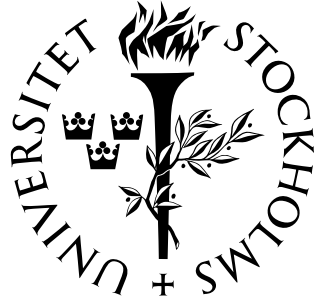


ISSN: 1401-5617



---

# The Poincaré series of the module of derivations of affine monomial curves

Ralf Fröberg  
Vincenzo Micale

---

RESEARCH REPORTS IN MATHEMATICS  
NUMBER 3, 2003

DEPARTMENT OF MATHEMATICS  
STOCKHOLM UNIVERSITY

Electronic versions of this document are available at  
<http://www.math.su.se/reports/2003/3>

Date of publication: April 14, 2003

2000 Mathematics Subject Classification: Primary 13D02, Secondary 13D07.

Keywords: Poincaré series, module of derivations, monomial curve.

Postal address:

Department of Mathematics

Stockholm University

S-106 91 Stockholm

Sweden

Electronic addresses:

<http://www.math.su.se/>

[info@math.su.se](mailto:info@math.su.se)

# The Poincaré series of the module of derivations of affine monomial curves

R. Fröberg  
Matematiska institutionen  
Stockholms Universitet  
10691 Stockholm  
Sweden  
*email* ralff@math.su.se

V. Micale  
Dipartimento di matematica  
Università di Catania  
Viale Andrea Doria 6  
95125 Catania  
Italy  
*email* vmicale@dmi.unict.it

## Abstract

Let  $A$  be a graded  $k$ -algebra and  $M$  be a finitely generated  $A$ -module. The Poincaré series  $P_M^A(z)$  is the formal power series  $\sum_i \dim_k \operatorname{Tor}_i^A(k, M)z^i$ . We study the Poincaré series of  $\operatorname{Der}_k k[S]$ , the module of derivations of a numerical semigroup ring  $k[S]$ , and we relate it to the Poincaré series of  $k$  over  $k[S]$  and to the type of  $S$ . We then use this in order to determine the Poincaré series of  $\operatorname{Der}_k k[S]$  or, at least, its rationality, for some classes of example. We finally give an example of non-rational  $P_{\operatorname{Der}_k k[S]}^{k[S]}(z)$ .

MSC: 13D02; 13D07

## 1 Introduction

If  $A$  is a commutative  $k$ -algebra, with  $k$  a field, the *module of derivations*,  $\operatorname{Der}_k(A) \subseteq \operatorname{hom}_k(A, A)$  is the set  $\{\rho \in \operatorname{hom}_k(A, A) \mid \rho(ab) = a\rho(b) + \rho(a)b \text{ for every } a, b \in A\}$ . This set has a natural  $A$ -module structure by multiplication from left by elements in  $A$ .

Let  $A$  be a graded  $k$ -algebra and  $M$  be a finitely generated  $A$ -module. The *Poincaré series*  $P_M^A(z)$  is the formal power series  $\sum_i \dim_k \operatorname{Tor}_i^A(k, M)z^i$ .

A graded  $k$ -algebra,  $A = \bigoplus_{i \geq 0} A_i$ , is called a *Koszul algebra* if the minimal graded  $A$ -resolution of  $k$  is linear, i.e., if  $(\operatorname{Tor}_i^A(k, k))_j = 0$  if  $i \neq j$ . If  $A$  is a Koszul algebra, then  $P_k^A(z) = 1/H_A(-z)$ , where  $H_A(z)$  is the Hilbert series  $\sum_{i \geq 0} \dim_k A_i z^i$  of  $A$  (cf. [12]).

### 1.1 Description of the content

We now make a closer description of the paper. In Section 2 we give some definitions and fundamentals about numerical semigroups  $S$  and we introduce

$k[S]$ , the numerical semigroup ring. Furthermore we give a minimal set of generators of  $\text{Der}_k^{k[S]}$  as a left  $k[S]$ -module (cf. Proposition 2.1) which represents the starting point of our paper.

In Section 3 we find the minimal free resolution of  $\text{Der}_k^{k[S]}$  for the class of two-generated numerical semigroups  $S$ .

In Section 4 we relate the Poincaré series of  $\text{Der}_k k[S]$  to the Poincaré series of  $k$  over  $k[S]$  and to the type of  $S$  (cf. Theorem 4.4).

In Section 5, we use Theorem 4.4 in order to determine (or just proving the rationality of) the Poincaré series of  $\text{Der}_k k[S]$  for some classes of example. We finally give an example of non-rational  $P_{\text{Der}_k k[S]}^{k[S]}(z)$ .

## 2 Preliminaries

Our object of study in this paper is the Poincaré series of the module of derivations on affine monomial curves, that is on numerical semigroup rings. Therefore we start with some definitions and fundamentals of numerical semigroups. For a general reference to properties of numerical semigroups and semigroup rings, see [4].

Let  $\mathbb{N}$  be the set of natural numbers (including zero). A subset  $S \subseteq \mathbb{N}$  that contains zero and is closed under addition is called a *numerical semigroup*. Every non-zero numerical semigroup is isomorphic to a numerical semigroup with finite complement to  $\mathbb{N}$ . Given such a numerical semigroup we always find a unique minimal finite set of generators  $\{g_1, g_2, \dots, g_m\}$  with  $g_1 < \dots < g_m$ . The numbers  $g_1$  and  $m$  are called *the multiplicity* and *the embedding dimension* of  $S$  respectively. We will use the notation  $S = \langle g_1, g_2, \dots, g_m \rangle = \{n_1 g_1 + \dots + n_m g_m \mid n_1, \dots, n_m \in \mathbb{N}\}$ . Now *the semigroup ring* associated to  $S$  over a field  $k$  is  $k[S] = k[t^{g_1}, \dots, t^{g_m}] \subseteq k[t]$ .

One of the most important invariants of a numerical semigroup  $S$  is the *Frobenius number*  $g(S)$ , that is the  $\max\{n \in \mathbb{Z} \mid n \notin S\}$ .

A numerical semigroup is called *symmetric* if for each  $n \in \mathbb{Z}$ , we have  $n \in S$  or  $g(S) - n \in S$ .

The *type* of  $S$  is  $|T(S)|$  where  $T(S) = \{n \in \mathbb{Z} \setminus S \mid n + s \in S \text{ for every } s \in S \setminus \{0\}\}$  (of course  $g(S) \in T(S)$  for every  $S$ ). The type of  $S$  equals the CM-type of  $k[S]$ . An equivalent condition for a numerical semigroup to be symmetric is that  $T(S) = \{g(S)\}$ , hence a numerical semigroup is symmetric if and only if its type is one. Thus  $k[S]$  is Gorenstein if and only if  $S$  is symmetric.

In general there is no formula for the Frobenius number in terms of the generators. However if the semigroup is 2-generated, say  $S = \langle g_1, g_2 \rangle$ , then the Frobenius number is  $g(S) = g_1 g_2 - g_1 - g_2$  and  $S$  is symmetric.

Throughout the rest of the paper we always assume that  $S \neq \mathbb{N}$  and that the characteristic of the field  $k$  is zero.

The starting point of our paper is the following proposition due to Eriksson in [10] and independently to Eriksen, see [9].

**Proposition 2.1.** *Let  $S$  be a numerical semigroup with  $S \neq \mathbb{N}$  and  $T(S) = \{a_1, a_2, \dots, a_h\}$ . Then the module of derivations of the numerical semigroup ring  $k[S]$ ,  $\text{Der}_k k[S]$ , is the left  $k[S]$ -module minimally generated by*

$$\{t\partial\} \cup \{t^{a_i+1}\partial, i = 1, \dots, h\}$$

where  $\partial = \frac{\partial}{\partial t}$ . In particular the number of minimal generators are  $|T(S)| + 1$ .

### 3 The 2-generated case

The aim of this paper is to calculate the Poincaré series of  $\text{Der}_k k[S]$  for some classes of examples. To this purpose we will use the Theorem 4.4 below. For the class of 2-generated numerical semigroups  $S$  we don't need to use that theorem as we are able to give the minimal free resolution of  $\text{Der}_k k[S]$ .

**Proposition 3.1.** *Let  $S = \langle a, b \rangle$  be a 2-generated numerical semigroup and let  $A = k[S]$ . Then the left  $A$ -module  $M = \text{Der}_k A$  has the following minimal free resolution:*

$$\dots \xrightarrow{\phi_3} A^2 \xrightarrow{\phi_2} A^2 \xrightarrow{\phi_1} A^2 \xrightarrow{\phi_1} A^2 \xrightarrow{\phi_0} M$$

where

$$\phi_0 = \begin{pmatrix} t\partial & t^{ab-a-b+1}\partial \end{pmatrix}, \phi_{2q+1} = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix},$$

$$\phi_{2r} = \begin{pmatrix} t^b & t^{a(b-1)} \\ -t^a & -t^{(a-1)b} \end{pmatrix},$$

with  $q \geq 0$  and  $r \geq 1$ . In particular the Poincaré series  $P_M^A(z) = \frac{2}{1-z}$ .

*Proof.* We know by Proposition 2.1 that  $M$  is a left  $A$ -module minimally generated by  $\{t\partial, t^{ab-a-b}\partial\}$ .

Let  $\phi_0 : A^2 \rightarrow M$  with  $\phi_0(\mathbf{e}_1) = t\partial$  and  $\phi_0(\mathbf{e}_2) = t^{ab-a-b}\partial$  and  $\{\mathbf{e}_1, \mathbf{e}_2\}$  the canonical base for  $A^2$ . In order to determine the kernel of  $\phi_0$ , we look for elements  $f(t)$  and  $g(t)$  in  $A$  such that  $f(t)\mathbf{e}_1 + g(t)\mathbf{e}_2 = 0$ . Since  $A$  is graded,  $\ker \phi_0$  is also graded. Furthermore  $\ker \phi_0$  is 1-dimensional in each degree, hence a generator of  $\ker \phi_0$  must be of the form  $(t^{s+ab-a-b}, -t^s)$  with  $s \in S$ . The two smallest values of  $s$  in order to obtain two independent generators are  $s = a$  and  $s = b$  which give  $((t^{(a-1)b}, -t^a), (t^{a(b-1)}, -t^b)) \subseteq \ker \phi_0$ . Furthermore, these two elements generate the kernel since  $a$  and  $b$  generate  $S$ . We also get

$$\phi_1 = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix}.$$

Now a generator for  $\ker \phi_1$  must be of the form  $(t^{s+b-a}, -t^s)$ . Let us consider  $\max\{l \mid lb \notin a\mathbb{N}\}$ . Since  $\gcd(a, b) = 1$ , then this number is  $a - 1$ . Now the two smallest values of  $s$  in order to obtain two independent generators are  $s = a$  and  $s = (a - 1)b$ . Therefore  $((t^b, -t^a), (t^{a(b-1)}, -t^{(a-1)b})) \subseteq \ker \phi_1$ . These elements

generate  $\ker \phi_1$ . Indeed, if  $(t^{s+b-a}, -t^s) \in \ker \phi_1$ , then  $s = a + s'$  for some  $s' \in S$ , or  $s = lb$  with  $l \geq a - 1$ . We also get  $\phi_2 = \begin{pmatrix} t^b & t^{a(b-1)} \\ -t^a & -t^{(a-1)b} \end{pmatrix}$ .

Let us consider  $\ker \phi_2$ . We look for generators of the form  $(t^{s+ab-a-b}, -t^s)$ . From  $s = a$  and  $s = b$ , we get, as above,  $\ker \phi_2 = ((t^{(a-1)b}, -t^a), (t^{a(b-1)}, -t^b))$  and  $\phi_3 = \begin{pmatrix} t^{(a-1)b} & t^{a(b-1)} \\ -t^a & -t^b \end{pmatrix}$ .

Using induction we have  $\phi_{2r} = \begin{pmatrix} t^b & t^{ab-a} \\ -t^a & -t^{(a-1)b} \end{pmatrix}$  with  $r \geq 1$  and

$\phi_{2q+1} = \begin{pmatrix} t^{(a-1)b} & t^{ab-a} \\ -t^a & -t^b \end{pmatrix}$  for every  $q \geq 0$ .

By construction the built complex is a free resolution of  $M$ . Furthermore, since the entries in the matrixes are in the graded maximal ideal of  $A$ , the resolution is minimal.

## 4 The main theorem

In this section we will prove a theorem which allows us to determine the Poincaré series of  $\text{Der}_k k[S]$  over  $k[S]$ , whenever we know the Poincaré series of  $k$  over  $k[S]$  and the type of  $S$ .

**Lemma 4.1.** *Let  $S = \langle g_1, g_2, \dots, g_m \rangle$  be a numerical semigroup with  $T(S) = \{a_1, a_2, \dots, a_h\}$  and let  $A = k[S]$  and  $\bar{A} = \frac{k[S]}{(t^{g_1})k[S]} = k[\bar{t}^{g_2}, \dots, \bar{t}^{g_m}]$ . Then  $\text{Soc} \bar{A} = \{\bar{t}^{a_1+g_1}, \dots, \bar{t}^{a_h+g_1}\}$ .*

*Proof.* Let  $\bar{t}^s \in \text{Soc} \bar{A}$  with  $\bar{t}^s \neq \bar{0}$  (hence  $s \notin g_1 + S$ ). By the definition of socle,  $\bar{t}^s \bar{t}^{g_i} = \bar{0}$  for every  $i = 2, \dots, m$  and this implies that  $s + g_i \in g_1 + S$  for every  $i = 1, \dots, m$ . Since  $s \in S$ ,  $s - g_1 \notin S$  and  $(s - g_1) + g_i \in S$  for every  $i = 1, \dots, m$ , then  $s - g_1 \in T(S)$ . Hence  $s - g_1 = a_j$  for some  $j = 1, \dots, h$  and  $\text{Soc} \bar{A} \subseteq \{\bar{t}^{a_1+g_1}, \dots, \bar{t}^{a_h+g_1}\}$ .

Let us consider now  $\bar{t}^{a_i+g_1}$  with  $i \in \{1, \dots, h\}$ . Since  $a_i \in T(S)$ , we get  $a_i + g_j \in S$  for every  $j = 1, \dots, m$  and  $a_i + g_j + g_1 \in g_1 + S$ . So  $\bar{t}^{a_i+g_1} \bar{t}^{g_j} = \bar{0}$  for every  $j = 1, \dots, m$ , that is  $\bar{t}^{a_i+g_1} \in \text{Soc} \bar{A}$ .

In [20], Levin introduces the idea of a large homomorphism of graded (or local) rings as a dual notion to small homomorphism of graded rings introduced in [1]. Namely if  $A$  and  $B$  are graded rings and  $f : A \rightarrow B$  is a graded homomorphism which is surjective, then  $f$  is large if  $f_* : \text{Tor}^A(k, k) \rightarrow \text{Tor}^B(k, k)$  is surjective.

If  $S$ ,  $A$  and  $\bar{A}$  are as in Lemma 4.1, then, as a particular case of [20, Theorem 2.1], we get that the homomorphism  $A \rightarrow \bar{A} = \frac{A}{(t^{g_1})A}$  is large.

Hence using [20, Theorem 1.1], we have the following lemma.

**Lemma 4.2.** *Let  $S$ ,  $A$ , and  $\bar{A}$  be as in Lemma 4.1 and let  $M$  be a finitely generated left  $A$ -module such that  $(t^{g_1})M = 0$ . Then  $P_M^A(z) = P_{\bar{M}}^{\bar{A}}(z)P_{\bar{A}}^A(z)$ .*

**Lemma 4.3.** *Let  $S$  be a numerical semigroup with  $T(S) = \{a_1, a_2, \dots, a_h\}$ . Then  $\text{Der}_k k[S]$  is isomorphic as a left  $k[S]$ -module to the ideal  $(t^{g_1}, t^{a_1+g_1}, \dots, t^{a_h+g_1})$  in  $k[S]$ .*

*Proof.* By Proposition 2.1,  $\text{Der}_k k[S]$  is minimally generated as left  $k[S]$ -module by  $(t\partial, t^{a_1+1}\partial, \dots, t^{a_h+1}\partial)$ . Since, in our context,  $\partial$  is just a symbol, we can delete it, and since  $t^i \in S$  if  $i \gg 0$ , we have

$$\begin{aligned} (t, t^{a_1+1}, \dots, t^{a_h+1}) &\simeq t^{L-1}(t, t^{a_1+1}, \dots, t^{a_h+1}) = \\ &= (t^L, t^{a_1+L}, \dots, t^{a_h+L}) = t^{L-g_1}(t^{g_1}, t^{a_1+g_1}, \dots, t^{a_h+g_1}) \simeq \\ &\simeq (t^{g_1}, t^{a_1+g_1}, \dots, t^{a_h+g_1}). \end{aligned}$$

**Theorem 4.4.** *Let  $S = \langle g_1, g_2, \dots, g_m \rangle$  be a numerical semigroup with  $T(S) = \{a_1, a_2, \dots, a_h\}$  and let  $A = k[S]$ . Then  $P_{\text{Der}_k A}^A(z) = 1 + hP_k^A(z)$ . In particular  $P_{\text{Der}_k A}^A(z)$  is rational if and only if  $P_k^A(z)$  is rational.*

*Proof.* By Lemma 4.3, we can replace  $\text{Der}_k A$  with  $I = (t^{g_1}, t^{a_1+g_1}, \dots, t^{a_h+g_1})$  in  $A$  and use the equality  $P_{\text{Der}_k A}^A(z) = P_I^A(z)$ . We finally note that, by Lemma 4.1,  $\bar{I}$ , the image of  $I$  in  $\bar{A} = \frac{k[S]}{(t^{g_1})k[S]}$ , is the Socle of  $\bar{A}$  and that  $P_{A/I}^A(z) = 1 + zP_{\bar{I}}^{\bar{A}}(z)$ .

We note that  $\frac{A}{I}$  and  $\frac{\bar{A}}{\bar{I}}$  are isomorphic as  $\bar{A}$ -modules, hence, using Lemma 4.2 with  $M = A/I$ , we have the following

$$P_I^A(z) = \frac{P_{A/I}^A(z) - 1}{z} = \frac{(1+z)P_{\bar{A}/\bar{I}}^{\bar{A}}(z) - 1}{z} = \frac{(1+z)(1 + zP_{\bar{I}}^{\bar{A}}(z)) - 1}{z}.$$

We note that the kernel of the homomorphism  $\varphi : \bar{A}^h \rightarrow \bar{I} = \text{Soc} \bar{A}$ , with  $\varphi(\mathbf{e}_i) = \bar{t}^{a_i+g_1}$  and  $\{\mathbf{e}_1, \dots, \mathbf{e}_h\}$  the canonical base for  $\bar{A}^h$ , is  $\bar{\mathfrak{m}} \oplus \dots \oplus \bar{\mathfrak{m}}$  (with  $h$  summands) where  $\bar{\mathfrak{m}}$  is the graded maximal ideal of  $\bar{A}$ . Since the minimal resolution of a direct sum is the sum of the minimal resolutions of the summands, we have

$$\begin{aligned} \frac{(1+z)(1 + zP_{\bar{I}}^{\bar{A}}(z)) - 1}{z} &= \frac{(1+z)(1 + z(h + zP_{\bar{\mathfrak{m}} \oplus \dots \oplus \bar{\mathfrak{m}}}^{\bar{A}}(z))) - 1}{z} = \\ &= \frac{(1+z)(1 + z(h + hzP_{\bar{\mathfrak{m}}}^{\bar{A}}(z))) - 1}{z} = \frac{(1+z)(1 + z(h + h(P_k^{\bar{A}}(z) - 1))) - 1}{z}. \end{aligned}$$

Now we can use e.g. Lemma 4.2 to get  $P_k^{\bar{A}}(z) = \frac{P_k^A(z)}{1+z}$ . Thus we get

$$\begin{aligned} \frac{(1+z)(1 + z(h + h(P_k^{\bar{A}}(z) - 1))) - 1}{z} &= \frac{(1+z)(1 + z(h + h(\frac{P_k^A(z)}{1+z} - 1))) - 1}{z} \\ &= \frac{(1+z)(1 + \frac{zhP_k^A(z)}{1+z}) - 1}{z} = 1 + hP_k^A(z). \end{aligned}$$

## 5 Examples

In this section we will use Theorem 4.4 in order to determine the Poincaré series of  $\text{Der}_k k[S]$  for some classes of examples.

### 5.1 The 3-generated case

Let us start with considering the class of rings  $A = k[S]$  where  $S = \langle g_1, g_2, g_3 \rangle$  is a 3-generated numerical semigroup. Then  $\bar{A} = \frac{A}{(t^{g_1})A}$  has embedding dimension 2, and  $\bar{A}$  is either a complete intersection or a Golod ring (c.f. [22, Satz 9]). If  $\bar{A}$  is a complete intersection, we have  $P_k^{\bar{A}}(z) = 1/(1-z)^2$  (cf. [23, Theorem 6]), so  $P_{\text{Der}_k A}^A(z) = \frac{2-z+z^2}{(1-z)^2}$  since the type is one. Otherwise  $\bar{A} = k[x_1, x_2]/(f_1, f_2, f_3)$  (c.f. [17, Theorem 3.7]), which gives  $P_k^{\bar{A}}(z) = \frac{(1+z)^2}{1-3z^2-2z^3}$  (cf. [22]), so  $P_{\text{Der}_k A}^A(z) = \frac{3+6z+3z^2}{1-3z^2-2z^3}$ , since the type equals 2.

### 5.2 The case $S = \langle a, a+1, \dots, a+d \rangle$

Let us now consider the class of ring  $A = k[S]$  with  $S = \langle a, a+1, \dots, a+d \rangle$  with  $2d \geq a-1$ . Let  $N := S \setminus \{0\}$  (the so called *maximal ideal of S*).

**Lemma 5.1.** *Let  $S = \langle a, a+1, \dots, a+d \rangle$  with  $2d \geq a-1$ . Then  $3N = a+2N$ .*

*Proof.* By  $2d \geq a-1$ , we get that  $3a+2d \geq 4a-1$ . This implies that  $\{3a, 3a+1, \dots, 4a-1\} \subseteq a + \{2a, 2a+1, \dots, 2a+d\}$ , which gives the proof.

Let  $\bar{A} = A/t^a A$ . Then  $\bar{A}$  has an induced  $t$ -grading from  $A$ , and  $\bar{A}$  exists (and is 1-dimensional) only in degrees  $S \setminus (a+S)$ . We can regard  $\bar{A}$  as  $k[x_1, \dots, x_d]/I$ , where  $I$  is the kernel of the epimorphism which sends  $x_i$  to  $\bar{t}^{a+i}$ , so  $t \deg(x_i) = a+i$ . We thus have  $x_i x_j \in I$  if  $(a+i) + (a+j) \in a+S$ ,  $x_i x_j - x_k x_l \in I$  if  $i+j = k+l$ , and, by Lemma 5.1,  $(x_1, \dots, x_d)^3 \subseteq I$ . Let  $H = \{d+1, d+2, \dots, a-1\}$ ,  $F = \{x_i x_j \mid i+j \notin H\} \cup \{x_i x_j - x_{n-d} x_d \mid i+j = n \in H\}$ , and  $\bar{B} = k[x_1, \dots, x_d]/(F)$ .

**Lemma 5.2.** *Let  $\bar{A} = k[x_1, \dots, x_d]/I$ . Then  $I = (F) + (x_1, \dots, x_d)^3$ . Thus  $\bar{A}$  is a standard graded algebra ( $\deg x_i = 1$  for each  $i$ ) with Hilbert series  $1 + dz + (a - (d-1))z^2$ .*

*Proof.* This follows since  $S = \{0, a, a+1, \dots, a+d, 2a, \dots\}$  and since, using Lemma 5.1,  $S \setminus (a+S) = \{0, a+1, \dots, a+d, 2a+d+1, \dots, 3a-1\}$ .

We will now show that the set  $F$  is a Gröbner basis of  $(F)$  in Degrevlex. We will prove it considering two different cases. We recall that  $a-1 \leq 2d$ . We start to consider the case  $a-1 < 2d$ .

**Lemma 5.3.** *Let  $a-1 < 2d$ . Then  $I = (F)$  and  $F$  is a Gröbner Bases of  $(F)$  (hence of  $I$ ) in Degrevlex.*



Proof. Let us denote the elements of  $F$  by  $f_1, \dots, f_r$  (in no special order), the Gröbner basis of  $(F)$  by  $G(F)$ , the initial term of  $f_i$  by  $\text{in}(f_i)$  and  $\frac{k[x_1, \dots, x_d]}{(\text{in}(f_1), \dots, \text{in}(f_r))}$  by  $\bar{C}$ .

Since  $H_{\bar{A}}(z) \leq H_{\bar{B}}(z) = H_{\frac{k[x_1, \dots, x_d]}{G(F)}}(z) \leq H_{\bar{C}}(z)$  (coefficientwise), we only need is to show that  $H_{\bar{A}}(z) = H_{\bar{C}}(z)$ .

By Lemma 5.2, we know that  $H_{\bar{A}}(z) = 1 + dz + (a - (d + 1))z^2$ . Since  $\text{in}(x_i x_j - x_{n-d} x_d) = x_i x_j$  (as we use Degrevlex), we get that the only monomials in  $\bar{C}$  different from zero in degree less than or equal two are  $1, \bar{x}_1, \dots, \bar{x}_d, \bar{x}_2 \bar{x}_d, \dots, \bar{x}_{(3a-1)-(2a+d+1)+1} \bar{x}_d = \bar{x}_{a-(d+1)} \bar{x}_d$ .

The only possible nonzero monomials of degree three are  $\bar{x}_i \cdot \bar{x}_j \bar{x}_d$  with  $1 \leq i \leq d$  and  $2 \leq j \leq a - (d - 1)$ . Since  $a - 1 < 2d$  we have  $j < d$ . If  $i = d$  we get  $\bar{x}_i \bar{x}_d = 0$ , if  $i < d$  we get  $\bar{x}_i \bar{x}_j = 0$ , hence all monomials of degree three are 0. Hence  $H_{\bar{C}}(z) = 1 + dz + (a - (d + 1))z^2 = H_{\bar{A}}(z)$ .

**Corollary 5.4.** *Let  $a - 1 < 2d$ . Then  $P_k^{\bar{A}}(z) = \frac{1}{1 - dz + (a - (d + 1))z^2}$ .*

Proof. By Lemma 5.3 and [7, Theorem 2.2],  $\bar{A}$  is a Koszul algebra.

**Theorem 5.5.** *Let  $S = \langle a, a + 1, \dots, a + d \rangle$  with  $2d > a - 1$  and  $A = k[S]$ . Then  $P_{\text{Der}_k A}^A(z) = \frac{a - d + (a - 2d - 1)z + (a + 1 - d)z^2}{1 - dz + (a - d - 1)z^2}$  if  $d < a - 1$  and  $P_{\text{Der}_k A}^A(z) = \frac{a + (1 - a)z}{1 - (a - 1)z}$  if  $d = a - 1$ .*

Proof. By [15, Corollary 3.4.2 (ii)],  $P_k^A(z) = (1 + z)P_k^{\bar{A}}(z) = \frac{1 + z}{1 - dz + (a - (d + 1))z^2}$ .

We note that for this kind of semigroups  $S$ , we have  $T(S) = \{a + d + 1, \dots, 2a - 1\}$  (hence  $h = |T(S)| = a - (d + 1)$ ) if  $d < a - 1$  and  $T(S) = \{1, \dots, a - 1\}$  (hence  $h = |T(S)| = a - 1$ ) if  $d = a - 1$ . Using Theorem 4.4, we get

$$P_{\text{Der}_k A}^A(z) = 1 + (a - (d + 1))P_k^A(z) = \frac{a + (a - 1)z + (a + 1 - d)z^2}{1 - dz + (a - d - 1)z^2}$$

if  $d < a - 1$  and

$$P_{\text{Der}_k A}^A(z) = 1 + (a - 1)P_k^A(z) = \frac{a}{1 - (a - 1)z}$$

if  $d = a - 1$ .

Let us consider now the remaining case  $a - 1 = 2d$ . In this case we cannot use the proof of Lemma 5.3 to show that the set  $F$  is a Gröbner basis as  $\bar{A}$  and  $\bar{B}$  have not the same Hilbert series (and, in particular,  $I \neq (F)$ ). This is the case, for example, for  $S = \langle 13, 14, 15, 16, 17, 18, 19 \rangle$ , where  $H_{\bar{A}}(z) = 1 + 6z + 6z^2$  (cf. Lemma 5.2) and  $H_{\bar{B}}(z) = 1 + 6z + 6z^2 + 6z^3 + \dots = \frac{1 + 5z}{1 - z}$ .

**Lemma 5.6.** *Let  $a - 1 = 2d$ . Then  $F$  is a Gröbner basis of in Degrevlex. In particular  $\bar{B} = k[x_1, \dots, x_d]/(F)$  is a Koszul algebra.*

Proof. We note that in this case  $H = \{d + 1, \dots, 2d\}$ . As in Lemma 5.3, let us denote the elements of  $F$  by  $f_1, \dots, f_r$  and the initial term of a polynomial

$f$  by  $\text{inf}$ . We need to show that the  $S$ -polynomials  $S_{i,j}$  of each pair  $(f_i, f_j)$  of the elements from  $F$  is zero modulo  $F$ .

Since this happens whenever  $f_i, f_j$  are both monomials or  $\text{gcd}(\text{inf}_i, \text{inf}_j) = 1$ , without loss of generality, we can restrict to consider only two cases, that is  $f_i = x_i x_l, f_j = x_i x_j - x_{n-d} x_d$  (with  $i+l \notin H$  and  $i+j = n \in H$ ) and  $f_i = x_i x_j - x_{n_1-d} x_d, f_j = x_i x_j - x_{n_2-d} x_d$  (with  $i+l = n_1 \in H, i+j = n_2 \in H$ ).

In the first case,  $S_{i,j} = x_j(x_i x_l) - x_l(x_i x_j - x_{n-d} x_d) = x_l x_{n-d} x_d$ . Since  $i+l \notin H$  and  $i, l \leq d$ , then  $i+l < d+1$ . By  $j \leq d$ , we get  $i+l+j < 2d+1$ , that is  $l+(n-d) < d+1$ . This implies  $l+(n-d) \notin H$  and  $x_l x_{n-d} \in F$ .

Let us now consider the second case. Here we have

$$S_{i,j} = x_l(x_i x_j - x_{n_1-d} x_d) - x_j(x_i x_l - x_{n_2-d} x_d) = -x_l x_{n_1-d} x_d + x_j x_{n_2-d} x_d.$$

By  $i+j = n_1$  and  $i+l = n_2$ , we get  $l+n_1 = n_2 - i + i + j = n_2 + j$ . Hence  $\bar{x}_l \bar{x}_{n_1-d} \bar{x}_d = \bar{x}_j \bar{x}_{n_2-d} \bar{x}_d$  modulo  $F$  as  $\bar{B}$  is graded and one-dimensional in each degree.

The second part of the lemma follows by [7, Theorem 2.2].

**Lemma 5.7.** *Let  $a-1 = 2d$  and  $\bar{B}$  be as above. Then the Hilbert series  $H_{\bar{B}}(z) = \frac{1+(d-1)z}{1-z}$ .*

*Proof.* We first note that  $|H| = |\{d+1, \dots, 2d\}| = d$ .

Since, by Lemma 5.6,  $F$  is a Gröbner basis in Degrevlex, then  $H_{\bar{B}}(z) = H_{\bar{C}}(z)$ . Moreover  $\text{in}(x_i x_j - x_{n-d} x_d) = x_i x_j$ , hence the only elements in  $\bar{C}$  of degree two different from zero are of the kind  $\bar{x}_{n-d} \bar{x}_d$  with  $n \in H$ .

Using induction it is easy to see that the only elements in  $\bar{C}$  of degree  $i$  different from zero are of the kind  $\bar{x}_{n-d} \bar{x}_d^{i-1}$  with  $n \in H$ .

This gives  $H_{\bar{B}}(z) = H_{\bar{C}}(z) = 1 + dz + dz^2 + dz^3 + \dots = \frac{1+(d-1)z}{1-z}$ .

**Theorem 5.8.** *Let  $S = \langle a, a+1, \dots, a+d \rangle$  with  $2d = a-1$  and  $A = k[S]$ . Then  $P_{\text{Der}_k A}^A(z) = \frac{1+d}{1-dz}$ .*

*Proof.* Since  $\bar{B}$  is a Koszul algebra (cf. Lemma 5.6) and  $\bar{A} = \frac{k[x_1, \dots, x_d]}{(F) + \mathfrak{m}^3}$ , where  $\mathfrak{m} = (x_1, \dots, x_d)$ , we get, as a particular case of [20, Theorem 1.5],

$$P_k^{\bar{A}}(-z) = \frac{z}{(z-1)H_{\bar{B}}(z) + H_{\bar{A}}(z)} = \frac{z}{(z-1)\frac{1+(d-1)z}{1-z} + 1 + dz + dz^2} = \frac{1}{1+dz}.$$

Finally, by [15, Corollary 3.4.2 (ii)],  $h = |T(S)| = a - (d+1) = d$  and Theorem 4.4, we have

$$P_{\text{Der}_k A}^A(z) = 1 + dP_k^A(z) = 1 + d\frac{1+z}{1-dz} = \frac{1+d}{1-dz}.$$

*Remark 5.9.* Let  $S_1 = \langle a, a+1, \dots, a+d \rangle$  and  $S_l = \langle a, a+l, \dots, a+ld \rangle$  with  $2d \geq a-1, l \geq 1$  and  $\text{gcd}(a, l) = 1$ .

Since  $\frac{k[S_1]}{(t^a)k[S_1]} = \bar{A} = \frac{k[S_l]}{(t^a)k[S_l]}$ , then Theorems 5.5 and 5.8 hold for  $S = S_l$ .

### 5.3 The case of multiplicity less than or equal 7

In [18] there is a classification of all possible algebras  $\bar{A}$ , for  $A = k[S]$  of multiplicity at most 7. We will show that  $P_{\text{Der}_k A}^A(z)$  is rational in all these cases.

Since  $P_k^R(z)$  is rational for all rings  $R$  of embedding dimension at most 3 (c.f. [24, Corollary 4.4]), we get that  $P_k^A(z)$ , and thus  $P_{\text{Der}_k A}^A(z)$ , is rational for all  $A = k[S]$ , where  $S$  has at most 4 generators. This takes care of all  $\bar{A}$  for semigroups of multiplicity at most 5, except one for which  $\bar{A} = k[x_1, \dots, x_5]/(x_i x_j, 1 \leq i, j \leq 5)$ , which is a Koszul algebra (cf. [11]).

If  $A$  has multiplicity 6, there are 5 different  $\bar{A}$  of embedding dimension 4 or 5. All these have relations of degree 2 which constitute a Gröbner basis, so they are Koszul algebras (c.f. [7, Theorem 2.2]).

Finally, if  $A$  has multiplicity 7, there are 25 (out of 55)  $\bar{A}$  of embedding dimension larger than 3. They are all of the form  $k[x_1, \dots, x_n]/I$ , with  $I = I_2 + J$ , where  $I_2$  (the part of the ideal in degree 2) is a Gröbner basis in all cases and  $J = 0$  (so the ring is Koszul), or  $J = (x_1, \dots, x_n)^3$ , or (in one case)  $J = (x_1, \dots, x_n)^4$ . In the first case  $\bar{A}$  is a Koszul algebra, so  $P_k^A(z)$ , and thus  $P_{\text{Der}_k A}^A(z)$ , is rational. For the last two cases we can use [20, Theorem 1.5] to conclude that  $P_k^{\bar{A}}(z)$ , and thus  $P_{\text{Der}_k A}^A(z)$ , is rational.

### 5.4 The case of maximal embedding dimension, maximal length, or the almost maximal length

A one-dimensional ring is of maximal embedding dimension if its embedding dimension equals its multiplicity (the same definition holds for the numerical semigroups and  $S$  is of maximal embedding dimension if and only if  $k[S]$  is of maximal embedding dimension). Let  $S = \langle g_1, \dots, g_m \rangle$ . If  $k[S]$  is of maximal embedding dimension, then  $\bar{A} = k[x_1, \dots, x_{m-1}]/(x_1, \dots, x_{m-1})^2$ , which is a Koszul algebra, so  $P_k^{\bar{A}}(z) = \frac{1}{1-(m-1)z}$ , and thus  $P_{\text{Der}_k A}^A(z) = \frac{m}{1-(m-1)z}$  since the type is  $m - 1$ .

For a one-dimensional ring  $R$  we have the inequality  $l(\bar{R}/R) \leq l(R/C)t(R)$ , where  $\bar{R}$  is the integral closure of  $R$  in its field of fraction,  $C$  is the conductor, and  $t(R)$  the CM-type (c.f. [13, Theorem 34]). The ring  $R$  is called of maximal length if there is equality, and it is called of almost maximal length if  $l(\bar{R}/R) = l(R/C)t(R) - 1$ .

If  $k[S]$  is of maximal length it is either Gorenstein or of maximal embedding dimension (c.f. [5] or [6]) thus, rings of maximal embedding dimension and with type at least two have rational  $P_{\text{Der}_k A}^A(z)$ .

If  $k[S]$  is of almost maximal length, then  $S = \langle 4, 5, 11 \rangle$ ,  $S = \langle 4, 7, 13 \rangle$ ,  $S = \langle 3, 3d+2, 3d+4 \rangle$  for some  $d \geq 1$ , or  $S = \langle p, dp+1, dp+2, \dots, dp+p-1 \rangle$  for some  $p \geq 3$  and  $d \geq 1$  (c.f. [5, Theorems 3 and 4]). For the first three examples  $\bar{A}$  is a Golod ring, so  $P_{\text{Der}_k A}^A(z) = \frac{3+6z+3z^2}{1-3z^2-2z^3}$  (see Subsection 5.1). The last class is of maximal embedding dimension.

## 5.5 The case of monomial semigroups

Let  $R$  be a one-dimensional Noetherian domain with  $k \subset R \subseteq k[[t]]$  and  $v$  be the natural valuation for nonzero elements of  $k((t))$ . Then  $v(R)$  is a numerical semigroup. If  $S = \langle g_1, \dots, g_m \rangle$ , then by  $k[[t^S]]$  we mean  $k[[t^{g_1}, \dots, t^{g_m}]]$ . An equivalent definition of semigroup ring for this kind of rings  $R$  is that  $R = k[[x^S]]$  for some  $x \in (t) \setminus (t^2)$ . In general if  $S$  is fixed and we consider all rings  $R$  as above with  $v(R) = S$ , it is not true that all these rings are semigroup rings.

A numerical semigroup  $S$  is called monomial if each ring  $R$  with  $v(R) = S$  is a semigroup ring.

If  $S$  is a monomial semigroup then  $S$  is one from the following list: (i)  $S$  is such that the only elements smaller than the Frobenius number are multiples of  $g_1$ , (ii)  $l \notin S$  only for one  $l > g_1$ , (iii)  $g_1 \geq 3$  and the only elements greater than  $g_1$  that are not in  $S$  are  $g_1 + 1$  and  $2g_1 + 1$  (cf [21, Theorem 3.12]).

The first class is of maximal embedding dimension.

Let us consider the second class. In this case  $S = \{0, g_1, \dots, g_1 + \alpha - 1, g_1 + \alpha + 1, \dots\}$ . If  $\alpha = 1$ , then  $S$  is of maximal embedding dimension. If  $\alpha = 2$ , then  $\bar{A} = \frac{k[x_1, \dots, x_{m-1}]}{I} = \frac{k[x_1, \dots, x_{m-1}]}{I_2 + \bar{m}^3}$ , where  $I_2$  is the part of  $I$  in degree two and  $\frac{k[x_1, \dots, x_{m-1}]}{I_2}$  is a Koszul algebra. Using the same argument as in the proof of Theorem 5.5, we get  $P_{\text{Der}A}^A(z)$ . Finally for  $\alpha > 2$ ,  $I = I_2$  and even in this case, using the same argument as in the proof of Theorem 5.8, we get  $P_{\text{Der}A}^A(z)$ .

Finally, let  $S$  be in the third class. If  $g_1 = 3$ , then  $S$  is two-generated. If  $g_1 > 3$ , then  $I = I_2$  and  $\bar{A}$  is Koszul.

## 5.6 Further rational cases

There are some more classes of semigroups for which we can say that  $P_{\text{Der}_k A}^A(z)$  is rational. If  $k[S]$  is a complete intersection, then  $P_k^A(z)$ , and thus  $P_{\text{Der}_k A}^A(z)$ , is rational (cf. [23]). Semigroups defining complete intersections are classified in [8].

Gorenstein rings  $R$  of codimension at most 4 have rational Poincaré series  $P_k^R(z)$  (c.f. [16]). They treat the case  $\text{char}(k) \neq 2$ , the general case is in [2]. Thus symmetric semigroups with at most 5 generators have rational series  $P_{\text{Der}_k A}^A(z)$ .

In [2] also other classes of rings are shown to have rational series, e.g. almost complete intersections of codimension 4. Thus, if  $S$  has 5 generators and  $k[S]$  (or  $\bar{A}$ ) 5 relations, then  $P_{\text{Der}_k A}^A(z)$  is rational. Also if  $\bar{A}$  has monomial relations, the series are rational, c.f. [3].

## 5.7 An example of non-rational $P_{\text{Der}_k k[S]}^{k[S]}(z)$

If  $S = \langle 18, 24, 25, 26, 28, 30, 33 \rangle$ , it is shown in [14] that  $P_k^{k[S]}(z)$  is not rational. Thus  $P_{\text{Der}_k k[S]}^{k[S]}(z)$  is not rational.

## References

- [1] L. L. Avramov, *Small Homomorphism of Local Rings*, J. Algebra, **50** (1978), 400–453.
- [2] L. L. Avramov, A. Kustin, M. Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), 162–204.
- [3] J. Backelin, *Les anneaux locaux à relations monomiales ont des séries de Poincaré-Betti rationnelles*, C.R. Acad. Sci. Paris Sér. **295**, 607–610.
- [4] V. Barucci-D. E. Dobbs-M. Fontana, *Properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. vol 125, **598** (1997).
- [5] W. C. Brown-F. Curtis, *Numerical Semigroups of Maximal and Almost Maximal Length*, Semigroup Forum, **42** (1991), 219–235.
- [6] W. C. Brown, J. Herzog, *One dimensional local rings of maximal and almost maximal length*, J. Algebra **51** (1992), 332–347.
- [7] W. Bruns, J. Herzog, U. Vetter, *Syzygies and walks*, in *Commutative algebra*, Trieste 1992, eds. Simis, Trung, Valla, World Scientific 1994.
- [8] C. Delorme, *Sous-monoïdes d'intersection complète de  $N$* , Ann. Sci. École Norm. Sup. **9** (1976), 145–154.
- [9] E. Eriksen, *Differential operators on monomial curves*, to appear in J. Algebra (preprint at [www.maths.warwick.ac.uk/~eriksen/index-papers.html](http://www.maths.warwick.ac.uk/~eriksen/index-papers.html)).
- [10] A. Eriksson, *Differential Operators on Numerical Semigroup Rings*, p. 63–92, in *Differential Operators on Some Classes of Rings*, dissertation, Stockholm University (2000).
- [11] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), 29–39.
- [12] R. Fröberg, *Koszul algebras*, Advances in commutative ring theory (Fez, 1997), 337–350, Lecture Notes in Pure and Appl. Math. **205**, Dekker, New York, 1999.
- [13] R. Fröberg-C. Gottlieb-R. Häggkvist, *Semigroups, Semigroup Rings and Analytically Irreducible Rings*, report No. 1, University of Stockholm, Sweden, 1986.
- [14] R. Fröberg, J. E. Roos, *An affine monomial curve with irrational Poincaré-Betti series*, J. Pure Appl. Algebra **152** (2000), 89–92.

- [15] T. H. Gulliksen and G. Levin, *Homology of Local Rings*, Queen's Papers in Pure and Applied Mathematics, Vol 20. (1969).
- [16] C. Jacobsson, A. Kustin, M. Miller, *The Poincaré series of a codimension four Gorenstein ring is rational*, J. Pure Appl. Algebra **38** (1985), 255–275.
- [17] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193.
- [18] E. Kunz, *Über die Klassifikation numerischer Halbgruppen*, Regensburger Mathematische Schriften **11**, Universität Regensburg, Fachbereich Mathematik, Regensburg (1987).
- [19] G. Levin, *Large Homomorphisms of Local Rings*, Math. Scand. **46** (1980), 209–215.
- [20] C. Löfwall, *On the subalgebra generated by one-dimensional elements in the Yoneda Ext-algebra*, in Algebra, Algebraic Topology and their Interactions, J.-E. Roos ed., Lect. Notes Math., **1183**, 291–338, Springer-Verlag, Berlin/New York, 1986.
- [21] V. Micale, *On Monomial Semigroups*, Comm. Algebra, **30**, no. 10 (2002), 4687–4698.
- [22] G. Scheja, *Über die Bettizahlen lokaler Ringe*, Math. Annalen **155** (1964), 155–172.
- [23] J. Tate, *Homology of Noetherian rings and local rings*, Illinois J. Math. **1** (1957), 14–27.
- [24] J. Weyman, *On the structure of free resolutions of length 3*, J. Algebra **126** (1980), 1–33.