ISSN: 1401-5617

Geometric Interplay between Function Subspaces and their Rings of Differential Operators

Rikard Bögvad
Rolf Källström

Research Reports in Mathematics
Number 6, 2003

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.math.su.se/reports/2003/6

Date of publication: June 5, 2003

Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

Geometric interplay between function subspaces and their rings of differential operators

by

Rikard Bögvad* and Rolf Källström †

5th June 2003

Abstract

We give, in a geometric setting, a general method for constructing differential operators with finite dimensional invariant spaces - the generalised Bochner problem [Tur94] is to characterise such. Conversely we also show that given a differential operator preserving a finite-dimensional vector space V, it comes (under some restrictions on V and the order of the differential operator) from the global sections of a line bundle. The methods are primarily taken from algebraic geometry, and includes a study of the principal bundle to simplify some proofs in the local study of this problem done in [KMO00]. Finally we show that several of the published examples of results characterising this type of differential operators come from our setup.

1 Introduction

In [Tur92] Turbiner had the sophisticated idea of finding solutions to the Schrödinger equation in the following way. Start with a second order differential equation (preferably with parameters) with a large set of solutions and transform it by gauge a transformation, change of coordinates, and specialisation of the parameters, to reveal a hidden underlying Lie algebraic structure. The validity of this approach was demonstrated by him in several cases (see the survey [GLKO94] and the introduction to [KMO00]). As part of this programme he posed the problem to find vector spaces of functions and differential operators preserving them (the generalised Bochner problem [Tur94] - Bochner was interested in finding eigenfunctions). This is the problem that we will be concerned with here. In a way it was generally

*Department of Mathematics, Stockholm University, S-106 91, Sweden, email:rikard@matematik.su.se

†Department of Mathematics, University College of Gävle, S-801 76, Sweden, email:rkm@hig.se

solved by [KMO00]. They proved in the setting of (real-valued) real-analytic functions A on a domain of some $\mathbf{R}^n$ that the algebra of differential operators $\mathcal{D}_A^V$ preserving a finite-dimensional vector space $V \subset A$ always acts irreducibly on V. We give below (section 2) a simpler commutative algebra argument for this theorem, in our algebraic or analytic setting, instead of using the Hodge operation, and relate it to the simplicity of the structure sheaf of a variety under the sheaf of differential operators. The technique is to use the bundle of principal parts, to analyse differential operators.

Some mystery remains, however, and several connections to other areas of mathematics are left to be explored. In particular, few higher dimensional examples have been studied [GLKO94]. In this paper we will show that many examples can be found by using geometrical constructions.

The enlightening example is the space V_k of monomials in n variables of degree less than a number k, which by computation was found to have a large subring of differential operators that preserve them [Tur94, Tur88]. This sub-algebra is generated by a Lie algebra isomorphic to $\mathfrak{sl}(n, \mathbf{C})$. In their local classification of finite-dimensional Lie algebras of first order differential operators on the complex plane $\mathbf{C}^2$, acting transitively on a Zariski open subset, it was noted in [GLHKO93] that this example has a geometric origin: V_k is the restriction of the global sections of the line bundle $\mathcal{O}(k)$ on $\mathbf{P}^n$ to the affine cell, and $\mathfrak{sl}(n, \mathbf{C})$ is the restriction of the Lie algebra of global (twisted) vector fields on $\mathcal{O}(k)$. These authors also found two other kinds of projective rational surfaces -Hirzebruch surfaces and $\mathbf{P}^1 \times \mathbf{P}^1$ that corresponded to known examples.

The example of V_k and $\mathbf{P}^n$ is also well-known in representation theory of Lie algebras. In a development starting with among others Gelfand, several authors have by means of sheaves of twisted rings of differential operators on homogeneous spaces, managed to give a geometric classification of all highest weight representations of complex simple Lie algebras(see[BlB81, BGK$^+$87]). Central to this theory was the theory of modules over rings of differential operators. Here the whole algebra of differential operators is used, not only the first order ones, and plays a central role, as a mediator between geometry and representation theory. Using this formalism it is easy to again recognise Turbiner's description of the differential operators (on $\mathbf{A}^n$) that preserve V_k, as the ring of globally defined $\mathcal{O}(k)$-twisted differential operators on $\mathbf{P}^n$.

This suggests that there is a rich geometric source of examples of solutions to the generalised Bochner problem. We describe the construction: Suppose that $X \subset \overline{X}$ is an affine open sub-variety of a projective complex algebraic variety, and that $\mathcal{L}$ is a line bundle on $\overline{X}$. There is a map

$$\rho : \Gamma(\overline{X}, \mathcal{L}) \to \Gamma(X, \mathcal{O}_X) =: A, \ \ s \mapsto s/s_0 \tag{1.1}$$

induced by choosing any non-vanishing local section $s_0 : A \cong \mathcal{L}_{|X}$. Hence we may identify the k vector space of global sections $\Gamma(\overline{X}, \mathcal{L})$ with a finite-dimensional vector subspace of the ring A. Let $\mathcal{D}_{\bar{X}}(\mathcal{L})$ denote the sheaf differential operators on $\mathcal{L}$ (as $\mathcal{O}_{\bar{X}} \times \mathcal{O}_{\bar{X}}$-bimodule we have $\mathcal{D}_{\bar{X}}(\mathcal{L}) \cong \mathcal{L} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{D}_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{L}^{-1}$), and $\mathcal{D}_X$ the sheaf of differential operators on X. We have an induced isomorphism

$$\eta : \Gamma(\overline{X}, \mathcal{D}_{\bar{X}}(\mathcal{L}))_{|X} \cong \mathcal{D}_X \subset Hom_{\mathbf{C}}(A, A), \tag{1.2}$$

defined by $\eta(P) * f = s_0^{-1}(P * s_0 f)$, if $P \in \Gamma(\overline{X}, \mathcal{D}_{\bar{X}}(\mathcal{L}))$ and $f \in \mathcal{O}_X = A$. We also have an obvious map

$$\theta : \Gamma(\overline{X}, \mathcal{D}_{\bar{X}}(\mathcal{L})) \to End_{\mathbf{C}}(\Gamma(\overline{X}, \mathcal{L})).$$

These maps are clearly compatible.

Thus whenever we have a ring of global differential operators on a line bundle on a compactification of an affine variety X, we will get a finite-dimensional vector space of functions on X that is preserved by a ring of differential operators. Of course we have to see that this gives us non-trivial examples, i.e. that there actually are examples of large algebras of global differential operators in $\Gamma(\overline{X}, \mathcal{D}_{\bar{X}}(\mathcal{L}))$. Preferably, 'large' means that $\Gamma(\bar{X}, \mathcal{L})$ is simple as $\Gamma(\bar{X}, \mathcal{D}_{\bar{X}}(\mathrm{L}))$-module. But there are several classes of compact varieties for which this is known: in addition to homogeneous spaces [BlB81], mentioned above, we also have toric varieties [MVdB98, Jon94, Mus94]. In these cases the ring of global differential operators will even act irreducibly on the global sections V, and will thus enable us to describe all local differential operators that preserve V modulo the annihilator ideal (*Th.* 3.2). We have complemented the results of the investigations cited on these varieties with a study of the bundle of principal parts, exemplifying the concepts which we have made use of (see sections 5,6). Taking restrictions to various open affine sub-varieties, of either toric or homogeneous varieties, thus produces a natural and abundant source of interesting examples, and we suggest that the presence of them may be viewed as a an explanation of the "significant mystery [that] is the connection [of the algebraic approach of[KMO00]] with the Lie algebraic approach of [Tur94] to quasi-exactly solvable modules" [KMO00, p.316]. As a strengthening of this suggestion in another way, we exemplify a partial converse in Proposition 7.2, where we give conditions on V, to ensure that it comes from a homogeneous space, by the above procedure.

These finite-dimensional vector space examples in themselves stem from a geometric context, but they do not in an immediately obvious way exhibit this geometric structure. For example, given the space of monomials V_k, where is $\mathbf{P}^n$? But there is a basic algebraic geometric technique, known to the ancients, to construct a projective algebraic variety X_V from an m-dimensional vector space of polynomials $V \subset A = \mathbf{C}[x_1, \dots, x_n]$, with a basis $f_0, \dots, f_m$. Define maps $j : X = \mathbf{A}^n \to \mathbf{P}(V^*)$ by $j(x) := [f_0(x); \dots; f_m(x)]$. This is a well-defined map if the functions $f_0, \dots, f_m$ have no common zero, or are basepoint free. Assume for simplicity that $f_0 = 1$. Then j maps X to $\mathbf{A}^m$, and this map is given on rings by

$$\mathbf{C}[y_1, \dots, y_m] \to \mathbf{C}[x_1, \dots, x_n], \quad y_i \mapsto f_i, \quad i = 1, 2, 3, \dots, m$$

Hence j will be a closed inclusion, i e the ring morphism is surjective, if and only if V generates A as a $\mathbf{C}$-algebra. Furthermore let $\bar{X}_V := \overline{j(X)}$. It is a closed, possibly singular sub-variety of $\mathbf{P}(V^*)$. Denote $\mathcal{O}(1)|_X$ by $\mathcal{L}_V$. Since the composed map

$$V \cong \Gamma(\mathbf{P}(V^*), \mathcal{O}(1)) \to \Gamma(X, \mathcal{O}(1)) \to A$$

(where the first isomorphism is standard, the second map is the restriction, the third is by choosing the local section $s_0 = 1$) just is the inclusion $V \subset A$, we get that $V \subset \Gamma(X, \mathcal{L}_V)$. Hence we have a situation similar to the previous setup. Observe that this construction is coordinate and gauge-invariant. It has the property that it unites many differing choices of embeddings of finite-dimensional vector spaces in affine rings.

To use these two constructions, restriction and completions, is the main idea of the present study, beside drawing attention to the examples described above. We hope that it will provide a framework for disparate results on quasi exactly solvable differential operators, with an emphasis on detecting possible underlying “hidden”geometry, as well as “hidden symmetries”.

We introduce two numerical constants associated to V, n_{inj} and n^1_{surj}, that express properties of V relative to its Taylor expansion. These conditions are on the one hand similar to the ones used by [KMO00], and on the other hand similar to the notions of k-jet ampleness [BS95]. In terms of these invariants it is possible to give numerically calculable conditions on the order of a local differential operator P that ensure that it comes from a global differential operator on $\mathcal{L}$(see Section 5), as well as study the action of differential operators on V.

All algebraic-geometric terminology conforms to the use in [Har85]. By a variety $(X/k, \mathcal{O}_X)$ over a field k we mean a separated integral scheme of finite type; we assume that the field k is infinite. Points in X are always rational over k, and by the normalisation theorem the points are dense in X. We do most of the time not need any restrictions on the characteristic of the ground field or the algebraic closedness — except when describing the generators of rings of global differential operators in particular examples— but the reader may profitably think of the ground field as $\mathbf{C}$ or $\mathbf{R}$, if so inclined.

2 Wronskians

2.1 Generalities

General references for the following material in this section are [Gro67, LT95]. Let $(X/k, \mathcal{O}_X)$ be a variety and M be a locally free $\mathcal{O}_X$-module. Let V be a finite-dimensional k-subspace of the space of global sections $\Gamma(X, M)$. Let $\Delta : X \to X \times_k X$, $x \mapsto (x, x)$, be the diagonal map, I_Δ be the kernel if the mapping $\Delta^*(\mathcal{O}_{X\times_k X}) \to \mathcal{O}_X$, and put, for each integer $n \geq 0$, $\mathcal{P}^n_X := \mathcal{P}^n_{X/k} = \Delta^*(\mathcal{O}_{X\times_k X})/I_\Delta^{n+1}$. The sheaf of $(\mathcal{O}_X, \mathcal{O}_X)$-bimodules $\mathcal{P}^n_X$ is the sheaf of nth order principal parts. Put also $\mathcal{P}^n_X(M) = \mathcal{P}^n_X \otimes_{\mathcal{O}_X} M$. There is a mapping $\mathrm{d}^n : M \to \mathcal{P}^n_X(M)$, $m \mapsto 1 \otimes m$, which is injective since M is locally free, and composing it with the injective map $V \to M$ (V is here regarded as a constant sheaf) we get an injective map $\mathrm{d}^n_V : V \to \mathcal{P}^n_X(M)$. Extending the scalars $\mathcal{V}_X = \mathcal{O}_X \otimes_k V$ we get a same noted map $\mathrm{d}^n_V : \mathcal{V}_X \xrightarrow{\mathrm{d}^n} \mathcal{P}^n_X(M)$, $\phi \otimes v \mapsto \phi \otimes v \mod I_\Delta^{n+1}$; this map need not be injective. Let $\mathcal{K}^n$ and $\mathcal{C}^n$ be the kernel and cokernel, respectively, of d^n_V, so we have the exact sequence (the Wronskian sequence)

$$0 \to \mathcal{K}^n \to \mathcal{V}_X \xrightarrow{\mathrm{d}^n_V} \mathcal{P}^n_X(M) \to \mathcal{C}^n \to 0. \tag{2.1}$$

Varying the integer n one gets different exact sequences (2.1), forming an inverse system. Then $\mathcal{P}^\infty_X(M) := \operatorname{proj\,lim}_{n\to\infty} \mathcal{P}^n_X(M)$ is provided with the I_Δ-adic topology which is used to define the sheaf of rings of differential operators $\mathcal{D}_X(M)$ on M as the bimodule of continuous maps $Hom_{\mathcal{O}_X, cont}(\mathcal{P}^\infty_X(M), M)$. The differential operators of order n is $\mathcal{D}^n_X(M) = Hom_{\mathcal{O}_X}(\mathcal{P}^n_X(M), M)$. One may notice that $\mathcal{D}_X(M)$ is 'matrix-valued' when the rank of M is > 1. The fibre of the sheaf of principal parts at a rational point x in X is

$$k \otimes_{\mathcal{O}_x} \mathcal{P}^n(M)_x \cong M_x/\mathfrak{m}_x^{n+1} M_x \quad \text{([Gro67, 16.4.11])}. \tag{2.2}$$

and if X is regular, so $\mathcal{P}^n_X(M)$ is locally free, then the fibre $k \otimes_{\mathcal{O}_x} \mathcal{D}^n_x(M) = (M_x/\mathfrak{m}_x^{n+1} M_x)^*$.

The map of fibres

$$\mathrm{d}^n_V(x) : V \to k \otimes_{\mathcal{O}_x} \mathcal{P}^n(M)_x$$

is the nth Taylor expansion map at x of the vectors in V.

Proposition 2.1. *Let x be a point in X. Then*

(1) the map of stalks $\mathrm{d}^n_{V,x} : \mathcal{V}_x \to \mathcal{P}^n_x(M)$ is injective when $n \gg 1$ and if $\operatorname{Char} k = 0$, then it is injective when $n \geq \dim V - 1$;

(2) the map of fibres $\mathrm{d}^n_V(x)$ is injective when $n \gg 1$ and the function $x \mapsto n_{inj}(x)$ is lower semi-continuous.

Proof. The first part of (2) follows from (2.2) since $\cap_{n=1}^{\infty} \mathfrak{m}_x^n M_x = 0$ by Krull's theorem and that V injects in M_x since M is locally free. Since the kernel $\mathcal{K}^n$ of d_V^n is coherent, its support will be closed, implying the semi-continuity of the function $n_{inj}(x)$. We now prove (1). The locus U^n of points where the fibre map $\mathrm{d}_V^n(x)$ is injective is open and non-empty by (2). One has $U^n \subset U^{n+1}$ and since X is a noetherian topological space there exists an integer n_0 such that $U = U^{n_0} = U^n$ when $n \geq n_0$. Then $X \setminus U$ is a proper closed subset. Since k is infinite the (rational) points are dense in $X \setminus U$ (by the normalisation theorem), hence by (2) we must have $U = X$. Then a section s of $\mathrm{Ker}(\mathrm{d}_V^n)$ will belong to $\mathfrak{m}_x M_x$ when $n \geq n_0$ and all points x in X. Since the rational points are dense in X and $\mathcal{V}_X$ is locally free we get $s = 0$. This proves that d_V^n is injective when $n \gg 1$. We refer to [LT95] for the remaining assertion that it suffices that $n \geq \dim V - 1$ in characteristic 0. $\square$

Definition 2.2. Let N_{inj} ($\leq \dim V - 1$) be the smallest integer such that d_V^n is injective and $n_{inj}(x)$ be the smallest integer such that $\mathrm{d}_V^n(x)$ is injective when $n \geq n_{inj}(x)$ (injectivity order at x). Define also $n_{inj} = \sup\{n_{inj}(x) : x \in X\}$ (injectivity order).

That $n_{inj} < \infty$ follows since X is quasi-compact and the function $x \mapsto n_{inj}(x)$ is lower semi-continuous. In the language of [KMO00] V is a "regular" subspace of M.

In local coordinates, and if $\mathrm{Char}\, k = 0$, the map $\mathrm{d}_{V,x}^n$ is described by the matrix with rows $(\partial^\alpha(m_i))$, if the m_i form a basis of V. If $n_{inj}(x) < N_{inj}$ then x can be regarded as a "Weierstrass point" on X for V. These are points where there exists a vectors v in V such that $\mathcal{D}^{N_{inj}}(M)_x(v) \subset \mathfrak{m}_x M_x$.

Definition 2.3. Let $n_{surj}(x)$ be the largest integer such that $n \leq n_{surj}(x)$ implies that $\mathrm{d}_V^n(x)$ is surjective (so $\mathcal{C}_x^n = 0$). The integer $n_{surj}(x)$ is the *jet ampleness degree* of V at x. Define $n_{surj} = \inf\{n_{surj}(x) : x \in X\}$

Evidently, $n_{surj}(x)$ is less than the biggest integer n such that $\dim M_x/\mathfrak{m}_x^{n+1} M_x \leq \dim V$.

We will have use for the following well-known splitting criterion.

Proposition 2.4. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be an injective map of locally free $\mathcal{O}_X$-modules. Then the following are equivalent at a point x in X:*

(1) the map of fibres $\phi(x) : k \otimes_{\mathcal{O}_x} \mathcal{F}_x \to k \otimes_{\mathcal{O}_x} \mathcal{G}_x$ is injective;

(2) the map of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is split injective.

Proof. That (2) $\Rightarrow$ (1) is evident. (1) $\Rightarrow$ (2): Since $\mathcal{F}$ and $\mathcal{G}$ are locally free one can reduce to the case when $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{G}_X = \oplus^m \mathcal{O}_X$. Let $\phi : \mathcal{O}_X \to \oplus^m \mathcal{O}_X$ send 1 to $(a_1, \ldots, a_m)$. (1) implies that one of the a_i is a unit, assume it is a_1. A split is then given by $(r_1, \ldots, r_m) \mapsto r_1/a_1$. $\square$

That a surjective map $\mathcal{F} \to \mathcal{G} \to 0$ is locally split when $\mathcal{G}$ is locally free is clear, and this split holds in affine subsets of X, by Serre's theorem. In [KMO00] they describe a nice way, using Hodge calculus, of choosing a split when $\mathcal{F}$ and $\mathcal{G}$ are free over the sheaf of *real-valued* analytic functions in open domains of $\mathbf{R}^n$.

Proposition 2.5. *Let $A : \mathcal{O}_X^n \to \mathcal{O}_X^r \to 0$ be a surjective homomorphism. Let $\omega = v_1 \wedge \cdots \wedge v_r \in \bigwedge_k^r \mathcal{O}_X^n$, where $\{v_1, \dots, v_r\} \subset \mathcal{O}_X(X)^n$ are the row vectors of the matrix $A(X)$. Assume that $\|\omega\|^2 = *(\omega \wedge *\omega)$ is a unit in $\mathcal{O}_X(X)$ (alt. $\omega \wedge *\omega = \phi e_1 \wedge \cdots \wedge e_n$, where e_i is the standard basis of $\mathcal{O}_X(X)^n$, and ϕ is a unit in $\mathcal{O}_X(X)$). Then the mapping of global sections is also surjective, $A(X) : \mathcal{O}_X(X)^n \to \mathcal{O}_X(X)^r \to 0$.*

In particular, if k is an ordered infinite field, then $\|\omega\|^2$ is a unit and the map $A(X)$ is surjective.

We first recall the Hodge *- operation. Let M be a free $\mathcal{O}_X$-module of rank n. Let $\{v_1, \dots, v_n\}$ be free generators. Define $*(v_{i_1} \wedge \cdots v_{i_p}) = \frac{1}{n-p}\epsilon_{i_1 \cdots i_p i_{p+1} \cdots i_n} v_{i_{p+1}} \wedge \cdots \wedge v_{i_n}$ and extend by linearity to a map $\bigwedge^p M \to \bigwedge^{n-p} M$. Then $* * \omega = (-1)^{p(n-p)}\omega$. We have $A(h) = (*(v_1 \wedge *h), *(v_2 \wedge *h), \cdots *(v_r \wedge *h)) \in \mathcal{O}_X(X)^r$, this follows since $v \cdot w = *(v \wedge *w)$.

Proof. Since $*(\omega \wedge *\omega)$ is a unit, given $e_i = (0, 0, \dots, 1, 0, \dots, 0) \in \mathcal{O}_X(X)^r$, the equations $Ah_i = e_i$ have the solution $h_i = *(v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots v_r \wedge *\omega)/*(\omega \wedge *\omega) \in \mathcal{O}_X(X)^n$.

Assume now that k is an ordered infinite field. Let x be a point in X. Then $\|\omega\|^2(x) \in k = \mathcal{O}_x/\mathfrak{m}_x$ is a sum of non-zero squares, hence since k is ordered, this sum is non-zero. Since the rational points in the zero locus of $\|\omega\|^2$ are dense, this zero locus must be empty, so $\|\omega\|^2$ is a unit. □

The following lemma is well-known.

Lemma 2.6. *(k is alg. closed) Assume that A is strongly quasi-projective where M is an invertible sheaf. Then if M is very ample it follows that in each point x in X we have $n_{surj}(x) \geq 1$.*

For the proof, see [Har85]

2.2 M is simple as $\mathcal{D}_X(M)$-module when X is regular

Let $n \geq N_{inj}$, so $K_n = 0$ and d_V^n is injective. Applying $Hom_{\mathcal{O}_X}(\cdot, M)$ to the Wronskian sequence (2.1) we get an exact sequence

$$\begin{aligned} 0 \to Hom_{\mathcal{O}_X}(\mathcal{C}_n, M) \to \mathcal{D}^n(M) \xrightarrow{W^n} Hom_{\mathcal{O}_X}(\mathcal{V}_X, M) \to \\ \to Ext^1_{\mathcal{O}_X}(\mathcal{C}_n, M) \to Ext^1_{\mathcal{O}_X}(\mathcal{P}^n_X(M), M) \to \end{aligned} \tag{2.3}$$

We can identify $Hom_{\mathcal{O}_X}(\mathcal{C}_n, M)$ with the annihilator $\mathrm{Ann}^n(V) = \{P \in \mathcal{D}^n(M) | P \cdot V = 0\}$. Let $\mathcal{D}^V = \{P \in \mathcal{D}_X | P \cdot V \subset V\} = (W_n^*)^{-1}(End_k(V))$

(this is a sheaf of k-algebras). Notice that $Hom_{\mathcal{O}_X}(\mathcal{V}_X, M) = Hom_k(V, M) = V^* \otimes_k M$, where $V^* = Hom_k(V, k)$, and $End_k(V) \subset Hom_{\mathcal{O}_X}(V, M)$.

Proposition 2.7. *Let X/k be a regular variety, M be a locally free $\mathcal{O}_X$-module and $\mathcal{D}_X(M)$ its ring of differential operators. Then*

(1) Let x be a point in X and V be a finite-dimensional sub-space of the stalk M_x. The mapping of stalks

$$W_x^n : \mathcal{D}_x^n \to Hom_{\mathcal{O}_x}(\mathcal{V}_x, M_x) = Hom_k(V, M_x)$$

is surjective when $n \gg 1$.

(2) The $\mathcal{D}_X(M)$-module M is simple.

Proof. (1): Since X is regular and M is locally free it follows that $\mathcal{P}_X^n(M)$ is locally free. By Propositions 2.1 and 2.4 there exists an integer $n_{inj}(x)$ (*Def.* 2.2) such that d_x^n is split injective, so $\mathcal{C}_x^n$ is free; hence, since $\mathcal{C}^n$ is coherent, $Ext^1_{\mathcal{O}_X}(\mathcal{C}^n, M)_x = Ext^1_{\mathcal{O}_x}(\mathcal{C}_x^n, M_x) = 0$.

(2): the $\mathcal{D}_X(M)$-module M is simple of all its stalks are simple. Let $m_x \in M_x$ and put $V = km_x$. Therefore (2) follows from (1). □

Remark 2.8. If X is not regular it is well-known that $\mathcal{O}_X$ need not be a simple $\mathcal{D}_X = \mathcal{D}_X(\mathcal{O}_X)$-module. Notice that (1) follows from the density theorem knowing that M_x is simple over $\mathcal{D}_x(M)$; if k is not algebraically closed we need that M_x is absolutely simple. Thus we have a counterpart of (1) also for simple coherent $\mathcal{D}_X(M)$-modules that need not be coherent over $\mathcal{O}_X$.

Definition 2.9. Let $s(x)$ be the smallest integer such that W_x^n is surjective when $n \geq s(x)$. Define also $s = \sup\{s(x) : x \in X\}$. The integer $s(x)$ is the differential order of V at x and s is its differential order (on X).

Again we can think of points x where $s(x) < s$ as "Weierstrass point" for V.

We state a result whose proof can be seen from the proof of Proposition 2.7.

Proposition 2.10. *If M is locally free, then $n_{inj}(x) \geq s(x)$, so $n_{inj} \geq s$. If $n \geq n_{inj}$, then one has a locally split short exact sequence*

$$0 \to \operatorname{Ann}^n V \to \mathcal{D}_X^n(M) \to V^* \otimes_k M \to 0. \tag{2.4}$$

3 The sheaf $\mathcal{D}^V(M)$

Letting n be an integer $\geq s$, one gets the short exact sequences (2.10). We have $End_k(V) = V^* \otimes_k V \subset V^* \otimes_k M$, so one can push out (2.3)to the short exact sequence

$$0 \to \mathrm{Ann}^n V \to \mathcal{D}^{V,n}(M) \xrightarrow{W_n^*} End_k(V) \to 0. \tag{3.1}$$

Assuming moreover that $n \geq n_{inj}$, then this sequence is locally split, so let us describe explicitly a splitting. Put $r = \dim V$ and let $\{\hat{L}^i\}$ be a basis of $End_k(V)$. Then select $L_x^i \in \mathcal{D}_x^{V,n}(M)$ such that $W_n^*(L_x^i) = \hat{L}^i$ and define a local splitting $\mathcal{D}_x^{V,n}(M) \to \mathrm{Ann}^n V$, $P_x \mapsto P_x - \sum_i \alpha_i L_x^i$ (the sum contains r^2 terms), where the coefficients $\alpha_i \in k$ satisfy the equation $\sum_i \alpha_i \hat{L}^i = W_n^*(P_x)$ in $End_k(V)$. Notice that the same $\alpha_i = \alpha_i(W_n^*(P_x))$ works for all x in affine subsets of X; see the proof of Theorem 3.1 below. We need to compute r^2 differential operators L_x^i to define a split. In [KMO00]) another better choice of split is used when $M = \mathcal{O}_X$, using a basis $\{v_1, \dots, v_r\}$ for V. By (*Prop.* 2.10) one can choose $R_x^i \in \mathcal{D}_x^n$, $i = 1, \dots, r$, so that $R_x^i(v_j) = \delta_{ij}$, and one defines a split by $P_x \mapsto P_x - \sum_{i=1}^r P_x(v_i) R_x^i$. In [loc. cit.] the differential operators $\{R_x^1, dots, R_x^r\}$ are called a dual basis for the basis $\{v_1, \dots, v_r\}$.

We collect our results in a theorem about splittings of global sections. It was proven in [loc. cit., Th. 4.8] when M is the sheaf of real-valued real-analytic functions on *arbitrary* open connected subsets of some $\mathbf{R}^n$ (contained in (2) below). The following is a tiny extension of their result to our algebraic situation.

Theorem 3.1. *Let X/k be regular and M be a locally free $\mathcal{O}_X$-module. Then we have a split short exact sequence*

$$0 \to \mathrm{Ann}^n(M)(X) \to \mathcal{D}^{n,V}(X) \to \mathrm{End}_k(V) \to 0 \tag{3.2}$$

and hence a split exact sequence

$$0 \to \mathrm{Ann}(M)(X) \to \mathcal{D}^V(X) \to \mathrm{End}_k(V) \to 0. \tag{3.3}$$

in the following cases:

(1) X is affine;

(2) M is free and X/k satisfies the assumptions in the last part of Proposition 2.5.

Proof. (1): By Proposition 2.10 the sequence (2.4) is locally split, hence if X is affine also globally split by Serre's vanishing cohomology theorem. Therefore the push-out (3.2) is also split exact. We get (3.3), since a split $End_k(V) \to \mathcal{D}^{n,V}(X)$ also gives a split $End_k(V) \to \mathcal{D}^V(X)$.

(2): By Proposition 2.5 the mapping $W^n(X)$ is surjective and split. □

In general it is difficult to decide when V is simple as $\mathcal{D}^V(X)$-module. Still there are interesting cases when one can prove simplicity. It is for example true both for toric varieties [MVdB98] and homogeneous spaces. Since in the case where the ground-field is algebraically closed, this is equivalent, by the density theorem, to θ being surjective we can describe the differential operators on X that preserve V, in an obvious way:

Proposition 3.2. *Let M be a quasi-coherent $\mathcal{O}_X$-module and V be a subspace of $V \subset \Gamma(X, M)$ that is simple as $\Gamma(X, \mathcal{D}_X^V(M))$-module. Then $\mathcal{D}_X^V(M) = \Gamma(X, \mathcal{D}_X^V(M)) + \mathrm{Ann}_{\mathcal{D}_{\mathrm{X}}(\mathrm{M})}(\mathrm{V})$ (equality of sheaves, where $\Gamma(X, \mathcal{D}_X^V(M))$ is regarded as the constant sheaf of global sections of $\mathcal{D}_X^V(M)$)*

Lemma 3.3. *If V is a subspace of $\bar{V}$ such that $\bar{V}$ is simple over $\mathcal{D}^{\bar{V}}$, then V is simple over $\mathcal{D}^{\bar{V},V} = \mathcal{D}^V \cap \mathcal{D}^{\bar{V}}$, the sub-sheaf of $\mathcal{D}_X^{\bar{V}}(M)$ consisting of sections that preserve V.*

We leave out its immediate proof.

Next two propositions exemplify the situation when V is simple as $\mathcal{D}^V(X)$-module. The first proves Theorem 3.1 in the case when $X = \mathbf{A}^n$ with a simple geometric argument.

Proposition 3.4. *Suppose that $V \subset \mathcal{O}_{\mathbf{A}^n}$ is a non-zero finite-dimensional vector space. Then $V \subset \mathcal{O}_{\mathbf{A}^n}(\mathbf{A}^n)$ is simple as $\mathcal{D}^V$-module, and furthermore V is contained in a subspace $\bar{V}$ such that $(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}, \bar{V} \subset \mathcal{O}_{\mathbf{A}^n}(\mathbf{A}^n), \mathcal{O}_{\mathbf{A}^n})$ is quasi-projective with completion $(\mathbf{A}^n \subset \mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}, \mathcal{O}_{\mathbf{P}^n}(m))$ for some positive integer m.*

Proof. Embed V in some $V_m =< \prod_{i=1}^n x_i^{k_i},\ 0 \leq k_i \leq m >$. Then $\bar{X}_V = \mathbf{P}^n$ and $\mathcal{L}_V = \mathcal{O}(m)$. It is known that the enveloping algebra $U(\mathfrak{sl}(n,k))$ [BlB81] gives naturally global differential operators on $\mathcal{O}(m)$ that makes V_n an irreducible module. Hence $U(\mathfrak{sl}(n,k))$ maps surjectively by θ to $End_k(V_n)$. A choice of a vector space complement K such that $V_n = V \oplus K$, gives an inclusion $End_k(V) \subset End_k(V_n)$. Then $S_V := \theta^{-1}(End_k(V))$ □

It is often the case that $\mathrm{Ann}^n V = 0$ for small values of n. This is illustrated in the following result, using a well-know description of $\Gamma(\mathbf{P}_k^n, \mathcal{D}(m))$, where $\mathcal{D}(m)$ is the ring of differential operators on $\mathcal{O}(m)$ [BGK+87].

Proposition 3.5. *[Tur94] Any differential operator that has order less than m and preserves the vector space V_m of polynomials of degree less than m, is a polynomial in the differential operators $\partial_{x_l}, x_k\partial_{x_l},\ k, l = 1, \dots, n$ together with $-\sum_{i=1}^n x_i x_k \partial_{x_i} + m x_k,\ k = 1, \dots, n$.*

Proof. Since $\mathbf{P}^n$ is a homogeneous space there is a map $\beta : U(\mathfrak{sl}_n) \to \Gamma(X, \mathcal{D}(\mathcal{L}))$([Jan87]) and taking the global sections V is the traditional way to construct (some of the) finite-dimensional irreducible $U(\mathfrak{sl}_n)$-modules. Hence V is also irreducible as a $\Gamma(X, \mathcal{D}(\mathcal{L}))$-module, so θ will be surjective, and Proposition 3.2 applies. Since $\mathrm{Ann}_{\mathcal{D}_{\mathrm{U}}}(\mathrm{V})$ is the left ideal generated by the derivations $\partial^{\alpha} = \Pi\partial_{x_i}^{\alpha_i}$, where $\alpha = (\alpha_1, \ldots, \alpha)_n$, and $\sum_1^n \alpha_i \geq n+1$, the result follows. In this case, by the way, β is also surjective. $\square$

4 Completions

To fix our situation we define two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows.

An object in $\mathcal{C}_1$ consists of the datum $(X, \mathcal{O}_X, V \xrightarrow{i} \Gamma(X, M), M)$ where $(X, \mathcal{O}_X)$ is a variety over the field k, V is a finite-dimensional vector space over k and i is a k-linear map to the global sections $\Gamma(X, M)$ of a locally free module M. In practice i will be an inclusion. Let $A = (X, \mathcal{O}_X, V \xrightarrow{i} \Gamma(X, M), M)$ and $B = (X', \mathcal{O}_{X'}, V' \xrightarrow{i'} \Gamma(X, M'), M')$ be objects in $\mathcal{C}_1$. A morphism $J : A \to B$ is a morphism of k-varieties $j : X \to X'$, an isomorphism of $\mathcal{O}_X$-modules $\psi : \phi^*(M') \to M$, and a surjective map of linear spaces $F : V' \to V$, requiring that the pair (ψ, F) forms a commutative diagram with (i, i') in the natural way.

The category $\mathcal{C}_2$ consists of morphisms $J : A \to B$ where $A, B \in \mathcal{C}_1$. Morphisms $J_1 \to J_2$ in $\mathcal{C}_2$ are given by obvious commutative diagrams. We will mostly be interested in cases when j is an open immersion.

We have an evident "restriction" functor

$$\begin{aligned} R : \mathcal{C}_2 &\to \mathcal{C}_1, \\ (J : A \to B) &\mapsto A. \end{aligned}$$

Definition 4.1. An object $A = (X, \mathcal{O}_X, i : V \to \Gamma(X, M), M)$ is *strongly quasi-projective* if there exists an object J in $\mathcal{C}_2$ such that $R(J) = A$, and

(1) $j : X \to X'$ is an open immersion into a projective variety X';

(2) $i' : V' \to \Gamma(X', M')$ is an isomorphism;

(3) $F : V' \to V$ is an isomorphism.

We say that J is a *completion* of A.

If it is clear from the situation what J is we also say that B is a completion of A. Thus a completion is a simultaneous extension of X to a projective variety and a locally free extension $\bar{M}$ of M, with the condition that $\Gamma(X', M') \cong V$.

It is in general difficult to see when completions exists, but below we show how standard methods give completions when $M = \mathcal{O}_X$ and X is affine in certain cases.

We will in fact only consider objects in $\mathcal{C}_1$ of the form $A = (X, \mathcal{O}_X, V \subset \Gamma(X, \mathcal{O}_X), \mathcal{O}_X)$ where X is an affine open subset of a variety of finite type over k; to be reasonably certain of a good supply of global differential operators it is even natural to let X be an open affine subset of some affine space $\mathbf{A}^r$, but here we do not need this assumption. The family of such affine objects A form a sub-category $\mathcal{C}_1^{aff}$ of $\mathcal{C}_1$. Let $\mathcal{C}_2^c$ be the sub-category in $\mathcal{C}_2$ of completions J, i.e. J satisfies (1-3) in Definition 4.1, such that $R(J) \in \mathcal{C}_1^{aff}$. We will study the possibility of defining a left adjoint of R, which then can be thought of as a "completion" functor $C : \mathcal{C}_1^{aff} \to \mathcal{C}_2^c$, so that $Hom_{\mathcal{C}_1^{aff}}(A, R(J)) = Hom_{\mathcal{C}_2}(C(A), J)$; thus the object $C(A) \in \mathcal{C}_2^c$ should represent the functor

$$\mathcal{C}_A : \mathcal{C}_2 \to \mathrm{Set}, \quad J \mapsto Hom_{\mathcal{C}_1^{aff}}(A, R(J)). \tag{4.1}$$

However, to represent this functor we need that certain extra conditions are satisfied for A (I-III below), so we are unable to find a functor defined an all of $\mathcal{C}_1^{aff}$.

To determine $C(A)$ we will use some facts from [Har85, GD61], and we also follow their notations for constructions pertaining to projective schemes.

Let Proj A be the projective scheme of a graded ring A. If V is a vector space over a field k and $\mathrm{S}[V]$ its symmetric algebra, then Proj $\mathrm{S}[V] = \mathbf{P}_k^{n-1}$, if $n = \dim V$. The grading of the polynomial ring $\mathrm{S}[V][t]$ with coefficients in $\mathrm{S}[V]$ gives Proj $\mathrm{S}[V][t] = \operatorname{Spec} \mathrm{S}(V) = \mathbf{A}^n$; its closed points can be identified with the dual space V^* (if k is algebraically closed). If X is any scheme we have a map $i : X \to \mathbf{A}^n$ induced by $\mathrm{S}[V] \to \mathcal{O}_X(X)$.

Let R be a k-algebra so that $X = \operatorname{Spec} R$. Then i is a closed embedding if R is generated by the k-subspace V.

Definition 4.2. Let $V \subset R$ be a finite-dimensional vector sub-space.

$$\gamma : \mathrm{S}[V][t] \to R[t].$$

Let B_V be the sub-algebra of $S[V][t]$ that is generated by tV and denote by A_V the image under γ of B_V. Then Proj $B_V = \mathbf{P}^n (= \mathbf{P}(V^*))$, if $n = \dim_k V$ and Proj $S[V][t]$ is the affine space $\mathbf{A}^n$. Define

$$X_V := \text{Proj } A_V$$

and let $\mathcal{L}_V = \mathcal{O}_{X_V}(1)$ be the associated line bundle.

Example 4.3. Consider $x := \mathbf{A}^n$ and $V =< \prod_{i=1}^n x_i^{k_i},\ 0 \leq k_i \leq m >$. we get $\bar{X}_V = \mathbf{P}^n$ and $\mathcal{L}_V = \mathcal{O}(m)$. For another example, let $V =< 1, x, x^3 >$. Then $A_V = k[t, tx, tx^3] \cong k[u, v, w]/(u^2 w - v^3)$. This is a cusp; on the open set $w \neq 0$, it is $k[u, v]/(u^2 - v^3)$.

We have a map $i : X \to \mathbf{A}^n$ induced by the homogeneous map $\mathrm{S}[V][t] \to R[t]$. Put $X^V = \{x \in X | V \nsubseteq \mathfrak{m}_x\}$, the base-point free locus for V in X, so $X^V = X$ means that V is base-point free on X. Then we have a map

$$X^V \to X_V$$

and a commutative diagram

$$\begin{tikzcd} X^V \arrow[r, hook, "i_0"] \arrow[d, "j"'] & \mathbf{A}^n \arrow[d] \\ X_V \arrow[r, hook, "\phi"] & \mathbf{P}^n \end{tikzcd} \tag{4.2}$$

where i_0 is the restriction of i. Here the horizontal arrows are closed immersions and the right vertical arrow is an open immersion. Let $V_1 \subset V_2$ be an inclusion of finite-dimensional sub-spaces of R. Put $V_{V_2}^{V_1} = \{x \in X_{V_2} | \ V_1 \not\subseteq \mathfrak{m}_x\}$, the base-point free locus of V_1 in X_{V_2}. Then the natural homogeneous map $A_{V_1} \to A_{V_2}$ induces a map $X_{V_2}^{V_1} \to X_{V_1}$. Notice that V is base-point free on X_V.

(I) Our first condition on A is that V is base-point free in X, so $X^V = X$. Then put $C(A) = J_A$, where $J_A : A \to B$, and

$$B = (X_V, \mathcal{O}_{X_V}, \Gamma(X_V, L_V), \mathcal{L}_V).$$

It should be clear what are the morphisms (j, ϕ, F). For $C(A)$ to satisfy (1) in Definition 4.1 we need a condition that the map $j : X \to X_V$ be an open immersion. Such conditions are well-known and we will use only a very simple one (II).

Proposition 4.4. *[Gro61, Chap 2,3.8.5] Let $X = \operatorname{Spec} R$ be affine, and $V \subset R$ a finite-dimensional vector space such that the map of algebras $\gamma : S[V] \to R$ is surjective. A sufficient condition that $j : X \to X_V$ be an open immersion is that*

$$((A_V)_{tv})_0 = R_v, \tag{4.3}$$

for a set of sections $v = v_i, \ i = 1, \dots, r$. In particular, if V contains a unit of R (so $X = X^V$), then this criterion is true.

The validity of the proposition is easily checked since Proj A_V is constructed by glueing together the affine spaces $\operatorname{Spec}((A_V)_{tv})_0 \cong \gamma(S[Vv^{-1}])$, for all sections $tv \in tV \subset A_V$. If $v \in V$ is a unit in R, it is clear that the subjectivity of γ implies that $((A_V)_{tv})_0 = R$, and hence X is isomorphic to an open set of $\bar{X}_V$.

There is now a last problem. In general the object $C(A)$ will not be a completion of A since we only have $V \subset \Gamma(X_V, \mathcal{O}_{X_V}(1))$, and not equality. Putting $\bar{V} = \Gamma(X_V, \mathcal{O}_{X_V}(1))$ we have a natural injective map $i : V \to \bar{V}$ and inclusions $A_V \subseteq A_{\bar{V}} \subset R[t]$. We can now iterate, $\bar{V} \subset \Gamma(X_{\bar{V}}, \mathcal{O}_{X_{\bar{V}}}(1))$ and so on. We do not know if this process stabilises. One condition (III) to ensure $V = \bar{V}$ is that A_V be a normal ring (one then says that the embedding of X_V in $\mathbf{P}(V^*)$ is projectively normal) because then the stronger assertion $A_V = \Gamma_*(X_V, \mathcal{O}_{X_V}) := \oplus_{i=0}^{\infty} \Gamma(\bar{X}, \mathcal{L}_V^i) t^i$ holds.

Let now $J : A \to B$ be a completion, where $B = (\bar{X}, \mathcal{O}_{\bar{X}}, \Gamma(X_V, \mathcal{L}), \mathcal{L})$. Does it follow that $B \cong (X_V, \mathcal{O}_{X_V}, \Gamma(X_V, \mathcal{L}_V), \mathcal{L}_V)$?

Definition 4.5. The object $B \in \mathcal{C}_1$ is projectively normal if $\mathcal{L}$ is very ample and its associated immersion $i : \bar{X} \to \mathbf{P}^l_k$ is projectively normal, i.e. its homogeneous coordinate ring $\mathrm{S}(\bar{X})$ is normal.

Proposition 4.6. *If $J : A \to B$ is a completion such that B is projectively normal, then $B \cong (X_V, \mathcal{O}_{X_V}, \Gamma(X_V, \mathcal{L}_V), \mathcal{L}_V)$.*

Proof. See also [Har85, II.5.16]. It suffices to prove $A_V \cong \mathrm{S}(\bar{X})$. Observe first that $R[t] \cong \oplus_{i=1}^{\infty} \Gamma(X, \mathcal{L}^i) t^i$, using the isomorphism $\psi : j^*(\mathcal{L}) \to \mathcal{O}_X$, and that the restriction map hence gives an injection $\Gamma_*(\bar{X}, \mathcal{O}_{\bar{X}}) \subset R[t]$, compatible with multiplication. Projective normality implies that $\Gamma_*(\bar{X}, \mathcal{O}_{\bar{X}}) \cong \mathrm{S}(\bar{X})$ and also that the natural map

$$\mathrm{S}(\bar{X}) \cong \Gamma_*(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to \Gamma_*(\bar{X}, \mathcal{O}_{\bar{X}}).$$

is surjective. Composing this map with the injection $\mathrm{S}(\bar{X}) \cong \Gamma_*(\bar{X}, \mathcal{O}_{\bar{X}}) \subset R[t]$ we get A_V. Therefore $A_V = \mathrm{S}(\bar{X})$. $\square$

Assuming (I-III) we now prove that $C(A) = J_A$ represents the functor $\mathcal{C}_A$ (4.1) on a sub-category of $\mathcal{C}_2^c$. Denote by $\mathcal{C}_2^{pc}$ the sub-category of objects $J \in \mathcal{C}_2^c$, where $J : A_1 \to B_1$, $A_1 = (X_1 = \operatorname{Spec} R_1, i_1 : V_1 \to \Gamma(X_1, \mathcal{L}), \mathcal{L})$, and $B_1 = (\bar{X}_1, i_2 : \bar{V} \cong \Gamma(\bar{X}_1, \bar{\mathcal{L}}_1), \bar{\mathcal{L}}_1)$ is projectively normal.

Proposition 4.7. *Assume that A satisfies (I-III). Then $C(A)$ is a completion of A and it represents the restriction of $\mathcal{C}_A$ to $\mathcal{C}_2^{pc}$.*

Proof. Let $G = (\phi_1, \psi_1, F)$ be a morphism $A \to R(J)$. We need to complement G to a morphism $G^c : C(A) \to J$. By Proposition 4.6 the homogeneous coordinate ring of $\bar{X}$ for the embedding determined by $\bar{\mathcal{L}}_1$ is A_{V_1}. The map $j : R_1 \to R$ therefore induces a surjective map $A_{V_1} \to A_V$, and since $F : V_1 \to V$ is surjective, V_1 has no base-points in X_V, so one gets a map $\bar{j} : X_V \to (X_1)_V = \bar{X}$. We leave out the remaining details to see that this gives us a morphism G^c. $\square$

5 Extending differential operators

Proposition 5.1. *Let $J : A \to B$ be a completion, where $A = (X, \mathcal{O}_X, i : V \to \Gamma(X, M), M)$ and $B = (\bar{X}, \mathcal{O}_X, \bar{i} : \bar{V} \cong \Gamma(\bar{X}, \bar{M}), \bar{M})$. If $\operatorname{codim} \operatorname{supp} \mathcal{C}^n \geq 1$, then the restriction mapping*

$$r_{X,X_0} : \mathcal{D}^n(X) \to \mathcal{D}^{n,V}(X_0)$$

is injective. Assume that d^n_V is surjective at points of height 1. Then r_{X,X_0} is surjective.

The surjectivity condition at points of height 1 can equivalently be phrased as $\operatorname{codim} \operatorname{supp} \mathcal{C}^n \geq 2$.

Proof. Apply $Hom_{\mathcal{O}_{\bar{X}}}(\cdot, \bar{M})$ to the sequence (2.10). Since $\mathcal{C}$ is a torsion sheaf and M is torsion free the injectivity of $r_{\bar{X},X}$ follows. Assume now that M has depth ≥ 2 (e.g. M is locally free) and that d_V^n is surjective at all points of height 1. An element P in $\mathcal{D}^{n,V}(X)$ gives an element $\tilde{P}$ in $\mathrm{Hom}_{\mathcal{O}_{\bar{X}}}(\mathcal{V}_{\bar{X}}, \bar{M}) = \mathrm{Hom}_k(V, \bar{M}))$. The fact that $\tilde{P}$ comes from P implies that generically the kernel K^n of d_V^n belongs to the kernel of $\tilde{P}$, but since $\mathcal{V}_{\bar{X}}$ and M are locally free, this gives $K^n \subset \mathrm{Ker}\, \tilde{P}$. Hence $\tilde{P}$ gives an element $\hat{P}$ in $\mathrm{Hom}_{\mathcal{O}_{\bar{X}}}(\mathrm{Im}\, \mathrm{d}_V^n, \bar{M})$. Since d_V^n is surjective at points of height 1, $\hat{P}$ induces a mapping $\hat{P}_x : \mathcal{P}^n(\bar{M})_x \to \bar{M}_x$ at points x of height 1. Since $\bar{M}$ has depth ≥ 2 at all points if height ≥ 2, the map $\hat{P}_x$ in fact gives an element P^n in $\mathrm{Hom}_{\mathcal{O}_{\bar{X}}}(\mathcal{P}^n_{\bar{X}}(\bar{M}), \bar{M})) = \mathcal{D}^n_{\bar{X}}(\bar{M})(\bar{X})$. □

Definition 5.2. Let n_s^1 be the largest integer k such that d_V^n is surjective at points of height 1.

Thus if P is a differential operators on M, defined in X and preserving V, then it has an extension to a global differential operator on $\bar{X}$. A similar condition to that of d_V^n being surjective (not just that $\mathrm{codim}\, \mathrm{Coker}\, \mathrm{d}_V^n \geq 2$) for a line bundle M, has been studied, under the name of k-jet ampleness [DR99, BS95, Dem96].

Remark 5.3. If X is a regular curve and X_0 a Zariski open subset, then r_{X,X_0} is always surjective. If X is a non-regular curve this no longer holds.

Example 5.4. Consider $\mathbf{A}^n \subset \mathbf{P}^n$ and Consider $V_m =< \prod_{i=1}^n x_i^{k_i},\ 0 \leq k_i \leq m >$. Then $n_s^1 = m$. and hence the above result obtains. To see that $\mathrm{d}^k_{V_m}$ is surjective if $k \leq m$ we use the fact that both $P^k_{\mathbf{P}^n}(\mathcal{O}(m))$ and $\mathcal{O}_{\mathbf{P}^n} \otimes V$ are $\mathrm{SL}(n,k)$-linearised sheaves(see [Jan87]) and hence the map will be surjective if it is surjective in one point; this is clearly true if $k \leq m$ for any point in $\mathbf{A}^n$. This can of course seen directly. Here we have a very symmetrical situation:

$$n^1_{surj} = n_{surj} = n_{inj}$$

6 Toric varieties

If $V \subset k[x_1, \ldots, x_n]$ is a finite-dimensional vector space generated by monomials, X_V will be a toric variety. Differential operators that preserve such V have been considered in an affine situation by [PT95], [Tur88], and [FK98], without using toric varieties. In this section we will calculate n_{inj} and n_{surj} for toric varieties, and describe the structure and behaviour of the ring of differential operators in some special cases. The point is to emphasise the importance of the completions, and the work that has been done on rings of differential operators in that context, by Musson [Mus94](see also [Jon94],[MVdB98]). We exemplify by the following result. By the work of Musson [Mus94](see also [Jon94],[MVdB98]), it is known that there are always lots of differential operators on an invertible sheaf on a toric variety. If the variety X is projective, $\Gamma(X, \mathcal{L})$ is simple as a $R = \Gamma(X, \mathcal{D}_{\mathcal{L}})$-module, by [Mus94], hence the following proposition is a consequence of Proposition 3.2; it means that every differential operator preserving V may be decomposed as the sum of an annihilator and a global differential operator.

Proposition 6.1. *Assume that $V = \oplus_{m \in P \cap M} kx^m$, P is a convex polytope (see below), satisfying the very ampleness condition at the end of this section. Then $V = \Gamma(X_V, \mathcal{L}_V)$. The map $\Gamma(X, \mathcal{D}_{\mathcal{L}}) \to \mathcal{D}^V$ is surjective.*

Musson describes the ring of global differential operators explicitly as a quotient of a ring of invariants of differential operators on an open subset of the affine space. In particular it is easy to give examples of smooth varieties where the global ring of differential operators is not generated by first order differential operators. These toric constructions corresponding to invariant V with a basis of monomials, do not seem to have been used before, even though Gonzalez-Lopez et al. [GLKO97] study Hirzebruch surfaces; this gives representations of the Lie algebra of global vector fields on a very ample line bundle that is the semi-direct product of sl_2 with an abelian Lie algebra.

Recall the construction of toric varieties, cf. [Ful93] for details. Suppose that $M = \mathbf{Z}^n$ is a lattice. If $m = (m_1, \dots, m_n) \in M$ define

$$x^m = x_1^{m_1} \dots x_n^{m_n} \in k[x_1, x_1^{-1}, ., \dots, x_n, x_1^{-1}] = k[T].$$

The torus in the nomenclature is $\operatorname{Spec} k[T]$. We will constantly go back and forth between elements in M and monomials in $k[T]$. The construction of X_V described before, works well, if V is the vector space generated by monomials x^m, such that m belongs to a strictly convex polytope $P \subset M$. Then X_V is usually called $X(P)$ and the associated line bundle $\mathcal{L}(P)$, cf. [Ful93, Section 1.5]. The construction is easily described the following way, streamlined to the monomial situation. Suppose that m_i is a vertex of P and let $M_i \subset M$ be the semigroup generated by $< p - m_i | p \in P >$ (or the elements of M that lie in the corresponding angle bounded by the codimension 1 faces of the polytope $F_j, j \in I_i$ that meet at m_i). Then define $k[M_i] \subset k[T]$ as the algebra generated by x^m $m \in M_i$ and $U_i = \operatorname{Spec} k[M_i]$. Furthermore define L_i as $k[M_i]x^{m_i} \subset k[T]$, with an obvious inclusion $P \to L_i$. These local data U_i and L_i now glue in a way that is uniquely determined by the given inclusions into $k[T]$, and this completes the construction. We also have that $V = \oplus_{m \in P \cap M} kx^m = H^0(X(P), L(P))$ ([Ful93, 3.4]). This line bundle will be very ample if the following condition is satisfied:

The polyhedron P is the convex hull of the points m_i, and for each i the semigroup generated by $< p - m_i | p \in P >$ is saturated([loc.cit.]).

6.1 An example: $\mathbf{P}^n$

We will now further exemplify the uses of toric varieties, and describe the ring of differential operators on $\mathbf{P}^n$, as an example of Musson's [Mus94] description of the twisted differential operators on an arbitrary toric variety.

Let $y_0, y_1, \ldots, y_n$ be coordinates of k^{n+1}, and consider $Y = \mathbf{C}^{n+1} - \{0\}$. If $T = k^*$ acts on Y by multiplication, then $\mathbf{P}^n = Y//T$. Hence y_i are the homogeneous coordinates on $\mathbf{P}^n$. Suppose that $\mathcal{L} = \mathcal{O}(m)$, where $m > 0$. Then $V = \Gamma(\mathbf{P}^n, \mathcal{L})$ is the vector space of homogeneous polynomials of degree m, in the homogeneous coordinates. Choose the section $s_0 = y_0^m$ of $\mathcal{L}$. Then $x_i := y_i/y_0,\ i = 1, 2, \ldots, n$ gives an isomorphism between the open set $X = \{y_0 \neq 0\} \subset \mathbf{P}^n$ and $\mathbf{A}^n$. The map ρ in (1.1) maps V to the subset of polynomials of degree at most m of $A = k[x_1, \ldots, x_n]$, by $p \mapsto p/y_0^n$. Consider the ring of differential operators

$$A_{n+1} = k[y_0, y_1, \ldots, y_n, \partial_{y_0}, \ldots, \partial_{y_n}]$$

on k^{n+1}. We use multi-index notation $y^\alpha \partial^\beta$ for monomials in the generators of A_{n+1}. The action of the torus T on A_{n+1} is given by

$$t y^\alpha \partial^\beta = \lambda^{<1, \alpha - \beta>} y^\alpha \partial^\beta,$$

where $1 = (1, 1, \ldots, 1) \in \mathbf{Z}^{n+1}$ and the scalar product is the usual one. The torus itself induces the Euler vector field $e := y_0 \partial_{y_0} + \cdots + y_n \partial_{y_n}$. Then

$$\Gamma(\mathbf{P}^n, \mathcal{D}_{\mathbf{P}^n}) \cong A_{n+1}^T/(e).$$

This is easily modified to the case of an arbitrary line bundle:

$$\Gamma(\mathbf{P}^n, \mathcal{D}(\mathcal{O}(m))) \cong A_{n+1}^T/(e - m).$$

The ring of invariants A_{n+1}^T, consists of all homogeneous differential operators of degree 0. Hence it is clear how they induce differential operators on $\mathbf{P}^n$. It is also easy to see that A_{n+1}^T as well as its quotients, is generated as an algebra by the vector fields $y_k \partial_{y_l}$.

We will now describe the restriction map η, which shows how these vector fields act on $A = k[x_1, \ldots, x_n]$. The function x_i corresponds to the global section $y_i y_0^{n-1}$, and hence, if $k, l \neq 0$,

$$\eta(y_k \partial_{y_l}) * x_i = \delta_{li} x_k$$

so that $\eta(y_k \partial_{y_l}) = x_k \partial_{x_l}$. Similarly it is easy to see that

$$\eta(y_0 \partial_{y_l}) = \partial_{x_l},$$

and

$$\eta(y_k \partial_{y_0}) = -\sum_{i=1}^{n} x_k x_i \partial_{x_i} + m x_k,$$

and

$$\eta(y_0\partial_{y_0}) = -\sum_{i=1}^{n} x_i\partial_{x_i} + m.$$

Hence we get from the proposition of the preceding section the following result on differential operators with polynomial coefficients, that preserve certain finite-dimensional vector spaces. It was first proved by Turbiner [Tur94] by algebraic means, and our proof serves as an illustration of the idea of the present exposition, that there is a "hidden" geometry in the situation. A variant, where the differential operators where identified with $U(\mathfrak{sl}(n,k))$, was given earlier.

Proposition 6.2. *[Tur94]The differential operators D_{V_0} that preserve $V_0 = \langle x^m \mid \sum_{i=1}^n m_i \leq m\rangle$, is the ring*

$$\Gamma(\mathbf{P}^n, \mathcal{D}(\mathcal{O}(m)) = k[x_k\partial_{x_l}, \partial_{x_k}, x_k(\pi - m)|i,k = 1,\dots,n],$$

where $\pi = \sum_{i=1}^n x_i\partial_{x_i}$ together with $\operatorname{Ann} V$*. Also* $\operatorname{Ann} V \cap \Gamma(\mathbf{P}^n, \mathcal{D}^r(\mathcal{O}(m)) = 0$*, if $r < m$.*

The last result of the proposition follows from the fact that all differential operators of order less than m, may be extended globally since $n_{surj} = m$. But if $P \in \operatorname{Ann}^r V$, $r < m$ then also $k[x_1,\dots,x_n]P \subset \operatorname{Ann}^r V$, and all these may not be extended to global sections, since they form an infinite-dimensional vector space, while the global differential operators of order less than m is a finite-dimensional vector space. It is of course easy to describe $\operatorname{Ann} V = (\partial_i^m| \quad i = 1,\dots,n)$.

6.2 Calculation of n_{inj} and n_{surj} for toric varieties in the smooth case

In the case that a toric variety is smooth and proper, Di Rocco [DR99], has studied the Wronskian, with the purpose to give criteria for k-jet-ampleness (a generally stronger, but similar condition to the surjectivity of the Wronskian.) In particular, she has given an easily calculated numerical criterion that implies that

$$\alpha_k : \mathcal{O}_X \otimes_k H^0(X,\mathcal{L}) \to \mathcal{P}_X^k(\mathcal{L})$$

is surjective. The criterion is almost evident in the situation we are most interested in, namely when we start with a convex polytope P, and consider the associated toric variety $X(P)$ and line bundle $\mathcal{L}_P$. It is formulated in terms of the geometry of the polytope.

Definition 6.3. [DR99] Assume that $P \subset M$ is a strictly convex polytope, that is the convex hull of the finite number of points $m_i, i \in I$. Along each edge E_{ij} of the polytope, connecting two vertices m_i and m_j, choose a minimal element $e_{ij} \in M$. If $m_i - m_j = l_{ij}e_{ij}$, the positive integer l_{ij} is called the length of the edge E_{ij}. The polytope P is said to be k-convex if the length of each edge of the polytope is larger than or equal to k. Denote by $k(P)$ the maximal k such that P is k-convex (i.e. the minimal length of an edge).

Di Rocco's result is that P is k-convex if and only if $\mathcal{L}(P)$ is k-jet ample, in the case when $X(P)$ is smooth. We will find that this is also equivalent to $n_{surj} = k$ Hence $n_{surj} = k(P)$ in this case. Her method is to use Cox's homogeneous coordinate ring; this is necessary for k-jet ampleness, but not in our simpler case, whence we present a proof.

That $X(P)$ is smooth is equivalent to the fact that any semigroup M_i is generated by a basis of M ([Ful93]). Since the e_{ij} will be precisely the generators of M_i , we get that there are n of them and that $k[M_i] \cong k[x_1, \dots, x_n]$. It is furthermore clear, by sophomore analysis,that locally the principal bundle may be described easily as $\mathcal{O}_{\mathbf{A}^n} \otimes V \xrightarrow{\sim} \mathcal{P}^k_{\mathbf{A}^n}$, where V is the vector space generated by all x^I, with $I = (i_1, i_2, \dots, i_n)$ with $0 \leq \sum_{j=1}^n i_j \leq k$, and the map is the generalised Wronskian. Let $S_i := \{\sum_j a_{ij} e_{ij}, \alpha_{ij} \in \mathbf{N}\}$, and let $P_i^k := S_i/(S(k)_i)$ be the quotient by $S(k)_i := \{\sum_j a_{ij} e_{ij}, \alpha_{ij} \in \mathbf{N},\ \sum_j a_{ij} \geq k+1\}$. Hence the generalised wronskian map

$$k[M_i] \otimes \langle x^p,\ p \in P\rangle \to \mathcal{P}^n_{U_i}(L(P))$$

may be described in M, by the effect on the basis $p \in P$ as

$$W_k : p \mapsto \sum_j a_{ij}(p - m_i) e_{ij} \in P_i^k.$$

By ordinary convexity of P, this map is clearly surjective if P is k-convex, since then any sum $\sum_j a_{ij} e_{ij}, \alpha_{ij} \in \mathbf{N},\ \sum_j a_{ij} \leq k \in P$. Conversely, assuming that the wronskian is surjective and tensoring with $k[M_i]/(x^{e_{ij}})$, we obtain that $P \to P_i^k$ is surjective, which is possible only if $re_{ij} + m_i \in P$ for $r = 1, \dots, k$, so that P is k-convex. Hence we get that Di Rocco's criterion, for jet ampleness also characterises n_{surj}.

We may also similarly study the injectivity of the generalised Wronskian map. It suffices to study this at the special closed and T-invariant point $p_i$in each open affine set U_i, since the set $\{x \in X(P) \mid \operatorname{rank} n(x) < |P|\}$ consists of T-orbits and is closed. This point is defined by the maximal ideal generated by $M_i \subset k[M_i]$. Clearly, n_{inj} will then be the least k such that P is contained in the set $S_i^k := \{\sum_j a_{ij} e_{ij} \mid \ a_{ij} \in \mathbf{N}, \sum_j a_{ij} \leq k\} + m_i$. It is enough to check this for the vertices m_j, $j \neq i$.

Hence we have proved the following proposition.

Proposition 6.4. *Suppose that $P \subset \mathcal{O}_{A^n}$ is a k-convex polytope, with vertices m_i, $i \in I$, and that that $X(P)$ is smooth and proper. Then*

$$n_{inj} = n(P)_{inj} := Max\{Min\{k : P \subset S_i^k\}, i \in I\},$$

and

$$n_{surj} \leq n(P)_{totalsurj} := Min\{Max\{k : S_i^k \subset P\}, i \in I\}.$$

6.2.1 $\mathbf{P}^n$ revisited

Let us calculate the integers of the preceding section for $\mathbf{P}_k^n$. Suppose that $m \geq 0$ is an integer and consider $M = \mathbf{Z}^n$ and the polytope given by $P(m) = \{b = (b_1, \ldots b_n) | b_i \geq 0, \ \sum_{i=1}^n b_i \leq m\}$. This is the convex hull of the points $m_i = (0, \ldots, 0, m, 0, \ldots, 0)$, and the minimal distance between two vertices is clearly m. Hence $P(m)$ is exactly m-convex, and hence also $n_{surj} = m$. Also clearly $n_{inj} = m$. The toric variety $X(P(m))$ will be $\mathbf{P}^n$ and the associated line bundle is $\mathcal{O}(m)$. In local coordinates, it follows from the considerations of the preceding section that the Wronskian $\mathcal{O}_{\mathbf{P}^n} \otimes P(m) \to \mathcal{P}_{\mathbf{P}^n}^k \otimes \mathcal{O}(m)$ actually is an isomorphism for $k = m$, and has a cokernel with support outside codimension 2 if $k < m$.

6.3 Calculation of n_{surj}^1 on general projective toric varieties

We next will consider the condition of surjectivity in codimension 2. In this case, smoothness will not be essential and we will describe a procedure below. The Wronskian (6.2) is a homogeneous homomorphism between T-homogeneous sheaves (on an open subset U_i this just means that there are natural gradings on the restrictions and that the homomorphism respect these gradings), and so the support of the cokernel and kernel will be unions of closures of orbits under the torus T. Hence to check the surjectivity in codimension 2, it suffices to check surjectivity at the orbits of codimension 0 or 1 orbits. (We refer to [Ful93] for information on orbits.) The orbits correspond to the faces F of the polytope, and each orbit $O(F)$ is generated by a well-defined and easily described point $p(F)$; codimension 1 orbits correspond to codimension 1 faces. In fact, it suffices to prove surjectivity for all orbits of codimension precisely 1, since the support is closed and the only open orbit T contains all codimension 1 orbits in its closure. Hence it suffices to check surjectivity at the point $p(F)$. Use the notation of the next to last section, that is: P is a strictly convex polytope with vertices at $m_i, \ i \in I$, and $X = X(P)$. Let F be a codimension 1 face of P, and suppose that m_i is a vertex of the polytope, that is contained in F. The affine open set that corresponds to m_i is $U_i = \operatorname{Spec} k[M_i]$. The codimension 1 face F of the polytope, defines a hyperplane H_F through origo parallel with F. The point $p(F) \subset U_i$ is defined by the map $p : M_i \to k$ given by $p(x) = 0$, if $x \notin H_F$, and $p(x) = 1$, if $x \in H_F$. These points are smooth points in U_{σ_F}, since any toric variety is normal, and these points are generic with respect to the torus action on $O(F)$.

Hence it suffices to compute the Taylor expansions in a system of local coordinates at these points. This gives us a finite number of mappings

$\alpha_{k,F} : V \to \mathcal{O}_{U_i}/m_{p(F)}^{k+1}$. For each face F let $n^1(F)$ be the maximal k such that $\alpha_{k,F}$ is surjective, for some i such that the vertex m_i is contained in F.

Proposition 6.5. $n^1_{X(P)} = Min\{n^1(F)|\ F$ *a codimension 1 face of* $P\}$.

It should also be noted that it is possible to estimate s, using the procedure described in [Jon94] to calculate the generators of the ring of global differential operators(see the example below).

6.4 Hirzebruch surfaces

Consider the finite-dimensional vector space of polynomials $V^r_{kl} := \{x^i y^j,\ 0 \leq i + rj \leq k,\ 0 \leq j \leq l\}$, where r, k, l are non-zero integers and $r \geq 1$. We will restrict ourselves to the truncated case $k - lr \geq 0$. As noted already in [GLHKO93], this vector space is the restriction of the global sections of the line-bundle $\mathcal{O}_{\Sigma_r}(k, l)$ on the Hirzebruch surface Σ_r to the affine $\mathbf{A}^2 \subset \Sigma_r$. The differential operators of order 1 that preserve the vector space are described in [loc.cit], and in [FK98] a graphic method is given to calculate the higher order differential operators that preserve V. This graphic method is just a use of the obvious bigrading, and as such a special case of the much more powerful methods of Jones/Musson. Even in this special case, the methods of the latter authors give fuller information on the whole ring of differential operators.

In particular it is known that $\mathcal{D}^{V^r_{kl}}$ is not generated by differential operators of order less than 1. Let us see what the preceding theory and the literature on toric varieties tells us. Firstly, we may for this compute $n_{inj}(V^r_{kl})$, and $n_{surj}(V^r_{kl})$. The vertices of the polytope are $(k, 0), (0, 0), (0, l), (k - lr, l)$; denote the face between the first two vertices by F_1, between the second and third by F_2, and so on, cyclically. The length of the edges are $k, l, k - lr$, and hence by Proposition 6.4, $n_{surj}(V^r_{kl}) = Min\{l, k - lr\}$.

Similarly $n_{inj}(V^r_{kl}) = k$, under the condition that $k - lr \geq 0$. Hence, by the general theory, locally there are differential operators of order less than k, that preserve V, and have the effect $L_{ab}(x^{m_i}) = \delta_{ai} x^{m_b}$, for $a, b \in P$.

Now consider $n^1_{surj}(V^r_{kl})$. At the edge from $(0, 0)$ to $(k, 0)$ F_1, local coordinates are $x = x^{(1,0)}$ and $y = x^{(0,1)}$, and $p(F_1)$ is defined by $x = 1, y = 0$. It is easy to see that $V^r_{kl} \to k[x, y]/(x-1, y)^{s+1}$ is surjective if and only if $s \leq l$, since in this case y^s must be in the image. In the same way $n^1(F_2) = l$, $n^1(F_3) = l$ and $n^1(F_4) = k - lr$. Hence $n_1 = n_{surj}(V^r_{kl}) = Min\{l, k - lr\}$. Moreover, since this is a toric situation, we know by Musson that $V := V^r_{kl}$ is an irreducible module over the finitely generated and Noetherian algebra of differential operators $\Gamma(X_V, \mathcal{D}_L)$. However we want a more explicit description and estimates of the order of the involved differential operators giving $End_k(V)$. In [Jon94] the restriction to $U_1 = \operatorname{Spec} k[x, y]$ of the global differential operators on the structure sheaf $\mathcal{L} = \mathcal{O}_{\Sigma_r}$ are calculated to be

$$\begin{aligned} R &= \Gamma(\Sigma_r, \mathcal{D}(\mathcal{L})) \\ &= k[\partial_x, x^j\partial_y, x\pi, \partial_x^j y(\nabla_y)\pi(\pi+1)\dots(\pi+a-j-1), x\partial_x, y\partial_y | j = 0, 1\dots, r]. \end{aligned} \tag{6.1}$$

Here $\pi := x\partial_x + ry\partial_y$ and $\nabla_y = y\partial_y$. One may either repeat his calculations for an arbitrary line-bundle — in his and Musson's framework this is an easy, if laborious exercise — or one may use [Jon94, Theorem 4.9], to see that redefining $\pi := x\partial_x + ry\partial_y - k$, and also $\nabla_y := y\partial_y - l$ in the expression $P(j) = \partial_x^j y(\nabla_y)\pi(\pi+1)\dots(\pi+a-j-1)$, will give that the above expression for the ring of differential operators is valid for any line-bundle $\mathcal{L} = \mathcal{L}_{V_{kl}^r}$. This follows since after redefinition the differential operators on the right hand side are easily seen to act on $\mathcal{L}$, and the associated graded rings to the filtration by differential operator order are equal. We also note that the method of [Mus94] makes it easy to describe what differential operators there are in each multi-degree (not only the generators).

We may now estimate the global s, indirectly. Consider the bigrading giving x^iy^j the degree (i, j). The degree-vectors of the elements in the algebra $R^0 = k[\partial_x, x^j\partial_y, x\pi, x\partial_x, y\partial_y | j = 0, 1\dots, r]$ all have non-positive second coordinate, and hence there is a non-trivial filtration $F_tV_{kl}^r = \{\sum k_{ij}x^iy^j \mid j \leq t\}$. This filtration is preserved by R^0. In particular V_{kl}^r is not an irreducible module. Using this filtration, and the extra elements in R, it is easy to give, at least a naive estimate of the order of a differential operator in R_0 that will correspond to any matrix with only one non-zero entry, namely $s \leq 3k + lr$.

Proposition 6.6. *Let $k - lr \geq 0$. Then*

$$X = X_{V_{kl}^r} = \Sigma_r = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(r))$$

and $L_{V_{kl}^r} = \mathcal{O}(l, k)$, see [DR99]. Furthermore $n_{surj}(X) = n^1_{surj}(X) = Min\{l, k - lr\}$ and $n_{inj}(V_{kl}^r) = k$. Also $\mathcal{D}^{V_{kl}^r}$ is given up to $\mathrm{Ann}_{V_{kl}^r}$ by (6.1).

7 Hidden Lie algebras

In this section we will consider the situation when $V \subset A = \mathcal{O}_{X_0}$ is invariant under a reductive Lie algebra of differential operators in $\mathcal{D}^1_{X_0}$. General references for representation theory and homogeneous spaces are [Jan87] and [Hum75].

Consider first representations of a reductive connected and simply-connected (not really necessary) group. Each irreducible representation is constructed as $V = \Gamma(G/B, \mathcal{L})$ for some ample line bundle on the Borel variety G/B, and is associated to an unique integral and dominant character λ of the torus $T \subset B$. We write $\mathcal{L} = \mathcal{L}(\lambda)$. The Borel variety G/B contains an open cell U that is a B-orbit, which is isomorphic to some $\mathbf{A}^n$, and hence there is an inclusion $V \subset \mathcal{O}_U = k[x_1, \dots, x_n] =: R$. Actually it is covered by affine cells gU, all isomorphic to $\mathbf{A}^n$ and there are many possible embeddings of V in $k[x_1, \dots, x_n]$. If $\lambda - \rho$ is integral, dominant, and regular, the line bundle is very ample and the procedure is invertible and $X_V \cong G/B$ and $\mathcal{L}_V \cong \mathcal{L}$. Hence also $\Gamma(X_V, \mathcal{L}_V) = V$. If λ is integral dominant, but not regular, there are simple roots α such that $< \lambda, \check{\alpha}_i >= 0$, $i \in I$, and they define a parabolic group P_I. Furthermore $V = \Gamma(G/B, \pi^*(\mathcal{L})) \xrightarrow{\sim} V = \Gamma(G/P, \mathcal{L})$, for a certain line-bundle $\mathcal{L}$. In this case $X_V = G/P$, even if we start with $V \subset \mathcal{O}_U$, the structure sheaf of a open cell in G/B. Note how the proper variety G/B unites many different possible choices of open sub-varieties and different vector spaces V.

Since $\mathfrak{g}$ will act as derivations on $\mathcal{O}_U$, and V is irreducible as $\mathfrak{g}$-module, the situation of Proposition 3.2 obtains. It is also well-known what the ring of global differential operators of $\mathcal{L}(\lambda)$ is, see [BlB81]. Hence the following is well-known.

Proposition 7.1. *Suppose that $V|_U \subset \mathcal{O}_U$ is the restriction of $V = \Gamma(G/B, \mathcal{L}(\lambda))$. Then $\Gamma(G/B, \mathcal{D}_{\mathcal{L}(\lambda)}) \cong U(\mathfrak{g})/(m_\lambda)$, where $m_\lambda \subset Z$ is a maximal ideal of the center Z of $U(g)$. Furthermore this ring maps surjectively onto $End(V)$, so that any differential operator P on U, that preserves V, may be written $P = P_1 + P_2$, where $P_1 \in U(\mathfrak{g})/(m_\lambda$ and $P_2 \in AnnV$.*

It is also easy to calculate n_{inj} and n_{surj}, depending on the observation that the Wronskian sequence (2.1) is a sequence of G-linearised vector bundles on G/B (since any invertible sheaf is G-linearised on G/B for a simply-connected group.) Hence it suffices to check injectivity and surjectivity at a single point, so take $x = B \in G/B$, and assume that is defined by $x_i = 0$, $i = 1, \dots, n$) in $U \cong U^-$, the unipotent group contained in the opposite Borel group B^-. Then the Wronskian sequence becomes

$$V \to M = k[x_1, \dots, x_n]/(x_i^{n+1} | i = 1, \dots, n) t_\lambda,$$

where t_λ has weight λ, and the map is now T-homogeneous, and also compatible with the action of $U(\mathfrak{u}^-)$. hence for surjectivity, it is enough that $\prod_{i=1}^n x_i^n$ which has weight $\lambda - 2n\rho$, is contained in the image, since this element generates M. There is a unique element w in the Weyl group, such that $w(\lambda - 2n\rho)$ is dominant. By [Hum78, 13.4 Lemma B], the Wronskian above will be surjective if $w(\lambda - 2n\rho) < \lambda$, in the partial ordering induced by positive roots. For example for $\mathbf{P}^1$, we see that $k\rho$ satisfies this criterion for $n = k$, since in this case w will just be multiplication by -1, and $-(k\rho - 2k\rho) = k\rho < k\rho$. As for the injectivity, it is easy to see that it suffices to consider the lowest weight of V, and that the criterion is that this lowest weight $w_0(\lambda) > \lambda - 2n\rho$.

Let us now consider the problem whether the above situation is in some sense the only case. We have the following result, which might be epistemologically interpreted as strengthening our general philosophy that the construction X_V is worthwhile to pursue since it (under some conditions, of course) detects hidden geometry, in this case the underlying homogeneous space.

Proposition 7.2. *Given a finite-dimensional $V \subset A$ with affine $A = \mathcal{O}_{X_0}$, such that $X_0 \subset X_V \subset \mathbf{P}(V^*)$. Assume that $\mathfrak{g} \subset \mathcal{D}^1_{X_0}$ is a reductive Lie algebra. Then the action of $\mathfrak{g}$ on V may be integrated to an action of an algebraic group G, whose associated Lie algebra is $\mathfrak{g}$. This action may be canonically extended to X_V, in such a way that $\mathcal{L}_V$ is an equivariant invertible sheaf. Furthermore, $\mathfrak{g} \subset \Gamma(X_V, \mathcal{D}_{X_V})$, and $V \subset \Gamma(X_V, \mathcal{L}_V)$. In the special case that $\mathfrak{g}$ is locally transitive on X_0 and there is a point $x_0 \in X_0$, such that the kernel of $\mathfrak{g} \to (\mathcal{D}_{X_0})_{x_0}$ is a parabolic sub-algebra, we have $X_V \cong G/P$ for some parabolic subgroup $P \subset G$ and $\mathcal{L}_V = \mathcal{O}(\lambda)$ for some dominant weight.*

Proof. (Sketch)The action on V may be integrated to an action of an simply-connected algebraic group G. The fact that $\mathfrak{g} \subset \mathcal{D}^V$, implies that this action may be extended to a compatible and homogeneous action on A_V. This means that there is an action of G on X_V. This gives the first part of the theorem. By choosing a point $x_0 \subset X_0$, we get a map $G \to X_V$, and the kernel hence has to be a parabolic subgroup P of G. Hence we have a series of closed immersions $\phi : G/P \to X_V \to \mathbf{P}^n$. The local transitivity implies that there is an open subset $U \subset X_0$ that is in the image of ϕ, and hence ϕ is an isomorphism. Since V is a subset of the irreducible G-module $\Gamma(X_V, \mathcal{L}_V)$, it must equal the last module, and we are in the situation described in the beginning of the section. ☐

References

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers. *Algebraic D-modules*, volume 2 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1987.

[BlB81] Alexandre Beĭ linson and Joseph Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties.* Walter de Gruyter & Co., Berlin, 1995.

[Dem96] Jean-Pierre Demailly. L^2 vanishing theorems for positive line bundles and adjunction theory. In *Transcendental methods in* *algebraic geometry (Cetraro, 1994)*, volume 1646 of *Lecture Notes in Math.*, pages 1–97. Springer, Berlin, 1996.

[DR99] Sandra Di Rocco. Generation of k-jets on toric varieties. *Math. Z.*, 231(1):169–188, 1999.

[FK98] Frederico Finkel and Niky Kamran. The Lie algebraic structure of differential operators admitting invariant spaces of polynomials. *Adv. in Appl. Math.*, 20(3):300–322, 1998.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[GD61] Alexander Grothendieck and Jean Dieudonne. Étude cohomolique des faisceaux cohérent. *Publ. IHES*, (4), 1961.

[GLHKO93] Artemio González-López, Jacques C. Hurtubise, Niky Kamran, and Peter J. Olver. Quantification de la cohomologie des algèbres de Lie de champs de vecteurs et fibrés en droites sur des surfaces complexes compactes. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(12):1307–1312, 1993.

[GLKO94] Artemio González-López, Niky Kamran, and Peter J. Olver. Quasi-exact solvability. In *Lie algebras, cohomology, and new applications to quantum mechanics (Springfield, MO, 1992)*, volume 160 of *Contemp. Math.*, pages 113–140. Amer. Math. Soc., Providence, RI, 1994.

[GLKO97] A. González-López, N. Kamran, and P. J. Olver. Quasi-exact solvability in the real domain. In *Field theory, integrable systems and symmetries (Quebec, PQ, 1995)*, pages 58–70. Univ. Montréal, Montreal, QC, 1997.

[Gro61] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[Gro67] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math. No.*, 32:361, 1967.

[Har85] Robin Hartshorne. *Algebraic Geometry.* Springer, 1885.

[Hum75] James E. Humphreys. *Linear algebraic groups.* Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

[Jan87] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 131 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1987.

[Jon94] A. G. Jones. Rings of differential operators on toric varieties. *Proc. Edinburgh Math. Soc. (2)*, 37(1):143–160, 1994.

[KMO00] Niky Kamran, Robert Milson, and Peter J. Olver. Invariant modules and the reduction of nonlinear partial differential equations to dynamical systems. *Adv. Math.*, 156(2):286–319, 2000.

[LT95] Dan Laksov and Anders Thorup. Weierstrass points on schemes. *J. Reine Angew. Math.*, 460:127–164, 1995.

[Mus94] Ian M. Musson. Differential operators on toric varieties. *J. Pure Appl. Algebra*, 95(3):303–315, 1994.

[MVdB98] Ian M. Musson and Michel Van den Bergh. Invariants under tori of rings of differential operators and related topics. *Mem. Amer. Math. Soc.*, 136(650):viii+85, 1998.

[PT95] Gerhard Post and Alexander Turbiner. Classification of linear differential operators with an invariant subspace of monomials. *Russian J. Math. Phys.*, 3(1):113–122, 1995.

[Tur88] A. V. Turbiner. Quasi-exactly-solvable problems and sl(2) algebra. *Comm. Math. Phys.*, 118(3):467–474, 1988.

[Tur92] A. Turbiner. On polynomial solutions of differential equations. *J. Math. Phys.*, 33(12):3989–3993, 1992.

[Tur94] Alexander Turbiner. Lie-algebras and linear operators with invariant subspaces. In *Lie algebras, cohomology, and new applications to quantum mechanics (Springfield, MO, 1992)*, volume 160 of *Contemp. Math.*, pages 263–310. Amer. Math. Soc., Providence, RI, 1994.