Integral closure of powers of the graded maximal ideal in a monomial ring

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Abstract

In this paper we study the integral closure of ideals of monomial sub-rings $R$ of $k[x_1, x_2, \ldots, x_d]$ spanned by a finite set of distinct monomials of the polynomial ring. We generalize a well known result for monomial ideals in the polynomial ring to rings $R$ as above proving that the integral closure of a monomial ideal is monomial and we give a geometric description of the integral closure of an ideal of $R$. Then we focus our attention to the study of the integral closure of powers of the graded maximal ideal of $R$ in two particular cases.

MSC: 13B25; 13F20

1 Introduction

Let $H = \{h_1, h_2, \ldots, h_m\}$ be a finite set of distinct monomials in $k[x_1, x_2, \ldots, x_d]$ and let $R = k[H] = k[h_1, h_2, \ldots, h_m] \subseteq k[x_1, x_2, \ldots, x_d]$ be the monomial subring spanned by $H$. Furthermore we suppose that the complement to $\mathbb{N}^d$ of the set of exponents of all monomials in $R$ is finite.

In this paper we study the integral closure of ideals in $R$. In Section 2 we give the concepts of multidegree of a monomial and of integral closure and normality of an ideal. In Section 3 we generalize a well known result for monomial ideals in the polynomial ring to rings $R$ as above proving that the integral closure of a monomial ideal is monomial and we give a geometric description of the integral closure of an ideal of $R$. In Section 4 we focus our attention to the study of the integral closure of powers of the graded maximal ideal of $R$ in two particular cases.

2 Preliminaries

Let $R$ be a ring as in the Introduction. We can associate to every monomial $u^{a_1} \cdots x_d^{a_d}$ in $R$, with $u \in k \setminus \{0\}$, the power product $x_1^{a_1} \cdots x_d^{a_d}$. Let $m$ be a monomial in $R$ and let $x_1^{a_1} \cdots x_d^{a_d}$ be the associated power product. We call

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The above equation has the same multidegree and it must be of the form $u \in k \setminus \{0\}$ such that $m_1 = um_2$.

Let $I$ be an ideal of $R$. We denote by $\overline{I}$ the integral closure \( \{ z \in R \mid z^n + r_1 z^{n-1} + \cdots + r_n = 0, \text{for some } r_i \in I^j \} \).

By definition $I \subseteq \overline{I}$ and, in general, it may happen that $I \subsetneq \overline{I}$. An ideal $I$ in $R$ is called normal if $I^j = \overline{I}^j$ for every $j \geq 1$.

## 3 Integral closure of monomial ideals

Now we generalize a well-known result for monomial ideals in the polynomial ring to rings $R$ as in the Introduction; we have used some ideas from [2].

**Theorem 3.1.** Let $I$ be a monomial ideal in $R$. Then $\overline{I}$ is a monomial ideal.

**Proof.** Since it is well known that $\overline{I}$ is an ideal in $R$, we only need to prove that $\overline{I}$ is generated by monomials. Let $z \in \overline{I}$, $z = m_1 + \cdots + m_s$, where the $m_i$'s are monomials with different multidegrees. We want to prove that $m_i \in \overline{I}$ for every $i = 1, \ldots, s$. By induction, it suffices to verify that $m_i \in \overline{I}$ for some $i$ since $\overline{I}$ is an ideal in $R$. To this aim we prove that there exists $N > 0$ such that $m_i^N \in I^N$ for some $i = 1, \ldots, s$ (hence $m_i \in \overline{I}$ since it is root of $Z^N - m_i^N = 0$).

From the definition of integral closure, we have

\[(m_1 + \cdots + m_s)^n + l_1(m_1 + \cdots + m_s)^{n-1} + \cdots + l_n = 0, \quad (l_i \in I^j).\]

Let us consider the multidegree of $m_i^N$. There must exist another term in the equation above which has the same multidegree and it must be of the form $b_i m_1^{j_{i,1}} \cdots m_s^{j_{i,s}}$, where $b_i \in I^{n_1}$ and $\sum_{k=1}^s j_{i,k} = n - i$ (we note that if $s = 1$, then $z = m_1 \in \overline{I}$).

Since the set of elements of the same multidegree as $m_i^N$ is a 1-dimensional vector space over $\mathbb{R}$ and since $b_i m_1^{j_{i,1}} \cdots m_s^{j_{i,s}}$ and $m_i^N$ have the same multidegree, we get $u b_i m_1^{j_{i,1}} \cdots m_s^{j_{i,s}} = m_i^N$ for some $u \in k \setminus \{0\}$. Using the same argument as above, we get that for every $v = 1, \ldots, s$, $m_v^N = c_i m_1^{j_{v,1}} \cdots m_s^{j_{v,s}}$ with $c_i \in I^{n_1}$, $\sum_{k=1}^s j_{v,k} = n - i$.

Since $m_v^N$ is a monomial, we get (after cancelling out common terms) that $m_v^{n,v} = c_i \prod_{k \neq v} m_k^{j_{v,k}}$ for some $n,v$. We note that $0 \leq j_{v,k} < n,v$ for every $v$ and $k$. Indeed if, for example, $j_{2,1} = n_{1,1}$, then $m_1^{n_{1,1}} = c_{i_1} m_2^{n_{1,2}}$ with $c_{i_1} \in k$, whence $\text{mdeg}(m_1^{n_{1,1}}) = \text{mdeg}(m_2^{n_{1,1}})$, that is $\text{mdeg}(m_1) = \text{mdeg}(m_2)$. A contradiction.

Hence we have a system of $s$ equalities

\[
\begin{align*}
m_1^{n_{1,1}} &= c_{i_1} \prod_{k \neq 1} m_k^{j_{1,k}} \\
m_2^{n_{1,2}} &= c_{i_2} \prod_{k \neq 2} m_k^{j_{2,k}} \\
&\vdots \\
m_s^{n_{1,s}} &= c_{i_s} \prod_{k \neq s} m_k^{j_{s,k}}
\end{align*}
\]
We use an induction on \( s \) to prove that there exists \( N > 0 \) such that \( m_i^N \in I^N \) for some \( i = 1, \ldots, s \). If \( s = 1 \), then \( m_1^{n_1} = c_{n_1} \in I^{n_1} \). Suppose now such \( N \) exists for systems as above with \( s - 1 \) equalities.

Consider the system above. For every \( v = 2, \ldots, s \), we first raise, \( m_v^{n_1} \) to \( n_1 \) and \( m_v^{n_1} \) to \( j_v, 1 \), then we substitute \( m_v^{n_1 + 1} j_v, 1 \) in \( m_v^{n_1 + 1} \) with \((c_i, \prod_{k \neq i} m_k^{j_k})^{v, 1} \) and finally cancel out common terms (it is easy to check that, by \( j_v < n_1, v \) for every \( v \) and \( k \), we never cancel out \( m_v^{n_1 + 1} \)). Hence, for every \( v = 2, \ldots, s \), we get \( m_v^{n_2} = d_v m_2^{k_2} \ldots m_s^{k_s} \) for some \( n_2, v \) and with \( d_v \in I^{n_2} \) and \( d_v \in I^{n_2} \) (where \( m_v^{k_v} \) means that we delete \( m_v^{k_v} \) in the product). Using induction we get the proof.

### 3.1 A geometric description

Our next aim is to have a geometric description of the integral closure of ideals in \( R \).

Let \( a_i \in \mathbb{N}^d, i = 1, \ldots, r \) and let

\[
\text{conv}(a_1, \ldots, a_r) = \{ \sum_{i=1}^{r} \lambda_i a_i \mid \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \in \mathbb{Q}_{\geq 0} \}
\]

be the convex hull (over the rational numbers) of \( a_1, \ldots, a_r \).

For \( \alpha = (a_1, \ldots, a_d) \in \mathbb{N}^d \) we set \( x^\alpha = x_1^{a_1} \ldots x_d^{a_d} \).

**Proposition 3.2.** Let \( I \) be a monomial ideal in \( R \) generated by \( x^a, \ldots, x^r \). Then the exponents \( a \) such that a monomial \( x^a \) in \( R \) belongs to the integral closure \( \overline{T} \) are the integer points in the convex hull of the union of the set \( b + \mathbb{N}^d \), where \( b \) is an exponent of an element in \( I \).

**Proof.** Let \( x^a \in R \) such that \( a \) is in the convex hull of the union of the set \( b + \mathbb{N}^d \), where \( b \) is an exponent of an element in \( I \). Hence \( a = \sum_{i=1}^{r} \lambda_i a_i \) with \( x^{a_i} \in I, \sum_{i=1}^{r} \lambda_i = 1 \) and \( \lambda_i \in \mathbb{Q}_{\geq 0} \). Let \( m \) an integer such that \( m \lambda_i \in \mathbb{N} \) for every \( i = 1, \ldots, r \). Then, by \( \sum_{i=1}^{r} m \lambda_i = m \), we get \( (x^a)^m = (x^{a_1})^m \ldots (x^{a_r})^m \in I^m \). So \( x^a \in \overline{T} \).

Vice versa if \( x^a \in \overline{T} \), then, by the proof of Theorem 3.1, we get \( x^{am} \in I^m \), that is \( x^{am} = x^{b_1} \ldots x^{b_m} \) with \( x^{b_i} \in I \). By \( a = \sum_{i=1}^{m} \frac{1}{n_i} b_i \), we get that \( a \) is in the convex hull of the union of the set \( b + \mathbb{N}^d \), where \( b \) is an exponent of an element in \( I \).

### 4 Integral closure of powers of the graded maximal ideal \( m \) in special cases

We recall that we are interested in monomial subrings \( R = k[h_1, h_2, \ldots, h_m] \) of \( k[x_1, x_2, \ldots, x_d] \) spanned by a finite set \( H \) of monomials and such that the complement to \( \mathbb{N}^d \) of the set of exponents of all power products in \( R \) is finite. \( R \) is a graded ring with graded maximal ideal \( m = (h_1, h_2, \ldots, h_m) \).
In this section we focus our attention to the study of the integral closure of powers of the graded maximal ideal \( m \) of \( R \) in two particular cases. We remark that, by definition of integral closure of an ideal, \( m = \overline{m} \).

4.1 The first case

In this first case we restrict to rings \( R = k + (x^{a_1}, \ldots, x^{a_i})k[x] \), subalgebras of \( k[x_1, x_2] = k[x] \) such that the complement to \( \mathbb{N}^2 \) of the set of exponents of all power products in \( R \) is finite. Let \( m \) denote the graded maximal ideal of \( R \). By \( m = (x^{a_1}, \ldots, x^{a_i})k[x] \), we get \( m^r = ((x^{a_1}, \ldots, x^{a_i})k[x])^r = (x^{a_1}, \ldots, x^{a_i})^r k[x] \).

Let
\[
\alpha_1 = \min\{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in m\},
\alpha_2 = \min\{\alpha_2 \neq 0 \mid x_1^{\alpha_1} x_2^{\alpha_2} \in m, \text{for some } \alpha < \alpha_1\}
\]

As the complement to \( \mathbb{N}^2 \) of the set of exponents of all power products in \( R \) is finite, such \( \alpha_i \) exists for \( i = 1, 2 \). By definition of \( \alpha_2 \), we get \( x_1^{\alpha_1} x_2^{\alpha_2} \in m \) for some positive integer \( \gamma < \alpha_1 \).

**Proposition 4.1.** Suppose that \( \alpha_i \geq 2 \), for \( i = 1, 2 \). If there exists \( \gamma \) as above such that \( \alpha_1 - \gamma \geq 2 \), then \( m^j \setminus m^j \neq \emptyset \) for every \( j \geq 2 \).

Proof. Let \( j \geq 2 \) and let us consider \( m_1 = x_1^{\alpha_1 - j} x_2^{\alpha_2 - j} \).

By definition of \( \alpha_1 \) and by \( j \geq 2 \), we have \( x_1^{\alpha_1 - j} x_2^{\alpha_2 - j} \in R \). Since \( \alpha_1 - \gamma \geq 2 \), \( x_1^{\alpha_1 - j} x_2^{\alpha_2 - j} \in m^j \). Finally by \( \alpha_2 \geq 2 \), \( x_1^{\alpha_1} x_2^{\alpha_2 - j} \in m^j \).

Since \((j_1 - 1, j_2 - 1) = \lambda_1(j_1 - 2, j_2) + \lambda_2(j_1, j_2 - 2)\) with \( \lambda_1 = \lambda_2 = 1/2 \), we get, by Proposition 3.2, that \( m_j \in \overline{m}^j \). But \( m_j \notin \overline{m}^j \) as, by definition of \( \alpha_2 \), \( x_1^{\alpha_1} x_2^{\alpha_2 - j} \in \overline{m}^j \) only if \( \alpha_1 - \gamma > j_1 \).

**Remark 4.2.** We can change the role of \( x_1 \) with that one of \( x_2 \) in Proposition 4.1. Indeed let
\[
b_2 = \min\{\beta_2 \neq 0 \mid x_2^{\beta_2} \in m\},
\beta_1 = \min\{\beta_1 \neq 0 \mid x_1^{\beta_1} x_2^{\beta_2} \in m, \text{for some } \beta < \beta_2\}
\]

As above such \( \beta_i \) exists for \( i = 0, 1 \). By definition of \( \beta_1 \), we get \( x_1^{\beta_1} x_2^{\beta_2} \in m \) for some \( \gamma < \beta_2 \).

Using the same argument as for \( \alpha_i \), we get that if there exists \( \gamma \) as above such that \( \beta_2 - \gamma \geq 2 \), then \( x_1^{\beta_1} x_2^{\beta_2 - j} \in \overline{m}^j \setminus \overline{m}^j \) for every \( j \geq 2 \).

**Example 4.3.** Let \( R = k + (x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3)k[x_1, x_2] \) and let us consider \( m = (x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3)k[x_1, x_2] \) the graded maximal ideal of \( R \) as in Figure 1. Then \( a_1 = 9, a_2 = 3, b_1 = 2, b_2 = 9 \) and \( \{x_1^{9-1}, x_2 x_3^{3-1}, x_2 x_4^{3-1}\} \subseteq \overline{m}^j \setminus \overline{m}^j \) for every \( j \geq 2 \).

**Remark 4.4.** We can generalize Proposition 4.1 to the \( d \)-dimensional case. Let \( R = k + (x^{a_1}, \ldots, x^{a_i})k[x] \) be a subalgebra of \( k[x_1, x_2, \ldots, x_d] = k[x] \) and let
\[
a_1 = \min\{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in m\}.
\]
Suppose $a_1 \geq 2$ and that there exists $i$ with $2 \leq i \leq d$ such that

$$a_i := \min\{\alpha_i \neq 0 \mid x_1^\alpha x_i^\beta \in \mathfrak{m}, \text{for some } \alpha < a_1\} \geq 2.$$ 

By definition of $a_i$ there exists a positive integer $\gamma_i < a_1$ such that $x_1^{\gamma_i} x_i^{a_i} \in \mathfrak{m}$.

Suppose $a_1 - \gamma_i \geq 2$. Since (for every $j \geq 2$)

$$(ja_1 - 1, 0, \ldots, 0, a_1 - 1, 0, \ldots, 0) = \frac{1}{2}(ja_1 - 2, 0, \ldots, 0, a_1, 0, \ldots, 0) + \frac{1}{2}(ja_1, 0, \ldots, 0, a_1 - 2, 0, \ldots, 0)$$

and by definition of $a_i$, we get that for every $j \geq 2$ the element $x_1^{ja_1 - 1} x_i^{a_i - 1} \in \mathfrak{m}^j \setminus \mathfrak{m}^j$.

It is straightforward to change the role of $x_1$ with that one of each $x_i$, $i \in \{2, \ldots, d\}$.

**Example 4.5.** Let $R = k + (x_1^8, x_2^3, x_3^{|x_3^5}, x_4^9)k[x_1, x_2, x_3]$ and let us consider $\mathfrak{m} = (x_1^8, x_2^3, x_3^{|x_3^5}, x_4^9)k[x_1, x_2, x_3]$ the graded maximal ideal of $R$.

Then $a_1 = 8$, $a_2 = 3$ and $a_3 = 5$ and, by Remark 4.4, $x_1^{8j - 1} x_2 x_1^{8j - 1} x_3^5 \in \mathfrak{m}^j \setminus \mathfrak{m}^j$ for every $j \geq 2$.

Let us come back to the 2-dimensional case, that is $R$ subalgebras of $k[x_1, x_2]$.

Our next aim is to show that if a power $t$ (with $t \geq 2$) of the graded maximal ideal $\mathfrak{m}$ is integrally closed, then every other power $l$, with $l \geq t$, of $\mathfrak{m}$ is integrally
closed (cf. Theorem 4.11). As corollaries to this we give a characterization and a sufficient condition for \( \mathfrak{m} \) to be normal (cf. Corollaries 4.13 and 4.15).

To this aim we need a little amount of work. From now on we always denote any power of \( \mathfrak{m} \) by \( J \).

We note that if \( x_1^{a_1} x_2^{b_1} \) and \( x_1^{a_2} x_2^{b_2} \) are different minimal generators of \( J \) as a \( k[x] \)-module, then \( a_1 \neq a_2 \) and \( b_1 \neq b_2 \) and, furthermore, \( a_1 < a_2 \) implies \( b_1 > b_2 \).

We say that \( (a_1, b_1) \ll (a_2, b_2) \) in \( \mathbb{N}^2 \) if \( a_1 < a_2 \).

Let \( x_1^{a_1} x_2^{b_1} \) and \( x_1^{a_2} x_2^{b_2} \) be two generators of \( J \) as a \( k[x] \)-module with \( (a_1, b_1) \ll (a_2, b_2) \) and with the property that if \( x_1^{a_1} x_2^{b_1} \) is any other element of \( J \), then
\[
(b_j - b_i)\alpha + (a_i - a_j)\beta + a_i(b_i - b_j) + b_i(a_j - a_i) \geq 0
\]
that is, \( (\alpha, \beta) \) is not under the straight line in \( \mathbb{R}^2 \) connecting \( (a_i, b_i) \) and \( (a_j, b_j) \).

We call the pair \( (a_i, b_i)(a_j, b_j) \) of elements of \( \mathbb{N}^2 \) as above special pair of generators of \( J \) as a \( k[x] \)-module, \( \text{spg}(J) \).

**Example 4.6.** Let \( R = k + (x_1^2, x_1^2 x_2, x_1 x_2, x_1^2 x_2, x_1^7)k[x_1, x_2] \) and consider \( \mathfrak{m} = (x_1^2, x_1^2 x_2, x_1 x_2, x_1^5 x_2, x_1^7)k[x_1, x_2] \) the graded maximal ideal of \( R \) as in Figure 2. It is easy to check that the only \( \text{spg}(\mathfrak{m}) \) are \( (0, 7)(5, 1) \) and \( (5, 1)(7, 0) \).

![FIGURE 2](image)

**Lemma 4.7.** Let \( J \) be generated by \( x_1^{a_1} x_2^{b_1}, x_1^{a_2} x_2^{b_2}, \ldots, x_1^{a_r} x_2^{b_r} \) as a \( k[x] \)-module with \( (a_1, b_1) \ll (a_2, b_2) \ll \cdots \ll (a_r, b_r) \). Then it is possible to choose \( (a_{i_1}, b_{i_1}) \ll (a_{i_2}, b_{i_2}) \ll \cdots \ll (a_{i_r}, b_{i_r}) \) among the elements of \( \mathbb{N}^2 \) as above such that
\[
(a_{i_1}, b_{i_1})(a_{i_2}, b_{i_2})(a_{i_3}, b_{i_3}), \ldots, (a_{i_{r-1}}, b_{i_{r-1}})(a_{i_r}, b_{i_r}) \text{ are spg}(J) \]
with \( (a_{i_1}, b_{i_1}) = (a_1, b_1) \) and \( (a_{i_r}, b_{i_r}) = (a_r, b_r) \).

Proof. Since the complement to \( \mathbb{N}^2 \) of the set of exponents of all power products in \( R \) is finite, then \( a_1 = 0 = b_r \). If \( (a_1, b_1)(a_r, b_r) \) is not a spg\((J) \) for
for every $i = 2, \ldots, r - 1$, then (by definition of $\text{spg}(J)$) $(a_1, b_1)(a_r, b_r)$ is a $\text{spg}(J)$ and we get the proof.

Hence suppose there exists $i_1 < r$ such that $(a_1, b_1)(a_{i_1}, b_{i_1})$ is a $\text{spg}(J)$. As above, if $(a_{i_1}, b_{i_1})(a_k, b_k)$ is not a $\text{spg}(J)$ for every $k = i_1 + 1, \ldots, r - 1$, then $(a_{i_1}, b_{i_1})(a_r, b_r)$ is a $\text{spg}(J)$ and we get the proof. If not, using the same argument as above we get, after a finite number of steps (as the number of generators of $J$ as a $k[x]$-module is finite), the proof.

**Lemma 4.8.** Let $(a_1, b_1)(a_2, b_2)$ be a $\text{spg}(m)$, then $(a_i, b_i)(a_j, b_j)$ is a $\text{spg}(m')$.

Proof. Since $(a_i, b_i) \subseteq (a_j, b_j)$ whenever $(a_i, b_i) \subseteq (a_j, b_j)$, to get the proof we need that if $x_i^2 x_j^2 \in m'$, then $(a_i, b_i)$ is not under the straight line in $\mathbb{R}^2$ connecting $(a_i, b_i)$ and $(a_j, b_j)$ and that $x_i^2 x_j^2$ and $x_i^2 x_j^2$ are generators for $m'$ as a $k[x]$-module.

Let $x_i^2 x_j^2 \in m'$, hence $(a, b) = \sum_{k=1}^l(a_k, b_k)$ and $x_i^2 x_j^2 \in m$. Since $(a_i, b_i)(a_j, b_j)$ is a $\text{spg}(m)$,

$$(b_j - b_i)a_k + (a_i - a_j)b_k + a_i(b_i - b_j) + b_i(a_j - a_i) \geq 0$$

for every $k = 1, \ldots, l$. Hence

$$(b_j - b_i)\sum_{k=1}^l a_k + (a_i - a_j)\sum_{k=1}^l b_k + a_i(b_i - b_j) + b_i(a_j - a_i) =$$

$l[(b_j - b_i)\sum_{k=1}^l a_k + (a_i - a_j)\sum_{k=1}^l b_k + l(a_i(b_i - b_j) + b_i(a_j - a_i)) =$$

$l[(b_j - b_i)a_1 + (a_i - a_j)b_1 + a_i(b_i - b_j) + b_i(a_j - a_i)] +$ $$(b_j - b_i)a_1 + (a_i - a_j)b_1 + a_i(b_i - b_j) + b_i(a_j - a_i) \geq 0.$$  

Suppose $x_i^2 x_j^2$ is not a generator for $m'$ as a $k[x]$-module, then there exists $x_i^2 x_j^2 \in m'$ such that either $a = a_i$ and $b < b_j$ or $a < a_j$ and $b = b_j$. Since, in this case, $(a, b)$ is under the straight line in $\mathbb{R}^2$ connecting $(a_i, b_i)$ and $(a_j, b_j)$, we get

$$(b_j - b_i)a + (a_i - a_j)b + a_i(b_i - b_j) + b_i(a_j - a_i) < 0$$

that is a contradiction to what we proved above, since if $x_i^2 x_j^2 \in m'$, then $(a_i, b_i)$ cannot be under the straight line in $\mathbb{R}^2$ connecting $(a_i, b_i)$ and $(a_j, b_j)$.

**Corollary 4.9.** If $(a_{i_1}, b_{i_1}) \subseteq (a_{i_2}, b_{i_2}) \subseteq \cdots \subseteq (a_{i_r}, b_{i_r})$ are as in Lemma 4.7 with $J = m$, then $(a_{i_1}, b_{i_1})(a_{i_2}, b_{i_2})(a_{i_3}, b_{i_3})(a_{i_4}, b_{i_4}), \ldots, (a_{i_{r-1}}, b_{i_{r-1}})(a_{i_r}, b_{i_r})$ are $\text{spg}(m')$ (with $(a_{i_1}, b_{i_1}) = (a_1, b_1)$ and $(a_{i_r}, b_{i_r}) = (a_r, b_r)$).

**Remark 4.10.** Let $(a_i, b_i)(a_j, b_j)$ be a $\text{spg}(m')$ and let $r(X, Y) = (b_j - b_i)X + (a_i - a_j)Y + l[a_i(b_j - b_i) + b_i(a_j - a_i)]$ be the straight line in $\mathbb{R}^2$ connecting $(a_i, b_i)$ and $(a_j, b_j)$. It is straightforward to prove that for every $k$ such that $0 \leq k < l$, the integer point $((l - k)a_i, (l - k)b_i) + (ka_j, kb_j) = ((l - k)a_i + ka_j, (l - k)b_i + kb_j)$ is on the straight line in $\mathbb{R}^2$ with equation $r(X, Y)$.  

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Let \( m = (x_1^{a_1}x_2^{b_1}, \ldots, x_1^{a_n}x_2^{b_n})k[x_1, x_2] \) and \( x^\gamma \in \overline{m^l} \) (hence \( l \geq 2 \)) with \( \gamma = (\gamma_1, \gamma_2) \). By Proposition 3.2, \( \gamma \) is an integer point in the convex hull of the union of the set \( b + \mathbb{N}d \), where \( b \) is an exponent of an element in \( m^l \). Hence, by Corollary 4.9, there exists \((a_i, b_i)\) such that \( (\gamma_1, \gamma_2) \) is not under the straight line in \( \mathbb{R}^2 \) connecting \((a_i, b_i)\) and \((a_{i+1}, b_{i+1})\) and such that \( a_i < \gamma_1 < a_{i+1} \).

So \( \gamma \) is in the triangle in \( \mathbb{R}^2 \) with vertices \((a_i, b_i)\), \((a_{i+1}, b_{i+1})\), \((a_{i+1}, b_i)\) (we note that \( \gamma \) cannot be out of the triangle since \( x^\gamma \notin m^l \) and \( x_1^{a_i}x_2^{b_i}, x_1^{a_{i+1}}x_2^{b_{i+1}} \) are generators for \( m^l \) as a \( k[x] \)-module by the proof of Lemma 4.8).

Finally, since \( \gamma \) is in the triangle in \( \mathbb{R}^2 \) with vertices \((a_i, b_i)\), \((a_{i+1}, b_{i+1})\), \((a_{i+1}, b_i)\) and since (cf. Remark 4.10), for every \( k \) with \( 1 \leq k < l \), \( ((l-k)a_i, (l-k)b_i) + (ka_j, kb_j) \) is in the straight line connecting \((a_i, b_i)\) and \((a_{i+1}, b_{i+1})\), we get

\[
\gamma = (l-i)a_i, (l-k)b_i) + (ka_j, kb_j), ((l-k)a_i + ka_j, lb_i)
\]

or

\[
\gamma = (l-i)a_i, (l-k)b_i) + (ka_j, kb_j), (l-k)a_i + ka_j, lb_i)
\]

**Theorem 4.11.** Let \( m = (x_1^{a_1}x_2^{b_1}, \ldots, x_1^{a_n}x_2^{b_n})k[x_1, x_2] \) and suppose there exists \( t \geq 2 \) such that \( m^t = \overline{m^t} \). Then \( m^t = \overline{m^t} \) for every \( l \geq t \).

Proof. It is enough to prove that if \( m^t = \overline{m^t} \) with \( t \geq 2 \), then \( m^{t+1} = \overline{m^{t+1}} \).

Suppose \( x^\gamma \in \overline{m^{t+1}} \) and let \( l = t + 1 \) (hence \( l \geq 3 \)). By what is written above, for a fixed \( k \) with \( 1 \leq k < l \) we have either

\[
\gamma = (l-i)a_i, (l-k)b_i) + (ka_j, kb_j), ((l-k)a_i + ka_j, lb_i)
\]

or

\[
\gamma = (l-i)a_i, (l-k)b_i) + (ka_j, kb_j), (l-a_j, (l-k)b_i + kb_j).
\]

Let us consider the first case. Hence

\[
(\gamma_1, \gamma_2) = \lambda_1 ((l-i)a_i, (l-k)b_i) + \lambda_2 ((l-k)a_i + ka_j, (l-k)b_i + kb_j) + \lambda_3 ((l-k)a_i + ka_j, lb_i),
\]

\[
\lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \in \mathbb{Q} \geq 0.
\]

Let \( \delta = (\delta_1, \delta_2) = (\gamma_1, \gamma_2) - (a_i, b_i) \). Since \( x^\delta \notin m^t \), necessary \( x^\delta \notin m^{t-1} \). But

\[
\delta = (\delta_1, \delta_2) = \lambda_1 ((l-1)a_i, (l-1)b_i) + \lambda_2 ((l-k-1)a_i + ka_j, (l-k-1)b_i + kb_j) + \lambda_3 ((l-k-1)a_i + ka_j, (l-1)b_i), \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \in \mathbb{Q} \geq 0,
\]

hence
\( \delta \) is in the triangle in \( \mathbb{R}^2 \) with vertices \((l - 1)a_i, (l - 1)b_i), ((l - k - 1)a_i, (l - k - 1)b_i)\), \((ka_j, kb_j), ((l - k - 1)a_i + ka_j, (l - 1)b_i)\),

that is \( \delta \) is an integer point in the convex hull of the union of the set \( b + \mathbb{N}^d \),

where \( b \) is an exponent of an element in \( m^{l-1} \). By \( x^{\delta} \in R \) and Proposition 3.2,

we get \( x^{\delta} \in m^{l-1} = m^{l-1} \). Absurd.

Similarly we get the proof for the other case.

The statement of Theorem 4.11 is, in general, not true for other kind of rings \( R \) as in the Introduction (cf. Remark 4.33).

\textbf{Remark 4.12.} We note that it is not true in general that if \( m^t = \overline{m} \), then \( m^l = \overline{m} \) for some \( l < t \). Indeed, let \( n \geq 1 \) and \( R = k + (x^n, x^{n-1}x_2, x_1x_2^{n-1}, x_2^n)k[x_1, x_2] \).

In Remark 4.20 we show that \( \overline{m^k} = m^k \) if and only if \( k \geq n - 2 \).

As corollary to Theorem 4.11 and by \( m = \overline{m} \), we get a criterion for \( m \) to be normal.

\textbf{Corollary 4.13.} The graded maximal ideal \( m \) is normal if and only if \( m^2 = \overline{m^2} \).

\textbf{Example 4.14.} Let \( R = k + (x^8, x_1x_2^6, x_1^2x_2^5, x_1^4x_2^4, x_1^8x_2^2, x_1x_2^9, x_1^{11})k[x_1, x_2] \) as in Figure 4. By Proposition 3.2,

\[
\begin{align*}
m^2 &= (x_1^{16}, x_1x_2^{14}, x_1^2x_2^{12}, x_1^3x_2^{11}, x_1^4x_2^{10}, x_1^6x_2^9, x_1^8x_2^8, x_1^{10}x_2^7, x_1^{11}x_2^6, x_1^{13}x_2^5, x_1^{15}x_2^4, x_1^{17}x_2^3, \\
x_1^{18}x_2^2, x_1^{20}x_2, x_1^{22})k[x_1, x_2] = \overline{m^2}
\end{align*}
\]

and, by Corollary 4.13, we get that \( m \) is normal.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{}
\end{figure}

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We note that in the Example 4.14, $x_1^7 x_2^3 \notin R$, while (7, 3) is an integer point in the convex hull of the union of the set $b + \mathbb{N}^d$, where $b$ is an exponent of an element in $m$.

By the proof of Theorem 4.11, we get the following corollary.

**Corollary 4.15.** If for every integer point $a$ in the convex hull of the union of the set $b + \mathbb{N}^d$, where $b$ is an exponent of an element in $m$, we get $x^a \in m$, then $m$ is normal.

**Example 4.16.** Let be $k + (x_1^7, x_1^2 x_2, x_1^3 x_2^6, x_1^3 x_2^5, x_1^7 x_2, x_1^1 x_2^5, x_1^1 x_2^2, x_1^18 x_2, x_1^22 x_2)[x_1, x_2]$.

Since for every integer point $a$ in the convex hull of the union of the set $b + \mathbb{N}^d$, where $b$ is an exponent of an element in $m = (x_1^7, x_1^2 x_2, x_1^3 x_2^6, x_1^3 x_2^5, x_1^7 x_2, x_1^1 x_2^5, x_1^1 x_2^2, x_1^18 x_2, x_1^22 x_2)[x_1, x_2]$, we get $x^a \in m$, then, by Corollary 4.15, $m$ is normal.

### 4.1.1 A class of examples

Let $n = (x_1, x_2)$ be the graded maximal ideal of the polynomial ring $k[x_1, x_2] = k[x]$. We look for rings $R = k + (x_1^{h_1}, \ldots, x_1^{h_l})k[x]$ with graded maximal ideal $m = (x_1^{h_1}, \ldots, x_1^{h_l})k[x]$ such that $m^2 = n^{2n}$ ($n \geq 1$). Indeed by Proposition 3.2 and by Corollary 4.13, we have that $m$ is normal.

Let $a_1 = \min \{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in m\}$ and $a_2 = \min \{\beta_2 \neq 0 \mid x_2^{\beta_2} \in m\}$ as at the beginning of Subsection 3.1. By $m^2 = n^{2n} = (x_1, x_2)^{2n}$, we necessarily have that $a_1 = n = b_2$ and that, if $x^r = x_1^{\alpha_1} x_2^{\beta_2}$ is a generator for $m$ as a $k[x]$-module, then $r_1 + r_2 \geq n$. Furthermore if $r_1 + r_2 > n$, then this generator is uninteresting in our discussion as if $(x_1^{\alpha_1}, x_2^{\beta_2})^2 k[x] = (x_1, x_2)^{2n}$ then $a := (\{x_1^{\alpha_1}, \ldots, x_1^{h_l}\} \setminus \{x^r\})^2 k[x]$ is equal to $(x_1, x_2)^{2n}$. Indeed if there exists $x_1 c_1 x_2^2 \in m^2 \setminus a$ with $x_1 x_2^2$ generator for $m^2$ as a $k[x]$-module, then $x_1 c_1 x_2^2 = (x_1 x_2^2)(x_1^{h_1} x_2^{h_2})$. This is absurd as $2n = c_1 + c_2 = r_1 + r_2 + b_1 + b_2 > 2n$. Hence we can assume $r_1 + r_2 = n$.

Finally by Proposition 4.1 and Remark 4.2, $x_1^{n-1} x_2, x_1 x_2^{n-1} \in R$. Moreover, since we can suppose $r_1 + r_2 = n$ whenever $x^r = x_1^{\alpha_1} x_2^{\beta_2}$ is a generator for $m$ as a $k[x]$-module, we have that $x_1^{n-1} x_2, x_1 x_2^{n-1}$ are generators of $m$ as a $k[x]$-module.

By what is written above we can translate the problem to a merely combinatorial problem just considering the powers of the $x_2$’s in the generators of $m$ as a $k[x]$-module.

Indeed we look for a class of sets $X$ with $\{0, 1, n - 1, n\} \subseteq X \subseteq \{0, 1, \ldots, n\}$ such that $2X := X + X = \{0, 1, \ldots, 2n\}$.

From now on, given two integers $a$ and $b$ with $a \leq b$, we denote the set of integers between $a$ and $b$ (included) by $[a, b]$.

**Proposition 4.17.** Let $X = \{0, 1, \ldots, h_1 - 1, h_1, h_2, \ldots, h_z = n - h_1, n - h_1 + 1, \ldots, n\}$ with $h_1 \geq 1$ and $h_{i+1} - h_i \leq h_1 + 1$ for every $i \in [1, z - 1]$ (*). Then $2X = [0, 2n]$.

Proof. We show that for every $x \in [0, 2n]$, there exist $x_1, x_2 \in X$ such that $x_1 + x_2 = x$.

If $x \in [0, h_1]$, then $x_1 = x$ and $x_2 = 0$.

If $x \in [h_1, h_2]$, then there exists $i$ such that $h_i \leq x \leq h_{i+1}$. If $x = h_i$ or $x = h_{i+1}$, then $x_1 = x$ and $x_2 = 0$. Suppose hence that $h_i < x < h_{i+1}$, that is
Proposition 4.19. Then we can assume \( pX \) maximal ideal. Theorem 4.11, this implies

So \( x = h_1 + h \) with \( 0 \leq h \leq h_1 \) and we can assume \( x_1 = h_1 \) and \( x_2 = h \).

If \( x \in [h_x, \alpha] \), then \( x_1 = x \) and \( x_2 = 0 \).

If \( x \in [n, n + h_1] \), then \( x = n + h \) with \( 0 \leq h \leq h_1 \). Hence \( x_1 = n \) and \( x_2 = h \).

If \( x \in [n + h_1, n + h_2] \), then there exists \( i \) such that \( n + h_i \leq x \leq n + h_{i+1} \). If

\[ x = n + h_i, \] then \( x_1 = n \) and \( x_2 = h_i \). Suppose hence that \( n + h_i < x \leq n + h_{i+1} \).

So \( n + h_i - h_{i+1} < x - h_{i+1} \leq n \). By (*) \( n - h_1 - 1 \leq n + h_i - h_{i+1} \) and this implies \( n - h_1 - 1 < x - h_{i+1} \leq n \). Hence \( x - h_{i+1} \in X \) and we can assume \( x_1 = h_{i+1} \) and \( x_2 = x - h_{i+1} \).

Finally if \( x \in [n + h_x, 2n] = [2n - h_1, 2n] \), then \( x = 2n - h_1 + h \) with \( 0 \leq h \leq h_1 \). Since \( n - h_1 \leq n - h_1 + h \leq n \), then \( n - h_1 - h \in X \). Hence \( x_1 = n - h_1 + h \) and \( x_2 = n \).

Corollary 4.18. The graded maximal ideal \( m \) of \( R = k + (x_1^n, x_2^m, \ldots, x_1^n, x_2^m, x_1^n, x_2^m, \ldots, x_1^n, x_2^m, x_1^n, x_2^m, \ldots, x_1^n, x_2^m) \subseteq k[x_1, x_2] \) with \( h_1 \geq 1 \) and \( h_{i+1} - h_i \leq h_1 + 1 \) for every \( i \in [1, z - 1] \) is normal.

We now generalize Proposition 4.17. Indeed we look for rings \( R \) with graded maximal ideal \( m \) such that \( m^n = n^m \), \( (n \geq 1) \). By Proposition 3.2 and by Theorem 4.11, this implies \( m^l = m^n \) for every \( l \geq k \).

As above we have that \( a_1 = n = b_2 \), that \( x_1^{n-1}, x_2, x_1^{n-1} \in R \) are generators of \( m \) as a \( k[x] \)-module and that if \( x_1^i x_2^j \) is a generator for \( m \) as a \( k[x] \)-module, then we can assume \( r_1 + r_2 = n \). Hence we look for a class of sets \( X \) with \( \{0, 1, n - 1, n\} \subseteq X \subseteq \{0, n\} \) such that \( kX = [0, kn] \).

Proposition 4.19. Let \( X \) be a set with \( \{0, 1, n - 1, n\} \subseteq X \subseteq \{0, n\} \) such that \( pX = [0, pn] \). Then for every \( q \geq p \), \( qX = [0, qn] \).

Proof. It is enough to prove the proposition for \( q = p + 1 \). By \( \{0, (p + 1)n\} = \{0\} + pX \cup \{\{n\} + pX\} \subseteq (p + 1)X \), we get the proof.

Remark 4.20. We note that if \( X = \{0, 1, n - 1, n\} \), then \( kX = [0, kn] \) if and only if \( k \geq n - 2 \). Indeed for every \( i = 0, \ldots, n - 2 \), \( [i(n - 1) + (n - i - 2) \cdot 0, i(n + (n - i - 2) \cdot 1)] = [i(n - i, i + 1) n - (i + 2)] \subseteq (n - 2)X \). Hence \( [0, (n - 2)n] = \bigcup_{i=0}^{n-2} [i(n - i, i + 1) n - (i + 2)] \subseteq (n - 2)X \). Moreover \( n - 2 \notin \alpha X \) when \( \alpha \leq n - 3 \).

In particular, if \( k \geq n - 2 \), then \( kX = [0, kn] \) for every set \( X \) such that \( \{0, 1, n - 1, n\} \subseteq X \subseteq \{0, n\} \). Hence if \( m \) is the graded maximal ideal of \( R \) for which \( x_1^n, x_2^{n-1}, x_3^{n-1}, x_4^n, x_5^n \) are part of a set of generators for \( m \) as a \( k[x] \)-module, then \( m^k = m^n \) for every \( k \geq n - 2 \).

As a particular case of Proposition 4.19 we get that the class of set \( X \) as in Proposition 4.17, satisfies \( kX = [0, kn] \) for every \( k \geq 2 \).

Now we find a class of sets \( X \) such that \( kX = [0, kn] \) but, in general, \( (k - 1)X \neq [0, (k - 1)n] \). To this aim we generalize the class \( X \) of Proposition 4.17.
Proposition 4.21. Let $X = \{0, 1, \ldots, h_1 - 1, h_1, h_2, \ldots, h_z = n - h_1, n - h_1 - 1, \ldots, n\}$ such that $h_1 \geq 2$ and $h_{i+1} - h_i \leq h_1 + k - 1$ for every $i \in [1, z - 1]$ (**). Then $kX = [0, kn]$. 

Proof. The case $k = 2$ is Proposition 4.17. We suppose hence $k \geq 3$.

Let $Y := \{0, 1, \ldots, h_1, n - h_1, \ldots, n\} \subseteq X$. So $kY = \{0, \ldots, kh_1, n - h_1, \ldots, n + (k - 1)h_1, 2(n - h_1), \ldots, 2n + (k - 2)h_1, \ldots, k(n - h_1), \ldots, kn\} \subseteq kX$.

To get the proof we need to cover all the holes in $kX$ between $in + (k - i)h_1$ and $(i + 1)(n - h_1)$ for every $0 \leq i \leq k - 1$.

We start covering all the holes in $kX$ between $kh_1$ and $n - h_1$. We note that $(k - 1)Y = \{0, \ldots, (k - 1)h_1, n - h_1, \ldots, n + (k - 2)h_1, 2(n - h_1), \ldots, 2n + (k - 3)h_1, \ldots, (k - 1)(n - h_1), \ldots, (k - 1)n\} \subseteq (k - 1)X$.

For every $l \in [1, z - 1]$ we have $\{h_l\} + [0, (k - 1)h_1] = [h_l, h_l + (k - 1)h_1] \subseteq kX$.

By (**) and by $h_1 \geq 2$ and $k \geq 3$, we get $h_{i+1} \leq h_1 + (k - 1)h_1$. Hence $\{h_1, h_2, \ldots, h_z\} + [0, (k - 1)h_1] = [h_1, h_1 + 1, \ldots, h_z + (k - 1)h_1] = [h_1, h_2, \ldots, (k - 1)h_1] \subseteq kX$.

By $k \geq 3$ and $h_z = n - h_1$, we get $h_1 \leq kh_1$ and $n - h_1 \leq h_2 + (k - 1)h_1$.

Making exactly the same sort of calculus as above you can check that $\{h_1, h_2, \ldots, h_z\} + [i(n - h_1), in + (k - i - 1)h_1]$ covers all the holes in $kX$ between $in + (k - i)h_1$ and $(i + 1)(n - h_1)$ for every $1 \leq i \leq k - 1$. Hence $[0, kn] = kX$.

Corollary 4.22. Let $R = k + (x_1^n, x_1^{n-1}x_2, \ldots, x_1^{n-h_1}, x_2^{n-h_2}, x_3^{n-h_3}, \ldots, x_1^{n-h_1}x_2^{h_2}, x_1^{n-h_1}x_2^{-1}x_2^{h_2} \ldots, z_1^{n-h_1}x_2^{h_2}, x_3^{n-h_3}, \ldots, z_2^{n-h_1}x_2^{h_2})k[x_1, x_2]$ with $h_1 \geq 2$, $h_{i+1} - h_i \leq h_1 + k - 1$ for every $i \in [1, z - 1]$. Then for every $l \geq k$, the $l$-th power of the graded maximal ideal $m$ of $R$ is integrally closed.

Remark 4.23. We note that $h_1$ in Proposition 4.21 must be greater than 1. Indeed if $X = \{0, 1, 4, 7, 9, 10\}$ (hence $n = 10$, $k = 3$ and $h_1 = 1$), then 22, 25 $\notin 3X$.

Remark 4.24. If $X$ is a set in the class as in Proposition 4.21, then we know $kX = [0, kn]$. Anyway in general $(k - 1)X \neq [0, (k - 1)n]$ (in particular the converse to Proposition 4.19 does not hold). Indeed let $X = \{0, 1, 2, 6, 10, 12, 13, 14\}$ (hence $n = 14$, $h_1 = 2$ and $k = 3$). By Proposition 4.21, $3X = [0, 3n] = [0, 42]$. Anyway 5, 9, 17 $\notin 2X$.

Let $V = [0, n]$ and $k \geq 2$ and let us consider the collection $\Delta_k$ of subset of $V$ such that $F \in \Delta_k$ if and only if 0, 1, $n - 1, n \notin F$ and $k(V \setminus F) = [0, n]$. Since $F \in \Delta_k$ whenever $F \subseteq G$ for some $G \in \Delta_k$ and since $\{i\} \in \Delta_k$ for every $i \in V \setminus \{0, 1, n - 1, n\}$, then $\Delta_k$ is a (finite) simplicial complex on $V \setminus \{0, 1, n - 1, n\}$.

We first find a lower and an upper bound for dim $\Delta_2 := \text{sup} \{\text{dim}(F) \mid F \in \Delta_2\}$, where dim$(F) := |F| - 1$.

Theorem 4.25. dim $\Delta_2 \leq n - \left\lfloor \frac{1 + \sqrt{n + 1}}{2} \right\rfloor$. Furthermore, if $n \geq 4$ then $n - \sqrt{n + 1} + 1 \leq \text{dim} \Delta_2$. 

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Proof. Let $F$ be a face in $\Delta_2$ with $|V \setminus F| = m$. As $F$ is a face, $|2(V \setminus F)| = \lfloor 0, 2n \rfloor = 2n + 1$. Since there are no more than $\lfloor (m - 1)m/2 \rfloor$ sums of mutually different numbers from $X$ along with no more than $m$ sums of equal numbers, we get that $2n + 1 \leq \frac{m(m + 1)}{2}$, hence $m \geq \left\lfloor \frac{-1 + \sqrt{1 + 4m + m^2}}{2} \right\rfloor$. So $\dim \Delta_2 \leq (n+1) - \left\lfloor \frac{-1 + \sqrt{1 + 4(n+1) + (n+1)^2}}{2} \right\rfloor - 1 = n - \left\lfloor \frac{-1 + \sqrt{1 + 4n + n^2}}{2} \right\rfloor$.

Let now $n \geq 4$ and let us consider $X = \{0, 1, \ldots, h_1 - 1, h_1, h_2, \ldots, h_z = n - h_1, n - h_1 + 1, \ldots, n\}$ with $h_1 \geq 1, h_{i+1} - h_i = 1 + 1$ for every $i \in [1, z - 2]$ (note that $z - 2 \geq 1$ as $n \geq 4$) and $h_z - h_{z-1} \leq h_{z+1}$. By Proposition 4.17, $2X = \{0, 2n\}$. Since in $X$ we exclude $2(h_1 + 1) + \left\lfloor \frac{n-2(h_1 + 1)}{h_1 + k -1} \right\rfloor - 2$ integers from $[0, n]$, we get $n - 2(h_1 + 1) - \left\lfloor \frac{n-2(h_1 + 1)}{h_1 + k -1} \right\rfloor$ number of vertices in a maximal face of $\Delta_2$.

Let us consider $f(h_1) := n - 2(h_1 + 1) - \frac{n}{h_1 + k -1} + 2$ as a function of $h_1$. The derivative $f'(h_1) = 0$ if and only if $h_1 = \sqrt{\frac{n}{2}} - 1$. Since $f(\sqrt{\frac{n}{2}} - 1) = n - \sqrt{8n} + 2$ and $-\left\lfloor \frac{n}{h_1 + k -1} \right\rfloor \geq -\left\lfloor \frac{n}{n + k} \right\rfloor \geq -\left\lfloor \frac{1}{n + k} \right\rfloor - 1$, we have $\dim \Delta_2 \geq n - \sqrt{8n} + 2$.

Remark 4.26. We note that since, for $n \gg 0$, $n - \sqrt{8n} + 2 \approx n - \sqrt{4n}$, the class of sets $X$ as in the second part of the proof of Theorem 4.25 is a almost extremal class of examples.

Now we generalize part of Theorem 4.25 finding a lower bound for $\Delta_k$.

Proposition 4.27. Let $n \geq 4$. Then $\dim \Delta_k \geq n - 2(\sqrt{\frac{n}{2}} + k - 2) - k - \frac{n + 2k - 4}{\sqrt{\frac{n}{2}} + k -2} - 3$

Proof. Let $X = \{0, 1, \ldots, h_1 - 1, h_1, h_2, \ldots, h_z = n - h_1, n - h_1 + 1, \ldots, n\}$ with $h_1 \geq 2, h_{i+1} - h_i = h_1 + k -1$ for every $i \in [1, z - 2]$ and $h_z - h_{z-1} \leq h_{z+1}$. By Proposition 4.21, $kX = \{0, kn\}$. Since in $X$ we exclude $2(h_1 + 1) + \left\lfloor \frac{n-2(h_1 + 1)}{h_1 + k -1} \right\rfloor - 2$ integers from $[0, n]$, we get $n - 2(h_1 + 1) - \left\lfloor \frac{n-2(h_1 + 1)}{h_1 + k -1} \right\rfloor$ number of vertices in a maximal face of $\Delta_k$.

Let us consider $f(h_1) := n - 2(h_1 + 1) - \frac{n-2(h_1 + 1)}{h_1 + k -1} + 2$ as a function of $h_1$. Using the same argument as in Theorem 4.25, we get the proof.

We note that for $n = 1, 2, 3$ then $\Delta_k = \{\emptyset\}$ for every $k \geq 2$; for $n = 4$, $\Delta_k = \{\emptyset, \{2\}\}$ for every $k \geq 2$; for $n = 5$, $\Delta_2 = \{\emptyset, \{2\}, \{3\}\}$ and $\Delta_k = \{\emptyset, \{2\}, \{3\}\}$ for every $k \geq 3$; for $n = 6$, $\Delta_2 = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 4\}\}$, $\Delta_3 = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, $\Delta_k = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ for every $k \geq 4$.

Our next aim is to prove that if $k \geq 3$, then $\Delta_k$ is connected for every $n$ and that $\Delta_2$ is connected for every $n \neq 5, 6$. We need to show that if $n \geq 7$ then for every $a, b \in V \setminus \{0, 1, n - 1, n\}$ there exist $\gamma_1, \ldots, \gamma_m \in V \setminus \{0, 1, n - 1, n\}$ such that $\{a, \gamma_1, \gamma_2, \ldots, \gamma_m, b\}$ are in $\Delta_k$ for every $k \geq 2$.

Lemma 4.28. If $F \in \Delta_i$, then $F \in \Delta_i$ for every $l \geq i$.

Proof. This follows by definition of face of $\Delta_i$ and by Proposition 4.19.
Lemma 4.29. Let \( n \geq 7 \) and \( a, b \in V \setminus \{0, 1, n - 1, n\} \) with \( a < b \). Then \( \{a, b\} \in \Delta_k \), with \( k \geq 3 \), and \( \{a, b\} \in \Delta_2 \) if and only if the pair \( (a, b) \) is different from \((2, 3)\) and from \((n - 3, n - 2)\).

Proof. Let \( (a, b) \) be different from \((2, 3)\) and \((n - 3, n - 2)\) (this is possible as \( n \geq 7 \)). We need to prove that \( \{a, b\} \) is a face in \( \Delta_2 \), that is \( 2V \setminus \{a, b\} = [0, 2n] \).

Let \( x \in [0, 2n] \) and suppose first that \( x \leq n \).

If \( x \neq a, b \), then \( x = x + 0 \in 2V \setminus \{a, b\} \). If \( x = a \), then \( x = (x - 1) + 1 \in 2V \setminus \{a, b\} \). Finally if \( x = b \), then \( x = (b - 1) + 1 \in 2V \setminus \{a, b\} \) if \( a \neq b - 1 \) and \( x = (b - 2) + 2 \in 2V \setminus \{a, b\} \) if \( a = b - 1 \).

Suppose now \( n < x \leq 2n - 4 \). Since \( x - n \leq n - 4 \), we get \( n > x - n \), \( n - 1 > x - n + 1 \) and \( n - 2 > x - n + 2 \). Hence we can write \( x \) in at least three different ways as a sum of two natural number less or equal to \( n \). Precisely \( x = n + (x - n) \), \( x = (n - 1) + (x - n + 1) \) and \( x = (n - 2) + (x - n + 2) \). This implies that in at least one of the sums above the two summands are both different from \( a \) and \( b \). Thus \( x \in 2V \setminus \{a, b\} \).

Suppose now \( 2n - 3 = x \leq 2n \). Since \( \{n - 2, n - 1, n\} \subseteq V \setminus \{a, b\} \) or \( \{n - 3, n - 1, n\} \subseteq V \setminus \{a, b\} \), then \( x \in 2V \setminus \{a, b\} \). Hence \( \{a, b\} \in \Delta_2 \).

To get the proof we need to show that \((2, 3)\) and \((n - 3, n - 2)\) are not in \( \Delta_2 \).

Indeed if \( X = [0, n] \setminus \{2, 3\} \) then \([0, 2n] \setminus 2X = \{3\} \) and if \( X = [0, n] \setminus \{n - 3, n - 2\} \) then \([0, 2n] \setminus 2X = \{2n - 3\} \).

Let now \( k \geq 3 \). By the first part of the proof and by Lemma 4.28 we know that if \( a, b \in V \setminus \{0, 1, n - 1, n\} \) with \( (a, b) \) different from \((2, 3)\) and from \((n - 3, n - 2)\), then \( \{a, b\} \in \Delta_k \). Let \( V \setminus \{2, 3\} = \{0, 1, 4, 5, 6, \ldots, n\} \). By \( 3[0, 1, 4, 5, 6] = [0, 12] \), by \( 3[4, n] = [12, 3n] \) and by Lemma 4.28, we have \( \{2, 3\} \in \Delta_k \). Finally let \( V \setminus \{n - 3, n - 2\} = \{0, 1, \ldots, n - 4, n - 1, n\} \). By \( 3[0, n - 4] = [0, 3n - 12] \), by \( 3[n - 5, n - 4, n - 1, n] = [3n - 12, 3n] \) and by Lemma 4.28, we have \( \{n - 3, n - 2\} \in \Delta_k \).

Theorem 4.30. If \( k \geq 3 \), then \( \Delta_k \) is connected for every \( n \). \( \Delta_2 \) is connected for every \( n \neq 5, 6 \).

Proof. By what is written before Lemma 4.28, we need to show that if \( n \geq 7 \), then \( \Delta_k \) is connected for every \( k \geq 2 \). The case \( k \geq 3 \) is immediate by Lemma 4.29.

Let \( k = 2 \). Given \( a, b \in V \setminus \{0, 1, n - 1, n\} \). If \( (a, b) \neq (2, 3), (n - 3, n - 2) \), then, by Lemma 4.29, \( \{a, b\} \in \Delta_2 \). Otherwise, since \( n \geq 7 \), there exists \( \gamma \in V \setminus \{0, 1, a, b, n - 1, n - 2\} \). By Lemma 4.29, \( \{a, \gamma\} \) and \( \{\gamma, b\} \) are in \( \Delta_2 \).

4.2 The second case

Let us denote the set of all power products in the indeterminates \( x_1, x_2, \ldots, x_d \) of degree \( i \) in \( k[x_1, x_2, \ldots, x_d] \), that is \( \{x_1^i, x_1^{i-1}x_2, \ldots, x_d^i\} \), by \( F_i \). Let \( g_1, g_2, \ldots, g_n \) be positive natural numbers with \( g_1 < g_2 < \cdots < g_n \) and \( \gcd(g_1, g_2, \ldots, g_n) = 1 \).

In this second case we study the integral closure of powers of the graded maximal ideal of \( R = k[F_{g_1}, F_{g_2}, \ldots, F_{g_n}] \).
We define the $k$-algebra homomorphism
\[
\psi : k[x_1, x_2, \ldots, x_d] \longrightarrow k[t]
\]
with $\psi(f(x_1, \ldots, x_d)) = f(t, \ldots, t)$.

We also define the $k$-algebra homomorphism
\[
\phi : k[t] \longrightarrow k[x_1, x_2, \ldots, x_d]
\]
with $\phi(l(t)) = l(x_1)$.

**Theorem 4.31.** Let $R = k[F_{g_1}, F_{g_2}, \ldots, F_{g_n}]$ and $T = k[t^{g_1}, t^{g_2}, \ldots, t^{g_n}]$. Let $m = (F_{g_1}, F_{g_2}, \ldots, F_{g_n})$ and $M = (t^{g_1}, t^{g_2}, \ldots, t^{g_n})$ denote the graded maximal ideal of $R$ and $T$ respectively. Then for every natural number $a$, $\overline{m^a} \neq \emptyset$ if and only if $\overline{M^a} \neq \emptyset$.

Proof. It is straightforward to verify that, since $\gcd(g_1, g_2, \ldots, g_n) = 1$, the complements to $\mathbb{N}^d$ and $\mathbb{N}$ respectively of the set of exponents of all power products in $R$ and in $T$ are finite.

Suppose there exists $z \in \overline{m^a}$ for some $a$. By Theorem 3.1, we can suppose that $z$ is a monomial in $k[x_1, x_2, \ldots, x_d]$, that is $z = x_1^{i_1} \cdots x_d^{i_d} 1 \leq i_1 < \cdots < i_d \leq d$.

We note that $\psi(z) = t^{j_1} \cdots t^{j_d} \notin M^a$ (since $z \notin m^a$). We will show that $\psi(z) \in \overline{M^a}$.

By $z \in \overline{m^a}$, there exist $c_1, c_2, \ldots, c_m$ with $c_i \in (m^a)^i$ such that $z^m + c_1 z^{m-1} + \cdots + c_m = 0$. Hence
\[
0 = \psi(z^m + c_1 z^{m-1} + \cdots + c_m) = \psi(z^m) + \psi(c_1) \psi(z^{m-1}) + \cdots + \psi(c_m) = 
\]
\[
\psi(z)^m + \psi(c_1) \psi(z)^{m-1} + \cdots + \psi(c_m).
\]
Since $c_i \in (m^a)^i$, we get $\psi(c_i) \in (M^a)^i$. Thus $\psi(z) \in \overline{M^a}$.

Suppose now $\overline{M^a} \neq \emptyset$ for some $a$. By Theorem 3.1 we can suppose $t^b \in \overline{M^a} \setminus M^a$ for some $b$. Let us consider $x_1^b$. Clearly $x_1^b \notin m^a$ (if not $\psi(x_1^b) = t^b \in M^a$). We will show that $x_1^b \in \overline{m^a}$.

By $t^b \in \overline{m^a}$ there exist $c_1, c_2, \ldots, c_m$ with $c_i \in (m^a)^i$ such that $(t^b)^m + c_1 (t^b)^{m-1} + \cdots + c_m = 0$. Hence
\[
0 = \phi((t^b)^m + c_1 (t^b)^{m-1} + \cdots + c_m) = \phi((t^b)^m) + \phi(c_1) \phi((t^b)^{m-1}) + \cdots + \phi(c_m) = 
\]
\[
(x_1^{b})^m + \phi(c_1) (x_1^b)^{m-1} + \cdots + \phi(c_m).
\]
Since $\phi(c_i) \in (m^a)^i$, we get $x_1^b \in \overline{m^a}$.

**Corollary 4.32.** Let $(R, m)$ and $(T, M)$ be rings as in Theorem 4.31. Then $m$ is normal if and only if $M$ is normal.

A study on normal graded maximal ideal for rings $k[t^{g_1}, t^{g_2}, \ldots, t^{g_n}]$ with $\gcd(g_1, g_2, \ldots, g_n) = 1$ can be found in [1].
Remark 4.33. Let \( R = k[F_6, F_7, F_{11}] \). Then, using Theorem 4.31, it is easy to check that \( m^2 = m^2 \), but \( x_1^2 \not\in m^3 \setminus m^3 \). Hence the statement of Theorem 4.11 is not true for this kind of rings.

Corollary 4.34. Let \( R \) be a ring as above with graded maximal ideal \( m = (F_{g_1}, F_{g_2}, \ldots, F_{g_n}) \).

(i) If \( m \) is normal, then \( g_2 = g_1 + 1 \) and \( g_n < 2g_1 \).

(ii) \( m \) is normal if and only if \( m^{a+1} = \overline{m}^a m \) for every \( a \geq 0 \).

(iii) Let \( m_i \) denote the maximal ideal of \( k[F_{g_1}, F_{g_2}, \ldots, F_{g_i}] \); if \( m_i \) is not normal for some \( i < n \), then \( m_n (= m) \) is not normal.

Proof. (i) This follows by Corollary 4.32 and [1, Proposition 3.1]).

(ii) Use Corollary 4.32 and [1, Theorem 3.5].

(iii) By Corollary 4.32 and [1, Theorem 3.14].

In the 3-generated case (\( n = 3 \)) we can give a concrete characterization for normal graded maximal ideal \( m \). Suppose \( g_2 = g_1 + 1 \) and \( g_3 < 2g_1 \) (if one of these two conditions is not satisfied, then \( m \) is not normal by (i) of Corollary 4.34) and let \( \alpha \) be the unique integer such that \( (\alpha - 1)g_3 < \alpha g_2 \) and \( \alpha g_3 \geq (\alpha + 1)g_1 \).

Corollary 4.35. Let \( R, g_2 \) and \( g_3 \) as above. Then \( m \) is normal if and only if \( \alpha g_3 \leq (\alpha + 1)g_2 \).

Proof. This follows by Corollary 4.32 and [1, Theorem 3.25].

Example 4.36. Let \( k[F_{10}, F_{11}, F_{17}] = k[x_1^{10}, x_1^{11}, x_1^{17}, x_2^{10}, \ldots, x_d^{10}, x_1^{11}, x_2^{11}, \ldots, x_d^{11}, x_1^{17}, x_2^{17}, \ldots, x_d^{17}] \).

Then \( \alpha = 2 \) and, by Corollary 4.35, we get \( m = (F_{10}, F_{11}, F_{17}) \) is not normal. Using the same argument as above it is easy to check that in \( k[F_{10}, F_{11}, F_{16}] \), \( m = (F_{10}, F_{11}, F_{17}) \) is normal.

References
