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On the Schrödinger equation involving a critical Sobolev exponent and magnetic field

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Abstract

We consider the semilinear Schrödinger equation $-\Delta_A u + V(x)u = Q(x)|u|^{2^*-2}u$. Assuming that V changes sign, we establish the existence of a solution $u \neq 0$ in the Sobolev space $H^1_{A,V^+}(\mathbb{R}^N)$. The solution is obtained by a min - max type argument based on a topological linking. We also establish certain regularity properties of solutions for a rather general class of equations involving the operator $-\Delta_A$.

1 Introduction

In this paper we consider the semilinear Schrödinger equation

$$-\Delta_A u + V(x)u = Q(x)|u|^{2^*-2}u, \qquad u \in H^1_{A,V}(\mathbb{R}^N), \tag{1.1}$$

where $-\Delta_A = \big(-i\nabla + A\big)^2$, $u : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, $2^* := 2N/(N-2)$ is the critical Sobolev exponent. The coefficient V is the scalar (or electric) potential and $A = (A_1, \dots , A_N) : \mathbb{R}^N \to \mathbb{R}^N$ the vector (or magnetic) potential. Throughout this paper we assume that $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Here V^- is the negative part of V, that is $V^-(x) = \max\big(-V(x), 0\big)$. It is assumed that the coefficient Q is positive, continuous and bounded on $\mathbb{R}^N$. Further assumptions on Q will be formulated later.

We now define some Sobolev spaces. By $D^{1,2}_A(\mathbb{R}^N)$ we denote the Sobolev space defined by

$$D^{1,2}_A(\mathbb{R}^N) = \{u;\ u \in L^{2^*}(\mathbb{R}^N),\ \ \nabla_A u \in L^2(\mathbb{R}^N)\},$$

where $\nabla_A = \big(\nabla + iA\big)$. The space $D^{1,2}_A(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\int_{\mathbb{R}^N} \nabla_A u \overline{\nabla_A v}\, dx.$$

It is known that the space $C_0^\infty(\mathbb{R}^N)$ is dense in $D^{1,2}_A(\mathbb{R}^N)$ [8]. Equivalently $D^{1,2}_A(\mathbb{R}^N)$ can be defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}_A} = \int_{\mathbb{R}^N} |\nabla_A u|^2\, dx.$$

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By $H^1_{A,V^+}(\mathbb{R}^N)$ we denote the Sobolev space obtained as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{H^1_{A,V^+}} = \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V^+(x)|u|^2\right) dx,$$

where $V^+(x) = \max(V(x), 0)$. $H^1_{A,V^+}(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\int_{\mathbb{R}^N} \left(\nabla_A u \overline{\nabla_A v} + V^+(x) u\bar{v}\right) dx.$$

Obviously, we have a continuous embedding $H^1_{A,V^+}(\mathbb{R}^N) \subset D^{1,2}_A(\mathbb{R}^N)$.

We shall frequently use in this paper the diamagnetic inequality (see [11])

$$|\nabla |u|| \leq |\nabla_A u| \quad \text{a.e. in } \mathbb{R}^N. \tag{1.2}$$

This inequality implies that if $u \in H^1_{A,V^+}(\mathbb{R}^N)$, then $|u| \in D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is the usual Sobolev space of real valued functions defined by

$$D^{1,2}(\mathbb{R}^N) = \{u;\ \nabla u \in L^2(\mathbb{R}^N) \text{ and } u \in L^{2^*}(\mathbb{R}^N)\}.$$

Therefore, as a consequence of the Sobolev inequality, we see that $|u| \in L^{2^*}(\mathbb{R}^N)$.

Solutions of (1.1) will be sought in the Sobolev space $H^1_{A,V^+}(\mathbb{R}^N)$ as critical points of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V(x)|u|^2\right) dx - \frac{1}{2^*}\int_{\mathbb{R}^N} Q(x)|u|^{2^*} dx.$$

It is easy to see that J is a C^1 - functional on $H^1_{A,V^+}(\mathbb{R}^N)$.

The paper is organized as follows. Section 2 is devoted to the regularity properties of solutions of (1.1). We show that solutions in $H^1_{A,V^+}(\mathbb{R}^N)$ are bounded and decay to 0 at infinity. In Section 3 we establish the Palais - Smale condition for the variational functional J. The existence results for (1.1) are given in Section 4. First we solve a weighted linear eigenvalue problem for the operator $-\Delta_A + V^+$. If the first eigenvalue $\mu_1 > 1$, then a solution is obtained through a constrained minimization. This situation has already been envisaged in the paper [1]. If $\mu_1 \leq 1$ we employ a topological linking argument.

Problem (1.1) with $A = 0$ has an extensive literature. However, the interest in the case $A \neq 0$ has arisen recently ([1], [7], [8], [10], [14]). The importance of problem (1.1) in physics has been discussed in the paper [1].

In this paper we use standard notations. In a given Banach space X weak convergence is denoted by " $\rightharpoonup$ " and strong convergence by " $\to$ ".

2 The regularity of solutions involving the operator Δ_A

Let V be a nonnegative function in $L^1_{\rm loc}(\mathbb{R}^N)$. We commence by establishing the integrability properties of solutions of the equation

$$-\Delta_A u + Vu = g(x)u \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

It is assumed that $g : \mathbb{R}^N \to \mathbb{R}$ is a measurable function satisfying

$$|g(x)| \le a + b(x) \quad \text{on } \mathbb{R}^N,$$

where $a \ge 0$ is a constant and b is a nonnegative function in $L^{\frac{N}{2}}(\mathbb{R}^N)$.

Let $\phi(x) = \eta(x)^2 u(x) \min\big(|u(x)|^{\beta-1}, L\big)$, where $\beta > 1$ and $L > 0$ are constants, $u \in H^1_{A,V}(\mathbb{R}^N)$ and η is a C^1- real valued function which is bounded together with its derivatives.

In what follows, χ_Ω denotes the characteristic function of the set Ω. By straightforward computations we have

$$\begin{aligned}\overline{\nabla_A \phi} &= 2\eta\nabla\eta\bar{u}\min\big(|u|^{\beta-1}, L\big) + \eta^2\overline{\nabla_A u}\min\big(|u|^{\beta-1}, L\big)\\ &+ (\beta-1)\eta^2\bar{u}|u|^{\beta-2}\nabla|u|\chi_{\{|u|^{\beta-1}<L\}}\end{aligned}$$

and

$$\begin{aligned}\nabla_A u\overline{\nabla_A \phi} &= |\nabla_A u|^2\eta^2\min\big(|u|^{\beta-1}, L\big) + 2\eta\nabla\eta\bar{u}\min\big(|u|^{\beta-1}, L\big)\nabla_A u\\ &+ (\beta-1)\eta^2\bar{u}|u|^{\beta-2}\nabla|u|\chi_{\{|u|^{\beta-1}<L\}}\nabla_A u.\end{aligned}$$

We now observe that

$$\mathrm{Re}\,(\bar{u}\nabla_A u) = \mathrm{Re}\,\big(\nabla u + iAu\big)\bar{u} = \mathrm{Re}\,(\bar{u}\nabla u) = |u|\mathrm{Re}\,\big(\frac{\bar{u}}{|u|}\nabla u\big) = |u|\nabla|u|.$$

Taking the real part of $\nabla_A u\overline{\nabla_A \phi}$ we obtain the following inequality:

$$\begin{aligned}\mathrm{Re}\,(\nabla_A u\overline{\nabla_A \phi}) &= |\nabla_A u|^2\eta^2\min\big(|u|^{\beta-1}, L\big) + 2\eta\nabla\eta\nabla|u||u|\min\big(|u|^{\beta-1}, L\big)\\ &+ (\beta-1)\eta^2|u|^{\beta-1}|\nabla|u||^2\chi_{\{|u|^{\beta-1}<L\}}\\ &\ge |\nabla_A u|^2\eta^2\min\big(|u|^{\beta-1}, L\big) + 2\eta\nabla\eta\nabla|u||u|\min\big(|u|^{\beta-1}, L\big).\end{aligned} \tag{2.2}$$

Lemma 2.1 *Solutions of equation (2.1) in $H^1_{A,V}(\mathbb{R}^N)$ belong to $L^p(\mathbb{R}^N)$ for every $p \in [2^*, +\infty)$.*

Proof We adapt to our case an argument which may be found e.g. in [16, Appendix B]. We test equation (2.1) with $\phi = u\min\big(|u|^{\beta-1}, L\big)$. It then follows from inequality (2.2), with $\eta = 1$, that for every constant $K > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_A u|^2 \min\big(|u|^{\beta-1}, L\big)\,dx &\le a\int_{\mathbb{R}^N} |u|^2\min\big(|u|^{\beta-1}, L\big)\,dx\\ &+ K\int_{b(x)\le K} |u|^2\min\big(|u|^{\beta-1}, L\big)\,dx\\ &+ \left(\int_{b(x)>K} b(x)^{\frac{N}{2}}\,dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left(|u|\min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\right)^{2^*}dx\right)^{\frac{N-2}{N}}\\ &\le (a+K)\int_{\mathbb{R}^N} |u|^2\min\big(|u|^{\beta-1}, L\big)\,dx\\ &+ \left(\int_{b(x)>K} b(x)^{\frac{N}{2}}\,dx\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left(|u|\min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\right)^{2^*}dx\right)^{\frac{N-2}{N}}.\end{aligned} \tag{2.3}$$

On the other hand, by the diamagnetic inequality we have

$$\int_{\mathbb{R}^N} |\nabla|u||^2 \min\big(|u|^{\beta-1}, L\big)\, dx \le \int_{\mathbb{R}^N} |\nabla_A u|^2 \min\big(|u|^{\beta-1}, L\big)\, dx. \tag{2.4}$$

We also have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\big(|u| \min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\big)|^2\, dx &\le 2\int_{\mathbb{R}^N} |\nabla|u||^2 \min\big(|u|^{\beta-1}, L\big)\, dx \\ &+ \frac{(\beta-1)^2}{2}\int_{\mathbb{R}^N} |\nabla|u||^2 |u|^{\beta-1}\chi_{\{|u|^{\beta-1}<L\}}\, dx \\ &\le \big(2+\frac{(\beta-1)^2}{2}\big)\int_{\mathbb{R}^N} |\nabla|u||^2 \min\big(|u|^{\beta-1}, L\big)\, dx.\end{aligned} \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\big(|u| \min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\big)|^2\, dx &\le (a+K)\big(2+\frac{(\beta-1)^2}{2}\big)\int_{\mathbb{R}^N} |u|^2 \min\big(|u|^{\beta-1}, L\big)\, dx \\ + \big(2+\frac{(\beta-1)^2}{2}\big)\Big(\int_{b(x)>K} b(x)^{\frac{N}{2}}\, dx\Big)^{\frac{2}{N}} &\Big(\int_{\mathbb{R}^N}\Big(|u| \min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\Big)^{2^*} dx\Big)^{\frac{N-2}{N}}.\end{aligned}$$

Since $\int_{b(x)>K} b(x)^{\frac{N}{2}}\, dx \to 0$ as $K \to \infty$, taking K sufficiently large and applying the Sobolev inequality to the left-hand side above, we obtain

$$\Big(\int_{\mathbb{R}^N}\Big(|u| \min\big(|u|^{\frac{\beta-1}{2}}, L^{\frac{1}{2}}\big)\Big)^{2^*} dx\Big)^{\frac{2}{2^*}} \le C_1(K,\beta)\int_{\mathbb{R}^N} |u|^2 \min\big(|u|^{\beta-1}, L\big)\, dx \tag{2.6}$$

for some constant $C_1(K,\beta) > 0$. We now set $\beta + 1 = 2^*$. Letting $L \to \infty$ we derive from the above inequality that

$$\Big(\int_{\mathbb{R}^N} |u|^{\frac{2^*N}{N-2}}\, dx\Big)^{\frac{2}{2^*}} \le C_1(K, 2^*)\int_{\mathbb{R}^N} |u|^{2^*}\, dx$$

and thus $u \in L^{\frac{2^*N}{N-2}}(\mathbb{R}^N)$. A standard application of a boot - strap argument to (2.6) completes the proof. □

Proposition 2.2 *If $u \in H^1_{A,V}(\mathbb{R}^N)$ is a solution of (2.1), then $u \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} u(x) = 0$.*

Proof We follow some ideas from the proof of Theorem 8.17 in [9] (in particular, we use Moser's iteration technique). Let η be a C^1-function in $\mathbb{R}^N$ with a compact support. Testing (2.1) with $\phi = \eta^2 u \min\big(|u|^{\beta-1}, L\big)$ and using inequality (2.2) we obtain the estimate

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla_A u|^2\eta^2 \min\big(|u|^{\beta-1}, L\big)\, dx + 2\int_{\mathbb{R}^N} \eta\nabla\eta\nabla|u||u| \min\big(|u|^{\beta-1}, L\big)\, dx \\ &\le \int_{\mathbb{R}^N} b|u|^2\eta^2 \min\big(|u|^{\beta-1}, L\big)\, dx + a\int_{\mathbb{R}^N} |u|^2\eta^2 \min\big(|u|^{\beta-1}, L\big)\, dx.\end{aligned}$$

Hence by the diamagnetic inequality and since

$$\frac{1}{2}\eta^2|\nabla|u||^2 - 2|u|^2|\nabla\eta|^2 \le \eta^2|\nabla|u||^2 + 2\eta|u|\nabla|u|\nabla\eta,$$

we get

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla|u||^2\eta^2\min\bigl(|u|^{\beta-1},L\bigr)\,dx \;\le\; \int_{\mathbb{R}^N} b|u|^2\eta^2\min\bigl(|u|^{\beta-1},L\bigr)\,dx$$
$$+2\int_{\mathbb{R}^N}|\nabla\eta|^2|u|^2\min\bigl(|u|^{\beta-1},L\bigr)\,dx + a\int_{\mathbb{R}^N}|u|^2\eta^2\min\bigl(|u|^{\beta-1},L\bigr)\,dx.$$

Letting $L\to\infty$ we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}|\nabla|u||^2\eta^2|u|^{\beta-1}\,dx &\le \int_{\mathbb{R}^N} b|u|^{\beta+1}\eta^2\,dx + 2\int_{\mathbb{R}^N}|\nabla\eta|^2|u|^{\beta+1}\,dx\\ &+ a\int_{\mathbb{R}^N}|u|^{\beta+1}\eta^2\,dx.\end{aligned}$$

Substituting $w=|u|^{\frac{\beta+1}{2}}$ in this inequality, we obtain

$$\begin{aligned}\frac{2}{(\beta+1)^2}\int_{\mathbb{R}^N}|\nabla w|^2\eta^2\,dx &\le \int_{\mathbb{R}^N} bw^2\eta^2\,dx + 2\int_{\mathbb{R}^N}|\nabla\eta|^2w^2\,dx\\ &+ a\int_{\mathbb{R}^N}w^2\eta^2\,dx.\end{aligned}\tag{2.7}$$

We now observe that

$$\int_{\mathbb{R}^N}|\nabla(w\eta)|^2\,dx \le 2\int_{\mathbb{R}^N}|\nabla w|^2\eta^2\,dx + 2\int_{\mathbb{R}^N}|\nabla\eta|^2w^2\,dx,$$

which combined with (2.7) gives

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla(w\eta)|^2\,dx &\le (\beta+1)^2\int_{\mathbb{R}^N} bw^2\eta^2\,dx + 2\bigl((\beta+1)^2+1\bigr)\int_{\mathbb{R}^N}|\nabla\eta|^2w^2\,dx\\ &+ (\beta+1)^2a\int_{\mathbb{R}^N}\eta^2w^2\,dx.\end{aligned}$$

It then follows from the Hölder and Sobolev inequalities that

$$\begin{aligned}S\Bigl(\int_{\mathbb{R}^N}(w\eta)^{2^*}\,dx\Bigr)^{\frac{N-2}{N}} &\le (\beta+1)^2\Bigl(\int_{\mathbb{R}^N} b^{\frac{N}{2}}\,dx\Bigr)^{\frac{2}{N}}\Bigl(\int_{\mathbb{R}^N}(w\eta)^{2^*}\,dx\Bigr)^{\frac{N-2}{N}}\\ &+ 2\bigl((\beta+1)^2+1\bigr)\int_{\mathbb{R}^N}|\nabla\eta|^2w^2\,dx + a(\beta+1)^2\int_{\mathbb{R}^N}\eta^2w^2\,dx,\end{aligned}\tag{2.8}$$

where $S=\inf\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx;\ u\in C_\circ^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{2^*}\,dx=1\}$ is the Sobolev constant. To proceed further we choose $R>0$ so that

$$(\beta+1)^2\Bigl(\int_{|x|>R} b^{\frac{N}{2}}\,dx\Bigr)^{\frac{2}{N}} \le \frac{S}{2}.$$

Assuming that $\operatorname{supp}\eta\subset(|x|>R)$ we derive from (2.8) that

$$\begin{aligned}S\Bigl(\int_{\mathbb{R}^N}(w\eta)^{2^*}\,dx\Bigr)^{\frac{N-2}{N}} &\le 4\bigl((\beta+1)^2+1\bigr)\int_{\mathbb{R}^N}|\nabla\eta|^2w^2\,dx\\ &+ 2a(\beta+1)^2\int_{\mathbb{R}^N}\eta^2w^2\,dx.\end{aligned}\tag{2.9}$$

We now make a more specific choice of η: $\eta \in C^1(\mathbb{R}^N,[0,1])$, $\eta(x)=1$ in $B(x_0,r_1)$, $\eta(x)=0$ in $\mathbb{R}^N - B(x_0,r_2)$, $|\nabla\eta(x)| \le \frac{2}{r_2-r_1}$ in $\mathbb{R}^N$, $1\le r_1<r_2\le 2$. It is also assumed that $B(x_0,r_2)\subset(|x|>R)$. It then follows from (2.9) that

$$\Big(\int_{B(x_0,r_1)} w^{2^*}\,dx\Big)^{\frac{1}{2^*}} \le \frac{A(\beta+1)}{r_2-r_1}\Big(\int_{B(x_0,r_2)} w^2\,dx\Big)^{\frac12},$$

where A is an absolute constant. Setting $\gamma=\beta+1=2^*$, $\chi=\frac{N}{N-2}$ we get

$$\Big(\int_{B(x_0,r_1)} |u|^{\gamma\chi}\,dx\Big)^{\frac{1}{\gamma\chi}} \le \Big(\frac{A\gamma}{r_2-r_1}\Big)^{\frac{2}{\gamma}}\Big(\int_{B(x_0,r_2)} |u|^{\gamma}\,dx\Big)^{\frac{1}{\gamma}}.$$

To iterate this inequality (which holds for any $\gamma\ge 2^*$), we take $s_m=1+2^{-m}$, $r_1=s_m$, $r_2=s_{m-1}$ and replace $\gamma=2^*$ by $\gamma\chi^{m-1}$, $m=1,2,\ldots$. Then we get

$$\begin{aligned}\Big(\int_{B(x_0,s_m)} |u|^{\chi^m\gamma}\,dx\Big)^{\frac{1}{\gamma\chi^m}} &\le \Big(\frac{A\gamma\chi^{m-1}}{s_{m-1}-s_m}\Big)^{\frac{2}{\gamma\chi^{m-1}}}\Big(\int_{B(x_0,s_{m-1})} |u|^{\chi^{m-1}\gamma}\,dx\Big)^{\frac{1}{\gamma\chi^{m-1}}}\\ &= (A\gamma)^{\frac{2}{\gamma\chi^{m-1}}}2^{\frac{2m}{\gamma\chi^{m-1}}}\chi^{\frac{2(m-1)}{\gamma\chi^{m-1}}}\Big(\int_{B(x_0,s_{m-1})} |u|^{\chi^{m-1}\gamma}\,dx\Big)^{\frac{1}{\gamma\chi^{m-1}}},\end{aligned}$$

and by induction,

$$\Big(\int_{B(x_0,s_m)} |u|^{\chi^m\gamma}\,dx\Big)^{\frac{1}{\gamma\chi^m}} \le (A\gamma)^{\frac{2}{\gamma}\sum_{j=0}^{m-1}\frac{1}{\chi^j}}2^{\frac{2}{\gamma}\sum_{j=0}^{m-1}\frac{j+1}{\chi^j}}\chi^{\frac{2}{\gamma}\sum_{j=0}^{m-1}\frac{j}{\chi^j}}\Big(\int_{B(x_0,s_0)} |u|^{\gamma}\,dx\Big)^{\frac{1}{\gamma}}$$

for each $m>1$. Since $s_0=2$ and $s_m\to 1$, we deduce the following estimate by letting $m\to\infty$: there exist constants $R>0$ and $C>0$ such that for every $B(x_0,2)\subset(|x|>R)$ we have

$$\sup_{B(x_0,1)}|u(x)| \le C\Big(\int_{B(x_0,2)} |u|^{\gamma}\,dx\Big)^{\frac{1}{\gamma}}.$$

This inequality yields $\lim_{|x|\to\infty}|u(x)|=0$. To prove the boundedness of u in the ball $B(0,R)$ we fix $\bar{x}\in B(0,R)$, choose $r>0$ so that

$$(\beta+1)^2\Big(\int_{B(\bar{x},r)} b^{\frac N2}\,dx\Big)^{\frac{2}{N}} \le \frac{S}{2},$$

and then let η have support in $B(\bar{x},r)$. We now repeat the previous argument with a suitable rescaling in the ball $B(\bar{x},r)$ to obtain the boundedness of u in $B(\bar{x},\frac r2)$. By a standard compactness argument we show that u is bounded in $B(0,R)$. This combined with the first part of the proof shows that $u\in L^\infty$. □

We now observe that any solution $u\in H^1_{A,V^+}(\mathbb{R}^N)$ of the equation

$$-\Delta_A u + V(x)u = f(x,|u|)u, \tag{2.10}$$

where $|f(x,|u|)|\le c(1+|u|^{2^*-2})$, satisfies

$$-\Delta_A u + V^+(x)u = \big(V^-(x)+f(x,|u|)\big)u \equiv g(x)u. \tag{2.11}$$

Since $|g(x)|\le c+(V^-(x)+c|u(x)|^{2^*-2})$ and $V^-\in L^{\frac N2}(\mathbb{R}^N)$, $|u|^{2^*-2}\in L^{\frac{2^*}{2^*-2}}(\mathbb{R}^N)=L^{\frac N2}(\mathbb{R}^N)$, we can state the following result:

Corollary 2.3 *Let $u \in H^1_{A,V^+}(\mathbb{R}^N)$, $N \geq 3$, be a solution of (2.10). Then $u \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} u(x) = 0$ (in the sense that $\lim_{R\to\infty} \|u\|_{L^\infty(\mathbb{R}^N - B(0,R))} = 0$).*

Remark 2.4 Let $N = 2$. If $u \in H^1_{A,V^+}(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, then $u \in L^p(\mathbf{R}^2)$ for all $p \in [2, +\infty)$ by the diamagnetic inequality and the Sobolev embedding theorem. Suppose $g(x)$ in (2.1) is such that $b \in L^q(\mathbf{R}^2)$ for some $q \in (1, 2)$ and $u \in H^1_{A,V^+}(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$ is a solution of (2.1). Then the conclusion of Proposition 2.2 remains valid. Indeed, the argument employed there applies except that the L^{2^*}-norm in (2.8) should be replaced by the $L^{q'}$-norm, where $q' = \frac{q}{q-1}$, and one needs to take $\gamma = \beta + 1 = q'$, $\chi = \frac{q'}{2}$. Also the conclusion of Corollary 2.3 remains valid if $u \in H^1_{A,V^+}(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, $V^- \in L^q(\mathbf{R}^2)$ and $|f(x, |u|)| \leq c(1 + |u|^r)$ for some $q \in (1, 2)$ and $r > 0$.

Note in particular that Corollary 2.3 (or Remark 2.4 if $N = 2$) applies to all solutions found in [1], [8] as well as to the solutions found in Theorems 4.1, 4.2 and Corollary 4.3 below.

As an application of Corollary 2.3 and Remark 2.4 we establish an exponential decay of solutions of (2.10). However, we need additional assumptions on V and f.

Proposition 2.5 *Suppose that $f \geq 0$, $f(x, 0) = 0$, $V^+ \in L^p_{\text{loc}}(\mathbb{R}^N)$ and $V^- \in L^p(\mathbb{R}^N)$ for some $p > \frac{N}{2}$. Moreover, assume that there exist constants $a > 0$ and $R > 0$ such that $V(x) \geq a$ for $|x| \geq R$. If $u \in H^1_{A,V^+}(\mathbb{R}^N)$ is a solution of (2.10), then*

$$|u(x)| \leq Ce^{-\alpha|x|} \quad \textit{a.e. in } \mathbb{R}^N,$$

where $\alpha^2 = \frac{a}{2}$.

Proof Since $V \geq a$ for $|x| \geq R$, it is easy to see that $u \in L^2(\mathbb{R}^N)$, and hence $u \in L^q(\mathbb{R}^N)$ for all $2 \leq q \leq +\infty$ according to Corollary 2.3 (or Remark 2.4). Therefore there exists a unique solution $v \in H^1(\mathbb{R}^N)$ of the equation

$$-\Delta v + V^+(x)v = \big(V^-(x) + f(x, |u|)\big)|u|,$$

and by standard regularity theory and the maximum principle v is continuous and ≥ 0. Moreover, it follows from (2.11) and Theorem B.13.2 in [15] that $|u| \leq v$ a.e. (more precisely, one obtains this inequality by integrating (B41) of [15] from $t = 0$ to $t = +\infty$; the hypothesis that $p > \frac{N}{2}$ is used in order to have v continuous and $V^+ \in K^{\text{loc}}_N$, $V^- \in K_N$ in the notation of [15]). Now it remains to establish the exponential decay of v. We follow the argument used in Proposition 4.4 from [17]. Since v satisfies

$$-\Delta v + V^+(x)v \leq \big(V^-(x) + f(x, |u|)\big)v \quad \text{in } \mathbb{R}^N,$$

we have

$$-\Delta v \leq \big(-V(x) + f(x, |u|)\big)v \leq -\frac{a}{2}v \quad \text{for } |x| \geq R$$

by taking R larger if necessary. Let

$$W(x) = Me^{-\alpha(|x|-R)} \quad \text{and } \Omega(L) = \{x;\, R < |x| < L \quad \text{and} \quad v(x) > W(x)\},$$

where a constant $M > 0$ is chosen so that $v(x) \le W(x)$ for $|x| = R$. If $\alpha^2 = \frac{a}{2}$, we get

$$\Delta(W - v) = \big(\alpha^2 - \frac{\alpha(N-1)}{|x|}\big)W - \Delta v \le \alpha^2(W - v) \le 0$$

on $\Omega(L)$. By the maximum principle

$$W(x) - v(x) \ge \min_{x \in \partial\Omega(L)} (W - v) \ge \min\big(0, \min_{|x|=L}(W - v)\big).$$

Since $\lim_{|x|\to\infty} v(x) = \lim_{|x|\to\infty} W(x) = 0$, letting $L \to \infty$, we deduce that

$$v(x) \le W(x) = Me^{-\alpha(|x|-R)}$$

for $|x| \ge R$. □

3 Palais-Smale sequences

The following result is well-known, but we include it for the sake of completeness:

Lemma 3.1 *Let $\{u_m\} \subset H^1_{A,V^+}(\mathbb{R}^N)$ be a sequence such that*

$$J'(u_m) \to 0 \ \text{ in } H^{-1}_{A,V^+}(\mathbb{R}^N) \quad \text{ and } J(u_m) \to c.$$

Then $\{u_m\}$ is bounded in $H^1_{A,V^+}(\mathbb{R}^N)$.

Proof Arguing by contradiction, assume that $\{u_m\}$ is unbounded in $H^1_{A,V^+}(\mathbb{R}^N)$. We set $v_m = \frac{u_m}{\|u_m\|_{H^1_{A,V^+}}}$. We may assume that $v_m \rightharpoonup v$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and $v_m \to v$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for each $2 \le p < 2^*$ and a.e. on $\mathbb{R}^N$. For every $\phi \in H^1_{A,V^+}(\mathbb{R}^N)$ we have

$$\text{(3.1)} \qquad \frac{1}{\|u_m\|^{2^*-2}_{H^1_{A,V^+}}} \int_{\mathbb{R}^N} \big(\nabla_A v_m \overline{\nabla_A \phi} + V v_m \bar{\phi}\big)\, dx = \int_{\mathbb{R}^N} Q|v_m|^{2^*-2} v_m \bar{\phi}\, dx + o(1).$$

Hence

$$\int_{\mathbb{R}^N} Q|v|^{2^*-2} v\bar{\phi}\, dx = 0$$

for every $\phi \in H^1_{A,V^+}(\mathbb{R}^N)$ and consequently $v = 0$ a.e. on $\mathbb{R}^N$ (recall $Q > 0$). Since $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$ we see that $\lim_{m\to\infty} \int_{\mathbb{R}^N} V^-|v_m|^2\, dx = 0$. Therefore substituting $\phi = v_m$ in (3.1) we get

$$\|v_m\|^2_{H^1_{A,V^+}} \equiv \int_{\mathbb{R}^N} \big(|\nabla_A v_m|^2 + V^+|v_m|^2\big)\, dx = \|u_m\|^{2^*-2}_{H^1_{A,V^+}} \int_{\mathbb{R}^N} Q|v_m|^{2^*}\, dx + o(1).$$

Since $J(u_m) \to c$, we also have

$$\frac{1}{2}\|v_m\|^2_{H^1_{A,V^+}} \equiv \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla_A v_m|^2 + V^+|v_m|^2\big)\, dx = \frac{\|u_m\|^{2^*-2}_{H^1_{A,V^+}}}{2^*} \int_{\mathbb{R}^N} Q|v_m|^{2^*}\, dx + o(1).$$

The last two relations imply that $\|v_m\|_{H^1_{A,V^+}} \to 0$, which is impossible. $\square$

In Proposition 3.2 below we determine the energy level of the functional J below which the Palais - Smale condition holds. Let

$$\tilde{Q} = \sup_{x\in\mathbb{R}^N} Q(x).$$

Proposition 3.2 *Let a sequence $\{u_m\} \subset H^1_{A,V^+}(\mathbb{R}^N)$ be such that*

$$J(u_m) \to c < \frac{S^{\frac{N}{2}}}{N\tilde{Q}^{\frac{N-2}{2}}} \quad \text{and } J'(u_m) \to 0 \quad \text{in } H^{-1}_{A,V^+}(\mathbb{R}^N).$$

Then $\{u_m\}$ is relatively compact in $H^1_{A,V^+}(\mathbb{R}^N)$.

Proof By Lemma 3.1 $\{u_m\}$ is bounded. Therefore we may assume $u_m \rightharpoonup u$ in $H^1_{A,V^+}(\mathbb{R}^N)$ and $u_m \to u$ a.e. Let $u_m = v_m + u$. Then

$$\int_{\mathbb{R}^N} \left(|\nabla_A u_m|^2 + V^+|u_m|^2\right) dx = \int_{\mathbb{R}^N} \left(|\nabla_A v_m|^2 + V^+|v_m|^2\right) dx + \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V^+|u|^2\right) dx + o(1),$$

$$\int_{\mathbb{R}^N} V^-|u_m|^2\, dx = \int_{\mathbb{R}^N} V^-|v_m|^2\, dx + \int_{\mathbb{R}^N} V^-|u|^2\, dx + o(1) = \int_{\mathbb{R}^N} V^-|u|^2\, dx + o(1)$$

and by the Brézis-Lieb lemma [2], [18],

$$\int_{\mathbb{R}^N} Q|u_m|^{2^*}\, dx = \int_{\mathbb{R}^N} Q|v_m|^{2^*}\, dx + \int_{\mathbb{R}^N} Q|u|^{2^*}\, dx + o(1).$$

As u is a solution of (1.1), it follows that

$$o(1) = \langle J'(u_m), u_m\rangle = \langle J'(v_m), v_m\rangle + \langle J'(u), u\rangle + o(1) = \langle J'(v_m), v_m\rangle + o(1),$$

and thus

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \left(|\nabla_A v_m|^2 + V^+|v_m|^2\right) dx = \lim_{m\to\infty} \int_{\mathbb{R}^N} Q|v_m|^{2^*}\, dx = l \tag{3.2}$$

after passing to a subsequence. It remains to show that $l = 0$. We have $J(u) = J(u) - \frac{1}{2}\langle J'(u), u\rangle = \frac{1}{N}\int_{\mathbb{R}^N} Q|u|^{2^*}\, dx \geq 0$ and

$$c = J(u_m) + o(1) = J(v_m) + J(u) + o(1) \geq J(v_m) + o(1).$$

Hence using (3.2),

$$\frac{l}{N} = (\frac{1}{2} - \frac{1}{2^*})l \leq c < \frac{S^{\frac{N}{2}}}{N\tilde{Q}^{\frac{N-2}{2}}}. \tag{3.3}$$

By the Sobolev and the diamagnetic inequalities,

$$\left(\int_{\mathbb{R}^N} Q|v_m|^{2^*}\, dx\right)^{\frac{2}{2^*}} \leq \tilde{Q}^{\frac{2}{2^*}} \left(\int_{\mathbb{R}^N} |v_m|^{2^*}\, dx\right)^{\frac{2}{2^*}} \leq S^{-1}\tilde{Q}^{\frac{2}{2^*}} \int_{\mathbb{R}^N} \left(|\nabla_A v_m|^2 + V^+|v_m|^2\right) dx.$$

Letting $m \to \infty$ we get

$$l^{\frac{2}{2^*}} \leq S^{-1} \tilde{Q}^{\frac{2}{2^*}} l,$$

so either

$$l \geq \frac{S^{\frac{N}{2}}}{\tilde{Q}^{\frac{N-2}{2}}}$$

which contradicts (3.3) or $l = 0$. □

4 Existence results - linking

First we study the linear eigenvalue problem

$$-\Delta_A u + V^+(x)u = \mu V^-(x)u \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

We assume that $V^- \neq 0$. Since the functional $u \mapsto \int_{\mathbb{R}^N} V^-|u|^2\,dx$ is weakly continuous in $H^1_{A,V^+}(\mathbb{R}^N)$, problem (4.1) has a sequence of eigenvalues $\mu_1 < \mu_2 \leq \mu_3 \leq \dots \mu_n \to \infty$. Let us denote the corresponding orthonormal system of eigenfunctions by $e_1(x), e_2(x), \dots$. The sequence is complete in $H^1_{A,V^+}(\mathbb{R}^N)$. Since the first eigenvalue is defined by the Rayleigh quotient

$$\mu_1 = \inf_{u \in H^1_{A,V^+}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V^+|u|^2\right) dx}{\int_{\mathbb{R}^N} V^-|u|^2\,dx},$$

we see that $\mu_1 > 0$. Indeed, the denominator is weakly continuous, so the infimum is attained at some $\bar{u} \neq 0$. It follows from Proposition 2.2 that $e_i \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x| \to \infty} e_i(x) = 0$, $i = 1, 2, \dots$.

Following the paper [6] we distinguish two cases: (i) $\mu_1 > 1$ and (ii) $0 < \mu_1 \leq \dots \leq \mu_{n-1} \leq 1 < \mu_n \leq \dots$.

In the proofs of the existence results in both cases, we shall use a family of instantons

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U\big(\frac{x-y}{\epsilon}\big),\ \epsilon > 0,\ y \in \mathbb{R}^N, \ \text{ where } U(x) = \frac{\big(N(N-2)\big)^{\frac{N-2}{4}}}{\big(1+|x|^2\big)^{\frac{N-2}{2}}}.$$

It is known [18] that

$$-\Delta U = U^{2^*-1} \ \text{ in } \mathbb{R}^N.$$

Moreover, we have $\int_{\mathbb{R}^N} |\nabla U|^2\,dx = \int_{\mathbb{R}^N} U^{2^*}\,dx = S^{\frac{N}{2}}$.

Let ψ be a C^1-function such that $\psi(x) = 1$ for $|x-y| \leq \frac{\delta}{2}$ and $\psi(x) = 0$ for $|x-y| > \delta$. We need the following asymptotic relations for $w_{\epsilon,y} = \psi U_{\epsilon,y}$:

$$\|w_{\epsilon,y}\|_{2^*}^{2^*} = S^{\frac{N}{2}} + O\big(\epsilon^N\big),\ \|\nabla w_{\epsilon,y}\|_2^2 = S^{\frac{N}{2}} + O\big(\epsilon^{N-2}\big),\ \|w_{\epsilon,y}\|_{2^*-1}^{2^*-1} = O\big(\epsilon^{\frac{N-2}{2}}\big). \tag{4.2}$$

Since $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$, we also have

$$\int_{\mathbb{R}^N} V^- w_{\epsilon,y}\, dx = O\big(\epsilon^{\frac{N-2}{2}} |\log \epsilon|^{\frac{N-2}{N}}\big). \tag{4.3}$$

Indeed, $\big|\int_{\mathbb{R}^N} V^- w_{\epsilon,y}\, dx\big| \le \|V^-\|_{N/2} \|w_{\epsilon,y}\|_{N/(N-2)}$ and

$$\begin{aligned}
\|w_{\epsilon,y}\|_{N/(N-2)}^{N/(N-2)} &= \int_{\mathbb{R}^N} |w_{\epsilon,y}|^{\frac{N}{N-2}}\, dx \\
&\le c_1 \int_{B(0,\delta)} \frac{\epsilon^{\frac{N}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N}{2}}}\, dx \;=\; c_1 \epsilon^{\frac{N}{2}} \int_{B(0,\delta/\epsilon)} \frac{1}{(1+|x|^2)^{\frac{N}{2}}}\, dx \;\le\; c_2 \epsilon^{\frac{N}{2}} |\log \epsilon|.
\end{aligned}$$

In case (i) $\left(\int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V|u|^2\big)\, dx\right)^{\frac{1}{2}}$ is an equivalent norm in $H^1_{A,V^+}(\mathbb{R}^N)$. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V^+|u|^2\big)\, dx &\ge \int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V|u|^2\big)\, dx \\
&\ge \big(1 - \frac{1}{\mu_1}\big) \int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V^+|u|^2\big)\, dx.
\end{aligned}$$

In this case the spectrum of the operator $-\Delta_A + V$ is contained in $(0,\infty)$. So we can obtain a solution of (1.1) as a multiple of a minimizer of the constrained minimization problem

$$S_Q = \inf_{u \in H^1_{A,V^+}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V|u|^2\big)\, dx}{\left(\int_{\mathbb{R}^N} Q(x)|u|^{2^*}\, dx\right)^{\frac{N-2}{N}}}. \tag{4.4}$$

In fact, we have the following existence result:

Theorem 4.1 *Let $N \ge 4$ and $\mu_1 > 1$. Suppose that there exists an $\bar{x} \in \mathbb{R}^N$ such that $Q(\bar{x}) = \tilde{Q}$, $V(x) < -c < 0$ in some neighbourhood of $\bar{x}$, A is continuous at $\bar{x}$ and*

$$|Q(x) - Q(\bar{x})| = o\big(|x - \bar{x}|^2\big)$$

for x close to $\bar{x}$. Then the infimum of (4.4) is attained at some $u \in H^1_{A,V^+}(\mathbb{R}^N)$ (and a multiple of u is a solution of (1.1)).

Proof First, we claim that

$$S_Q < \frac{S}{\tilde{Q}^{\frac{N-2}{N}}}.$$

Without loss of generality we may assume that $\bar{x} = 0$. Let $\vartheta(x) = -\sum_{j=1}^N A_j(0) x_j$. Then $\big(A + \nabla\vartheta\big)(0) = 0$ and by the continuity $|\big(A + \nabla\vartheta\big)(x)|^2 \le c' < c$ for all $|x| < \delta$ and sufficiently small δ. Let $u_\epsilon(x) = w_{\epsilon,0}(x) e^{i\vartheta(x)}$. Letting $U_\epsilon = U_{\epsilon,0}$ and using (4.2) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \big(|\nabla_A u_\epsilon|^2 + V|u_\epsilon|^2\big)\, dx &\le \int_{\mathbb{R}^N} \big(|\nabla(\psi U_\epsilon)|^2 + \psi^2 U_\epsilon^2 |\nabla\vartheta + A|^2 - c\psi^2 U_\epsilon^2\big)\, dx \\
&\le S^{\frac{N}{2}} + (c' - c) \int_{B(0,\delta/2)} U_\epsilon^2\, dx + O\big(\epsilon^{N-2}\big).
\end{aligned}$$

It follows from the assumption on Q that

$$\int_{\mathbb{R}^N} Q w_\epsilon^{2^*}\, dx = \int_{\mathbb{R}^N} Q|u_\epsilon|^{2^*}\, dx = S^{\frac{N}{2}}\tilde{Q} + o\big(\epsilon^2\big), \tag{4.5}$$

where $w_\epsilon = w_{\epsilon,0}$. For small $\epsilon > 0$ we have

$$\int_{B(0,\delta/2)} U_\epsilon^2\, dx \geq \begin{cases} C\epsilon^2|\log\epsilon| & \text{if } N = 4 \\ C\epsilon^2 & \text{if } N \geq 5. \end{cases} \tag{4.6}$$

Combining the last three relations our claim easily follows. Let $\{u_m\}$ be a minimizing sequence for S_Q such that $\int_{\mathbb{R}^N} Q|u_m|^{2^*}\, dx = 1$. Let $v_m = S_Q^{\frac{N-2}{4}} u_m$. The rescaled sequence $\{v_m\}$ is a Palais - Smale sequence for the functional J at the level $c = \frac{1}{N} S_Q^{\frac{N}{2}} < \frac{S^{\frac{N}{2}}}{N\tilde{Q}^{\frac{N-2}{2}}}$ (cf. Theorem 2.1 in [13] or Lemma 8.2.1 in [3]). By Proposition 3.2 $\{v_m\}$ is relatively compact in $H^1_{A,V^+}(\mathbb{R}^N)$ and the result easily follows. □

Therefore, it remains to consider the case (ii). In this case we use the topological linking. Let $Y = \text{span}\,\big(e_1, e_2, \dots, e_{n-1}\big)$, $Z = Y^\perp$ and let $z \in Z - \{0\}$. Obviously, we have $H^1_{A,V^+}(\mathbb{R}^N) = Y \oplus Z$. Define

$$M = \{u = y + \lambda z;\ y \in Y,\ \|u\|_{H^1_{A,V^+}} \leq R,\ \lambda \geq 0\},$$

$$M_\circ = \{u = y + \lambda z;\ y \in Y,\ \|u\|_{H^1_{A,V^+}} = R,\ \lambda \geq 0\} \cup \{u \in Y;\ \|u\|_{H^1_{A,V^+}} \leq R\},$$

$$N = \{u \in Z;\ \|u\|_{H^1_{A,V^+}} = r\}.$$

First we check that

$$\max_{u \in M_\circ} J(u) = 0 < \inf_{u \in N} J(u) \tag{4.7}$$

provided $0 < r < R$ are suitably chosen. To show (4.7) we note that on Z

$$\begin{aligned} J(u) &\geq \big(1 - \mu_n^{-1}\big) \int_{\mathbb{R}^N} \big(|\nabla_A u|^2 + V^+|u|^2\big)\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} Q(x)|u|^{2^*}\, dx \\ &\geq \frac{1}{2}\big(1 - \mu_n^{-1}\big)\|u\|^2_{H^1_{A,V^+}} - \frac{S^{-\frac{2^*}{2}}}{2^*}\tilde{Q}\|u\|^{2^*}_{H^1_{A,V^+}}. \end{aligned}$$

Taking $r > 0$ sufficiently small we get

$$\inf\{J(u);\ \|u\|_{H^1_{A,V^+}} = r,\ u \in Z\} > 0.$$

Since $Y \oplus \mathbb{R}z$ is finite dimensional and $2^* > 2$, it is easy to see that $J(u) \to -\infty$ as $\|u\|_{H^1_{A,V^+}} \to \infty$, $u \in Y \oplus \mathbb{R}z$. We choose $R > 0$ so that $\max_{u \in M_\circ} J(u) = 0$.

We now state and prove the existence theorem for problem (1.1) in case (ii).

Theorem 4.2 *Suppose that* $\sup_{x\in\mathbb{R}^N} Q(x) = Q(\bar{x})$ *for some* $\bar{x} \in \mathbb{R}^N$ *and* $Q(x)-Q(\bar{x}) = o\big(|x-\bar{x}|^2\big)$ *for* x *close to* $\bar{x}$. *Further assume that* $V(x) \le -c < 0$ *in some neighbourhood of* $\bar{x}$ *and that* A *is continuous at* $\bar{x}$.

(i) If $\mu_{n-1} = 1$, *then problem (1.1) has a solution for* $N \ge 7$,

(ii) If $\mu_{n-1} < 1$, *then problem (1.1) has a solution for* $N \ge 5$,

(iii) If $\mu_{n-1} < 1$ *and* $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^q(B(\bar{x},\delta))$ *for some* $q > \frac{N}{2}$ *and* $\delta > 0$, *then problem (1.1) has a solution for* $N = 4$.

Proof Without loss of generality we assume that $\tilde{Q} = Q(0)$, that is, $\bar{x} = 0$. Let

$$c = \min_{\gamma\in\Gamma} \max_{u\in M} J(\gamma(u)),$$

where

$$\Gamma = \{\gamma;\ \gamma \in C\big(M, H^1_{A,V^+}(\mathbb{R}^N)\big),\ \gamma\,|_{M_\circ} = \mathrm{id}\}.$$

According to (4.7) and the linking theorem [18], $c > 0$ and there exists a Palais - Smale sequence for J at the level c. So by Proposition 3.2 it suffices to show that

$$c < \frac{S^{\frac{N}{2}}}{N\tilde{Q}^{\frac{N-2}{2}}}. \tag{4.8}$$

We follow a modified argument from pp. 51 - 52 in [18] and from [5]. For $u \in H^1_{A,V^+}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + V|u|^2\big)\,dx > 0$, we have

$$\max_{s\ge 0} J(su) = \frac{1}{N}\frac{\Big(\int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + V|u|^2\big)\,dx\Big)^{\frac{N}{2}}}{\Big(\int_{\mathbb{R}^N} Q|u|^{2^*}\,dx\Big)^{\frac{N-2}{2}}}. \tag{4.9}$$

As in the proof of Theorem 4.1, let $\vartheta(x) = -\sum_{j=1}^N A_j(0)x_j$. Then $V(x) \le -c$ and $|(A+\nabla\vartheta)(x)|^2 \le c' < c$ for $|x| < \delta$, if $\delta > 0$ is sufficiently small. Let u_ϵ be the function introduced in the proof of Theorem 4.1 and take $z = u_\epsilon^+$ in the definition of the set M, where u_ϵ^+ is the projection of u_ϵ on Z. Then $Y \oplus \mathbb{R}u_\epsilon = Y \oplus \mathbb{R}u_\epsilon^+$. According to (4.9) it is enough to show that

$$\max_{\substack{u\in Y\oplus\mathbb{R}^+u_\epsilon \\ \int_{\mathbb{R}^N} Q|u|^{2^*}dx=1}} \int_{\mathbb{R}^N}\big(|\nabla_A u|^2 + V|u|^2\big)\,dx < \frac{S}{\tilde{Q}^{\frac{N-2}{N}}}. \tag{4.10}$$

Suppose that the maximum above is attained at $u = y + tu_\epsilon = \tilde{y} + tu_\epsilon^+$. It is clear that $t > 0$, and since Y is finite dimensional and $e_i \in L^\infty(\mathbb{R}^N)$, all L^p-norms on Y are equivalent for $2 \le p \le \infty$. Therefore $\|u_\epsilon^-\|_{2^*} \le c_1\|u_\epsilon^-\|_2 \le c_1\|u_\epsilon\|_2 \to 0$, so $\|u_\epsilon^+\|_{2^*}^{2^*} \to S^{\frac{N}{2}}$ as $\epsilon \to 0$. Moreover, since Q is bounded away from 0 on compact sets and $\operatorname{supp} u_\epsilon \subset B(0,\delta)$, $c_2\|u\|_{2^*}^{2^*} \le \int_{\mathbb{R}^N} Q|u|^{2^*}\,dx \le c_3\|u\|_{2^*}^{2^*}$ for all $u \in Y \oplus \mathbb{R}u_\epsilon^+$ and all $\epsilon > 0$. Using the inequality $\|\tilde{y}\|_{2^*} \le c_4\|u\|_{2^*}$ it is now easy to see that $\int_{\mathbb{R}^N} Q|y|^{2^*}\,dx$ and t are bounded, uniformly in ϵ. Since $y \in Y$, we have $y = \sum_{i=1}^{n-1} \alpha_i e_i$ and by straightforward computations we get

$$
\begin{aligned}
(4.11)\quad \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V|u|^2\right) dx \;&=\; \int_{\mathbb{R}^N} \left(|\nabla_A y|^2 + V|y|^2\right) dx + \int_{\mathbb{R}^N} \left(|\nabla_A (tu_\epsilon)|^2 + V|tu_\epsilon|^2\right) dx \\
&+\; 2\mathrm{Re}\left(\int_{\mathbb{R}^N} \left(\nabla_A y \overline{\nabla_A (tu_\epsilon)} + V y \overline{tu_\epsilon}\right) dx\right) \\
&\le\; \left(1 - \mu_{n-1}^{-1}\right) \int_{\mathbb{R}^N} \left(|\nabla_A y|^2 + V^+|y|^2\right) dx \\
&+\; \int_{\mathbb{R}^N} \left(|\nabla_A (tu_\epsilon)|^2 + V|tu_\epsilon|^2\right) dx \\
&+\; O(\epsilon^{\frac{N-2}{2}} |\log \epsilon|^{\frac{N-2}{N}}) \|y\|_{H^1_{A,V+}}.
\end{aligned}
$$

In estimating the last term on the right-hand side of the equality above we have used the identity

$$
\int_{\mathbb{R}^N} (\nabla_A e_i \overline{\nabla_A u_\epsilon} + V^+ e_i \bar{u}_\epsilon)\, dx = \mu_i \int_{\mathbb{R}^N} V^- e_i \bar{u}_\epsilon\, dx,
$$

the fact that the L^∞- and the H^1_{A,V^+}-norms are equivalent on Y and (4.3). Recalling that $u_\epsilon = w_\epsilon e^{i\vartheta(x)}$, we see that

$$
\begin{aligned}
(4.12)\quad \int_{\mathbb{R}^N} (|\nabla_A u_\epsilon|^2 + V|u_\epsilon|^2)\, dx \;&\le\; \int_{\mathbb{R}^N} (|\nabla w_\epsilon|^2 + w_\epsilon^2 |\nabla \vartheta + A|^2 - c w_\epsilon^2)\, dx \\
&\le\; S^{\frac{N}{2}} + (c' - c) \int_{\mathbb{R}^N} w_\epsilon^2\, dx + O(\epsilon^{N-2}).
\end{aligned}
$$

Combining (4.11) and (4.12) we get

$$
\begin{aligned}
(4.13)\quad \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V|u|^2\right) dx \;&\le\; (1 - \mu_{n-1}^{-1}) \|y\|^2_{H^1_{A,V+}} + t^2 S^{\frac{N}{2}} + t^2 (c' - c) \int_{\mathbb{R}^N} w_\epsilon^2\, dx \\
&+\; O(\epsilon^{\frac{N-2}{2}} |\log \epsilon|^{\frac{N-2}{N}}) \|y\|_{H^1_{A,V+}}.
\end{aligned}
$$

Moreover, by the convexity of the mapping $s \mapsto |sy + tu_\epsilon|^{2^*}$,

$$
\begin{aligned}
1 \;&=\; \int_{\mathbb{R}^N} Q|u|^{2^*}\, dx \ge \int_{\mathbb{R}^N} Q (tw_\epsilon)^{2^*}\, dx - 2^* \int_{\mathbb{R}^N} Q|y| (tw_\epsilon)^{2^*-1}\, dx \\
&\ge\; \int_{\mathbb{R}^N} Q (tw_\epsilon)^{2^*}\, dx - O(\epsilon^{\frac{N-2}{2}}) \|y\|_{H^1_{A,V+}},
\end{aligned}
$$

and hence, using (4.2), (4.5) and (4.6),

$$
\begin{aligned}
(4.14)\quad t^2 S^{\frac{N}{2}} + t^2 (c' - c) \int_{\mathbb{R}^N} w_\epsilon^2\, dx &= \frac{\left(S^{N/2} + (c' - c) \int_{\mathbb{R}^N} w_\epsilon^2\, dx\right) \left(\int_{\mathbb{R}^N} Q (tw_\epsilon)^{2^*}\, dx\right)^{2/2^*}}{\left(\int_{\mathbb{R}^N} Q w_\epsilon^{2^*}\, dx\right)^{2/2^*}} \\
&\le\; \frac{\left(S^{N/2} + (c' - c) \int_{\mathbb{R}^N} w_\epsilon^2\, dx\right) \left(1 + O(\epsilon^{(N-2)/2}) \|y\|_{H^1_{A,V+}}\right)}{\left(S^{N/2} \tilde{Q} + o(\epsilon^2)\right)^{2/2^*}} \\
&=\; \frac{S}{\tilde{Q}^{(N-2)/N}} - d \int_{\mathbb{R}^N} w_\epsilon^2\, dx + o(\epsilon^2) + O(\epsilon^{\frac{N-2}{2}}) \|y\|_{H^1_{A,V+}},
\end{aligned}
$$

where $d > 0$. If $\mu_{n-1} \leq 1$ and $N \geq 7$, the conclusion easily follows from (4.13), (4.14) and (4.6). Suppose $\mu_{n-1} < 1$. Since

$$(4.15) \qquad (1-\mu_{n-1}^{-1})\|y\|^2_{H^1_{A,V^+}} + O(\epsilon^{\frac{N-2}{2}}|\log\epsilon|^{\frac{N-2}{N}})\|y\|_{H^1_{A,V^+}} \leq O(\epsilon^{N-2}|\log\epsilon|^{\frac{2(N-2)}{N}})$$

in this case, the conclusion remains valid for $N = 5$ and 6. If $N = 4$ and $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)\cap L^q(B(0,\delta))$, then $\int_{\mathbb{R}^N} V^- u_\epsilon\, dx = O(\epsilon^{\frac{N-2}{2}}) = O(\epsilon)$ by an argument similar to that of (4.3), so the right-hand side above is $O(\epsilon^2)$ and the conclusion follows again. □

We remark that if $\mu_{n-1} < 1$ and $N = 4$ in Theorem 4.2, then we may assume $Q(x) - Q(\bar{x}) = O(|x-\bar{x}|^2)$ because in this case it suffices to have $O(\epsilon^2)$ instead of $o(\epsilon^2)$ in (4.14).

Corollary 4.3 *Suppose that* $\sup_{x\in\mathbb{R}^N} Q(x) = Q(\bar{x})$ *for some* $\bar{x} \in \mathbb{R}^N$ *and* $Q(x)-Q(\bar{x}) = O\big(|x-\bar{x}|^2\big)$ *for* x *close to* $\bar{x}$. *Further assume that there are* $\alpha > 0$, $c > 0$ *such that* $V^-(x) \geq \frac{c}{|x-\bar{x}|^\alpha}$ *in a neighbourhood of* $\bar{x}$ *and* A *is continuous at* $\bar{x}$. *Then problem (1.1) has a solution* $u \neq 0$ *in each of the following cases:*

(i) $\mu_{n-1} = 1$, $N \geq 6$ *and* $0 < \alpha < 2$,

(ii) $\mu_{n-1} = 1$, $N = 3$, 4 *or* 5 *and* $\frac{6-N}{2} < \alpha < 2$,

(iii) $\mu_{n-1} < 1$, $N \geq 4$ *and* $0 < \alpha < 2$,

(iv) $\mu_{n-1} < 1$, $N = 3$ *and* $1 \leq \alpha < 2$.

Note that since $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $0 < \alpha < 2$.

Proof We may asssume $\bar{x} = 0$. A small change is needed in the argument of Theorem 4.2. Now in (4.12) we have

$$(4.16) \qquad \int_{\mathbb{R}^N}(|\nabla_A u_\epsilon|^2 + V|u_\epsilon|^2)\,dx \leq S^{\frac{N}{2}} - \frac{c}{2}\int_{\mathbb{R}^N}\frac{w_\epsilon^2}{|x|^\alpha}\,dx + O(\epsilon^{N-2}).$$

Moreover,

$$(4.17) \qquad \int_{\mathbb{R}^N}\frac{w_\epsilon^2}{|x|^\alpha}\,dx \geq \int_{B(0,\delta/2)}\frac{U_\epsilon^2}{|x|^\alpha}\,dx \geq \begin{cases} c_1\epsilon^{2-\alpha} + c_2\epsilon^{N-2} & \text{if } N \neq 3 \text{ or } \alpha \neq 1 \\ c_1\epsilon|\log\epsilon| & \text{if } N = 3 \text{ and } \alpha = 1. \end{cases}$$

So the conclusion follows using (4.13), (4.14), (4.15) and taking into account the changes prompted by (4.16), (4.17). Note that since $Q(x) - Q(0) = O(|x|^2)$, $o(\epsilon^2)$ is replaced by $O(\epsilon^2)$ in (4.14). □

As a final remark we would like to mention that combining the above estimates with those appearing in [4], it is possible to show the existence of a nontrivial solution of (1.1) also if $\tilde{Q} = \lim_{|x|\to\infty} Q(x)$ and $Q(x) < \tilde{Q}$ for all $x \in \mathbb{R}^N$. However, since the assumptions we would need to make on V, Q, A and the dimension N are rather restrictive (in particular, we need A globally Lipschitzian and $V^-(x) \geq \frac{c}{|x|^\alpha}$ for some $\alpha > 2$ and all large $|x|$), we omit the details.

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