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Rikard Bögvad
Thomas Meyer

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Department of Mathematics
Stockholm University

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Postal address:
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden

Electronic addresses:
http://www.math.su.se/
info@math.su.se

On algorithmically checking whether a Hilbert series comes from a complete intersection

by

Rikard Bögvad and Thomas Meyer [1]

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[1]Department of Mathematics, Stockholm University, S-106 91, Sweden, email:rikard@matematik.su.se

Abstract

It is not possible to determine from the Hilbert series whether a graded Noetherian algebra is a complete intersection. Nevertheless, such Hilbert series satisfy very stringent conditions and we define a concept CI-type for formal power series, that embodies some of these necessary properties. This definition works for algebras that are not standard i. e. not generated in degree 1. For the class of formal power series that occur as the Hilbert series of Noetherian Cohen-Macualay algebras, the main result is a criterion for a series to be of CI-type, that is formulated in terms of properties of truncated power series. Hence it can be used as the basis for an algorithm that provides in a finite number of steps either a rational function expression of the formal power series, or the information that the truncated power series is not of CI-type. Also sample computations using this algorithm on some non-standard graded invariant algebras are described.

1 Introduction

The present study is motivated by the problem how to recognize whether a ring is a complete intersection or not, in terms of computable invariants of the ring. There has been several similar studies e. g. Avramov-Herzog [3], characterizing complete intersections in homological terms, as well as work done with classes of special types of rings, e. g. rings of invariants [12], and semi-group rings [7].

We take a different approach; in the spirit of Macaulay and Stanley, we make our starting point the Hilbert series of a graded algebra. If the algebra is Noetherian the Hilbert series may be expressed as a rational function. Stanley showed for example that the condition of a standard algebra (i. e. all generators in degree 1) being Gorenstein, could be determined from the knowledge of its Hilbert series. Macaulay characterized Cohen-Macaulay algebras in a similar way, also in the case of standard algebras. Other authors have pursued this theme, both in order to characterize different properties of a ring in terms of their Hilbert series – e. g. integrality is discussed in [16], as well as the related problem of which formal power series occur as Hilbert series of certain types of rings [8].

As noted already by Stanley [15, Example 3.7], it is not possible to decide whether an algebra is a complete intersection only by considering its Hilbert series. But on the positive side Stanley showed that it is possible, for standard algebras, to describe completely which formal power series occur as Hilbert series of complete intersections [15, Theorems 3.5-6]. The possible Hilbert series will belong to a very restricted class of formal power series. Knowledge whether a Hilbert series belongs to this class, is then a strong necessary condition on an algebra for it to be a complete intersection.

However, many naturally occurring commutative algebras are not generated in degree 1, for example invariant rings or algebras that arise in algebraic geometry as section rings associated to line bundles. Hence it is important to see what can be done in the more general case.

We define a concept CI-type in this more general situation. It is formulated in terms of the existence of a special kind of rational function expression of a formal power series, and it embodies the most easily visible consequences for the Hilbert series of the fact that a graded algebra is a complete intersection. That the Hilbert series is CI-type is thus a necessary condition for a ring to be a CI, but not sufficient.

There is however a practical problem with the definition, in that it assumes the full knowledge of the Hilbert series as a rational function. In many calculations what is given is a partial knowledge of the Hilbert series, as a truncated series, and hence what is needed is a method for using this partial information effectively.

Complete intersections are always Cohen-Macaulay algebras, and this class of algebras has Hilbert series that have a precise description (see below) so we consider only this class of formal power series, called *CM-type.* Our main result is a characterization of CI-type CM-type formal power series in terms of their truncated Hilbert series. This is algorithmic in character and we formulate it below as an algorithm. In practice, its use will be to prove that a particular graded algebra cannot be a complete intersection, and as such we have illustrated its use on rings of invariants.

We will now give a more precise description of the contents of the paper. Suppose we have a truncated Hilbert series

$$H(z) = 1 + a_1 z + a_2 z^2 + \dots, \qquad a_i \in \mathbf{N},$$

of a noetherian graded algebra. It is natural to try to naively use our partial information to construct an expression of the series as a rational function by starting with

$$H_1(z) := (1-z)^{-a_1} = 1 + a_1 z + \binom{-a_1}{2} z^2 + \dots,$$

which equals the given Hilbert series up to degree 1, and then continuing with

$$H_2(z) = (1-z)^{-a_1}(1-z^2)^{\binom{-a_1}{2}-a_2} = 1 + a_1 z + a_2 z^2 + \dots,$$

which is equal to $H(z)$ up to degree 2, and so on. This gives an algorithmic construction of a formal power series

$$H_n(z) := \prod_{j=1}^{n} (1-z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z},$$

which equals $H(z)$ up to degree n. In the limit this uniquely expresses $H(z)$ as an infinite product (well-defined as a formal power series)

$$H(z) = \prod_{j=1}^{\infty} (1-z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z}.$$

For a general series, this product is infinite, but if the Hilbert series stems from a (graded) complete intersection, only a finite product is needed, and the method produces a rational expression in a finite number of steps. There are strict restrictions on the finite products that can occur for a complete intersection R. Namely, in a presentation of R, the number of relations up to an arbitrary degree n must be less than the number of generators up to the same degree n. We give a name to the class of formal power series that satisfy a slightly more sophisticated version of this restriction.

Definition 1. *For an infinite product*

$$H(z) = \prod_{j=1}^{\infty} (1-z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z},$$

define $\Delta_n(H)$ *as the sum*

$$\Delta_n(H) := \sum_{j=1}^{n} \delta_j, \quad n = 1, 2, \dots .$$

Definition 2. *A formal power series* $H(z)$ *with integer coefficients that is a finite product*

$$H(z) = \prod (1 - z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z}$$

and such that each sum $\Delta_n(H)$ *is non-positive, is said to be a power series of* CI-type*.*

We emphasize that these conditions do not characterize Hilbert series of complete intersections, but that they are interesting as necessary criterions.

The characterization of CI-type series, in terms of the formal power series, is as follows. First we give a definition. Formal power series of *CM-type* are precisely those that arise as the Hilbert series of some graded Cohen-Macaulay ring (see Lemma 4). They can be characterized in the following way.

Definition 3. *A formal power series* $H(z)$ *with integer coefficients that is a finite product*

$$H(z) = \frac{p(z)}{\prod (1 - z^j)^{\gamma_j}} = \frac{p(z)}{q(z)}, \quad \gamma_j \in \mathbb{N} ,$$

where $p(z)$ *is a polynomial with non-negative integer coefficients, is said to be a power series of* CM-type*.*

The following theorem is proved using elementary algebraic number theory and some analysis.

Theorem 1. *If* $H(z)$ *is of CM-type, then either* $H(z)$ *is of CI-type or there is some* $\Delta_n(H) > 0$*.*

This theorem justifies the following algorithm that will, in a finite number of steps, either find a rational CI-type expression of the formal power series, or recognize that the series is not of CI-type:

Algorithm 1.
Input: A formal power series $H(z)$ with coefficients in $\mathbb{N}$.
Output: The sequence $H_n(z)$, or the statement that $H(z)$ is not of CI-type.
Compute in each step $H_n(z)$ and Δ_n.
If $\Delta_n > 0$ then state that $H(z)$ is not of CI-type and stop
Else print $H_n(z)$ and let $n \to n + 1$.

The behaviour of the algorithm is then the following: on input a CI-type formal power series it will after a finite number of steps produce a non-changing output consisting of a rational function expression of the series, while on input a non-CI-type formal power series it will in a finite number of steps state that this series is non-CI. It might be considered as a defect of the above algorithm that it will, on input a CI-type formal power series, just continue to give constant output. It is possible to remedy this (assuming that the degree of the denominator and numerator of a rational function expression of the series is bounded), so that the algorithm will in this case also finish after a finite number of steps . We have in Algorithm 3 constructed a modified procedure that will in a finite number of steps, decide whether a formal power series is of CI-type or not. This works irrespective of any hypothesis of CM-type, and the proof is independent of the proof of the validity of Algorithm 1, but uses the bound above in an essential way.

In practice, however, the interesting use of the algorithm seems to be to show that algebras are not CI in a situation where bounds on the degree of the denominator and numerator of a rational function expression of the series are unrealistically large, as for example is the case with invariant rings, so these modifications are less relevant.

It seems possible to show that Algorithm 1 is effective in the following sense. Assume that there is given an a priori bound N of the degrees of $p(z)$ and $q(z)$, where $H(z) = p(z)/q(z)$. Then there is a number $E = E(N)$, such that the above algorithm will have stated that $H(z)$ is not of CI-type, before step E, if this is true, or else stated that $H(z) = H_E(z)$ is of CI-type. The reason why one might expect this is that the asymptotic expression of Δ_i that is obtained below depends on the maximum value $M(p) = max\{|\alpha^{-1}|\}$ of the roots α of $p(z) = 0$. This value is always larger than 1, if the series is of non CI-type (and 1 if the series is of CI-type). If it is close to 1, we expect that the algorithm will have to work for a long time, while if it is large it should produce a negative answer quickly. Considering the set of non-cyclotomic polynomials of degree less than N, with integer coefficients and constant term 1, it is easy to see that there is a constant c_N such that $M(p) \geq c_N > 1$. In particular c_N represents the slowest convergence.

There are however other parts to the asymptotic expressions, in particular the use of the van der Monde determinant will give trouble [9], and anyhow, $E(N)$ will be unusably large; in fact if we take the bound seriously there are other algorithms that are more suitable. For example, in the case where the Hilbert series stems from an Artinian ring, implying that it is a polynomial $p(z)$, we want to check whether a polynomial of bounded degree is cyclotomic, and that we can of course do by evaluating the polynomial on all possible roots of unity. However this will clearly involve much computation, and misses the chance to decide the issue already by using the initial coefficients of the polyomial. For example the polynomial $p(z) = 1 + 3z + 2z^2 + \ldots + z^n$ can never represent the Hilbert series of a complete intersection, since $\Delta_2 = 1 > 0$. Here we have used only the two first coefficients, and do not need to use n-th roots of unity.

Hence we have not pursued the question of global effective bounds, and have instead given some sample calculations on Hilbert series of invariant rings of classical groups. Here the degrees of generators and relations grow very quickly; but the algorithm above detects rings that are not complete intersections quickly, at least for some small representations.

We would like to express thanks to Ralf Fröberg, and Clas Löfwall for useful information on Hilbert series, and Mathematica, respectively.

2 Product decompositions of formal power series

We lack a reference for the following well known and easily proved result:

Proposition 1. *Every formal power series*

$$S(z) = 1 + \sum_{j=1}^{\infty} a_j z^j, \quad a_j \in \mathbb{Z}\,,$$

may be written in an unique way as an infinite rational function

$$S(z) = \prod_{j=0}^{\infty} (1 - z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z}\,.$$

The rational function

$$S_n(z) := \prod_{j=0}^{n} (1 - z^j)^{\delta_j}, \quad \delta_j \in \mathbb{Z}\,,\ n \in \mathbb{N} \tag{1}$$

is uniquely determined by the condition that $S_n(z) = S(z) + O(z^{n+1})$.

Proof. Use induction: Let $S_0(z) = 1$ and assume that

$$S_{i-1}(z) = \prod_{j=1}^{i-1} (1 - z^j)^{\delta_j} = 1 + b_1 z + b_2 z^2 + \cdots + b_{i-1} z^{i-1} + \cdots$$

is a rational function that is identical with $S(z)$ up to the power $i-1$. Let $\delta_i = a_i - b_i$ be the coefficient of z^i in $S(z) - S_{i-1}(z)$, and define $S_i(z) = S_{i-1}(z) \cdot (1 - z^i)^{\delta_i}$. Then $S_i(z)$ equals $S(z)$ up to the power i, because

$$\begin{aligned} S_i(z) &= S_{i-1}(z) \cdot (1 - z^i)^{a_i - b_i} \\ &= \left[1 + b_1 z + \cdots + b_i z^i + \mathrm{O}(z^{i+1})\right]\left[1 + (a_i - b_i) z^i + \mathrm{O}(z^{i+1})\right] \\ &= 1 + b_1 z + \cdots + b_{i-1} z^{i-1} + a_i z^i + \mathrm{O}(z^{i+1})\,. \end{aligned}$$

The limit $S(z) = \lim_{i\to\infty} S_i(z)$ gives a well defined product decomposition of $S(z)$. □

If $r(z)$ is a rational function of the type in the proposition, we will sometimes for clarity, denote by $H_n(r,z)$ the approximation $r(z) = H_n(r,z) + O(z^{n+1})$, given by the proposition. The following lemma is clear.

Lemma 1. *If $HK(z)$ is the product of the two formal power series $H(z)$ and $K(z)$ of the type considered in the preceding proposition, then*

$$H_n(z)K_n(z) = (HK)_n(z).$$

2.1 A criterion on the formal power series for having $H(z) = H_n(z)$

Suppose we start the process of the preceding section with a series expression of a rational function, and that we after a while get $H_n(z) = H_{n+1}(z) = \dots$. When can we conclude that actually $H(z) = H_n(z)$? Clearly it is necessary to first have a known bound of the complexity of the involved power series in some way. In the following result we have assumed that the degree of the denominator and numerator of a rational function expression of the series is bounded. We will use this bound to construct an algorithm that has the complementary property of Algorithm 1, i. e. it will on input a formal power series of CI-type, in a finite number of steps state that this series is of CI-type, or else continue calculating forever. Combining this algorithm with Algorithm 1, we will in the next section solve the CI-type decision problem for formal power series, n. b. for a priori bounded formal power series.

Proposition 2. *Assume that there is given an a priori bound N of the degrees of $p(z)$ and $q(z)$, where $H(z) = p(z)/q(z)$. In each step n of the calculation of $H_n(z) = P_n(z)/Q_n(z)$, define $M(n) := \mathrm{Max}\{\deg P_n, \deg Q_n\}$, where $P_n(z)$ and $Q_n(z)$ are relatively prime. If we, for some n on, get the constant repetition of the same result $H_n(z) = H_{n+1}(z) = \dots$, then it suffices to check that*

$$H_n(z) = \dots = H_{M(n)+N}(z),$$

in order to conclude that $H(z) = H_n(z)$. Conversely, if $H(z) = p(z)/q(z)$ where $\deg p \le N$, $\deg q \le N$, and there is an n such that $H(z) = H_n(z)$. Then $H(z) = H_{\phi^{-1}(N)}(z)$ where $\phi^{-1}(N) = \mathrm{Max}\{M : \phi(M) \le N\}$.

Proof. The first part is just calculating degrees. Since $H(z) - H_n(z) = z^{M(n)+N+1}k(z)$, we have that $pQ_n - P_n q = z^{M(n)+N+1}k(z)qQ_n$, and this is, by reasons of degree, possible only if $pQ_n - P_n q = 0$. The second part follows by noting that, by assumption, each irreducible (over the rationals) factor $p_1(z)$ of $p(z)$ is a cyclotomic polynomial, of degree $N_1 \le N$. Hence it belongs to a primitive M_1:th root of unity, for some M_1, such that (Euler's) $\phi(M_1) = N_1$, and $p_1(z) = H_{M_1} p_1, z)$ (see the explicit formula in [13]). Doing this for all irreducible factors, and multiplying together the result (using the lemma of the preceding section), gives $p(z) = H_{\phi^{-1}(N)}(p,z)$, and since we know that $q^{-1}(z) = H_N(q^{-1},z) = H_{\phi^{-1}(N)}(q^{-1},z)$, we get the result by multiplication. $\square$

In the interval from 1 to the product of the first k prime numbers $N = p_1 \dots p_k$, $\phi^{-1}(N) \leq \prod_{i=1}^{k}(1-p_i^{-1})^{-1}N$, so this function will not grow very fast compared with N (an asymptotic description is given in [10, 18.4]). But this is unfortunately not true of the other term $M(n)$.

For clarity let us formulate the corresponding algorithm. The algorithm will on input a formal power series $H(z)$ of the type that $H(z) = H_n(z)$ for some n, find the least such n and stop after a finite number of step with a proof of this fact. Else it will run forever.

Algorithm 2.
Input: A formal power series $H(z) = p(z)/q(z)$ with coefficients in $\mathbb{N}$, where the degrees of $p(z)$ and $q(z)$ are bounded by N.
Output: The statement that $H(z) = H_n(z)$.
Compute in each step $H_n(z)$ and $M(n)$.
If there is $k < n$ such that $n = M(k) + N$ and $H_n(z) = H_k(z)$ state that $H(z) = H_k(z)$ and stop.
Else let $n \to n+1$.

2.2 A formal power series algorithm that decides whether a formal power series is of CI-type

Using a known bound N on the degrees of $p(z)$ and $q(z)$ for a formal power series $H(z) = p(z)/q(z)$, as in the preceding section enables us to easily construct an algorithm that (in an unspecified finite number of steps) decides whether a formal power series is of CI-type or not. Compared to Algorithm 1, this procedure will have the great advantage that it works without the hypothesis of having a CM-type formal power series. However, it has the drawback that a bound must be known, and in realistic applications– as for example rings of invariants– this means having bounds on the degrees of relations and generators, and these grow very quickly with the dimension of the representation, cf. [14]. Algorithm 1 is thus the part that we consider most useful.

We will now describe the procedure. We start by combining the two algorithms. Once we have passed step $\phi^{-1}(N)$, we have the additional knowledge, by the preceding proposition, that each change $H_n(z) \neq H_{\phi^{-1}(N)}(z)$ implies that $H(z)$ is not CI. So we add a check of this to the procedure, and then we only have to continue calculation until at most step $K := M(\phi^{-1}(N)) + N$. If $H_K(z) = H_{\phi^{-1}(N)}(z)$, we may by the preceding proposition conclude that $H(z) = H_{\phi^{-1}(N)}(z)$, and if also all $\Delta_k \leq 0$, for $k \leq K$, then $H(z)$ is of CI-type. Hence we are done in a finite number of steps. Again we formulate this in pseudo-code.

Algorithm 3.
Input: A formal power series $H(z) = p(z)/q(z)$ with coefficients in $\mathbb{N}$, where the degrees of $p(z)$ and $q(z)$ are bounded by $N > 1$.
Output: Either the statement that $H(z)$ is of CI-type and a rational function expression of $H(z)$, or the statement that it is not of CI-type.
Compute in each step $H_n(z)$, Δ_n and $M(n)$.
If $\Delta_n > 0$ or if $n > \phi^{-1}(N)$ and $H_n(z) \neq H_{n-1}(z)$
then state that $H(z)$ is not of CI-type and stop.
Else
If there is $k < n$ such that $n = M(k) + N$ and $H_n(z) = H_k(z)$
then state that $H(z) = H_k(z)$ is of CI-type and stop.
Else let $n \to n + 1$.

3 Hilbert series of complete intersections

Our main object of interest in this note is the *Hilbert series* of a graded ring $R = \bigoplus_{d=0}^{\infty} R_d$, defined as the generating function

$$H(R, z) = \sum_{d=0}^{\infty} \dim(R_d) \cdot z^d.$$

A standard reference is [15]. We want to give conditions on Hilbert series which are necessarily satisfied when the ring is a complete intersection. If the ring is factored by a non-zero divisor x of degree β, the effect on the Hilbert series is easy to describe (cf.[6],[15]). There is the following exact sequence of graded vectorspaces. (Note that the second map has degree β, while the rest have degree 0.)

$$0 \to R \to R \to R/xR \to 0.$$

By adding dimensions of vectorspaces we get

$$\dim((R/xR)_{d+\beta}) + \dim(R_d) = \dim(R_{d+\beta}).$$

This translates into the following well-known relation between Hilbert series.

Lemma 2. If x is a homogeneous non-zero divisor of degree β in R, then

$$H(R/xR, z) = H(R, z)(1 - z^{\beta}).$$

Let now $R = \mathbb{C}[x_1, \ldots, x_n]$ be a free polynomial ring over the field $\mathbb{C}$, with a minimal set $\{x_1, \ldots, x_n\}$, $\deg(x_i) = \alpha_i$, $1 \leq i \leq n$, of homogeneous generators. The Hilbert series of R is given by

$$H(R, z) = \prod_{i=1}^{n} (1 - z^{\alpha_i})^{-1}$$

Let furthermore $P = \{y_1, \ldots, y_m\}$, $\deg(y_i) = \beta_i$, $1 \le i \le m$, be a set of homogeneous elements in R that generates an ideal $\langle P \rangle$ in R. If P is a regular sequence in R, so that the quotient ring $Q = R/\langle P \rangle$ is a complete intersection (CI), then the Hilbert series $H(Q, z)$ may, by iterated use of Lemma 2, be expressed as a rational function [15].

$$H(Q,z) = \frac{\prod_{i=1}^{m}(1 - z^{\beta_i})}{\prod_{i=1}^{n}(1 - z^{\alpha_i})}. \tag{2}$$

Collecting together factors of the same degree in (2) and letting

$$\delta_j = |\{y_i;\ \beta_i = j\}| - |\{x_i;\ \alpha_i = j\}|$$

and $\omega := \max(\alpha_i, \beta_j)$, $i \le n$, $j \le m$, we have

$$H(Q,z) = \prod_{j=1}^{\omega}(1 - z^j)^{\delta_j}. \tag{3}$$

Important for us is that, when Q is a CI, we must have $\Delta_\omega(H(Q,z)) = \sum_{j=1}^{\omega} \delta_j \le 0$. This just says that the length of the regular sequence $y_1, \ldots, y_m$ can be at most equal to the dimension n of the ring in which it is contained [6, 15]. In fact, it is possible to squeeze out more information on the behaviour of Hilbert series of complete intersections:

Proposition 3. *Let Q be a CI, with Hilbert series $H(Q,z)$ and assume that*

$$H(Q,z) = \prod_{j=1}^{\omega}(1 - z^j)^{\delta_j}.$$

Then each element in the sequence $\Delta_k(H(Q,z)) = \sum_{j=1}^{k} \delta_j$, $\ k = 1, 2, \ldots, \omega$ is non-positive.

For the proof we need the following easy lemma:

Lemma 3. *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a free polynomial ring over the field $\mathbb{C}$, with a minimal set $\{x_1, \ldots, x_n\}$, $deg(x_i) = \alpha_i$, $1 \le i \le n$, of homogeneous generators. Let $P = \{y_1, \ldots, y_m\}$, $deg(y_i) = \beta_i$, $1 \le i \le m$, be a set of homogeneous elements in the subring $R_l = \mathbb{C}[x_i; \deg(x_i) \le l] \subset R$. Then if P is a regular sequence in R, it is also a regular sequence of the subring R_l.*

Proof. This follows from the fact that the inclusion $R_l \subset R$ is faithfully flat(with respect to the grading). □

Proof of the proposition Let $Q_k = R_k/\langle y_i | \deg(y_i) \le k \rangle$, which by the lemma is a CI. Then

$$H(Q_k, z) = \frac{\prod_{\beta_j \le k}(1 - z^{\beta_j})}{\prod_{\alpha_j \le k}(1 - z^{\alpha_j})} = \prod_{j=1}^{k}(1 - z^j)^{\delta_j},$$

and hence $H(Q_k, z)$ equals $H_k(Q, z)$ (defined in Proposition 1). Since Q_k is a CI, we have that $\Delta_k(H(Q, z)) = \Delta_k(H(Q_k, z)) = \sum_{j=1}^{k} \delta_j$ must be non-positive. $\square$

This motivates the definition, given in the introduction, that a formal power series of the form (3) that satisfies the necessary condition of Proposition 3 is said to be a series of *CI-type*.

Note that solely an equality like (3), where $\delta_j \in \mathbb{Z}$, $\omega \in \mathbb{N}$, does not imply that Q is a CI, as is seen in the following example. So it is impossible to use the Hilbert series to decide whether the ring is a CI. (This is well known, e.g. it has been analyzed what rings have the Hilbert series $1/(1 - z)$ [2].) We give an example below, cf. also [15, Example 3.8], which is an example which is a ring of invariants, hence integral.

Example: The polynomial ring $\mathbb{C}[x]$ (generator in degree one) has the Hilbert series $H(\mathbb{C}[x], z) = 1/(1 - z)$. Consider the rings $A = \mathbb{C}[x]/\langle x^2\rangle$ and $B = \mathbb{C}[x^2, x^3] \subset \mathbb{C}[x]$. Then B equals $\mathbb{C}[x]_i$, $i \neq 1$, i.e. it contains all homogeneous elements in $\mathbb{C}[x]$ except those of degree 1, and thus has the Hilbert series $H(B, z) = 1 + z^2 + z^3 + z^4 + \cdots$. The ring A obviously has the Hilbert series $H(A, z) = 1 + z$. The example is the ring $M \subset A \oplus B$ where we define $M_0 := \mathbb{C}(1, 1)$ and $M_i := A_i \oplus B_i$, $i > 0$. Now M has the Hilbert series $H(M, z) = 1 + z + z^2 + \cdots = 1/(1 - z) = H(\mathbb{C}[x], z)$, but it is easy to see that M is not a CI, since it has too many relations. In fact it is $\mathbb{C}[x, y, z]/(x^2, xy, xz, y^3 - z^2)$. Hence it is not possible even to see from the Hilbert series whether a ring has syzygies or not.

4 CM-type formal power series

Finally, we note that any polynomial $p(z) = 1 + a_1z + \ldots + a_nz^n$ will occur as the Hilbert series of an Artinian ring. Take a graded vector space $V = \oplus_{d=1}^{n} V_d$, such that $\dim(V_d) = a_d$, $d \geq 1$, and consider the ring $R = \mathbb{C} \oplus V$, with multiplication given by $VV = 0$. Then $H_R(z) = p(z)$. By taking the free polynomial algebra $S = R[x]$, where $\deg x = \beta$ we get the Hilbert series $p(z)/(1 - z^\beta)$. By adjoining more free variables it is clear that any Hilbert series of CM-type occurs, as the Hilbert series of some graded Cohen-Macaulay ring. Conversely, in a graded CM-ring, there is an ideal I, which is generated by a homogeneous regular sequence, such that S/I is a finite-dimensional vector space. By Lemma 2, this means that the Hilbert series of S/I is of CM-type. We state this as a proposition.

Proposition 4. *The set of CM-type formal power series is precisely the set of Hilbert series of graded Cohen-Macaulay rings.*

It should be noted that already Macaulay characterized Cohen-Macaulay rings, through their Hilbert series, in the much less trivial case of standard algebras[15].

5 The Main Theorem

The following is a more precise version of the theorem given in the introduction. It has as a corollary that if $H(z)$ is a Hilbert series of CM-type, and if $H(z)$ is not of CI-type, then this will be detected by Algorithm 1 described in the introduction in a finite number of steps.

Theorem 2. *Let $H(z) = 1 + \cdots + a_n z^n + \ldots$, $a_i \in \mathbb{N}$. Assume that $H(z)$ is of CM-type. According to proposition 1, $H(z)$ may be written as*

$$H(z) = \prod_{j=1}^{\infty} (1 - z^j)^{\delta_j},$$

where $\delta_j \in \mathbb{Z}$. If the number of the δ_j's such that $\delta_j \neq 0$ is infinite, then

$$\limsup_{m\to\infty} \sum_{j=1}^{m} \delta_j = +\infty\,,$$
$$\liminf_{m\to\infty} \sum_{j=1}^{m} \delta_j = -\infty\,.$$

The proof is contained in a series of lemmas and occupies the rest of this section. Start by noting that we may restrict ourselves to the study of polynomials with integer coefficients. This is seen as follows: Since $H(z)$ is of CM-type, then, by definition 3,

$$H(z) = \prod_{j=1}^{\infty} (1 - z^j)^{\delta_j} = \frac{p(z)}{\prod_{j=1}^{r} (1 - z^j)^{\gamma_j}},$$

where $p(z)$ is a polynomial with non-negative integer coefficients. Hence

$$\begin{aligned} p(z) &= H(z) \cdot \prod_{j=1}^{r} (1 - z^j)^{\gamma_j} \\ &= \prod_{j=1}^{\infty} (1 - z^j)^{\delta_j} \cdot \prod_{j=1}^{r} (1 - z^j)^{\gamma_j}, \end{aligned}$$

which implies that $\Delta_k(p(z))$ and $\Delta_k(H(z))$ differs only by a constant if k is large enough, and thus we have

$$\begin{aligned} \limsup_{k\to\infty} \Delta_k(H(z)) = +\infty \quad &\Leftrightarrow \quad \limsup_{k\to\infty} \Delta_k(p(z)) = +\infty, \\ \liminf_{k\to\infty} \Delta_k(H(z)) = -\infty \quad &\Leftrightarrow \quad \liminf_{k\to\infty} \Delta_k(p(z)) = -\infty. \end{aligned}$$

This means that we may study the polynomial $p(z)$ instead of $H(z)$.

First we make some observations on the polynomial $p(z)$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $p(z) = 0$, let $\hat{\alpha}_i := 1/\alpha_i$ and let $M :=\max|\hat{\alpha}_i|$, $i = 1, 2, \ldots, n$.

Lemma 4. *A polynomial h with roots β_j, $j = 1, 2, \dots, n$, may be factored as*

$$h(z) = \prod_{i=1}^{n} (1 - z^i)^{\delta_i}, \ \delta_i \in \mathbb{Z} \quad \textit{if and only if} \quad |\beta_i| = 1, \ i = 1, \dots, n.$$

Proof. ($\Rightarrow$) is trivial. For the converse, assume that β is one of the roots. The minimal polynomial $q(z)$ (with rational coefficients) of β then has all its roots among the β_i's, and hence all its roots have absolute value 1. The celebrated theorem on units of Dirichlet then implies that β must be a root of $1 - z^k = 0$ for some $k \in \mathbb{N}$ and that hence $q(z)$ divides $1 - z^k$ [4, p.105]. This means that $q(z)$ is a cyclotomic polynomial and it is wellknown [4] that this implies that $q(z)$ may be written as a quotient of products of factors of the type $1 - z^i$. Hence $h(z)$ is a product of the desired form. □

Lemma 5. *If $p(z) = \prod_{i=1}^{n} (1 - \hat{\alpha}_i z)$ does not equal a finite product*

$$\prod (1 - z^j)^{\delta_j}, \ \delta_j \in \mathbb{Z},$$

then the maximum $M = max|\hat{\alpha}_i| > 1$.

Proof. Consider

$$p(z) = \prod_{i=1}^{n} (1 - \hat{\alpha}_i z) = 1 + a_1 z + \cdots + a_n z^n \Rightarrow a_n = (-1)^n \prod_{i=1}^{n} \hat{\alpha}_i.$$

If $|\hat{\alpha}_i| \leq 1$ for all i, then $|a_n| = \prod |\hat{\alpha}_i| \leq 1$. Since a_n is a non-zero integer, we have $\prod |\hat{\alpha}_i| = 1$ and hence all $|\hat{\alpha}_i| = 1$. This is a contradiction by Lemma 4, so we must have $|\hat{\alpha}_i| > 1$ for some i. □

Lemma 6. *If $\alpha_i \in \mathbb{Z}$, then $\alpha_j < 0$.*

Proof. This is clear since the coefficients of

$$p(z) = 1 + a_1 z + \cdots + a_n z^n$$

are non-negative. □

We shall now find an expression of $\Delta_k(p(z))$ in terms of the α_i's. We have

$$p(z) = \prod_{j=1}^{\infty} (1 - z^j)^{\delta_j} = \prod_{i=1}^{n} (1 - \hat{\alpha}_i z).$$

The logarithmic derivatives of these two expressions are

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\log\prod_{j=1}^{\infty}(1-z^j)^{\delta_j}\right] = \sum_{j=1}^{\infty}\frac{-j\delta_j z^{j-1}}{1-z^j} \quad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\log\prod_{i=1}^{n}(1-\hat{\alpha}_i z)\right] = \sum_{i=1}^{n}\frac{-\hat{\alpha}_i}{1-\hat{\alpha}_i z}$$

$$\Rightarrow$$

$$\sum_{j=1}^{\infty}\frac{j\delta_j z^j}{1-z^j} = \sum_{i=1}^{n}\frac{\hat{\alpha}_i z}{1-\hat{\alpha}_i z}.$$

When both sides of the last equality are expanded in power series, we get

$$\sum_{j=1}^{\infty} j\delta_j(z^j + z^{2j} + z^{3j} + \cdots) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}\hat{\alpha}_i^k\right)z^k$$

$$\Leftrightarrow$$

$$\sum_{j|k} j\delta_j = \sum_{i=1}^{n}\hat{\alpha}_i^k.$$

Apply Moebius inversion to get the δ_j's [11, p. 20]: Let $q_k := \sum_{i=1}^{n}\hat{\alpha}_i^k$. Then

$$k\delta_k = \sum_{d|k}\mu(d)q_{k/d}$$

and so

$$\sum_{k=1}^{m}\delta_k = \sum_{k=1}^{m}\left(\sum_{d|k}\frac{\mu(d)q_{k/d}}{k}\right) = \sum_{k=1}^{m}\frac{q_k}{k} + R_m,$$

where

$$R_m = \sum_{k=1}^{m}\left(\sum_{\substack{d|k\\ d\neq 1}}\frac{\mu(d)q_{k/d}}{k}\right).$$

The remainder R_m is easy to estimate. Recall that $M = \max|\hat{\alpha}_i|$, note that $k/d \leq m/2$ for every k/d in R_m, and that the number of divisors d such that $d|k$ is less than or equal to k. Then

$$|q_k| \leq \sum_{i=1}^{n}\left|\hat{\alpha}_i\right|^k \leq nM^k$$

$$\Rightarrow$$

$$|R_m| \leq \sum_{k=1}^{m}\left(\sum_{\substack{d|k\\ d\neq 1}}\frac{|q_{k/d}|}{k}\right) \leq \sum_{k=1}^{m}\frac{knM^{m/2}}{k} \leq nmM^{m/2}.$$

Thus the following result has been proved:

Proposition 5. *Assume that*

$$p(z) = \prod_{k=1}^{\infty}(1-z^k)^{\delta_k} = \prod_{i=1}^{n}\left(1-\hat{\alpha}_i z\right),$$

is a polynomial with non-negative integer coefficients. Then

$$\sum_{k=1}^{m}\delta_k = \sum_{i=1}^{n}\sum_{k=1}^{m}\frac{\hat{\alpha}_i^k}{k} + R_m, \tag{4}$$

where $|R_m| \leq nmM^{m/2}$.

Next we will estimate the magnitude of the dominating term in (4). Accepting a helpful hint by Mathematica, we have

$$\sum_{k=1}^{m}\frac{z^k}{k} = \frac{-z^{m+1}}{m+1}\,{}_2F_1(1,1+m,2+m,z) - \log(1-z), \tag{5}$$

where $z \in \mathbb{C}$ and ${}_2F_1$ is the hypergeometric function that is defined by analytic continuation from the series

$${}_2F_1(1,1+m,2+m,z) = \sum_{n=0}^{\infty}\frac{m+1}{n+m+1}z^n,$$

to the complex plane, cut along the real interval $[1,\infty[$.(More seriously, the identity is clear, by considering power series expansions of the involved functions for $z=0$ and then using analytic continuation.) The log function is cut along the real negative axis, and is bounded for $z=\hat{\alpha}_i$ since the roots of the polynomial are non-positive (Lemma 6). We did not find the precise result we needed, and so have done some elementary calculations with the hypergeometric function.

Lemma 7. *Let* $z \in \mathbb{C}\backslash[1,\infty[$. *Then*

$$\lim_{m\to\infty}\,{}_2F_1(1,1+m,2+m,z) = \frac{1}{1-z}\,.$$

Proof. We have the identities [1, p. 68]

$${}_2F_1(1+m,1,2+m,z) = \frac{1}{1-z}\,{}_2F_1(1,1,2+m,\frac{z}{z-1})$$

and [1, p. 65]

$${}_2F_1(1,1,2+m,z) = (m+1)\int_0^1\frac{(1-t)^m}{1-zt}\mathrm{d}t.$$

Thus we need to prove

$$\lim_{m\to\infty}(m+1)\int_0^1 \frac{(1-t)^m}{1-zt}\mathrm{d}t = 1.$$

We have

$$\begin{aligned}(m+1)&\int_0^1 \frac{(1-t)^m}{1-zt}\mathrm{d}t - 1\\ &= (m+1)\int_0^1 \frac{(1-t)^m}{1-zt} - (1-t)^m\mathrm{d}t\\ &= (m+1)\int_0^1 (1-t)^m \frac{zt}{zt-1}\mathrm{d}t\end{aligned}$$

In the last integral, the function $z/(zt-1)$ will be bounded for a fixed z outside the real interval $[1,\infty[$ and $t\in[0,1]$, so that $|zt/(zt-1)|\le rt$ for some $r\in\mathbb{R}^+$ and $t\in[0,1]$. Thus

$$\left|(m+1)\int_0^1 (1-t)^m\frac{zt}{zt-1}\mathrm{d}t\right| \le (m+1)\int_0^1 rt(1-t)^m\mathrm{d}t = \frac{r}{m+2}.$$

Since $r/(m+2)\to 0$ as $m\to\infty$, the lemma is proved. $\square$

By lemma 7 it is clear that, for any $\epsilon>0$, it is possible to choose some m_0 such that $m>m_0$ implies

$${}_2F_1(1+m,1,2+m,\hat{\alpha}_i) = \frac{1}{1-\hat{\alpha}_i} + \epsilon(\hat{\alpha}_i,m),$$

where $\epsilon(\hat{\alpha}_i,m)$ depends on $\hat{\alpha}_i$ and m, and $|\epsilon(\hat{\alpha}_i,m)|<\epsilon$. Hence lemma 7 together with (4) and (5) implies

Lemma 8. *For any $\epsilon>0$, there exists $m_0\in\mathrm{N}$ such that $m>m_0$ implies*

$$\Delta_m(p) = \sum_{k=1}^m \delta_k = \sum_{i=1}^n\sum_{k=1}^m \frac{\hat{\alpha}_i^k}{k} + R_m =$$
$$\sum_{i=1}^n \frac{-\hat{\alpha}_i^{m+1}}{(m+1)}\left(\frac{1}{1-\hat{\alpha}_i} + \epsilon(\hat{\alpha}_i,m)\right) + R_m + c,$$

where $\sum_{i=1}^n|\epsilon(\hat{\alpha}_i,m)|<\epsilon$, and the constant $c=-\sum_{i=1}^n\log(1-\hat{\alpha}_i)$ only denpends on the roots of the polynomial p.

We can now state a final result on the asymptotic behaviour of $\Delta_k(p)$. Simply multiply both sides of the equation in lemma 8 by $(m+1)/M^{m+1}$ and take the limit, to obtain

Proposition 6. *For any $\epsilon > 0$, there exists $m_0 \in \mathrm{N}$ such that $m > m_0$ implies*

$$\left| \frac{m+1}{M^{m+1}} \Delta_m(p) - \frac{1}{M^{m+1}} \sum_{i=1}^{n} \frac{-\hat{\alpha}_i^{m+1}}{1-\hat{\alpha}_i} \right| < \epsilon$$

Proof. By lemma 8

$$\left| \frac{m+1}{M^{m+1}} \Delta_m(p) - \frac{1}{M^{m+1}} \sum_{i=1}^{n} \frac{-\hat{\alpha}_i^{m+1}}{1-\hat{\alpha}_i} \right| =$$

$$\left| \frac{1}{M^{m+1}} \sum_{i=1}^{n} -\hat{\alpha}_i^{m+1} \cdot \epsilon(\hat{\alpha}_i, m) + \frac{m+1}{M^{m+1}} (R_m + c) \right| \leq$$

$$\left| \frac{1}{M^{m+1}} \sum_{i=1}^{n} \hat{\alpha}_i^{m+1} \cdot \epsilon(\hat{\alpha}_i, m) \right| + \left| \frac{m+1}{M^{m+1}} (R_m + c) \right|$$

Consider the two terms in the last expression. We have

$$\left| \frac{1}{M^{m+1}} \sum_{i=1}^{n} \hat{\alpha}_i^{m+1} \cdot \epsilon(\hat{\alpha}_i, m) \right| \leq \left| \frac{1}{M^{m+1}} \sum_{i=1}^{n} M^{m+1} \cdot \epsilon(\hat{\alpha}_i, m) \right| =$$

$$\left| \sum_{i=1}^{n} \epsilon(\hat{\alpha}_i, m) \right|$$

which, by lemma 8, can be made arbitrarily small. The second term, by proposition 5,

$$\left| \frac{m+1}{M^{m+1}} (R_m + c) \right| \leq \left| \frac{m+1}{M^{m+1}} \cdot nmM^{m/2} \right| + \left| \frac{m+1}{M^{m+1}} \cdot c \right|$$

obviously tends to zero as m becomes large. Thus the proposition is proved. $\square$

Now the following lemma will finish the proof of Theorem 2.

Lemma 9. *There exists a subsequence $\{m_j\}$ in $\mathbb{N}$ such that*

$$\lim_{m_j \to \infty} \left(\frac{1}{M^{m_j}} \sum_{i=1}^{n} \frac{-\hat{\alpha}_i^{m_j}}{(1-\hat{\alpha}_i)} \right) = \begin{cases} x_1 > 0 & \textit{if } j \textit{ is even} \\ x_2 < 0 & \textit{if } j \textit{ is odd} \end{cases}$$

Proof. It is of course just a question of finding two subsequences that converge to strictly positive and negative real numbers respectively. (Note that the left hand expression in the lemma above is a rational number, since it is invariant under each Galois-transformation of the polynomial p.)

The polynomial

$$f(z) := z^n p(1/z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

has $\hat{\alpha}_i$ as roots. Hence

$$\hat{\alpha}_i^n = -(a_1\hat{\alpha}_i^{n-1} + \cdots + a_{n-1}\hat{\alpha}_i + a_n)$$
$$\Leftrightarrow$$
$$\frac{\hat{\alpha}_i^{m+n}}{1-\hat{\alpha}_i} = -\left(a_1\frac{\hat{\alpha}_i^{m+n-1}}{1-\hat{\alpha}_i} + \cdots a_{n-1}\frac{\hat{\alpha}_i^{m+1}}{1-\hat{\alpha}_i} + a_n\frac{\hat{\alpha}_i^{m}}{1-\hat{\alpha}_i}\right)$$

for all $m \in \mathbb{N}$. Let

$$f_m = \sum_{i=1}^{n} \frac{\hat{\alpha}_i^m}{1-\hat{\alpha}_i}.$$

Then we have the nice recursion formula

$$\frac{f_{m+n}}{M^{m+n}} = -\left(\frac{a_1}{M}\frac{f_{m+n-1}}{M^{m+n-1}} + \cdots + \frac{a_{n-1}}{M^{n-1}}\frac{f_{m+1}}{M^{m+1}} + \frac{a_n}{M^n}\frac{f_m}{M^m}\right), \tag{6}$$

By the hypothesis that $p(z)$ is of CM-type we know that all coefficients in the formula are non-negative real numbers. Note that $\frac{f_m}{M^m}$ is uniformly bounded for all m, and that hence the sequence

$$a(m) := \left(\frac{f_{m+n-1}}{M^{m+n-1}}, \frac{f_{m+n-2}}{M^{m+n-2}}, \ldots, \frac{f_m}{M^m}\right),\ m = 1, 2, \ldots$$

lies in a compact subset of $\mathbb{R}^n$, and so must have a convergent subsequence $a(m_k)$ that converges to some point $(x_1, \ldots, x_n)$ as $m \to \infty$. There are three possibilities:

(i) $x_i > 0$ for some i and $x_j < 0$ for some j.
(ii) $x_i \geq 0$ for all i and some $x_i > 0$, or $x_i \leq 0$ for all i and some $x_i < 0$.
(iii) The whole sequence converges to $(0, \ldots, 0)$.

If (i) is the case, then f_{m_k+n-i}/M^{m_k+n-i} and f_{m_k+n-j}/M^{m_k+n-j} fulfill the requirements.

If (ii), consider the equation (6). If all $x_i \geq 0$, with strict inequality for some i, then the recursion formula (6) for f_{m_k+n}/M^{m_k+n} shows that it converges to a strictly negative real number, since the coefficients in the recursion formula are negative. Hence there are subsequences whose real values converge to both strictly positive and negative numbers. The argument in the case when the limit point has coordinates with negative real value is entirely similar.

The final case (iii) will be proven to not occur. First we have to take account of possible multiple roots of f. Therefore, let $\hat{\alpha}_1, \ldots, \hat{\alpha}_r$ be the inverses of a set of distinct representatives of all the roots. Let k_j be the multiplicity of α_j. Then

$$\frac{f_m}{M^m} = \sum_{j=1}^{r} \frac{1}{1-\hat{\alpha}_j}\cdot\frac{k_j\hat{\alpha}_j^m}{M^m} \quad \text{and by assumption} \quad \lim_{m\to\infty}\frac{f_m}{M^m} = 0.$$

Consider the matrix

$$F := \begin{pmatrix} \frac{1}{1-\hat{\alpha}_1} & \frac{1}{1-\hat{\alpha}_2} & \cdots & \frac{1}{1-\hat{\alpha}_r} \\ \frac{\hat{\alpha}_1}{(1-\hat{\alpha}_1)M} & \frac{\hat{\alpha}_2}{(1-\hat{\alpha}_2)M} & \cdots & \frac{\hat{\alpha}_r}{(1-\hat{\alpha}_3)M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\hat{\alpha}_1^{n-1}}{(1-\hat{\alpha}_1)M^{n-1}} & \frac{\hat{\alpha}_2^{n-1}}{(1-\hat{\alpha}_2)M^{n-1}} & \cdots & \frac{\hat{\alpha}_r^{n-1}}{(1-\hat{\alpha}_r)M^{n-1}} \end{pmatrix}$$

This matrix defines a map $F : \mathbb{C}^r \to \mathbb{C}^n$, that is continuous and injective since the first r rows of the matrix form a Vandermonde matrix. Hence there is a right inverse $G : \mathbb{C}^n \to \mathbb{C}^r$ such that $G \circ F = \mathrm{id}$. Then

$$\lim_{m\to\infty} F\left(\frac{k_1\hat{\alpha}_1^m}{M^m}, \frac{k_2\hat{\alpha}_2^m}{M^m}, \dots, \frac{k_r\hat{\alpha}_r^m}{M^m}\right) = \lim_{m\to\infty} (\frac{f_m}{M^m}, \dots, \frac{f_{m+n-1}}{M^{m+n-1}}) = (0, \dots, 0)$$

$$\Rightarrow$$

$$\lim_{m\to\infty} G \circ F\left(\frac{k_1\hat{\alpha}_1^m}{M^m}, \frac{k_2\hat{\alpha}_2^m}{M^m}, \dots, \frac{k_r\hat{\alpha}_r^m}{M^m}\right) =$$

$$\lim_{m\to\infty} \left(\frac{k_1\hat{\alpha}_1^m}{M^m}, \frac{k_2\hat{\alpha}_2^m}{M^m}, \dots, \frac{k_r\hat{\alpha}_r^m}{M^m}\right) = (0, \dots, 0).$$

But this is a contradiction; since $M = \max|\hat{\alpha}_i|$, we must have $|k_i\hat{\alpha}_i^m/M^m| = k_i \neq 0$ for some i and all m. Hence case (iii) cannot occur and the proof of the lemma is finished. □

Proof of Theorem 2: By the preceding lemma 9 and proposition 6, limes superior of $(m+1)M^{-m}\Delta_m(p)$ is strictly positive and limes inferior strictly negative. This clearly implies that limes superior and limes inferior of $\Delta_m(p)$ is $+\infty$ and $-\infty$, respectively, since $M > 1$. This is the conclusion of the theorem. □

6 Some calculations with invariant rings

The algorithm will now be applied to the Hilbert series of certain invariant rings. The truncated Hilbert series have been calculated by the algorithms in [5, section 4.6] and [17].

Let $S_d\mathbb{C}^2$ denote the vector space of homogeneous polynomials of degree d in two variables. Let $G := SL_2(\mathbb{C})$ and let $\mathbb{C}[S_d\mathbb{C}^2]^G$ denote the invariant ring under the action of G.

We get the following by computing the Hilbert series up to a hundred terms (in Mathematica). As an example the Hilbert series of $\mathbb{C}[S_7\mathbb{C}^2]^G$ has

$$\begin{aligned} &H_{30}(\mathbb{C}[S_7\mathbb{C}^2]^G, z) \\ &= \frac{(1-z^{20})(1-z^{24})^{10}(1-z^{26})^{25}(1-z^{28})^{20}}{(1-z^4)(1-z^8)^3(1-z^{12})^6(1-z^{14})^4(1-z^{16})^2(1-z^{18})^9(1-z^{22})}. \end{aligned}$$

Here $\Delta_{26} = 10$, so this is clearly not a CI.

In the table below are stated the results of computations for $d \leq 7$.

$$H_{100}(\mathbb{C}[S_2\mathbb{C}^2]^G, z) = \frac{1}{1-z^2}$$
$$H_{100}(\mathbb{C}[S_3\mathbb{C}^2]^G, z) = \frac{1}{1-z^4}$$
$$H_{100}(\mathbb{C}[S_4\mathbb{C}^2]^G, z) = \frac{1}{(1-z^2)(1-z^3)}$$
$$H_{100}(\mathbb{C}[S_5\mathbb{C}^2]^G, z) = \frac{1-z^{36}}{(1-z^4)(1-z^8)(1-z^{12})(1-z^{18})}$$
$$H_{100}(\mathbb{C}[S_6\mathbb{C}^2]^G, z) = \frac{1-z^{30}}{(1-z^2)(1-z^4)(1-z^6)(1-z^{10})(1-z^{15})}$$
$$H_{100}(\mathbb{C}[S_7\mathbb{C}^2]^G, z) \quad \Delta_{26} = 10$$

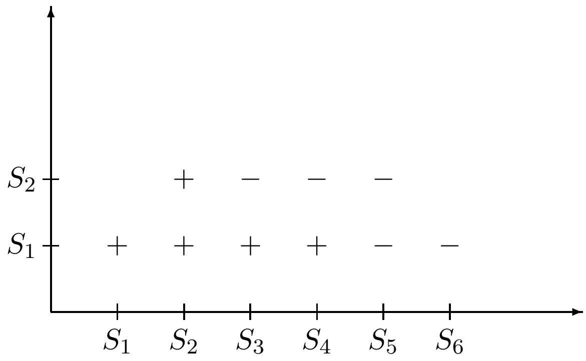
The results on the rings that are not CI are in precise accord with Popov [14], who proves that $\mathbb{C}[S_d\mathbb{C}^2]^G$ is a CI iff $d \leq 6$. The given Hilbert series are those tabled e. g. in [5].

Nakajima [12] states complete results for representations of simple groups. The case of representations of semi-simple groups is still not clarified but it is known (but apparently not published) that only a finite number of representations of this kind have invariant rings that are CI.

We calculate some of the invariant rings of $K := SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$.

$$H_{100}(\mathbb{C}[S_1\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) = \frac{1}{(1-z^2)}$$
$$H_{100}(\mathbb{C}[S_2\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) = \frac{1}{(1-z^4)}$$
$$H_{100}(\mathbb{C}[S_3\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) = \frac{1}{(1-z^2)(1-z^6)}$$
$$H_{100}(\mathbb{C}[S_4\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) = \frac{1-z^{36}}{(1-z^4)^2(1-z^8)(1-z^{12})(1-z^{18})}$$
$$H_{100}(\mathbb{C}[S_5\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) \quad \Delta_{72} = 8$$
$$H_{100}(\mathbb{C}[S_6\mathbb{C}^2 \otimes S_1\mathbb{C}^2]^K, z) \quad \Delta_{24} = 109$$
$$H_{100}(\mathbb{C}[S_2\mathbb{C}^2 \otimes S_2\mathbb{C}^2]^K, z) = \frac{1}{(1-z^2)(1-z^3)(1-z^4)}$$
$$H_{100}(\mathbb{C}[S_3\mathbb{C}^2 \otimes S_2\mathbb{C}^2]^K, z) \quad \Delta_{26} = 1$$
$$H_{100}(\mathbb{C}[S_4\mathbb{C}^2 \otimes S_2\mathbb{C}^2]^K, z) \quad \Delta_{21} = 64$$

These results are also displayed in the diagram below, where a plus sign indicates a probable CI and a minus sign indicates a non-CI.



7 Appendix

The following is an unsophisticated Mathematica program that implements the algorithm. We start by determining a value of $\mathbf{r}$ such that all computations are made mod $z^{\mathbf{r}+1}$. Furthermore, for simplicity, we determine the number of iterations **it** =; we will use in the loop of the program(more clever would have been to build this into the program). This should be less than $\mathbf{r}$.

Input is a (truncated) formal power series $\mathbf{j}$, mod z^{r+1} Output is a rational function $\mathbf{n} = \prod(1-t^j)^{\delta_j}$, such that $\mathbf{n} = \mathbf{j}(mod z^{\mathbf{r}+1})$ and the graph **DDelta** of the function $\Delta_{\mathbf{k}}$. The formal power series $\mathbf{m}$ is the series expansion of $\mathbf{n}$ and is introduced, since Mathematica calculates faster with power series, than with rational functions.

$\mathbf{r} =;$

$\mathbf{it} =;$

$\mathbf{m} = \mathbf{1};$

$\mathbf{n} = \mathbf{1};$

$\mathbf{Delta} = \mathbf{0};$

$\mathbf{DDelta} = \{\};$

$\mathbf{Do}[\mathbf{BB} = (\mathbf{j} - \mathbf{m})[[\mathbf{3}]][[\mathbf{1}]],$

$\mathbf{DD} = \mathbf{Part}[\mathbf{j} - \mathbf{m}, -\mathbf{3}]/\mathbf{Last}[\mathbf{j}],$

$\mathbf{m} = \mathbf{Series}[\mathbf{m} * (\mathbf{1} - \mathbf{t}^{\mathbf{D}}\mathbf{D})\hat{\ }(-\mathbf{BB}), \{\mathbf{t}, \mathbf{0}, \mathbf{r}\}],$

$\mathbf{n} = \mathbf{n} * (1 - \mathbf{t}^{\mathbf{D}}\mathbf{D})\hat{\ }(-\mathbf{BB}),$

$\mathbf{Delta} = \mathbf{Delta} - \mathbf{BB},$

$\mathbf{DDelta} = \mathbf{Append}[\mathbf{DDelta}, \{\mathbf{DD}, \mathbf{Delta}\}], \{\mathbf{it}\}]$

$\mathbf{n}$

$\mathbf{DDelta}$

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