij}(p, x) \tag{35}$$

$$g(x) \stackrel{\text{def}}{=} (g_1(x), g_2(x))\,. \tag{36}$$

Inserting this definition in the requirement (33) and normalizing, we obtain the requirement

$$g_i(x) =_i g_i(x)\,\text{true}\ (i : I,\, x : A)$$

which is proved by $\mathrm{refl}_{A_i g_i(x)}$.

We also have to prove that g is extensional, i.e., that

$$g(x) = g(y)\,\text{true}\ (x, y : A,\, p : x =_A y)\,.$$

The proposition $g(x) = g(y)$ normalizes to $(\forall i : I)(g_i(x) =_i g_i(y))$, which is proved by $\lambda i.\mathrm{ext}_i(p, x, y)$.

Finally, we need to prove uniqueness. So assume that h is another function satisfying the properties, i.e.,

$$\pi_i(h(x)) =_i g_i(x)\,\mathrm{true}\ (i : I,\ x : A).$$

Then $\pi_i(h(x)) =_i \pi_i(g(x))$ and by λ-abstraction we get

$$(\forall i : I)(\pi_i(h(x)) =_i \pi_i(g(x)))\,\mathrm{true}\ (x : A),$$

which is definitionally equal to $h(x) = g(x)\,\mathrm{true}\ (x : A)$.

4 Colimits

The inductive limit of inductive systems is the *colimit* of the functor $I_* \to$ **Setoid** (or $I^* \to$ **Setoid**, depending on taste). Again, the difference between I_* and I^* has no effect, because the equivalence relations on the Hom-sets are not used. Hence we stick to preorder notation also in the treatment of colimits. This time, it is convenient to use a little more than the graph structure of $(I, \leq)$, namely reflexivity (12), (20). This is no essential restriction since every graph can be extended by adding an endoarrow refl_i for every node i; while corresponding diagrams are extended by mapping the new arrows to substitution maps: if $(I, \leq)$ is a set with relation (a graph), then $(I, \mathsf{Id} \vee \leq)$ is the reflexive closure

$$(\mathsf{Id} \vee \leq)(i, j) \stackrel{\mathrm{def}}{=} \mathsf{Id}(I, i, j) \vee i \leq j\,,$$

and a diagram $D : (I, \leq) \to \mathbf{Setoid}$, mapping $p : i \leq j$ to $f_{ij}(p)$, is extended to $\mathsf{Id} \vee D : (I, \mathsf{Id} \vee \leq) \to \mathbf{Setoid}$ by letting

$$f'_{ij}(p, a) \stackrel{\mathrm{def}}{=} \mathsf{when}(p, (x)\,\mathsf{subst}(x, a), (x) f_{ij}(x, a))\,,$$

with the usual definition $\mathsf{subst}(x, a) \stackrel{\mathrm{def}}{=} \mathsf{app}(\mathsf{idpeel}(x, (y)\lambda z.z), a)$. Moreover, every cone from D extends to a cone from $\mathsf{Id} \vee D$, since

$$\mathsf{idpeel}(p, (x)\,\mathrm{refl}_A(g_x(a))) : g_j(\mathsf{subst}(p, a)) =_A g_i(a)\ (i, j : I,\ p : \mathsf{Id}(I, i, j),\ a : A_i)\,.$$

Consider the set $\Sigma(I, A)$ and define on it a relation $a\ R\ b$ as

$$(\exists k : I,\ p : \pi_\ell(a) \leq k,\ q : \pi_\ell(b) \leq k)(f_{\pi_\ell(a)k}(p, \pi_r(a)) =_k f_{\pi_\ell(b)k}(q, \pi_r(b)))\,, \tag{37}$$

where π_r is the right projection $(z)\,\mathsf{split}(z, (x, y)y)$.

It is obviously reflexive and symmetric, but the proof-terms are long (and omitted). In general it is not transitive, as for instance in the construction of the pushout.

$$\begin{tikzcd} & \{a\} \\ \{*\} \arrow[ur] \arrow[dr] & \\ & \{b\} \end{tikzcd}$$

Therefore, let $=$ be the transitive closure of R, defined by letting $a = b$ mean[5]

$$
\begin{aligned}
&(\exists n : \mathsf{N},\, s : (\Sigma x : \mathsf{N})(x \le n) \to \Sigma(I, A))\\
&\qquad(\quad \mathsf{Id}(\Sigma(I,A), a, \mathsf{app}(s, (0, \mathrm{p1}(n))))\\
&\qquad\ \wedge\ \mathsf{Id}(\Sigma(I,A), b, \mathsf{app}(s, (n, \mathrm{p2}(n))))\\
&\qquad\ \wedge\ (\forall k : \mathsf{N},\, p : \mathsf{succ}(k) \le n)(\mathsf{app}(s, (k, q(k,n,p)))\ R\ \mathsf{app}(s, (\mathsf{succ}(k), p)))\ )\,,
\end{aligned}
$$

where we have defined $a \le b$ on natural numbers as $(\exists x : \mathsf{N})(a + x = b)$, and we have $\mathrm{p1}(n) : 0 \le n$, $\mathrm{p2}(n) : n \le n$, and $q(k,n,p) : k \le n$ $(k, n : \mathsf{N},\, p : \mathsf{succ}(k) \le n)$.

When $(I, \le)$ is a directed preorder, R is transitive, hence equivalent to $=$. This, however, needs the principle (26) of proof irrelevance, or some related principle.[6]

We let $\varinjlim$ be $\Sigma(I, A)$ with equivalence relation $=$. We claim that it is the colimit. We have to verify the universal property, but first let

$$\iota_i(x) \stackrel{\text{def}}{=} (i, x)\,.$$

This defines extensional functions $A_i/{=_i} \to \varinjlim$: if $a =_i b$, then $\iota_i(a)\ R\ \iota_i(b)$, because this means nothing but $(\exists k : I,\, p : i \le k,\, q : i \le k)(f_{ik}(p,x) =_k f_{ik}(q,y))$, which is proved by letting $k \stackrel{\text{def}}{=} i$, $p \stackrel{\text{def}}{=} q \stackrel{\text{def}}{=} \mathrm{refl}_i$: then $f_{ik}(p,x) =_k f_{ik}(q,y)$ becomes $f_{ii}(\mathrm{refl}_i, x) =_i f_{ii}(\mathrm{refl}_i, y)$ which is proved using (23). We then prove $\iota_i(a) = \iota_i(b)$ by letting $n \stackrel{\text{def}}{=} 1$, $s \stackrel{\text{def}}{=} \lambda x.\, \mathsf{natrec}(\pi_\ell(x), \iota_i(a), (y,z)\iota_i(b))$.

The maps ι_i are compatible with the functions f_{ij}:

$$\iota_j(f_{ij}(p,x))\ R\ \iota_i(x)\,\text{true}\ (i, j : I,\, p : i \le j,\, x : A_i)\,. \tag{38}$$

To prove this, use the definition (37) of R, replace k by j, p by refl_j, q by p, and use (20).

Now to the universal property of $\varinjlim$. We treat uniqueness before existence. Assume that we have a cone from $\overrightarrow{D}$ to $A/{=_A}$, i.e. a compatible family of extensional functions $g_i : A_i/{=_i} \to A/{=_A}$, and an extensional function $g : \varinjlim \to A$ compatible with this cone:

$$g(\iota_i(x)) =_A g_i(x)\,\text{true}\ (i : I,\, x : A_i),$$

which is to say that

$$g((i,x)) =_A g_i(x)\,\text{true}\ (i : I,\, x : A_i)\,.$$

Thus g is determined up to extensionality. We define $g((i,x)) \stackrel{\text{def}}{=} g_i(x)$, or to be precise,

$$g(a) \stackrel{\text{def}}{=} \mathsf{split}(a, (i,x)g_i(x)) \tag{39}$$

[5] Again, one could give an inductive definition, which is in many respects more natural, but would require more explanations here.

[6] Per Martin-Löf pointed out to me that the principle really needed is the following one:

$$
\begin{gathered}
(\forall i,j,k : I,\, p : i \le j,\, q : i \le k,\, x : A_i)\\
(\exists \ell : I,\, r : j \le \ell,\, s : k \le \ell)(f_{j\ell}(r, f_{ij}(p,x)) =_\ell f_{k\ell}(s, f_{ik}(q,x)))\,.
\end{gathered}
$$

or, alternatively,

$$g(a) \stackrel{\text{def}}{=} g_{\pi_\ell(a)}(\pi_r(a)) \,. \tag{40}$$

It remains to check that g is extensional. Let us use a semi-formal style, in the hope that it is easier to understand. So assume $a = b$ in $\varinjlim$. This means that there is a finite sequence $a =_{\mathsf{Id}} s_0 \; R \; s_1 \; R \cdots R \; s_n =_{\mathsf{Id}} b$. We shall prove that $g(a) =_A g(b)$, and by transitivity it suffices to prove $g(s_k) =_A g(s_{k+1})$ for each $k < n$. We may easily do that using that we have

$$\mathsf{app}(s, (k, q(k, n, p))) \; R \; \mathsf{app}(s, (\mathsf{succ}(k), p))$$

by assumption. It then remains to show that if elements are R-equal, then their image under g are $=_A$-equal. It suffices to consider elements in canonical form, so assume $(i, a) \; R \; (j, b)$ and prove $g((i, a)) =_A g((j, b))$. Spelled out, we are assuming there are k, p, q with

$$f_{ik}(p, a) =_k f_{jk}(q, b) \,\mathrm{true}$$

and we have to prove

$$g_i(a) =_A g_j(b) \,\mathrm{true} \;.$$

We do that by using transitivity, proving

$$g_i(a) =_A g_k(f_{ik}(p, a)) =_A g_k(f_{jk}(q, b)) = g_j(b) \,,$$

where we in the first and last step use the cone-property of g_i (and symmetry), and in the middle step the extensionality of g_k applied to the proof of $f_{ik}(p, a) =_k f_{jk}(q, b)$.

5 Products and Disjoint Unions of Setoids

In the special case when the index graph $(I, \leq)$ is a setoid, that is, when $\leq$ is an equivalence relation, we use to say *products* and *coproducts* instead of *limits* and *colimits*. In the case with setoids, it is common to say *disjoint union* instead of *coproduct*.

It has been proposed [3, Chap. 5.2] (and earlier in [4]) that a family indexed by a setoid should be functorial also with respect to the symmetry proofs. That is not needed for our purposes, since our constructions do not make use of symmetry at all.

There is also the case when no relation $\leq$ is present, but we have a family of setoids indexed by a set I. We may construct products and disjoint unions in this case too.

The construction of products is immediate, because a family indexed by $I : \mathsf{set}$ is automatically a diagram from the graph $(I, (i, j)\bot)$. Then the definition (27) of the limit normalizes to

$$(\Sigma f : \Pi(I, A))(\forall i, j : I, \, p : \bot)(f_{ij}(p, \mathsf{app}(f, i)) =_j \mathsf{app}(f, j)) \,.$$

which, because $p : \bot$ is impossible, is an improper subsetoid of $\Pi(I, A)$. Thus the product in the categorical sense amounts to what is called the 'product' also in type theory, but with an equivalence relation being extensional equality.

In the construction of the disjoint union, we need to do as indicated in the beginning of the section about colimits: we need to construct the reflexive closure.

We get, using that construction, the definition $i \leq j \stackrel{\text{def}}{=} \mathsf{Id}(I, i, j) \vee \bot$. It is of course natural in this special case to omit the second (always false) disjunct, defining $\leq$ to be nothing but the identity. The corresponding definition for the functions then becomes

$$f_{ij}(p, a) \stackrel{\text{def}}{=} \mathsf{subst}(p, a) \,.$$

Now, $(I, \leq)$ is a setoid and the family of setoids under consideration is a diagram over this setoid. The disjoint union is nothing but the colimit of this diagram. The definition (37) of $a\ R\ b$ becomes

$$(\exists k : I,\ p : \mathsf{Id}(I, \pi_\ell(a), k),\ q : \mathsf{Id}(I, \pi_\ell(b), k))(\mathsf{subst}(p, \pi_r(a)) =_k \mathsf{subst}(q, \pi_r(b)))\,. \tag{42}$$

Since we may always take k to be $\pi_\ell(b)$ and q to be $\mathsf{id}(\pi_\ell(b))$, we may replace this relation by the equivalent and simpler, but less symmetric, relation

$$(\exists p : \mathsf{Id}(I, \pi_\ell(a), \pi_\ell(b)))(\mathsf{subst}(p, \pi_r(a)) =_{\pi_\ell(b)} \pi_r(b))\,. \tag{43}$$

It is transitive, so it is equivalent with the complicated relation $=$ which was constructed in the previous section. Thus the disjoint union is $\Sigma(I, A)$, which is called ‘disjoint union’ also in type theory, but with equivalence relation given by (42) or (43).

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