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ON GENERALIZED LAGUERRE POLYNOMIALS WITH REAL AND COMPLEX PARAMETER.

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Abstract

In this paper we consider families of polynomial eigenfunctions of the hypergeometric type operator $T_Q = (\alpha z + \beta)\frac{d}{dz} + (\gamma z + \delta)\frac{d^2}{dz^2}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. The object is to study the location of the zeros of these polynomials and to say something about the asymptotic zero distribution. The classical Laguerre polynomials will appear as a special case, and some well-known results about these will therefore be recovered and generalized.

1 Introduction

Let $Q_0, \ldots, Q_k$ be polynomials in one complex variable satisfying $\deg Q_j \leq j$ $\forall j$. Consider the differential operator

$$T_Q(f) = \sum_{j=0}^{k} Q_j f^{(j)}$$

where $f^{(j)}$ denotes the jth derivative of f. We will be interested in the eigenvalue problem $T_Q(p_n) = \lambda_n p_n$ and we call p_n the nth degree polynomial eigenfunction of the operator T_Q. The case with $\deg Q_k < k$, which will be referred to as the *degenerate case*, turns out to be much more complicated than the generic case with $\deg Q_k = k$, which has previously been studied in [3], where it was proved that for sufficiently large integers n there is a unique constant λ_n and a unique monic polynomial p_n of degree n satisfying $T_Q(p_n) = \lambda_n p_n$. Also it was proved that asymptotically as $n \to \infty$, the zeros of p_n are distributed according to a certain probability measure which depends only on the leading polynomial Q_k.

In this paper we will be interested in the eigenvalue problem $T_Q(p_n) = \lambda_n p_n$ with $\deg Q_j \le 1\ \forall j$ with equality when $j = 1$. We first prove that T_Q as defined above and with these restrictions on the Q_j has a unique eigenvalue λ_n and a unique monic polynomial eigenfunction p_n for *every* value of n, see Corollary 1. This is actually true for an even wider class of operators which includes this T_Q, see Theorem 1. The main goal of this paper is to study the simplest degenerate case. Below we restrict our study to the operator

$$T_Q = (\alpha z + \beta)\frac{d}{dz} + (\gamma z + \delta)\frac{d^2}{dz^2}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha, \gamma \neq 0$. Obviously by an appropriate affine transformation of z any such operator can be rewritten as

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]; \quad \delta, \kappa \in \mathbb{C}$$

and throughout our paper this form of T_Q will be used. Then our eigenvalue equation $T_Q(p_n) = \lambda_n p_n$ becomes a *confluent hypergeometric equation* [1]. More precisely, our eigenpolynomials p_n will be particular solutions to what is known as *Kummer's hypergeometric equation*, which is given by

$$zy^{''} + (\alpha + 1 - z)y^{'} - \beta y = 0$$

with $\alpha, \beta \in \mathbb{C}$, see [23] or [19]. Observe that this equation has a degree n polynomial solution if and only if $\beta = n$. Below we will study the sequence of polynomials $\{p_n\}$ being solutions to the above equation as $\beta \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. For certain choices of the parameter α ($\alpha \in \mathbb{R}$ with $\alpha > -1$) we get the *Laguerre polynomials* as solutions [2]. One of the most important properties of the Laguerre polynomials is that they constitute an orthogonal polynomial system with respect to the weight function $e^{-x}x^{\alpha}$ on the interval $[0, \infty)$. It is well-known that the Laguerre polynomials are all hyperbolic (i.e. all their roots are real) and that they have interlacing roots. For other choices of the complex parameter α in Kummer's equation the sequence $\{p_n\}$ is in general *not* an orthogonal system of polynomials, and it can therefore not be studied by means of the theory known for such systems.

One of the results in this paper is the characterization of the exact choices on α for which T_Q has hyperbolic polynomial eigenfunctions. Namely,

Theorem 2. *The following two conditions are equivalent:*
(i) *there exists a* real *affine transformation* $z \to az + b$ *such that our operator can be written as*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

[1] Various familiar functions of mathematical analysis, such as the Hermite polynomials, the Laguerre polynomials, the Whittaker functions, the Bessel functions and the cylinder functions, are special cases of the confluent hypergeometric functions, i.e. solutions of the confluent hypergeometric equations.

[2] With the same conditions on α, but with β not necessarily an integer, we obtain the *Laguerre functions.*

where $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa > -1$,
(ii) T_Q *has hyperbolic polynomial eigenfunctions* p_n *for* all n.

Remark. Each p_n is actually *strictly* hyperbolic here, i.e. has all its roots real and simple (see Corollary 3). Note that (i)$\Rightarrow$(ii) for $\kappa > 0$ also follows from the general theory of orthogonal polynomial systems, the p_n being normalized [3] *Laguerre polynomials.* By definition the Laguerre polynomials satisfy the following differential equation:

$$zy^{''} + (\alpha + 1 - z)y^{'} + ny = 0$$

where $\alpha \in \mathbb{R}$, $\alpha > -1$ and $n \in \mathbb{N}$. Making the transformation $z \to -z$ it is easy to see that this equation corresponds to our eigenvalue equation $zp_n^{''} + (z+\kappa)p_n^{'} - \lambda_n p_n = 0$ where $\lambda_n = n$ (see Corollary 1) and $\kappa \in \mathbb{R}$ with $\kappa > 0$.

Theorem $2^{'}$**.** *The following two conditions are equivalent:*
$(\mathrm{i})^{'}$ *there exists a* real *affine transformation* $z \to az + b$ *such that our operator can be written as*

$$T_Q = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa > -1$ *or* $\kappa = -1, -2, -3, \ldots, -(n-1)$,
$(\mathrm{ii})^{'}$ *the polynomial eigenfunction* p_n *of* T_Q *is hyperbolic.*

Remark. Thus if κ is a negative integer, then all p_n with $n > |\kappa|$ are hyperbolic. Note that in the limit as $n \to \infty$, p_n is hyperbolic for all negative integer values of κ.

These results imply the following corollaries:

Corollary 2. *The following two conditions are equivalent:*
(i) *there exists a* complex *affine transformation* $z \to \alpha z + \beta$ *such that our operator can be written as*

$$T_Q = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa > -1$,
(ii) *the roots of the polynomial eigenfunction* p_n *of* T_Q *lie on a straight line in* $\mathbb{C}$ *for* all n.

Corollary $2^{'}$**.** *The following two conditions are equivalent:*
$(\mathrm{i})^{'}$ *there exists a* complex *affine transformation* $z \to \alpha z + \beta$ *such that our operator can be written as*

$$T_Q = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

[3] The nth degree Laguerre polynomial becomes monic when multiplied by $n!(-1)^n$.

where $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa > -1$ *or* $\kappa = -1, -2, -3, \ldots, -(n-1)$,
(ii)$'$ *the roots of the polynomial eigenfunction* p_n *of* T_Q *lie on a straight line in* $\mathbb{C}$.

Remark. Thus if κ is a negative integer, then the roots of all p_n with $n > |\kappa|$ lie on straight lines in $\mathbb{C}$.

It is moreover possible to count the exact number of real and complex roots respectively for any real value of κ. We prove the following theorems:

Theorem 5. *Let*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *such that* $\kappa < -(n-1)$. *Then the polynomial eigenfunction* p_n *of* T_Q *has no real roots if* n *is even, and it has exactly one real root if* n *is odd.*

Theorem 6. *Let*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta \in \mathbb{C}$ *and* $\kappa \in \mathbb{R}$ *such that* $-(n-1) < \kappa < -1$, *and* κ *is* not *an integer. Let* $[\kappa]$ *denote the integer part of* κ. *Then the number of real roots of the polynomial eigenfunction* p_n *of* T_Q *equals* $\begin{cases} n+[\kappa]+1 & \text{if } [\kappa] \text{ is odd} \\ n+[\kappa] & \text{if } [\kappa] \text{ is even.} \end{cases}$

Next we prove *the interlacing property*:

Theorem 7. *Assume that our operator, after some* complex *affine transformation, can be written as*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta, \kappa \in \mathbb{C}$. *Then the roots of two consecutive polynomial eigenfunctions* p_n *and* p_{n+1} *are interlacing if* $\kappa \in \mathbb{R}$ *with* $\kappa = 0, -1, -2, \ldots, -(n-1)$. *If the eigenfunctions are hyperbolic then the meaning of this is obvious, while if they are not hyperbolic the roots are interlacing along the straight line on which they lie (see Corollary* $2'$*).*

Remark. For $\kappa > 0$ our polynomial eigenfunctions coincide with the Laguerre polynomials, whose interlacing property is classical.

The final part of this paper deals with the asymptotic zero distribution. When suitably scaled, it is possible to find a limiting expansion (as $n \to \infty$) for the polynomial eigenfunctions that is closely related to a Bessel function. Because of the scaling however, the convergence to this Bessel function only gives information about the asymptotic behaviour of the polynomial eigenfunctions in an infinitesimal neighbourhood of the origin. Although other methods must be used to get information elsewhere, it is interesting that on the infinitesimal scale, our polynomial eigenfunctions mimic the global behaviour of this particular Bessel function. Or, letting $z/n = w$ in Theorem 8 below, we can also say that the asymptotic behaviour of our polynomial eigenfunctions reflects the behaviour of the Bessel function at infinity. We have the following theorem, where $J_{\kappa-1}$ denotes the Bessel function of the first kind of order $(\kappa - 1)$:

Theorem 8. *Let $p_n(\kappa, z)$ be a monic polynomial eigenfunction of the operator*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$ and κ is not a negative integer. We then have the following limit formula:

$$\lim_{n\to\infty} \frac{n^{1-\kappa}}{n!} p_n(\kappa, z/n) = (-z)^{(1-\kappa)/2} J_{\kappa-1}(2i\sqrt{z}),$$

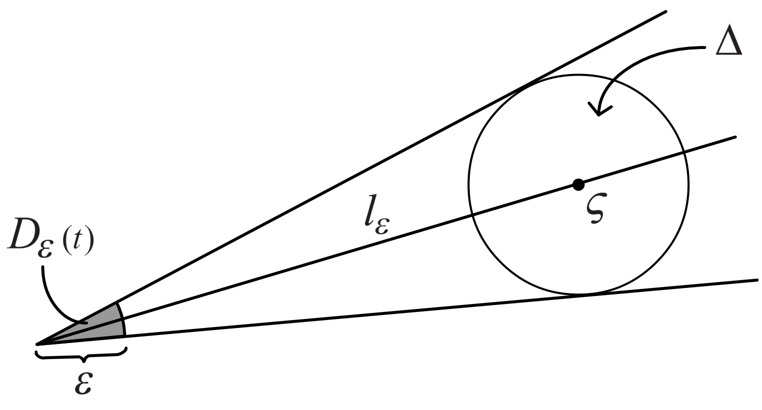
the convergence holding for all $z \in \mathbb{C}$ and uniformly on compact z-sets.

From Theorem 8 we have the following corollary:

Corollary 4. *Let*

$$F_\kappa(z) := \lim_{n\to\infty} \frac{n^{1-\kappa}}{n!} p_n(\kappa, z/n).$$

Suppose that $F_\kappa(\zeta) = 0$ for some nonzero complex $\zeta = re^{it}$ and denote by l_ζ the ray connecting the origin and ζ, as illustrated in the picture below, where the origin is located at the vertex of the angle.



Then, for every $\epsilon > 0$, all except possibly a finite number of the p_n vanish *in the domain* $D_\epsilon(t) := \{z : t - \epsilon < \arg z < t + \epsilon, |z| < \epsilon\}$.

Some history. In this paper the Laguerre polynomials appear normalized as polynomial eigenfunctions of a confluent hypergeometric type operator. The classical Laguerre polynomials are defined as

$$L_n(z, \alpha) \equiv z^\alpha e^z \frac{d^n}{dz^n}[e^{-z} z^{n+\alpha}]$$

where $\alpha \in \mathbb{R}$ with $\alpha > -1$. They satisfy the orthogonality relation

$$\int_0^\infty e^{-z} z^\alpha L_n(z) L_m(z) dz = 0; \quad m, n = 0, 1, \ldots, m \neq n,$$

from which it can be shown that all the zeros of the functions $L_n(z, \alpha)$ are real, distinct and lie inside $(0, \infty)$. The following differential equation for $L_n(z, \alpha)$ is well-known:

$$zL_n^{''}(z, \alpha) + (\alpha + 1 - z)L_n^{'}(z, \alpha) + nL_n(z, \alpha) = 0$$

and still holds if $\alpha \leq -1$. In this case, however, the orthogonality relations do not hold since the integrals involved do not exist. When α is arbitrary and real, the polynomials $L_n(z, \alpha)$ are referred to as *generalized Laguerre polynomials*. Some properties of the zeros when $\alpha \leq -1$ have been studied in [18]. In [23] the same results, and several others, are derived by considering the Laguerre polynomials as a limiting case of the Jacobi polynomials. In this paper we recover some of these results by yet another method.

The asymptotic zero distributions for the generalized Laguerre (and several other) polynomials with real and degree dependent parameter α_n ($\alpha_n/n \to \infty$) have been found in [6] using a continued fraction technique; the same results are derived in [11] via a differential equation approach. It is known that the zeros of $L_n(z, \alpha)$ for $\alpha \leq -1$ accumulate along certain contours in the complex plane. More recent results on this can be found in [14], where a Riemann-Hilbert formulation for the Laguerre polynomials together with the steepest descent method (introduced in [6]), is used to obtain asymptotics for the polynomials, from which the zero behaviour follows. The asymptotic location of the zeros depends on $A = \lim_{n\to\infty} -\frac{\alpha_n}{n} > 0$, and the results show a great sensitivity of the zeros to α_n's proximity to the integers. For $A > 1$ the contour is an open arc. For $0 < A < 1$ the contour consists of a closed loop together with an interval on the positive real axis. In the intermediate case $A = 1$ the contour is a simple closed contour. The case $A > 1$ is well-understood (see [21]), and uniform asymptotics for the Laguerre polynomials as $A > 1$ were obtained more recently, see [9], [15] and [26]. For fixed n and decreasing $\alpha \leq -1$, the dynamics of the zeros of $L_n(z, \alpha)$ is similar to the dynamics of the zeros of certain hypergeometric polynomials studied recently in [7] and [8]. In this paper we obtain some results on the asymptotic zero distribution for any complex α.

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2 Proofs.

We start with the following general statement:

Theorem 1. *Let*

$$T_Q = Q_k \frac{d^k}{dz^k} + Q_{k-1} \frac{d^{k-1}}{dz^{k-1}} + \ldots + Q_1 \frac{d}{dz} + Q_0$$

be such that $\deg Q_0 = 0$, $\deg Q_j = j$ *for exactly one* $j \in [1, k]$ *and* $\deg Q_m < m$ $\forall m \neq 0, j$. *Then, for every value of* n *there exists a unique polynomial solution to the eigenvalue equation* $T_Q(p_n) = \lambda_n p_n$ *with* p_n *monic. Also, using the notation* $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$, *we have*

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\ldots(n-j+1)} = q_{j,j}.$$

Proof. In [3] we proved the following lemma for the above operator T_Q with the weaker restriction $\deg Q_j \leq j$ $\forall j \in [0, n]$:

Lemma 1. *For* $n \geq 1$ *the coefficient vector* X *of* $p_n = (a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ *satisfies the linear system* $MX = Y$, *where* Y *is a vector and* M *is an upper triangular matrix, both with entries expressible in the coefficients* $q_{m,j}$.

This lemma is obviously also valid for our operator T_Q as defined in Theorem 1. Moreover, in [3] we computed the matrix M with respect to the basis of monomials $1, z, z^2, \ldots$. A diagonal element $M_{i+1,i+1}$ at the position $(i+1, i+1)$ in M, where $0 \leq i \leq n-1$, is given by

$$M_{i+1,i+1} = \sum_{0 \leq m \leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n$$

where

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!},$$

where $Q_m = \sum_{j=0}^{m} q_{m,j} z^j$. For our operator T_Q as defined in Theorem 1 we have $\deg Q_m < m$ $\forall m \neq 0, j$ and so we get $q_{m,m} = 0$ $\forall m \neq 0, j$. Inserting this in the expression for λ_n we obtain

$$\lambda_n = \sum_{m=0}^{k} q_{m,m} \cdot \frac{n!}{(n-m)!} = q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} =$$
$$= q_{0,0} + q_{j,j} \cdot n(n-1)\ldots(n-j+1)$$

whence

$$\lim_{n\to\infty} \frac{\lambda_n}{n(n-1)\ldots(n-j+1)} = \lim_{n\to\infty} \left(\frac{q_{0,0}}{n(n-1)\ldots(n-j+1)} + q_{j,j} \right) = q_{j,j}.$$

For the uniqueness of a monic polynomial solution we consider the determinant of the matrix M in the system $MX = Y$ corresponding to our eigenvalue equation $T_Q(p_n) = \lambda_n p_n$. Since the matrix is upper triangular by Lemma 1, its determinant equals the product of the diagonal elements. Thus, if we prove that all diagonal elements are nonzero, then M is invertible for every n and the system $MX = Y$ has a unique solution for every n. Inserting $q_{m,m} = 0$ $\forall m \neq 0, j$ in the expression $M_{i+1,i+1}$ for a diagonal element we get

$$\begin{aligned} M_{i+1,i+1} &= \sum_{0 \leq m \leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n = \\ &= \sum_{0 \leq m \leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \left(q_{0,0} + q_{j,j} \cdot \frac{n!}{(n-j)!} \right) = \\ &= q_{j,j} \cdot \left(\frac{i!}{(i-j)!} - \frac{n!}{(n-j)!} \right) \neq 0 \end{aligned}$$

since $q_{j,j} \neq 0$ and $i < n$. For $i < j$ one sets $i!/(i-j)! = 0$. $\square$

Corollary 1. *Consider the operator*

$$T_Q = Q_k \frac{d^k}{dz^k} + Q_{k-1} \frac{d^{k-1}}{dz^{k-1}} + \ldots + Q_1 \frac{d}{dz} + Q_0$$

with $\deg Q_0 = 0$ *and* $\deg Q_m \leq 1$ $\forall m > 0$ *with equality when* $m = 1$. *Then, for* every *value of* n *there exists a unique polynomial solution to* $T_Q(p_n) = \lambda_n p_n$ *with* p_n *monic. Also, using the same notations as in Theorem 1, we have*

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = q_{1,1}.$$

Proof. This operator corresponds to the operator in Theorem 1 with $j = 1$. We have

$$\lambda_n = q_{0,0} + q_{1,1} \cdot n,$$

whence

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \lim_{n \to \infty} \left(\frac{q_{0,0}}{n} + q_{1,1} \right) = q_{1,1}.$$

To show uniqueness we use the result from Theorem 1, according to which the diagonal elements of M are nonzero for all n. Explicitly we have

$$\begin{aligned} M_{i+1,i+1} &= \sum_{0 \leq m \leq min(i,k)} q_{m,m} \cdot \frac{i!}{(i-m)!} - \lambda_n = \\ &= q_{0,0} + q_{1,1} \cdot \frac{i!}{(i-1)!} - q_{0,0} - q_{1,1} \cdot n = q_{1,1} \cdot (i - n) \neq 0 \end{aligned}$$

since $q_{m,m} = 0 \; \forall m \neq 0, 1$, $q_{1,1} \neq 0$ and $i < n$. The determinant is thus given by

$$\det M = q_{1,1}^n \prod_{i=0}^{n-1} (i - n) \neq 0.$$

□

From now on we restrict our study to the operator with $k = 2$ and $\deg Q_1 = \deg Q_2 = 1$. We first prove that our operator can be written in a more convenient form after some suitable transformations[4].

Lemma 2. *Any operator*

$$T_Q = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *can be transformed to an operator of the form*

$$T_Q = \delta\left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta, \kappa \in \mathbb{C}$.

Proof. Dividing $T_Q = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$ by β we obtain

$$T_Q^* = T_Q/\beta = \left(\frac{\alpha}{\beta}z + \frac{\gamma}{\beta}\right)\frac{d}{dz} + \left(z + \frac{\delta}{\beta}\right)\frac{d^2}{dz^2}$$

and making the translation $\tilde{z} = z + \frac{\delta}{\beta}$ we have

$$\begin{aligned} T_{\tilde{Q}}^* &= \left(\frac{\alpha}{\beta}\left(\tilde{z} - \frac{\delta}{\beta}\right) + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \left(\tilde{z} - \frac{\delta}{\beta} + \frac{\delta}{\beta}\right)\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2}. \end{aligned}$$

Finally with $\tilde{\tilde{z}} = \frac{\alpha}{\beta}\tilde{z} \Leftrightarrow \tilde{z} = \frac{\beta}{\alpha}\tilde{\tilde{z}}$ we have $d\tilde{\tilde{z}}/d\tilde{z} = \alpha/\beta$ and so

$$\begin{cases} \frac{d}{d\tilde{z}} = \frac{\alpha}{\beta}\frac{d}{d\tilde{\tilde{z}}} \\ \frac{d^2}{d\tilde{z}^2} = \frac{d}{d\tilde{\tilde{z}}}\left(\frac{d}{d\tilde{z}}\right)\frac{d\tilde{\tilde{z}}}{d\tilde{z}} = \frac{d}{d\tilde{\tilde{z}}}\left(\frac{\alpha}{\beta}\frac{d}{d\tilde{\tilde{z}}}\right)\frac{\alpha}{\beta} = \frac{\alpha^2}{\beta^2}\frac{d^2}{d\tilde{\tilde{z}}^2} \end{cases}$$

and we get

$$\begin{aligned} T_{\tilde{\tilde{Q}}}^* &= \left(\frac{\alpha}{\beta}\tilde{z} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{z}} + \tilde{z}\frac{d^2}{d\tilde{z}^2} = \\ &= \left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}\tilde{\tilde{z}} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{\alpha}{\beta}\frac{d}{d\tilde{\tilde{z}}} + \frac{\beta}{\alpha}\tilde{\tilde{z}}\cdot\frac{\alpha^2}{\beta^2}\frac{d^2}{d\tilde{\tilde{z}}^2} = \\ &= \frac{\alpha}{\beta}\left[\left(\tilde{\tilde{z}} - \frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}\right)\frac{d}{d\tilde{\tilde{z}}} + \tilde{\tilde{z}}\frac{d}{d\tilde{\tilde{z}}^2}\right] = \delta\left[(\tilde{\tilde{z}} + \kappa)\frac{d}{d\tilde{\tilde{z}}} + \tilde{\tilde{z}}\frac{d}{d\tilde{\tilde{z}}^2}\right] \end{aligned}$$

[4]compare [10] p. 249

where $\delta = \frac{\alpha}{\beta}$ and $\kappa = -\frac{\alpha\delta}{\beta^2} + \frac{\gamma}{\beta}$. $\square$

Clearly any such operator with all coefficients real can transformed to

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

with $\delta, \kappa \in \mathbb{R}$.

We will now study hyperbolicity of the polynomial eigenfunctions of our operator $T_Q = (\alpha z + \gamma)\frac{d}{dz} + (\beta z + \delta)\frac{d^2}{dz^2}$ in all details. Note that performing the transformations above with real coefficients does not affect hyperbolicity of the polynomial eigenfunctions and so we can apply Lemma 2 in the proof of Theorems 2 and $2'$ below.

Proof of Theorems 2 and $2'$. Theorems 2 and $2'$ are proved using the following corollary (see [2]):

Corollary of Sturm's Theorem. *All roots of a monic and real polynomial are real if and only if the nonzero polynomials in its Sturm sequence have positive leading coefficients.*

Here the Sturm sequence is defined as follows. Let $p = p_0$ be a given real polynomial. Define $p_1 = p'$ (the derivative of p) and choose the p_i to satisfy

$$\begin{aligned} p_0 &= p_1q_1 - p_2, \quad \deg p_2 < \deg p_1 \\ p_1 &= p_2q_2 - p_3, \quad \deg p_3 < \deg p_2 \\ p_2 &= p_3q_3 - p_4, \quad \deg p_4 < \deg p_3 \\ &\vdots \end{aligned}$$

where the q_i are polynomials, and so on until a zero remainder is reached. That is, for each $i \geq 2$, p_i is the negative of the remainder when p_{i-2} is divided by p_{i-1}. Then the sequence $(p_0, p_1, p_2, \dots)$ is called the *Sturm sequence* of the polynomial p.

We now calculate the Sturm sequence for a monic and real polynomial eigenfunction $p_n = p$ of the operator $T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$, where $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{R}$. Note that p is real if $\kappa \in \mathbb{R}$ and any two operators differing by a complex constant have identical polynomial eigenfunctions. Since our eigenpolynomials by assumption are monic, the first two elements in the Sturm sequence, p and p', clearly have positive leading coefficients 1 and n. Define $R(i) = p_{i+1}$ in the Sturm sequence above. Then $R(1)$ is the negative of the remainder when p is divided by p'. With $\deg p = n$ we have $\deg R(i) = n - i - 1$. The last element in the Sturm sequence (if it has not already stopped) will be the constant $R(n-1)$.

We claim that for every n and every $i \geq 1$ we have

$$\begin{cases} R(i) = A\sum_{j=0}^{n-i-1}\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}z^j & \text{if } i \text{ is } \textit{odd} \\ R(i) = B\sum_{j=0}^{n-i-1}\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}z^j & \text{if } i \text{ is } \textit{even} \end{cases} \tag{1}$$

where

$$\begin{cases} A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\dots(n-i)(\kappa+n-i), \\ B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\dots(n-i)(\kappa+n-i). \end{cases}$$

It is obvious that with $\kappa = 0$ the leading coefficients of all the $R(i)$ are positive and p will be hyperbolic. For $\kappa \in \mathbb{R}$ and $\kappa \neq 0$ we have the following conditions for the leading coefficients $R(i)_{lc}$ of the $R(i)$ to be positive:

$$\begin{cases} R(1)_{lc} > 0 & \Rightarrow & \kappa > 1-n \\ R(2)_{lc} > 0 & \Rightarrow & \kappa > 2-n \\ R(3)_{lc} > 0 & \Rightarrow & \kappa > 3-n \\ \vdots & & \\ R(i)_{lc} > 0 & \Rightarrow & \kappa > i-n \\ \vdots & & \\ R(n-1)_{lc} > 0 & \Rightarrow & \kappa > -1 \end{cases}$$

and these conditions together yield $\kappa > -1$. But we must also consider the equality case, since if some factor $(\kappa + n - j) = 0$, then not only the leading coefficient is zero, but the whole polynomial $R(i)$ equals zero. So for $\kappa = j - n$ with $j \in [1, n-1]$ we also get hyperbolic p_n, since the Sturm sequence by definition stops when a zero remainder is reached, and it is easy to see that the leading coefficients of the previous components of the Sturm sequence will be positive. Thus, by the corollary of Sturm's Theorem, p_n is hyperbolic for *all* n if and only if $\kappa > -1$, and p_n is hyperbolic for a *particular* n if and only if $\kappa > -1$ or $\kappa = -1, -2, \dots, -(n-1)$.

We prove by induction that the Sturm sequence polynomials are of the form claimed in (1). For detailed calculations see Appendix.

It is obvious that if the roots of some p_n lie on a straight line, then they can be transformed to the real axis by some complex affine transformation, and the operator T_Q must then be on the form claimed by Theorems 2 or $2'$, and so Corollaries 2 and $2'$ follow easily from Theorems 2 and $2'$ respectively.

From now on we adopt the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ is the Gamma function[5].

[5] see e.g. [19]

Lemma 3. *Let $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ be the nth degree monic polynomial eigenfunction of the operator $T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$ with $\delta, \kappa \in \mathbb{C}$. Note that T_Q and δT_Q have identical polynomial eigenfunctions p_n. Then the coefficients $a_{n,j}$ of p_n are given by*

$$a_{n,j} = \binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}, \quad \forall j \in [0,n].$$

Proof. Inserting $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ in our eigenvalue equation we have

$$T_Q(p_n) = \lambda_n p_n$$
$$\Leftrightarrow$$
$$(z+\kappa)p_n^{'} + zp_n^{''} = np_n$$
$$\Leftrightarrow$$
$$(z+\kappa)\sum_{j=1}^{n} ja_{n,j}z^{j-1} + z\sum_{j=2}^{n} j(j-1)a_{n,j}z^{j-2} = n\sum_{j=0}^{n} a_{n,j}z^j$$
$$\Leftrightarrow$$
$$\sum_{j=1}^{n} ja_{n,j}z^j + \sum_{j=1}^{n} \kappa ja_{n,j}z^{j-1} + \sum_{j=2}^{n} j(j-1)a_{n,j}z^{j-1} = \sum_{j=0}^{n} na_{n,j}z^j$$
$$\Leftrightarrow$$
$$\sum_{j=1}^{n} ja_{n,j}z^j + \sum_{j=0}^{n-1} \kappa(j+1)a_{n,j+1}z^j + \sum_{j=1}^{n-1} j(j+1)a_{n,j+1}z^j = \sum_{j=0}^{n} na_{n,j}z^j.$$

Comparing coefficients we get

$$ja_{n,j} + \kappa(j+1)a_{n,j+1} + j(j+1)a_{n,j+1} = na_{n,j}$$
$$\Leftrightarrow$$
$$a_{n,j} = \frac{(j+1)(\kappa+j)}{(n-j)} \cdot a_{n,j+1}$$

Applying this iteratively and using $a_{n,n} = 1$ (by the monicity of p_n) we arrive at

$$a_{n,j} = \binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}, \quad \forall j \in [0,n].$$

□

Proposition 1. *Let $p_n(\kappa, z)$ denote the* nth *degree monic polynomial eigenfunction of the operator*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]$$

where $\delta, \kappa \in \mathbb{C}$. Then, using the explicit representation of p_n in Lemma 3, we derive the following identity:

$$p_n^{(m)}(\kappa, z) = \frac{n!}{(n-m)!} p_{n-m}(\kappa + m, z), \quad n = 0, 1, \dots; m = 1, 2, \dots$$

and the following recurrence formula:

$$p_n(\kappa, z) = (z + 2n + \kappa - 2)p_{n-1}(\kappa, z) - (n-1)(n + \kappa - 2)p_{n-2}(\kappa, z),$$

where $p_0(\kappa, z) = 1$ and $p_1(\kappa, z) = z + \kappa$.

Theorem 3. *Let*

$$T_Q = \delta\left[(z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}\right]; \quad \delta, \kappa \in \mathbb{C}.$$

Then the roots of every polynomial eigenfunction p_n of T_Q are all simple, unless $\kappa \in \mathbb{R}$ with $\kappa = -1, -2, \dots, -(n-1)$.

Proof. Let $\alpha \neq 0$ be a root of p_n that is not simple. Then, by repeatedly differentiating our eigenvalue equation $zp_n^{''} + (z+\kappa)p_n^{'} = np_n$ and inserting $z = \alpha$, we get $p_n^j(\alpha) = 0 \; \forall j$, and the multiplicity of α would be infinite, which is absurd. Thus, for all $\kappa \in \mathbb{C}$, any root $\alpha \neq 0$ of p_n is simple [6].

It remains to prove that if $\kappa \neq -1, -2, \dots, -(n-1)$ and if $\alpha = 0$ is a root of p_n then it must be simple too. Let $\alpha = 0$ be a root of p_n of multiplicity m and write $p_n(z) = z^m q(z)$ where $\alpha = 0$ is not a root of $q(z)$. Then $p_n^{(1)}(z) = mz^{m-1}q(z) + z^m q^{'}(z)$ and $p_n^{(2)}(z) = m(m-1)z^{m-2}q(z) + mz^{m-1}q^{'}(z) + mz^{m-1}q^{'}(z) + z^m q^{''}(z)$. Inserting this in our eigenvalue equation we obtain

$$\begin{aligned}
\lambda_n p_n(z) =& zp_n^{''}(z) + (z+\kappa)p_n^{'}(z) \\
&\Leftrightarrow \\
z^{m-1}[\lambda_n z q(z)] =& m(m-1)z^{m-1}q(z) + mz^m q^{'}(z) \\
&+ mz^m q^{'}(z) + z^{m+1}q^{''}(z) + mz^m q(z) + z^{m+1}q^{'}(z) \\
&+ \kappa m z^{m-1} q(z) + \kappa z^m q^{'}(z) \\
=& z^{m-1}[m(m-1)q(z) + mzq^{'}(z) + mzq^{'}(z) \\
&+ z^2 q^{''}(z) + mzq(z) + z^2 q^{'}(z) + \kappa m q(z) + \kappa z q^{'}(z)].
\end{aligned}$$

Equating the expressions in the brackets and setting $z = 0$ we arrive at the relation $m(m-1)q(0) + \kappa m q(0) = 0 \Leftrightarrow m(m-1+\kappa) = 0$. Thus $m = 0$ or $m = 1 - \kappa$ for the multiplicity m of the root $\alpha = 0$. But if $m = 0$ then $\alpha = 0$ is not a root of p_n whence all roots of p_n will be simple and we are done. If $\kappa = 0$ then $m = 0$ or $m = 1$ (it will soon be proved that the latter is true, see below),

[6] This also follows from the uniqueness theorem for a second order differential equation.

and so either we have no root at all at the origin or we have a simple root at the origin for $\kappa = 0$.

Now let $\kappa \neq 0, -1, -2 \ldots, -(n-1)$. Then either $m = 0$ and we are done, or $m = 1 - \kappa$. But we have $m \in \mathbb{Z}$ and $m > 0$, since m is the multiplicity of the root, and therefore $m = 1 - \kappa$ is impossible if $\kappa > -1$ and $\kappa \neq 0$. Thus $\alpha = 0$ is not a root at all, i.e. $m = 0$, for any p_n if $\kappa > -1$ and $\kappa \neq 0$. Also, $m = 1 - \kappa$ is absurd if $\kappa \notin \mathbb{Z}$. Thus $m = 0$ for $\kappa \notin \mathbb{Z}$. The cases $\kappa = -1, -2, .., -(n-1)$, i.e. $\kappa \in \mathbb{Z}$ with $-n < \kappa < 0$, will be treated in Theorem 4, and so here it remains to consider the cases $\kappa \in \mathbb{Z}$ with $\kappa \leq -n$. From our relation we have either $m = 0$ or $m = 1 - \kappa$. By Lemma 3 the constant term $a_{n,0}$ of p_n equals

$$a_{n,0} = \frac{(\kappa - 1 + n)!}{(\kappa - 1)!} = (\kappa - 1 + n)(\kappa - 2 + n)(\kappa - 3 + n) \ldots (\kappa + 2)(\kappa + 1)\kappa.$$

But this cannot be equal to zero if $\kappa \in \mathbb{Z}$ and $\kappa \leq -n$, and therefore $m = 0$. Thus all roots of p_n are simple for all $\kappa \in \mathbb{C} \setminus \{-1, -2, \ldots, -(n-1)\}$. □

Note that if $\kappa = 0$, then from the relation $m(m - 1 + \kappa) = 0$ we have either $m = 0$ or $m = 1$. But $\kappa = 0 \Rightarrow a_{n,0} = 0$ by Lemma 3 and therefore $m = 1$ for $\kappa = 0$.

Corollary 3. *The polynomial eigenfunctions p_n of T_Q are* strictly hyperbolic *(all roots are real and simple) for all n if and only if there exists a* real *affine transformation $z \to az + b$ such that our operator can be written as*

$$T_Q = \delta \left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2} \right]$$

with $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{R}$ with $\kappa > -1$.

Proof. Hyperbolicity follows from Theorem 2 and simplicity of the roots follows from Theorem 3. □

Theorem 4. *Let*

$$T_Q = \delta \left[(z + \kappa)\frac{d}{dz} + z\frac{d^2}{dz^2} \right]$$

where $\delta \in \mathbb{C}$ and $\kappa \in \mathbb{R}$ with $\kappa = 0, -1, -2, \ldots, -(n-1)$. Then the eigenpolynomial p_n has $(n + \kappa)$ distinct roots, all of which are simple except one root which lies at the origin and has multiplicity $(1 - \kappa)$. Note that for $\kappa = 0$ all roots are simple.

Proof. Let p_n be a polynomial eigenfunction of the operator T_Q with no restrictions on κ. By the proof of Theorem 3 all nonzero roots of p_n are simple, and for the root at the origin we have the relation $m(m - 1 + \kappa) = 0$, where m is the multiplicity of the root. Now if $m \neq 0$ then $m = 1 - \kappa$ and we are done. Thus we have to prove that $m \neq 0$, i.e that we do have a root at the origin for

$\kappa = 0, -1, -2, \dots, -(n-1)$. But this is only possible if the constant term $a_{n,0}$ equals zero. By Lemma 3 we have

$$a_{n,0} = (\kappa - 1 + n)(\kappa - 2 + n)(\kappa - 3 + n) \dots (\kappa + 2)(\kappa + 1)\kappa,$$

and obviously $a_{n,0} = 0$ if $\kappa = 0, -1, -2, \dots, -(n-1)$. Thus $m \neq 0$ and therefore $m = 1 - \kappa$ for $\kappa = 0, -1, -2, \dots, -(n-1)$. We have a total of n roots of p_n, all of which are simple except for the root at the origin which has multiplicity $1 - \kappa$. We thus have $n - (1 - \kappa) + 1 = n + \kappa$ distinct roots for $\kappa = 0, -1, -2, \dots, -(n-1)$ respectively. □

As stated in Theorems 5 and 6, it is possible to count the exact number of real and complex roots respectively of a polynomial eigenfunction of T_Q for any real value of κ. With Sturm's Theorem it is possible to count the number of real roots in any interval. We have (see [2]):

Sturm's Theorem. *Let $(p_0(t), p_1(t), p_2(t), \dots)$ be the Sturm sequence of a polynomial $p(t)$ (as defined in the proof of Theorems 2 and $2'$). Let $u < v$ be real numbers. Suppose that U is the number of sign changes in the sequence $(p_0(u), p_1(u), p_2(u), \dots)$ and that V is the number of sign changes in the sequence $(p_0(v), p_1(v), p_2(v), \dots)$. Then the number of real roots of $p(t)$ between u and v (with each multiple root counted exactly once) is exactly $U - V$.*

Remark. Combining Sturm's Theorem with Theorems 3 and 4 it is possible to recover Theorems 2 and $2'$ in the directions $(i) \Rightarrow (ii)$ and $(i)' \Rightarrow (ii)'$. Namely, let $\kappa > -1$ and let p_n be the nth degree monic polynomial eigenfunction of T_Q. Then the Sturm sequence of p_n has $(n + 1)$ nonzero elements, all with positive leading coefficients. With $u = -\infty$ and $v = \infty$ we then have $U = n$ and $V = 0$, and therefore the number of real roots of p_n is $U - V = n$, whence p_n is hyperbolic (Theorem 2).

Now let $\kappa = -1, -2, \dots, -(n-1)$. Since the Sturm sequence stops as soon as the zero remainder is reached, it has $(n + \kappa + 1)$ nonzero elements, all with positive leading coefficients. Therefore, with $u = -\infty$ and $v = \infty$, we have $U = n + \kappa$ and $V = 0$. By Theorem 4 all roots of p_n are simple except the root at the origin which has multiplicity $1 - \kappa$. Thus, counted with multiplicity, p_n has $U - V + (-\kappa) = n$ real roots and is therefore hyperbolic (Theorem $2'$) .

We already know that if $\kappa \neq -1, -2, \dots, -(n-1)$, then all roots of p_n are simple and no element in the Sturm sequence of p_n is identically zero. The leading coefficients of the elements of the Sturm sequence are, by the proof of

Theorems 2 and $2'$, given by

$$\begin{cases} p_{lc} = 1 \\ p'_{lc} = n \\ R(1)_{lc} = (n-1)(\kappa+n-1) \\ R(2)_{lc} = n(n-2)(\kappa+n-2) \\ R(3)_{lc} = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3) \\ R(4)_{lc} = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4) \\ R(3)_{lc} = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)(n-5)(\kappa+n-5) \\ R(4)_{lc} = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)(n-6)(\kappa+n-6) \\ \vdots \\ R(n-1)_{lc} = \dots \end{cases}$$

We now use Sturm's Theorem to prove Theorems 5 and 6:

Proof of Theorem 5. Let p_n be a monic polynomial eigenfunction of the operator T_Q where $\kappa \in \mathbb{R}$. We have $\kappa < -(n-1) \Leftrightarrow \kappa+n-1 < 0$, and therefore $\kappa + n - j < 0$ for every $j \geq 1$. Thus we have, for the leading coefficients of the Sturm sequence elements,

$$\begin{cases} p_{lc} = 1 > 0 \\ p'_{lc} = n > 0 \\ R(1)_{lc} < 0 \\ R(2)_{lc} < 0 \\ R(3)_{lc} > 0 \\ R(4)_{lc} > 0 \\ R(5)_{lc} < 0 \\ R(6)_{lc} < 0 \\ \vdots \end{cases}$$

and this pattern continues up to the last element $R(n-1)$ of the sequence. Inserting $v = \infty$ in the Sturm sequence of p_n we find that there is a sign change at every $R(i)$ where i is odd. Therefore the number of sign changes V in this sequence equals the number of $R(i)$ where i is odd. Thus

$$V = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Inserting $u = -\infty$ in the Sturm sequence we find that there is a sign change between the first two elements in the sequence and then at every $R(i)$ where i is even. Therefore the number of sign changes U equals 1+ [the number of $R(i)$ where i is even]. Thus

$$U = \begin{cases} \frac{n-2}{2} + 1 = \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} + 1 = \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

From Theorem 3 all roots of p_n are simple and thus the number of real roots of p_n equals $U - V = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$ $\square$

Proof of Theorem 6. Let p_n be a monic polynomial eigenfunction of the operator T_Q where $\kappa \in \mathbb{R}$ and $j-n < \kappa < j-n+1$ for some $j \in [1, n-2]$. Then $(\kappa+n-j) > 0$, $(\kappa+n-j-1) < 0$ and $[\kappa] = j-n$. Again we consider the leading coefficients in the Sturm sequence of p_n. Clearly $p_{lc} = 1 > 0$, $p'_{lc} = n > 0$ and $R(i)_{lc} > 0$ $\forall i \in [1, j]$. For the remaining leading coefficients we have

$$\begin{cases} R(j+1)_{lc} < 0 \\ R(j+2)_{lc} < 0 \\ R(j+3)_{lc} > 0 \\ R(j+4)_{lc} > 0 \\ R(j+5)_{lc} < 0 \\ R(j+6)_{lc} < 0 \\ \qquad \vdots \end{cases}$$

and this pattern continues up to the last element $R(n-1)$ in the sequence. Consider the sequence we obtain by inserting $v = \infty$ in this Sturm sequence. We have sign changes at every $R(j+l)$ where l is odd. Our last element is $R(n-1) = R(j+(n-j-1))$. Also note that if $n-j-1 = n-n-[\kappa]-1 = -[\kappa]-1$ is even then $[\kappa]$ is odd, and if $n-j-1$ is odd then $[\kappa]$ is even. Thus the number of sign changes V in this sequence is

$$V = \begin{cases} \frac{n-j-1}{2} & \text{if } [\kappa] \text{ is odd,} \\ \frac{n-j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Now insert $u = -\infty$ in the Sturm sequence. The number of sign changes from the first element p in the sequence till the element $R(j)$ is $(1+j)$. For the remaining $n-j-1$ elements of this sequence we have a change of sign at every $R(j+l)$ where l is even. Thus the number of sign changes is $(n-j-1)/2$ if $(n-j-1)$ is even $\Leftrightarrow$ $[\kappa]$ is odd, and the number of sign changes is $(n-j-2)/2$ if $(n-j-1)$ is odd $\Leftrightarrow$ $[\kappa]$ is even. Thus for the total number of sign changes U in this sequence we get

$$U = \begin{cases} (1+j) + \frac{n-j-1}{2} = \frac{n+j+1}{2} & \text{if } [\kappa] \text{ is odd} \\ (1+j) + \frac{n-j-2}{2} = \frac{n+j}{2} & \text{if } [\kappa] \text{ is even.} \end{cases}$$

Therefore the number of real roots $U - V$ of p_n, counted with multiplicity, is precisely

$$U - V = \begin{cases} \frac{n+j+1}{2} - \frac{n-j-1}{2} = j+1 = n+[\kappa]+1 & \text{if } [\kappa] \text{ is odd} \\ \frac{n+j}{2} - \frac{n-j}{2} = j = n+[\kappa] & \text{if } [\kappa] \text{ is even.} \end{cases}$$

since all roots of p_n are simple by Theorem 3. $\square$

Proof of Theorem 7. The proof of the interlacing property consists of a sequence of five lemmas. Lemmas 4 and 8 are well-known. Lemmas 4,5 and 6 are used in the proof of Lemma 7, which is proved using an idea of S. Shadrin in [20]. The five lemmas used in the proof of Theorem 7 are the following:

Lemma 4. *If R_n and R_{n+1} are strictly hyperbolic polynomials of degrees n and $n+1$ respectively, then $R_n + \epsilon R_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Lemma 5. *Let p_n and p_{n+1} be two polynomial eigenfunctions of the operator $T_Q = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \ldots, -(n-1)$. Then $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ.*

Proof of Lemma 5. By Theorem 4 we have that p_n and p_{n+1} have all their roots simple except for the root at the origin which for both polynomials has multiplicity $1-\kappa$. Thus we can write $p_n + \epsilon p_{n+1} = z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa})$, where $R_{n+\kappa-1}$ and $R_{n+\kappa}$ are strictly hyperbolic polynomials of degrees $n+\kappa-1$ and $n+\kappa$ respectively. By Lemma 4, $R_{n+\kappa-1} + \epsilon R_{n+\kappa}$ is hyperbolic for any sufficiently small ϵ, and then clearly $z^{1-\kappa}(R_{n+\kappa-1} + \epsilon R_{n+\kappa}) = p_n + \epsilon p_{n+1}$ is also hyperbolic for any sufficiently small ϵ. $\square$

Lemma 6. *Let $T_Q = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ with $\kappa = 0, -1, -2, \ldots, -(n-1)$, and let p_n and p_{n+1} be two consecutive polynomial eigenfunctions of T_Q. Then the application of T_Q to any linear combination $\alpha p_n + \beta p_{n+1}$ with $\alpha, \beta \in \mathbb{R}$ that is hyperbolic (i.e. has all its roots real) results in a hyperbolic polynomial.*

Proof of Lemma 6. Note that the operators $T_Q = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ and $T_Q = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ have identical eigenpolynomials. Let $f = \alpha p_n + \beta p_{n+1}$ be a hyperbolic linear combination with real coefficients of two consecutive polynomial eigenfunctions of T_Q. Then $f^{'}$ is a hyperbolic polynomial by Gauss-Lucas Theorem. By Rolle's Theorem f and $f^{'}$ have interlacing roots and so by the well-known Lemma 8 below, $(f + f^{'})$ is a hyperbolic polynomial. By Theorem 4 both p_n and p_{n+1} have a root at the origin of multiplicity $1-\kappa$. Thus $f = \alpha p_n + \beta p_{n+1}$ has a root at the origin of multiplicity at least $1-\kappa$, and $f^{'}$ has a root at the origin of multiplicity at least $-\kappa$. Thus the polynomial $(f + f^{'})$ has a root at the origin of multiplicity at least $(-\kappa)$ and we can write $(f + f^{'}) = z^{-\kappa} g$ for some hyperbolic polynomial g. Now $z^{\kappa}(f + f^{'}) = g$ is a hyperbolic polynomial. But $D[z^{\kappa}(f + f^{'})] = \kappa z^{\kappa-1}(f + f^{'}) + z^{\kappa}(f^{'} + f^{''}) = z^{\kappa-1}[\kappa f + (z+\kappa)f^{'} + z f^{''}] = z^{\kappa-1} T_Q(f)$ where $T_Q(f) = \kappa f + (z+\kappa)f^{'} + z f^{''}$. By Gauss-Lucas Theorem one has that $D[z^{\kappa}(f + f^{'})]$ is a hyperbolic polynomial and therefore $T_Q(f) = z^{1-k} D[z^{\kappa}(f + f^{'})]$ is a hyperbolic polynomial. $\square$

Lemma 7. *Let $T_Q = \kappa + (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$. Any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients of the polynomial eigenfunctions of T_Q with $\kappa = 0, -1, -2, \ldots, -(n-1)$ is a hyperbolic polynomial.*

Proof of Lemma 7. Applying to $\alpha p_n + \beta p_{n+1}$ some high power T_Q^{-N} of the inverse operator one gets

$$T_Q^{-N}(\alpha p_n + \beta p_{n+1}) = \frac{\alpha}{\lambda_n^N} p_n + \frac{\beta}{\lambda_{n+1}^N} p_{n+1} =$$

$$= \frac{\alpha}{\lambda_n^N}(p_n + \epsilon p_{n+1}),$$

where ϵ is arbitrarily small for the appropriate choice of N (since $0 < \lambda_n < \lambda_{n+1}$). Thus, by Lemma 5, the polynomial $T_Q^{-N}(\alpha p_n + \beta p_{n+1})$ is hyperbolic for sufficiently big N. Assume that $\alpha p_n + \beta p_{n+1}$ is nonhyperbolic and take the largest N_0 for which $R_{N_0} = T_Q^{-N_0}(\alpha p_n + \beta p_{n+1})$ is still nonhyperbolic. Then $R_{N_0} = T_Q(R_{N_0+1})$ where $R_{N_0+1} = T_Q^{-N_0-1}(\alpha p_n + \beta p_{n+1})$. Note that R_{N_0+1} is hyperbolic and that if $\kappa = 0, -1, -2, \dots, -(n-1)$ then the application of T_Q to any hyperbolic linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients results in a hyperbolic polynomial by Lemma 6. Contradiction. $\square$

Lemma 8. If R_n and R_{n+1} are any real polynomials of degrees n and $n+1$, respectively, then saying that every linear combination $\alpha R_n + \beta R_{n+1}$ with real coefficients is hyperbolic is equivalent to saying that
(i) both R_n and R_{n+1} are hyperbolic, and
(ii) their roots are interlacing.

We now prove Theorem 7. Consider the operator $T_Q = (z+\kappa)\frac{d}{dz} + z\frac{d^2}{dz^2}$ where $\kappa \in \mathbb{R}$ with $\kappa = 0, -1, -2, \dots, -(n-1)$, and let p_n and p_{n+1} be two consecutive polynomial eigenfunctions of this operator. (Recall that by Corollary $2'$ the roots of these polynomial eigenfunctions lie on straight lines in $\mathbb{C}$.) By Lemma 5 the linear combination $p_n + \epsilon p_{n+1}$ is hyperbolic for any sufficiently small ϵ. Using Lemmas 5 and 6 we can therefore apply Lemma 7 which says that any linear combination $\alpha p_n + \beta p_{n+1}$ with real coefficients α and β is a hyperbolic polynomial. By Lemma 8 this implies that the roots of p_n and p_{n+1} are interlacing and we are done. $\square$

Remark. Note that we recover the interlacing property of the Laguerre polynomials using the same proof as in Theorem 7, but with a small modification of Lemma 6. Namely, if $\kappa > 0$, then the application of T_Q to *any* hyperbolic polynomial results in a hyperbolic polynomial. For if f is a hyperbolic polynomial, then f' is hyperbolic by Gauss-Lucas Theorem, f and f' have interlacing roots by Rolle's Theorem, and by the well-known Lemma 8 the linear combination $(f + f')$, and therefore $z^\kappa(f + f')$, is a hyperbolic polynomial. Finally $D[z^\kappa(f + f')] = z^{\kappa-1}T_Q(f)$ is hyperbolic by Gauss-Lucas Theorem.

In the analysis leading to Theorem 8 we rule out the case when κ is a negative integer. We begin by introducing the *Bessel function of the first kind* of order

κ, which is defined by the series

$$J_\kappa(z) \equiv \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (z/2)^{\kappa+2\nu}}{\nu!\Gamma(\kappa+\nu+1)}$$

where $z, \kappa \in \mathbb{C}$ and $|z| < \infty$. Clearly $z^{-\kappa} J_\kappa(z)$ is an entire analytic function for all $z \in \mathbb{C}$ if κ is not a negative integer. This Bessel function is a solution to *Bessel's equation* [7] *of order* κ, which is the second-order linear differential equation given by

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + (z^2 - \kappa^2)y = 0.$$

In order to prove Theorem 8, we will need the following technical lemma:

Lemma 9.

$$\lim_{n\to\infty} \binom{n+\kappa-1}{n-\nu} n^{1-\kappa-\nu} = \frac{1}{\Gamma(\kappa+\nu)}$$

where $n, \nu \in \mathbb{R}$ *and* $\kappa \in \mathbb{C} \setminus \{-1, -2, \dots\}$.

Proof. Using the following well-known asymptotic formula:

Corollary of the Stirling formula. [8]

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)} n^{-\alpha} = 1$$

where $\alpha \in \mathbb{C}$ *and* $n \in \mathbb{R}$,

we get

$$\begin{aligned} \lim_{n\to\infty} \binom{n+\kappa-1}{n-\nu} n^{1-\kappa-\nu} &= \frac{1}{\Gamma(\kappa+\nu)} \lim_{n\to\infty} \frac{\Gamma(n+\kappa)}{\Gamma(n-\nu+1)} n^{1-\kappa-\nu} \\ &= \frac{1}{\Gamma(\kappa+\nu)} \lim_{n\to\infty} \frac{\Gamma(n+\kappa)}{\Gamma(n)} n^{-\kappa} \lim_{n\to\infty} \frac{\Gamma(n)}{\Gamma(n-\nu+1)} n^{1-\nu} \\ &= \frac{1}{\Gamma(\kappa+\nu)}. \end{aligned}$$

Proof of Theorem 8. By Lemma 3 our polynomial eigenfunctions have the explicit representation

$$p_n(\kappa, z) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(\kappa+n-1)!}{(\kappa+\nu-1)!} z^\nu = \sum_{\nu=0}^{n} \binom{n+\kappa-1}{n-\nu} \frac{n!}{\nu!} z^\nu$$

[7] Bessel's equation is encountered in the study of boundary value problems in potential theory for cylindrical domains. The solutions to Bessel's equation are referred to as *cylinder functions*, of which the Bessel functions are a special kind.

[8] see e.g.[25]

where $\kappa \in \mathbb{C}$.

Thus, with the scaling $z \to z/n$ and using Lemma 9, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^{1-\kappa}}{n!} p_n(\kappa, z/n) &= \lim_{n\to\infty} \sum_{\nu=0}^{n} \binom{n+\kappa-1}{n-\nu} n^{1-\kappa} \frac{1}{\nu!} \left(\frac{z}{n}\right)^{\nu} \\
&= \lim_{n\to\infty} \sum_{\nu=0}^{n} \binom{n+\kappa-1}{n-\nu} n^{1-\kappa-\nu} \frac{z^{\nu}}{\nu!} \\
&= \sum_{\nu=0}^{\infty} \frac{z^{\nu}}{\Gamma(\kappa+\nu)\nu!} = (-z)^{(1-\kappa)/2} J_{\kappa-1}(2i\sqrt{z}).
\end{aligned}$$

□

Remark. It is easy to prove that the power series $\sum_{\nu=0}^{\infty} \frac{z^{\nu}}{\Gamma(\kappa+\nu)\nu!}$ does indeed satisfy the differential equation $zu^{''} + \kappa u^{'} - u = 0$, which arises by the limiting procedure $n \to \infty$ from our eigenvalue equation $zu^{''} + (z+\kappa)u^{'} - nu = 0$, after scaling the variables.

Proof of Corollary 4. We will need the following well-known theorem [9]:

Hurwitz's Theorem: *Let $f_1, f_2, \ldots \in A(U)$, $f_n \to f$ uniformly on compact subsets of U. Suppose that $\bar{D}(z_0, r) \subset U$ and f is not zero on $\{z : |z - z_0| = r\}$. Then there is a positive integer N such that for $n \geq N$, f_n and f have the same number of zeros in $D(z_0, r)$.*

Fix $\epsilon > 0$. Let $\triangle$ denote the open disc centered at ζ and lying inside the wedge $\{t - \epsilon < \arg z < t + \epsilon\}$. For $z \in \triangle$, $z/n \in D_{\epsilon}(t)$ for large enough n. Therefore, if there is an infinite sequence $n_i \to \infty$ such that p_{n_i} does not vanish at any point of $D_{\epsilon}(t)$, the corresponding functions $p_{n_i}(z/n_i)$, for large i, are zero-free when z is in $\triangle$ (because of the factor $\frac{1}{n_i} \to 0$) and consequently $F_{\kappa}(\zeta)$ must be different from zero, contradiction. □

[9] see [1] p. 162

Appendix: Proof of (1) in Section 2.

Note that we have adopted the notational convention $\Gamma(n+\kappa) = (n+\kappa-1)!$ for $\kappa \in \mathbb{C}$, where Γ denotes the Gamma function. I start by calculating $R(1)$ and $R(2)$ and so the hypothesis (actually there are two hypotheses, one for even i and one for odd i) is true for one case of even i and one case of odd i. With the nth degree eigenpolynomial $p_n = \sum_{j=0}^{n} a_{n,j} z^j$ we have by Lemma 3 that

$$a_{n,j} = \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \quad \Rightarrow \quad p_n = \sum_{j=0}^{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j.$$

Calculation of $R(1) =$ [the negative of the remainder when the eigenpolynomial p_n is divided by $p_n^{'}$]:

$$\begin{array}{r|l}
 & \dfrac{z}{n} + \dfrac{(n-1+\kappa)}{n} \\ \hline
\displaystyle\sum_{j=1}^{n} j \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^{j-1} & \displaystyle\sum_{j=0}^{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j \\ \hline
 & \displaystyle -\Big[\sum_{j=1}^{n} \frac{j}{n} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} z^j\Big] \\ \hline
 & \displaystyle = \sum_{j=0}^{n-1} \binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \left[1 - \frac{j}{n}\right] z^j \\
 & \displaystyle -\Big[\sum_{j=0}^{n-1} \frac{j+1}{n} \binom{n}{j+1} (\kappa+n-1) \frac{(\kappa+n-1)!}{(\kappa+j)!} z^j\Big] \\ \hline
 & \displaystyle = \sum_{j=0}^{n-2} \left[\binom{n}{j} \frac{(\kappa+n-1)!}{(\kappa+j-1)!} \left(1 - \frac{j}{n}\right) - \frac{(j+1)}{n} \binom{n}{j+1} (\kappa+n-1) \frac{(\kappa+n-1)!}{(\kappa+j)!}\right] z^j
\end{array}$$

and it remains to prove that the negative of this remainder equals

$$R(1) = (n-1)(\kappa+n-1) \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(\kappa+n-2)!}{(\kappa+j)!} z^j.$$

Developing the coefficient in front of z^j in our remainder we obtain

$$\binom{n}{j}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}\left(1-\frac{j}{n}\right)-\frac{(j+1)}{n}\binom{n}{j+1}(\kappa+n-1)\frac{(\kappa+n-1)!}{(\kappa+j)!}$$

$$=\frac{n!}{(n-j)!j!}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{n!}{(n-j)!j!}\frac{j}{n}\frac{(\kappa+n-1)!}{(\kappa+j-1)!}-\frac{(j+1)}{n}\frac{n!(\kappa+n-1)}{(j+1)!(n-j-1)!}\frac{(\kappa+n-1)!}{(\kappa+j)!}$$

$$=\frac{n(n-1)(n-2)!}{(n-j-2)!(n-j-1)(n-j)j!}\frac{(\kappa+n-2)!(\kappa+n-1)(\kappa+j)}{(\kappa+j)!}$$

$$-\frac{(n-1)(n-2)!(\kappa+n-2)!(\kappa+n-1)j(\kappa+j)}{(n-j-2)!(n-j-1)(n-j)(\kappa+j)!j!}-\frac{(n-1)(n-2)!(\kappa+n-1)^2(\kappa+n-2)!}{j!(n-j-2)!(n-j-1)(\kappa+j)!}$$

$$=(n-1)(\kappa+n-1)\frac{(n-2)!}{j!(n-j-2)!}\frac{(\kappa+n-2)!}{(\kappa+j)!}\left[\frac{n(\kappa+j)}{(n-j-1)(n-j)}-\frac{j(\kappa+j)}{(n-j-1)(n-j)}\right.$$

$$\left.-\frac{(\kappa+n-1)(n-j)}{(n-j-1)(n-j)}\right]$$

$$=(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\left[\frac{n\kappa+nj-j\kappa-j^2-\kappa n+\kappa j-n^2+nj+n-j}{n^2-nj-nj+j^2-n+j}\right]$$

$$=-(n-1)(\kappa+n-1)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!},$$

and we are done.

Calculation of $R(2)=$ [the negative of the remainder when p_n' is divided by $R(1)$]:

$$\frac{nz}{(n-1)(\kappa+n-1)}+\frac{n(2n-3+\kappa)}{(n-1)(\kappa+n-1)}$$

$$\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}(n-1)(\kappa+n-1)z^j \;\Big|\; \sum_{j=0}^{n-1}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}z^j$$

$$-\left[\sum_{j=1}^{n-1}n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}z^j\right]$$

$$=\sum_{j=0}^{n-2}\left[(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\right]z^j$$

$$-\left[\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}n(2n-3+\kappa)z^j\right]$$

$$=\sum_{j=0}^{n-3}\left[\frac{(\kappa+n-2)!}{(\kappa+j-1)!}\left((j+1)\binom{n}{j+1}\frac{(\kappa+n-1)!}{(\kappa+j)!}-n\binom{n-2}{j-1}\right)-n(2n-3+\kappa)\binom{n-2}{j}\frac{(\kappa+n-2)!}{(\kappa+j)!}\right]z^j$$

and it remains to prove that the negative of this remainder equals

$$R(2)=n(n-2)(\kappa+n-2)\sum_{j=0}^{n-3}\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}z^j.$$

Developing the coefficient in front of z^j in our remainder we have

$$
\begin{aligned}
&\frac{(\kappa+n-2)!}{(\kappa+j-1)!}(j+1)\binom{n}{j+1}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}n\binom{n-2}{j-1} - \frac{(\kappa+n-2)!}{(\kappa+j)!}\binom{n-2}{j}n(2n-3+\kappa) \\
&= \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n!}{j!(n-j-1)!}\frac{(\kappa+n-1)}{(\kappa+j)} - \frac{(\kappa+n-2)!}{(\kappa+j-1)!}\frac{n(n-2)!}{(j-1)!(n-j-1)!} \\
&- \frac{(\kappa+n-2)!}{(\kappa+j)!}\frac{(n-2)!}{(j!(n-j-2)!}n(2n-3+\kappa) \\
&= \frac{(\kappa+n-3)!(\kappa+n-2)(n-3)!(n-2)(n-1)n(\kappa+n-1)}{(\kappa+j)!j!(n-j-3)!(n-j-2)(n-j-1)} \\
&- \frac{(\kappa+n-3)!(\kappa+n-2)n(n-2)(n-3)!j(\kappa+j)}{j!(n-j-3)!(n-j-2)(n-j-1)(\kappa+j)!} \\
&- \frac{(\kappa+n-3)!(\kappa+n-2)(n-2)(n-3)!n(2n-3+\kappa)}{(\kappa+j)!j!(n-j-2)(n-j-3)!} \\
&= \frac{(\kappa+n-3)!(n-3)!}{(\kappa+j)!j!(n-j-3)!}n(n-2)(\kappa+n-2)\left[\frac{(n-1)(\kappa+n-1)}{(n-j-2)(n-j-1)}\right. \\
&\left. - \frac{j(\kappa+j)}{(n-j-2)(n-j-1)} - \frac{(2n-3+\kappa)(n-j-1)}{(n-j-2)(n-j-1)}\right] \\
&= n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!}\left[\frac{-n^2+nj+n+nj-j^2-j+2n-2j-2}{n^2-nj-n-nj+j^2+j-2n+2j+2}\right] \\
&= -n(n-2)(\kappa+n-2)\binom{n-3}{j}\frac{(\kappa+n-3)!}{(\kappa+j)!},
\end{aligned}
$$

and we are done.

To prove the induction hypotheses we divide $R(i)$ by $R(i+1)$ to obtain $R(i+2)$. Here it is assumed that i is odd. The proof with even i differs only in small details from this proof and is therefore omitted here. For simplicity we use the notations

$$
\begin{cases}
A = (n-1)(\kappa+n-1)(n-3)(\kappa+n-3)\dots(n-i)(\kappa+n-i), \\
B = n(n-2)(\kappa+n-2)(n-4)(\kappa+n-4)\dots(n-i)(\kappa+n-i).
\end{cases}
$$

Dividing $R(i)$ by $R(i+1)$:

$$\frac{A}{B}z + \frac{A}{B}(2n-2i-3+\kappa)$$

$$B\sum_{j=0}^{n-i-2}\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j \quad \Big) \quad A\sum_{j=0}^{n-i-1}\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!}z^j$$

$$-\left[A\sum_{j=1}^{n-i-1}\binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}z^j\right]$$

$$= A\sum_{j=0}^{n-i-2}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}\right]z^j$$

$$-\left[A\sum_{j=0}^{n-i-2}(2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}z^j\right]$$

$$= A\sum_{j=0}^{n-i-3}\left[\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}\right]z^j$$

and it remains to prove that the negative of this remainder equals the excpected (by hypothesis)

$$R(i+2) = A(n-i-2)(\kappa+n-i-2)\sum_{j=0}^{n-i-3}\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}z^j,$$

i.e. we have to prove the following equality:

$$\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= -(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.$$

But

$$\binom{n-i-1}{j}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \binom{n-i-2}{j-1}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!} - (2n-2i-3+\kappa)\binom{n-i-2}{j}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= \frac{(n-i-1)!}{j!(n-i-j-1)!}\frac{(\kappa+n-i-1)!}{(\kappa+j)!} - \frac{(n-i-2)!}{(j-1)!(n-i-j-1)!}\frac{(\kappa+n-i-2)!}{(\kappa+j-1)!}$$
$$-(2n-2i-3+\kappa)\frac{(n-i-2)!}{j!(n-i-j-2)!}\frac{(\kappa+n-i-2)!}{(\kappa+j)!}$$
$$= \frac{(n-i-3)!(n-i-2)(n-i-1)(\kappa+n-i-3)!(\kappa+n-i-2)(\kappa+n-i-1)}{j!(n-i-j-1)(n-i-j-2)(n-i-j-3)!(\kappa+j)!}$$
$$- \frac{(n-i-3)!(n-i-2)j(\kappa+n-i-3)!(\kappa+n-i-2)(c+j)}{j!(n-i-j-3)!(n-i-j-2)(n-i-j-1)(\kappa+j)!}$$
$$-(2n-2i-3+\kappa)\frac{(n-i-3)!(n-i-2)(\kappa+n-i-3)!(\kappa+n-i-2)}{j!(n-i-j-3)!(n-i-j-2)(\kappa+j)!}$$
$$= (n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}$$
$$\cdot\left[\frac{(n-i-1)(\kappa+n-i-1)-j(\kappa+j)-(2n-2i-3+\kappa)(n-i-j-1)}{(n-i-j-1)(n-i-j-2)}\right]$$
$$= (n-i-2)(\kappa+n-i-2)\frac{(n-i-3)!}{j!(n-i-j-3)!}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}(-1)$$
$$= -(n-i-2)(\kappa+n-i-2)\binom{n-i-3}{j}\frac{(\kappa+n-i-3)!}{(\kappa+j)!}.$$

□

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