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$\mathrm{EM} + \mathrm{Ext}_{-} + \mathrm{AC}_{\mathrm{int}} \iff \mathrm{AC}_{\mathrm{ext}}$

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Abstract

It is well known that the extensional axiom of choice ($\mathrm{AC}_{\mathrm{ext}}$) implies the law of excluded middle (EM). In this note it is proved that the converse holds as well if we have the intensional ('type-theoretical') axiom of choice $\mathrm{AC}_{\mathrm{int}}$, which is provable in Martin-Löf's type theory, and a weak extensionality principle (Ext_{-}), which is provable in Martin-Löf's *extensional* type theory. In particular, $\mathrm{EM} \Leftrightarrow \mathrm{AC}_{\mathrm{ext}}$ holds in extensional type theory.

The following is the principle $\mathrm{AC}_{\mathrm{int}}$ of *intensional* choice: if A, B are sets and R a relation such that $(\forall y : B)(\exists x : A)R(x, y)$ is true, there is a function $g : B \to A$ such that $(\forall y : B)R(g(y), y)$ is true. It is provable in Martin-Löf's type theory [7, p. 50].

We may from this principle derive that surjective functions have right inverses: If $=_B$ is an equivalence relation on B and $f : A \to B$, we say that f is surjective if $(\forall y : B)(\exists x : A)(f(x) =_B y)$ is true. If we take $R(x, y) \stackrel{\text{def}}{=} (f(x) =_B y)$, we see that surjectivity is an instance of the premise needed to apply intensional choice. It gives us that there is a function $g : B \to A$ such that $(\forall y : B)(f(g(y)) =_B y)$ is true, that is, a right inverse of f.

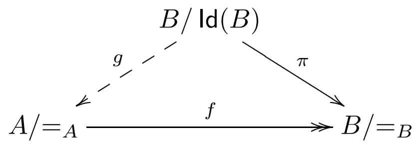
This, however, does not mean that g is *extensional*, i.e., that it preserves equivalence relations. If $=_A$ is an equivalence relation on A and $=_B$ is an equivalence relation on B, it might very well happen that f preserves them but g does not. The principle $\mathrm{AC}_{\mathrm{ext}}$ of *extensional* choice states precisely that the g obtained *does* preserve the equivalence relations. To be precise, it states that if R is an extensional relation (i.e., it respects the equivalence relations) and $(\forall y : B)(\exists x : A)R(x, y)$ is true, there is an *extensional* function $g : B \to A$ such that $(\forall y : B)R(g(y), y)$ is true.

One cannot justify $\mathrm{AC}_{\mathrm{ext}}$ from a constructive point of view, since it implies the principle of excluded middle.[1]

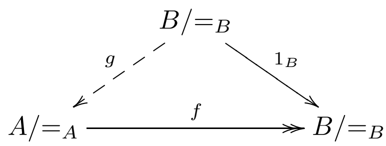
In theories with sufficiently strong axioms for quotient sets, like ZF and other theories with a suitable powerset axiom, extensional choice is obviously equivalent with intensional choice. Therefore, we are used to hear the name "axiom of choice", with little attention paid to the fact that *extensionality* of the choice is important.

[1] This was left as an exercise by Bishop [1, p. 58, pb. 2]. It was proved for toposes by Diaconescu [3], for intuitionistic set theory by Goodman–Myhill [4], and for type theory e.g. by Chicli–Pottier–Simpson [2] (there are also related results in [5, 6]). We include an alternative proof in the proof of the theorem of this note.

Proposition (well-known). $\mathrm{AC}_{\mathrm{int}}$ *is equivalent with the principle that every surjective function $f : A \to B$ has a right inverse g.*



$\mathrm{AC}_{\mathrm{ext}}$ *is equivalent with the principle that every surjective and extensional function $f : A/{=_A} \to B/{=_B}$ has an extensional right inverse g.*



Proof. We have already commented that $\mathrm{AC}_{\mathrm{int}}$ and $\mathrm{AC}_{\mathrm{ext}}$ imply the corresponding principles. It remains to prove the converse implications. Here is a sketch, the details are left to the reader. It suffices to consider $\mathrm{AC}_{\mathrm{ext}}$, since $\mathrm{AC}_{\mathrm{int}}$ can be seen as the special case when the equalities are $\mathsf{Id}(A)$ and $\mathsf{Id}(B)$.

Given an extensional relation R between sets A, B. Form the set $\{(x, y) \in A \times B \mid R(x, y)\}$,[2] with equality inherited from $A \times B$.

$$(a, b) =_\times (a', b') \iff a =_A a' \wedge b =_B b'$$

Let f be the right projection $(x, y) \mapsto y$, which is surjective and extensional. Hence there is an extensional right inverse g. Compose it with the left projection $(x, y) \mapsto x$, which is also extensional, and you have the function which is asserted to exist by $\mathrm{AC}_{\mathrm{ext}}$. □

Let us define also the other principles we will consider.

- EM is the principle of excluded middle, i.e., that if A is a proposition, $A \vee \neg A$ is true.
- Ext is the principle which, expressed in type-theoretical terms, says the following. Let A, B be sets and $f, g : A \to B$. We define extensional equality as

  $$(f \overset{\mathrm{ext}}{=} g) \overset{\mathrm{def}}{=} (\forall x : A)\, \mathsf{Id}(B, \mathsf{app}(f, x), \mathsf{app}(g, x))\,.$$

  Ext says that if $f \overset{\mathrm{ext}}{=} g$, $\mathsf{Id}(A \to B, f, g)$ is true. This principle is provable in extensional type theory [8, pp. 76–77]. That is generally considered as a drawback of this theory, because there is no constructive evidence for Ext. It is not derivable in Martin-Löf's intensional type theory, since if it was, we could decide if number-theoretic functions are extensionally equal [8, p. 76].[3]

[2] In type theory, the set should be $(\Sigma z : A \times B)R(\pi_\ell(z), \pi_r(z))$, where π_ℓ and π_r are the left and right projections, respectively.

[3] Thanks to Per Martin-Löf for reminding me of this argument.

- Ext_- was invented for the proof of the theorem below. In categorical terms, it expresses that if A, B are sets, then $(A \to B)/\overset{\text{ext}}{=}$ is projective in the category of sets with equivalence relations (setoids). In elementary terms, it says that for any sets A, B, there is an endofunction $\hat{\cdot}$ on $A \to B$ such that $f \overset{\text{ext}}{=} \hat{f}$ for every f and $f \overset{\text{ext}}{=} g \Rightarrow \mathsf{Id}(A \to B, \hat{f}, \hat{g})$. It is a weakening of Ext since if we have Ext we can take $\hat{f} \overset{\text{def}}{=} f$. It is also a weakening of AC_{ext}, since it says that the projection $(A \to B)/\mathsf{Id}(A \to B) \to (A \to B)/\overset{\text{ext}}{=}$ has an extensional right inverse. Also Ext_- is impossible to derive in type theory, by the same argument as for Ext.

Our proof will actually use Ext_- only in the case when B is Bool, so we could have weakened it further.

Theorem. $\mathrm{EM} + \mathrm{Ext}_- + \mathrm{AC}_{\text{int}} \Leftrightarrow \mathrm{AC}_{\text{ext}}$

Proof. ($\mathrm{AC}_{\text{ext}} \Rightarrow \mathrm{EM} + \mathrm{Ext}_- + \mathrm{AC}_{\text{int}}$) We have already remarked that Ext_- is a weakening of AC_{ext}, and of course, so is AC_{int}. So it remains to prove $\mathrm{AC}_{\text{ext}} \Rightarrow \mathrm{EM}$. The proofs in [3, 4, 2] can all be used but we include one which is more natural in the present setting.

Fix a proposition P. We shall prove that it is decidable, using AC_{ext}.

Let, for $a, b : \mathsf{Bool}$,

$$R(a, b) \overset{\text{def}}{=} \mathsf{Id}(\mathsf{Bool}, a, b) \vee P\,.$$

R is then an equivalence relation and it is, trivially, extensional with respect to the equality $\mathsf{Id}(\mathsf{Bool})$ in the first argument and with respect to itself in the second argument. Further, $(\forall y : \mathsf{Bool})(\exists x : \mathsf{Bool})R(x, y)$ is true, since we can take y for x. Hence we may apply AC_{ext}.

We get an extensional function $g : \mathsf{Bool}\,/R \to \mathsf{Bool}\,/\,\mathsf{Id}(\mathsf{Bool})$ with $R(g(b), b)$ true for every $b : \mathsf{Bool}$. In particular, if $\mathsf{Id}(\mathsf{Bool}, g(a), g(b))$ is true, so is $R(a, b)$. On the other hand, since g preserves the equalities, $R(a, b) \Rightarrow \mathsf{Id}(\mathsf{Bool}, g(a), g(b))$. So $R(a, b) \Leftrightarrow \mathsf{Id}(\mathsf{Bool}, g(a), g(b))$, hence R is decidable.

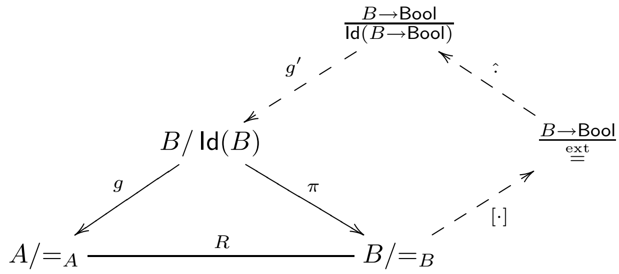
Now observe that $R(0, 1) \Leftrightarrow P$ (using a universe reflecting $\bot$ and $\top$), so also P is decidable.

($\mathrm{EM}+\mathrm{Ext}_-+\mathrm{AC}_{\text{int}} \Rightarrow \mathrm{AC}_{\text{ext}}$) The idea of the proof is very simple: if we have excluded middle, we can prove that $\mathsf{prop}\,/\!\Leftrightarrow$ is isomorphic to $\mathsf{Bool}\,/\mathsf{Id}(\mathsf{Bool})$. Hence subsets correspond to boolean characteristic functions. Now, because we have a principle saying that functions f, g, which are pointwise identical, correspond, via the operation $\hat{\cdot}$, to identical functions, we can conclude that extensionally equal subsets correspond to *identical* boolean functions. Hence we can pick representatives out of equivalence classes in an extensional way. This is sufficient to prove AC_{ext} in a few steps.

Let us turn this idea into a rigorous proof.

Suppose A, B are sets and $=_A$ and $=_B$ equivalence relations on them. Suppose R is extensional and that $(\forall y : B)(\exists x : A)R(x, y)$ is true. We shall construct an extensional function $g : B \to A$ which satisfies $(\forall y : B)R(g(y), y)$. We suppose in the following that B is inhabited, since the case when B is empty is trivial and because we have EM we may decide which is the case.

First apply intensional choice, so that we get a function $g : B \to A$ with $(\forall y : B)R(g(y), y)$ true. This g need not be extensional, but we will construct a new one which is. The idea is to compose g with another function which picks unique representatives from equivalence classes. This function will be built in three parts, called $[\cdot]$, $\hat{\cdot}$ and g'.



Define a valuation $v : \mathsf{prop} \to \mathsf{Bool}$ which takes true propositions to 1 and false propositions to 0. That can be done in Martin-Löf's type theory using $\mathrm{em}(P) : P \vee \neg P$ $(P : \mathsf{prop})$, which exists by EM:

$$v(P) \stackrel{\text{def}}{=} \mathsf{when}(\mathrm{em}(P), (x)1, (x)0) : \mathsf{Bool}\ .$$

The inverse $v^{-1} : \mathsf{Bool} \to \mathsf{prop}$ is defined by

$$v^{-1}(p) \stackrel{\text{def}}{=} \mathsf{Id}(\mathsf{Bool}, p, 1)\,.$$

These functions are extensional in the sense that they preserve the equalities $\Leftrightarrow, \mathsf{Id}(\mathsf{Bool})$ and they are inverses in the sense that $\mathsf{Id}(\mathsf{Bool}, v(v^{-1}(p)), p)$ is true for every $p : \mathsf{Bool}$ and $v^{-1}(v(P)) \Leftrightarrow P$ is true for every $P : \mathsf{prop}$.

Let us denote by $[b]$ the characteristic boolean function

$$\lambda x.v(x =_B b) : B \to \mathsf{Bool}\ .$$

We will construct a left inverse to the function $[\cdot] : B \to (B \to \mathsf{Bool})$ using intensional choice. Let, for $b : B$ and $f : B \to \mathsf{Bool}$,

$$R'(b, f) \ \stackrel{\text{def}}{=}\ (\exists x : B)v^{-1}(\mathsf{app}(f, x)) \to v^{-1}(\mathsf{app}(f, b))$$

and note the following fact, which will be very useful:

$$\begin{aligned} R'(b, [b']) &\stackrel{\text{def}}{=} (\exists x : B)v^{-1}(\mathsf{app}([b'], b)) \to v^{-1}(\mathsf{app}([b'], b)) \\ &\stackrel{\text{def}}{=} (\exists x : B)v^{-1}(v(x =_B b')) \to v^{-1}(v(b =_B b')) \\ &\Leftrightarrow (\exists x : B)(x =_B b') \to (b =_B b') \\ &\Leftrightarrow (b =_B b'). \end{aligned}$$

The proposition $(\forall y : B \to \mathsf{Bool})(\exists x : B)R'(x, y)$ is easily proved using EM and that B is inhabited: For every $f : B \to \mathsf{Bool}$, make a case analysis on $(\exists x : B)v^{-1}(\mathsf{app}(f, x))$. If it is true, say $v^{-1}(\mathsf{app}(f, b))$ is true, then $R'(f, b)$ is true and hence $(\exists x : B)R'(x, f)$ is true. If $(\exists x : B)v^{-1}(\mathsf{app}(f, x))$ is false, $R'(b, f)$ is vacuously true for any $b : B$, and hence, since B is inhabited, $(\exists x : B)R'(x, f)$ is true in this case too.

Hence intensional choice gives us a function $g' : (B \to \mathsf{Bool}) \to B$ such that $(\forall y : B \to \mathsf{Bool})R'(g'(y), y)$ is true.

It is clear that all functions in the diagram above are indeed extensional in the sense that they preserve the equalities indicated. For $\hat{\cdot}$ and g this is true by construction. For g' and π it follows from the fact that all functions preserve Id-equalities. For $[\cdot]$ it is true by the definition of $\overset{\text{ext}}{=}$.

It remains to prove $(\forall y : B)R(g(g'(\widehat{[y]})), y)$. So take an arbitrary $b : B$ and prove $R(g(g'(\widehat{[b]})), b)$. By construction of g, we have $R(g(g'(\widehat{[b]})), g'(\widehat{[b]}))$ true and so, since R is extensional in the second argument, it suffices to prove $g'(\widehat{[b]}) =_B b$. Our 'useful fact' gives us that this is equivalent to $R'(g'(\widehat{[b]}), [b])$, which in turn is equivalent to $R'(g'(\widehat{[b]}), \widehat{[b]})$ (just plug this into the definition of R' and use $[b] \overset{\text{ext}}{=} \widehat{[b]}$). But this is true since $(\forall y : B \to \mathsf{Bool})R'(g'(y), y)$ is true by construction of g'. □

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