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SOME RELATIONS FOR ONE-PART DOUBLE HURWITZ NUMBERS

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1. In this note we state some new relations for double Hurwitz numbers. These relations come from the intersection theory of the moduli spaces of curves and generalize the results announced in [3] and proved in [4].

The Hurwitz numbers studied here have recently been considered in details in [2]. In particular, we refer to [2] as to a good survey in this area with the most complete collection of references.

2. Recall the definition of Hurwitz numbers. Fix integers $g \geq 0$, $n \geq 1$, and two unordered partitions of the number n, $n = a_1 + \cdots + a_p = b_1 + \cdots + b_q$. Consider $\mathbb{C}\mathrm{P}^1$ with marked 0, ∞, and $2g+p+q-2$ points more, $z_1, \dots, z_{2g+p+q-2}$.

There is a finite number of functions $f\colon C \to \mathbb{C}\mathrm{P}^1$ of degree n defined on curves of genus g such that $(f) = -\sum_{i=1}^{p} a_i x_i + \sum_{j=1}^{q} b_j y_j$ (here x_i, y_j are pairwise distinct points of C), and $z_1, \dots, z_{2g+p+q-2}$ are simple critical values of f.

The number of such functions is called *Hurwitz number* and denoted by $H_g(a_1, \dots, a_p | b_1, \dots, b_q)$. Here we count each function $f\colon C \to \mathbb{C}\mathrm{P}^1$ weighted by $1/|\mathrm{aut}(f)|$. For instance, $H_g(2|2) = 1/2$ for any g.

We will consider only the numbers $H_g(n|b_1, \dots, b_q)$. For convenience we introduce the following notation:

$$\widehat{H}_{g,n}(b_1, \dots, b_q) := \frac{|\mathrm{aut}(b_1, \dots, b_q)|}{n^{2g+q-2}(2g+q-1)!} H_g(n|b_1, \dots, b_q). \tag{1}$$

3. Let us state the main theorem:

Theorem 1. *For any $g \geq 0$, $n \geq 1$, $b_1 + \cdots + b_q = n$, we have:*

$$\binom{g}{0}\widehat{H}_{g,n}(b_1, \dots, b_q) - \binom{g}{1}\widehat{H}_{g,n+1}(b_1, \dots, b_q, 1) + \\ + \cdots + (-1)^g \binom{g}{g}\widehat{H}_{g,n+g}(b_1, \dots, b_q, \underbrace{1, \dots, 1}_{g}) = \frac{(-1)^g}{24^g}. \tag{2}$$

Let us check this theorem in a special case. Let $g = 2$. Then there is a fomula from [2]:

$$\widehat{H}_{g,n}(b_1, \dots, b_q) = \frac{1}{24^2}\left(\frac{1}{2}A^2 - \frac{1}{5}B\right), \tag{3}$$

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where $A=-1+b_1^2+\dots+b_q^2$, $B=-1+b_1^4+\dots+b_q^4$.

Therefore, in this special case our theorem is equivalent to the following equation:

$$(4)\quad \left(\frac{1}{2}A^2-\frac{1}{5}B\right)-2\left(\frac{1}{2}(A+1)^2-\frac{1}{5}(B+1)\right)+ \\ +\left(\frac{1}{2}(A+2)^2-\frac{1}{5}(B+2)\right)=1.$$

One can check this by direct calculation.

4. Let us explain, how one can obtain such relation for Hurwitz numbers.

Consider the moduli space $\overline{\mathcal{M}}_{g,q+1}$ of curves of genus g with $q+1$ marked points. Let L_1 be the line bundle over $\overline{\mathcal{M}}_{g,q+1}$ with the fiber $T^*_{x_1}C$ at a point $(C,x_1,\dots,x_{q+1})\in\overline{\mathcal{M}}_{g,q+1}$. By ψ_1 denote $c_1(L_1)$.

Consider the subvariety $V_g^\circ(b_1,\dots,b_q)\subset\mathcal{M}_{g,q+1}$ consisting of smooth curves $(C,x_1,\dots,x_{q+1})$ such that $-(\sum_{i=1}^q b_i)x_1+b_1x_2+\dots+b_qx_{q+1}$ is a divisor of a meromorphic function. By $V_g(b_1,\dots,b_q)$ denote the closure of $V_g^\circ(b_1,\dots,b_q)$ in $\overline{\mathcal{M}}_{g,q+1}$.

Theorem 1 is the direct corollary of two relations for intersection numbers on the moduli spaces of curves looking like follows:

Lemma 1. *For any* $b_1,\dots,b_q$,

$$(5)\quad \frac{(-1)^g}{24^g}=(-1)^g g!\int_{\overline{\mathcal{M}}_{g,3}}\psi_1^{3g}=\binom{g}{0}\int_{V_g(b_1,\dots,b_q)}\psi_1^{2g+q-2}- \\ \binom{g}{1}\int_{V_g(b_1,\dots,b_q,1)}\psi_1^{2g+q-1}+\dots+(-1)^g\binom{g}{g}\int_{V_g(b_1,\dots,b_q,1,\dots,1)}\psi_1^{3g+q-2}.$$

Lemma 2. *For any* $b_1,\dots,b_q$,

$$(6)\qquad \widehat{H}_{g,n}(b_1,\dots,b_q)=\int_{V_g(b_1,\dots,b_q)}\psi_1^{2g+q-2}$$

Lemma 1 is a generalization of the similar statement in [3, 4]. Lemma 2 is just one of the theorems proved in [4].

5. There is a 'cut-and-join' type equation for the generating function of Hurwitz numbers considered here. (for the similar equations for some other Hurwitz numbers, see [1, 5]).

Consider the function $F(\theta,x_1,x_2,\dots)$ defined like follows:

$$(7)\quad F(\theta,x_1,x_2,\dots)= \\ =\sum_{g\ge0}\sum_{(b_1,\dots,b_q)}H_g(n|b_1,\dots,b_q)\frac{\theta^{(2g+q-1)}}{(2g+q-1)!}x_{b_1}\dots x_{b_q}.$$

Theorem 2. *The function $F = F(\theta, x_1, x_2, \dots)$ satisfies the following:*

$$\frac{\partial F}{\partial \theta} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijx_{i+j}\frac{\partial^2 F}{\partial x_i \partial x_j} + (i+j)x_i x_j \frac{\partial F}{\partial x_{i+j}}\right). \tag{8}$$

Generally speaking, this theorem is more or less obvious from the point of view of geometry of ramified coverings. Surely, this theorem has the similar purely combinatorial proof as its analog in [1]. But this theorem is also a direct corollary of relations for intersection numbers obtained in [4].

Really, one can rewrite this theorem as a relation for Hurwitz numbers like follows:

$$\begin{aligned}(9)\quad & H'_g(n|b_1,\dots,b_q) = \\ & \frac{1}{2}\sum_{i=1}^{q}\sum_{b_i^{(1)}+b_i^{(2)}=b_i} b_i^{(1)}b_i^{(2)}H'_{g-1}(n|b_1,\dots,\widehat{b_i},\dots,b_q,b_i^{(1)},b_i^{(2)})+ \\ & \qquad \frac{1}{2}\sum_{i\neq j}(b_i+b_j)H'_g(n|b_1,\dots,\widehat{b_i},\dots,\widehat{b_j},\dots,b_q,b_i+b_j),\end{aligned}$$

where by $H'_g(n|b_1,\dots,b_q)$ we denote

$H_g(n|b_1,\dots,b_q)\cdot|\mathrm{aut}(b_1,\dots,b_q)| = \widehat{H}_{g,n}(b_1,\dots,b_q)\cdot n^{2g+q-2}(2g+q-1)!$

Using Lemma 2, we can rewrite Eqn. (9) as a relation for the integrals $\int_{V_g(b_1,\dots,b_q)}\psi_1^{2g+q-2}$. Thus we obtain exactly a special case of Theorem 12.2 from [4].

References

[1] I. P. Goulden, D. M. Jackson, A. Vainshtein, Ann. Comb. 4 (2000),27-46
[2] I. P. Goulden, D. M. Jackson, R. Vakil, arXiv: math.AG/0309440
[3] S. V. Shadrin, Russ. Math. Surv, 58:1, 195-196 (2003)
[4] S. V. Shadrin, Int. Math. Res. Not. 2003, no. 38, 2051-2094
[5] D. Zvonkine, arXiv: math.AG/0304251

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