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Vincenzo Micale*

Abstract

In this paper we study the integral closure of ideals of monomial subrings R of $k[x_1, x_2, \ldots, x_d]$ spanned by a finite set of distinct monomials of the polynomial ring. We generalize a well known result for monomial ideals in the polynomial ring to rings R as above proving that the integral closure of a monomial ideal is monomial and we give a geometric description of the integral closure of an ideal of R. Then we focus our attention to the study of the integral closure of powers of the graded maximal ideal of R in two particular cases.

MSC: 13B25; 13F20

1 Introduction

Let $H = \{h_1, h_2, \ldots, h_m\}$ be a finite set of distinct monomials in $k[x_1, x_2, \ldots, x_d]$ and let $R = k[H] = k[h_1, h_2, \ldots, h_m] \subseteq k[x_1, x_2, \ldots, x_d]$ be the monomial subring spanned by H. Furthermore we suppose that the complement to $\mathbb{N}^d$ of the set of exponents of all monomials in R is finite.

In this paper we study the integral closure of ideals in R. In Section 2 we give the concepts of multidegree of a monomial and of integral closure and normality of an ideal. In Section 3 we generalize a well known result for monomial ideals in the polynomial ring to rings R as above proving that the integral closure of a monomial ideal is monomial and we give a geometric description of the integral closure of an ideal of R. In Section 4 we focus our attention to the study of the integral closure of powers of the graded maximal ideal of R in two particular cases.

2 Preliminaries

Let R be a ring as in the Introduction. We can associate to every monomial $ux_1^{a_1} \cdots x_d^{a_d}$ in R, with $u \in k \setminus \{0\}$, the power product $x_1^{a_1} \cdots x_d^{a_d}$. Let m be a monomial in R and let $x_1^{a_1} \cdots x_d^{a_d}$ be the associated power product. We call

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$(a_1, a_2, \dots, a_d) \in \mathbb{N}^d$ *the multidegree* of m and we denote it by $\mathrm{mdeg}(m)$. If m_1 and m_2 are monomials in R, then m_1 and m_2 have the same multidegree if there exists $u \in k \setminus \{0\}$ such that $m_1 = um_2$.

Let I be an ideal of R. We denote by $\overline{I}$ the integral closure $\{z \in R \mid z^n + r_1 z^{n-1} + \cdots + r_n = 0, \text{for some } r_i \in I^i\}$.

By definition $I \subseteq \overline{I}$ and, in general, it may happen that $I \subsetneq \overline{I}$. An ideal I in R is called *normal* if $I^j = \overline{I^j}$ for every $j \geq 1$.

3 Integral closure of monomial ideals

Now we generalize a well known result for monomial ideals in the polynomial ring to rings R as in the Introduction; we have used some ideas from [2].

Theorem 3.1. *Let I be a monomial ideal in R. Then $\overline{I}$ is a monomial ideal.*

Proof. Since it is well known that $\overline{I}$ is an ideal in R, we only need to prove that $\overline{I}$ is generated by monomials. Let $z \in \overline{I}$, $z = m_1 + \cdots + m_s$, where the m_i's are monomials with different multidegrees. We want to prove that $m_i \in \overline{I}$ for every $i = 1, \dots, s$. By induction, it suffices to verify that $m_i \in \overline{I}$ for some i since $\overline{I}$ is an ideal in R. To this aim we prove that there exists $N > 0$ such that $m_i^N \in I^N$ for some $i = 1, \dots s$ (hence $m_i \in \overline{I}$ since it is root of $Z^N - m_i^N = 0$).

From the definition of integral closure, we have

$$(m_1 + \cdots + m_s)^n + l_1(m_1 + \cdots + m_s)^{n-1} + \cdots + l_n = 0, \ (l_i \in I^i).$$

Let us consider the multidegree of m_1^n. There must exist another term in the equation above which has the same multidegree and it must be of the form $b_{i_1} m_1^{j_{1,1}} \cdots m_s^{j_{1,s}}$, where $b_{i_1} \in I^{i_1}$ and $\sum_{k=1}^s j_{1,k} = n - i_1$ (we note that if $s = 1$, then $z = m_1 \in \overline{I}$).

Since the set of elements of the same multidegree as m_1^n is a 1-dimensional vector space over $\mathbb{R}$ and since $b_{i_1} m_1^{j_{1,1}} \cdots m_s^{j_{1,s}}$ and m_1^n have the same multidegree, we get $ub_{i_1} m_1^{j_{1,1}} \cdots m_s^{j_{1,s}} = m_1^n$ for some $u \in k \setminus \{0\}$. Using the same argument as above, we get that for every $v = 1, \dots, s$, $m_v^n = c_{i_v} m_1^{j_{v,1}} \cdots m_s^{j_{v,s}}$ with $c_{i_v} \in I^{i_v}$, $\sum_{k=1}^s j_{v,k} = n - i_v$.

Since m_v^n is a monomial, we get (after cancelling out common terms) that $m_v^{n_{1,v}} = c_{i_v} \prod_{k \neq v} m_k^{j_{v,k}}$ for some $n_{1,v}$. We note that $0 \leq j_{v,k} < n_{1,v}$ for every v and k. Indeed if, for example, $j_{2,1} = n_{1,1}$, then $m_1^{n_{1,1}} = c_{i_1} m_2^{n_{1,1}}$ with $c_{i_1} \in k$, whence $\mathrm{mdeg}(m_1^{n_{1,1}}) = \mathrm{mdeg}(m_2^{n_{1,1}})$, that is $\mathrm{mdeg}(m_1) = \mathrm{mdeg}(m_2)$. A contradiction.

Hence we have a system of s equalities

$$\begin{cases} m_1^{n_{1,1}} = c_{i_1} \prod_{k \neq 1} m_k^{j_{1,k}} \\ m_2^{n_{1,2}} = c_{i_2} \prod_{k \neq 2} m_k^{j_{2,k}} \\ \cdots \\ m_s^{n_{1,s}} = c_{i_s} \prod_{k \neq s} m_k^{j_{s,k}} \end{cases}$$

We use an induction on s to prove that there exists $N > 0$ such that $m_i^N \in I^N$ for some $i = 1, \dots s$. If $s = 1$, then $m_1^{n_{1,1}} = c_{n_{1,1}} \in I^{n_{1,1}}$. Suppose now such N exists for systems as above with $s - 1$ equalities.

Consider the system above. For every $v = 2, \dots, s$, we first raise, $m_v^{n_{1,v}}$ to $n_{1,1}$ and $m_1^{n_{1,1}}$ to $j_{v,1}$, then we substitute $m_1^{n_{1,1}j_{v,1}}$ in $m_v^{n_{1,v}n_{1,1}}$ with $(c_{i_1} \prod_{k \neq 1} m_k^{j_{1,k}})^{j_{v,1}}$ and finally we cancel out common terms (it is easy to check that, by $j_{v,k} < n_{1,v}$ for every v and k, we never cancel out $m_v^{n_{1,v}n_{1,1}}$). Hence, for every $v = 2, \dots, s$, we get $m_v^{n_{2,v}} = d_{i_v} m_2^{k_{v,2}} \cdots \widehat{m_v^{k_{v,v}}} \cdots m_s^{k_{v,s}}$ for some $n_{2,v}$ and with $d_{i_v} \in I^{n_{2,v} - (k_{v,2} + \cdots + k_{v,v-1} + k_{v,v+1} + \cdots + k_{v,s})}$ (where $\widehat{m_v^{k_{v,v}}}$ means that we delete $m_v^{k_{v,v}}$ in the product). Using induction we get the proof.

3.1 A geometric description

Our next aim is to have a geometric description of the integral closure of ideals in R.

Let $a_i \in \mathbb{N}^d$, $i = 1, \dots, r$ and let

$$\operatorname{conv}(a_1, \dots, a_r) = \{\sum_{i=1}^{r} \lambda_i a_i \mid \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \in \mathbb{Q}_{\geq 0}\}$$

be the *convex hull* (over the rational numbers) of $a_1, \dots, a_r$.

For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Proposition 3.2. *Let I be a monomial ideal in R generated by $x^{a_1}, \dots, x^{a_r}$. Then the exponents a such that a monomial x^a in R belongs to the integral closure $\overline{I}$ are the integer points in the convex hull of the union of the set $b + \mathbb{N}^d$, where b is an exponent of an element in I.*

Proof. Let $x^a \in R$ such that a is in the convex hull of the union of the set $b + \mathbb{N}^d$, where b is an exponent of an element in I. Hence $a = \sum_{i=1}^r \lambda_i a_i$ with $x^{a_i} \in I$, $\sum_{i=1}^r \lambda_i = 1$ and $\lambda_i \in \mathbb{Q}_{\geq 0}$. Let m an integer such that $m\lambda_i \in \mathbb{N}$ for every $i = 1, \dots, r$. Then, by $\sum_{i=1}^r m\lambda_i = m$, we get $(x^a)^m = (x^{\sum_{i=1}^r \lambda_i a_i})^m = (x^{a_1})^{m\lambda_1} \cdots (x^{a_r})^{m\lambda_r} \in I^m$. So $x^a \in \overline{I}$.

Vice versa if $x^a \in \overline{I}$, then, by the proof of Theorem 3.1, we get $x^{am} \in I^m$, that is $x^{am} = x^{b_1} \cdots x^{b_m}$ with $x^{b_i} \in I$. By $a = \sum_{i=1}^m \frac{1}{m} b_i$, we get that a is in the convex hull of the union of the set $b + \mathbb{N}^d$, where b is an exponent of an element in I.

4 Integral closure of powers of the graded maximal ideal $\mathfrak{m}$ in special cases

We recall that we are interested in monomial subrings $R = k[h_1, h_2, \dots, h_m]$ of $k[x_1, x_2, \dots, x_d]$ spanned by a finite set H of monomials and such that the complement to $\mathbb{N}^d$ of the set of exponents of all power products in R is finite. R is a graded ring with graded maximal ideal $\mathfrak{m} = (h_1, h_2, \dots, h_m)$.

In this section we focus our attention to the study of the integral closure of powers of the graded maximal ideal $\mathfrak{m}$ of R in two particular cases. We remark that, by definition of integral closure of an ideal, $\mathfrak{m} = \overline{\mathfrak{m}}$.

4.1 The first case

In this first case we restrict to rings $R = k + (x^{\delta_1}, \ldots, x^{\delta_t})k[x]$, subalgebras of $k[x_1, x_2] = k[x]$ such that the complement to $\mathbb{N}^2$ of the set of exponents of all power products in R is finite. Let $\mathfrak{m}$ denote the graded maximal ideal of R. By $\mathfrak{m} = (x^{\delta_1}, \ldots, x^{\delta_t})k[x]$, we get $\mathfrak{m}^r = ((x^{\delta_1}, \ldots, x^{\delta_t})k[x])^r = (x^{\delta_1}, \ldots, x^{\delta_t})^r k[x]$.

Let

$$a_1 = \min\{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in \mathfrak{m}\},$$

$$a_2 = \min\{\alpha_2 \neq 0 \mid x_1^{\alpha} x_2^{\alpha_2} \in \mathfrak{m}, \text{for some } \alpha < a_1\}$$

As the complement to $\mathbb{N}^2$ of the set of exponents of all power products in R is finite, such a_i exists for $i = 1, 2$. By definition of a_2, we get $x_1^{\gamma} x_2^{a_2} \in \mathfrak{m}$ for some positive integer $\gamma < a_1$.

Proposition 4.1. *Suppose that $a_i \geq 2$, for $i = 1, 2$. If there exists γ as above such that $a_1 - \gamma \geq 2$, then $\overline{\mathfrak{m}^j} \setminus \mathfrak{m}^j \neq \emptyset$ for every $j \geq 2$.*

Proof. Let $j \geq 2$ and let us consider $m_j = x_1^{ja_1 - 1} x_2^{a_2 - 1}$.

By definition of a_1 and by $j \geq 2$, we have $x_1^{ja_1-1} x_2^{a_2-1} \in R$. Since $a_1 - \gamma \geq 2$, $x_1^{ja_1-2} x_2^{a_2} = x_1^{(j-1)a_1}(x_1^{a_1-2} x_2^{a_2}) \in \mathfrak{m}^j$. Finally by $a_2 \geq 2$, $x_1^{ja_1} x_2^{a_2-2} \in \mathfrak{m}^j$.

Since $(ja_1 - 1, a_2 - 1) = \lambda_1(ja_1 - 2, a_2) + \lambda_2(ja_1, a_2 - 2)$ with $\lambda_1 = \lambda_2 = 1/2$, we get, by Proposition 3.2, that $m_j \in \overline{\mathfrak{m}^j}$. But $m_j \notin \mathfrak{m}^j$ as, by definition of a_2, $x_1^{\alpha} x_2^{a_2-1} \in \mathfrak{m}^j$ only if $\alpha \geq ja_1$.

Remark 4.2. We can change the role of x_1 with that one of x_2 in Proposition 4.1. Indeed let

$$b_2 = \min\{\beta_2 \neq 0 \mid x_2^{\beta_2} \in \mathfrak{m}\},$$

$$b_1 = \min\{\beta_1 \neq 0 \mid x_1^{\beta_1} x_2^{\beta} \in \mathfrak{m}, \text{for some } \beta < b_2\}.$$

As above such b_i exists for $i = 0, 1$. By definition of b_1, we get $x_1^{b_1} x_2^{\gamma} \in \mathfrak{m}$ for some $\gamma < b_2$.

Using the same argument as for a_i, we get that if there exists γ as above such that $b_2 - \gamma \geq 2$, then $x_1^{b_1-1} x_2^{jb_2-1} \in \overline{\mathfrak{m}^j} \setminus \mathfrak{m}^j$ for every $j \geq 2$.

Example 4.3. Let $R = k + (x_2^9, x_1^2 x_2^7, x_1^3 x_2^6, x_1^5 x_2^3, x_1^9)k[x_1, x_2]$ and let us consider $\mathfrak{m} = (x_2^9, x_1^2 x_2^7, x_1^3 x_2^6, x_1^5 x_2^3, x_1^9)k[x_1, x_2]$ the graded maximal ideal of R as in Figure 1 . Then $a_1 = 9, a_2 = 3, b_1 = 2, b_2 = 9$ and $\{x_1^{j9-1} x_2^2, x_1 x_2^{j9-1}\} \subseteq \overline{\mathfrak{m}^j} \setminus \mathfrak{m}^j$ for every $j \geq 2$.

Remark 4.4. We can generalize Proposition 4.1 to the d-dimensional case. Let $R = k + (x^{\delta_1}, \ldots, x^{\delta_t})k[x]$ be a subalgebra of $k[x_1, x_2, \ldots, x_d] = k[x]$ and let

$$a_1 = \min\{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in \mathfrak{m}\}.$$

Suppose $a_1 \geq 2$ and that there exists i with $2 \leq i \leq d$ such that

$$a_i := \min\{\alpha_i \neq 0 \mid x_1^{\alpha} x_i^{\alpha_i} \in \mathfrak{m}, \text{for some } \alpha < a_1\} \geq 2.$$

By definition of a_i there exists a positive integer $\gamma_i < a_1$ such that $x_1^{\gamma_i} x_i^{a_i} \in \mathfrak{m}$.

Suppose $a_1 - \gamma_i \geq 2$. Since (for every $j \geq 2$)

$$(ja_1-1, 0, \dots, 0, a_i-1, 0, \dots, 0) = \frac{1}{2}(ja_1-2, 0, \dots, 0, a_i, 0, \dots, 0) + \frac{1}{2}(ja_1, 0, \dots, 0, a_i - 2, 0 \dots, 0)$$

and by definition of a_i, we get that for every $j \geq 2$ the element $x_1^{ja_1-1} x_i^{a_i-1} \in \overline{\mathfrak{m}^j} \setminus \mathfrak{m}^j$.

It is straightforward to change the role of x_1 with that one of each x_i, $i \in \{2, \dots, d\}$.

Example 4.5. Let $R = k + (x_1^8, x_2^3, x_1^4 x_3^5, x_3^9)k[x_1, x_2, x_3]$ and let us consider $\mathfrak{m} = (x_1^8, x_2^3, x_1^4 x_3^5, x_3^9)k[x_1, x_2, x_3]$ the graded maximal ideal of R.

Then $a_1 = 8$, $a_2 = 3$ and $a_3 = 5$ and, by Remark 4.4, $x_1^{8j-1} x_2^2, x_1^{8j-1} x_3^4 \in \overline{\mathfrak{m}^j} \setminus \mathfrak{m}^j$ for every $j \geq 2$.

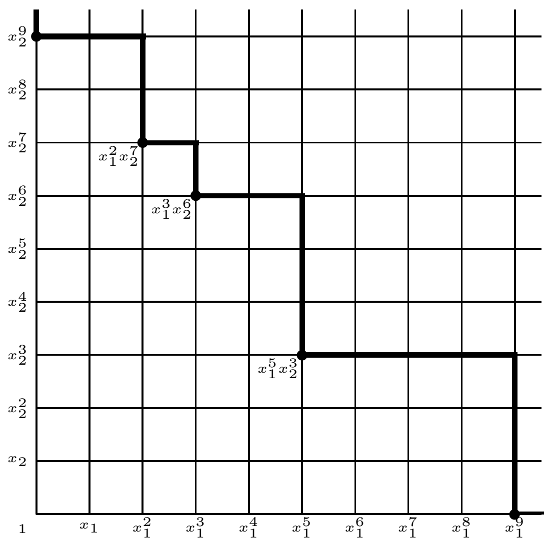

FIGURE 1

Let us come back to the 2-dimensional case, that is R subalgebras of $k[x_1, x_2]$.

Our next aim is to show that if a power t (with $t \geq 2$) of the graded maximal ideal $\mathfrak{m}$ is integrally closed, then every other power l, with $l \geq t$, of $\mathfrak{m}$ is integrally closed (cf. Theorem 4.11). As corollaries to this we give a characterization and a sufficient condition for $\mathfrak{m}$ to be normal (cf. Corollaries 4.13 and 4.15).

To this aim we need a little amount of work. From now on we always denote any power of $\mathfrak{m}$ by J.

We note that if $x_1^{a_i}x_2^{b_i}$ and $x_1^{a_j}x_2^{b_j}$ are different minimal generators of J as a $k[x]$-module, then $a_i \neq a_j$ and $b_i \neq b_j$ and, furthermore, $a_i < a_j$ implies $b_i > b_j$. We say that $(a_i, b_i) \lessdot (a_j, b_j)$ in $\mathbb{N}^2$ if $a_i < a_j$.

Let $x_1^{a_i}x_2^{b_i}$ and $x_1^{a_j}x_2^{b_j}$ be two generators of J as a $k[x]$-module with $(a_i, b_i) \lessdot (a_j, b_j)$ and with the property that if $x_1^{\alpha}x_2^{\beta}$ is any other element of J, then

$$(b_j - b_i)\alpha + (a_i - a_j)\beta + a_i(b_i - b_j) + b_i(a_j - a_i) \geq 0$$

that is, (α, β) is not under the straight line in $\mathbb{R}^2$ connecting (a_i, b_i) and (a_j, b_j).

We call the pair $(a_i, b_i)(a_j, b_j)$ of elements of $\mathbb{N}^2$ as above *special pair of generators of J as a $k[x]$-module*, (spg(J)).

Example 4.6. Let $R = k + (x_2^7, x_1^2x_2^5, x_1^3x_2^4, x_1^5x_2, x_1^7)k[x_1, x_2]$ and consider $\mathfrak{m} = (x_2^7, x_1^2x_2^5, x_1^3x_2^4, x_1^5x_2, x_1^7)k[x_1, x_2]$ the graded maximal ideal of R as in Figure 2. It is easy to check that the only spg$(\mathfrak{m})$ are $(0, 7)(5, 1)$ and $(5, 1)(7, 0)$.

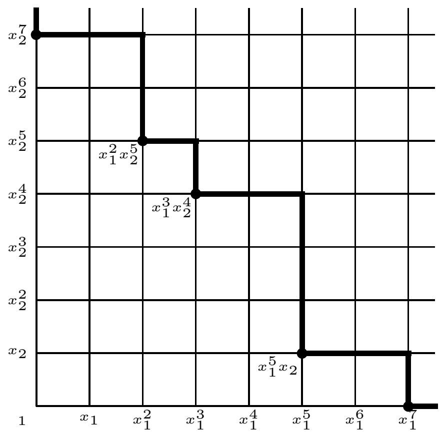

FIGURE 2

Lemma 4.7. *Let J be generated by $x_1^{a_1}x_2^{b_1}, x_1^{a_2}x_2^{b_2}, \ldots, x_1^{a_r}x_2^{b_r}$ as a $k[x]$-module with $(a_1, b_1) \lessdot (a_2, b_2) \lessdot \cdots \lessdot (a_r, b_r)$. Then it is possible to choose $(a_{i_1}, b_{i_1}) \lessdot (a_{i_2}, b_{i_2}) \lessdot \cdots \lessdot (a_{i_s}, b_{i_s})$ among the elements of $\mathbb{N}^2$ as above such that $(a_{i_1}, b_{i_1})(a_{i_2}, b_{i_2}), (a_{i_2}, b_{i_2})(a_{i_3}, b_{i_3}), \ldots, (a_{i_{s-1}}, b_{i_{s-1}})(a_{i_s}, b_{i_s})$ are spg(J) with $(a_{i_1}, b_{i_1}) = (a_1, b_1)$ and $(a_{i_s}, b_{i_s}) = (a_r, b_r)$*

Proof. Since the complement to $\mathbb{N}^2$ of the set of exponents of all power products in R is finite, then $a_1 = 0 = b_r$. If $(a_1, b_1)(a_i, b_i)$ is not a spg(J) for every $i = 2, \ldots, r-1$, then (by definition of spg(J)) $(a_1, b_1)(a_r, b_r)$ is a spg(J) and we get the proof.

Hence suppose there exists $i_1 < r$ such that $(a_1, b_1)(a_{i_1}, b_{i_1})$ is a spg(J). As above, if $(a_{i_1}, b_{i_1})(a_k, b_k)$ is not a spg(J) for every $k = i_1 + 1, \ldots, r-1$, then $(a_{i_1}, b_{i_1})(a_r, b_r)$ is a spg(J) and we get the proof. If not, using the same argument as above we get, after a finite number of steps (as the number of generators of J as a $k[x]$-module is finite), the proof.

Lemma 4.8. *Let $(a_i, b_i)(a_j, b_j)$ be a spg$(\mathfrak{m})$, then $(la_i, lb_i)(la_j, lb_j)$ is a spg$(\mathfrak{m}^l)$.*

Proof. Since $(la_i, lb_i) \lessdot (la_j, lb_j)$ whenever $(a_i, b_i) \lessdot (a_j, b_j)$, to get the proof we need that if $x_1^\alpha x_2^\beta \in \mathfrak{m}^l$, then (α, β) is not under the straight line in $\mathbb{R}^2$ connecting (la_i, lb_i) and (la_j, lb_j) and that $x_1^{la_i} x_2^{lb_i}$ and $x_1^{la_j} x_2^{lb_j}$ are generators for $\mathfrak{m}^l$ as a $k[x]$-module.

Let $x_1^\alpha x_2^\beta \in \mathfrak{m}^l$, hence $(\alpha, \beta) = \sum_{k=1}^{l} (\alpha_k, \beta_k)$ and $x_1^{\alpha_k} x_2^{\beta_k} \in \mathfrak{m}$.

Since $(a_i, b_i)(a_j, b_j)$ is a spg$(\mathfrak{m})$,

$$(b_j - b_i)\alpha_k + (a_i - a_j)\beta_k + a_i(b_i - b_j) + b_i(a_j - a_i) \geq 0$$

for every $k = 1, \ldots, l$. Hence

$$(lb_j - lb_i)\sum_{k=1}^{l} \alpha_k + (la_i - la_j)\sum_{k=1}^{l} \beta_k + la_i(lb_i - lb_j) + lb_i(la_j - la_i) =$$

$$l[(b_j - b_i)\sum_{k=1}^{l} \alpha_k + (a_i - a_j)\sum_{k=1}^{l} \beta_k + l[a_i(b_i - b_j) + b_i(a_j - a_i)]] =$$

$$l[(b_j - b_i)\alpha_1 + (a_i - a_j)\beta_1 + a_i(b_i - b_j) + b_i(a_j - a_i) + \cdots +$$
$$(b_j - b_i)\alpha_l + (a_i - a_j)\beta_l + a_i(b_i - b_j) + b_i(a_j - a_i)] \geq 0.$$

Suppose $x_1^{la_j} x_2^{lb_j}$ is not a generator for $\mathfrak{m}^l$ as a $k[x]$-module, then there exists $x_1^a x_2^b \in \mathfrak{m}^l$ such that either $a = la_j$ and $b < lb_j$ or $a < la_j$ and $b = lb_j$. Since, in this case, (a, b) is under the straight line in $\mathbb{R}^2$ connecting (la_i, lb_i) and (la_j, lb_j), we get

$$(lb_j - lb_i)a + (la_i - la_j)b + la_i(lb_i - lb_j) + lb_i(la_j - la_i) < 0$$

that is a contradiction to what we proved above, since if $x_1^\alpha x_2^\beta \in \mathfrak{m}^l$, then (α, β) can not be under the straight line in $\mathbb{R}^2$ connecting (la_i, lb_i) and (la_j, lb_j).

Corollary 4.9. *If $(a_{i_1}, b_{i_1}) \lessdot (a_{i_2}, b_{i_2}) \lessdot \cdots \lessdot (a_{i_s}, b_{i_s})$ are as in Lemma 4.7 with $J = \mathfrak{m}$, then $(la_{i_1}, lb_{i_1})(la_{i_2}, lb_{i_2}), (la_{i_2}, lb_{i_2})(la_{i_3}, lb_{i_3}), \ldots, (la_{i_{s-1}}, lb_{i_{s-1}})(la_{i_s}, lb_{i_s})$ are spg$(\mathfrak{m}^l)$ (with $(la_{i_1}, lb_{i_1}) = (la_1, lb_1)$ and $(la_{i_s}, lb_{i_s}) = (la_r, lb_r)$) .*

Remark 4.10. Let $(la_i, lb_i)(la_j, lb_j)$ be a spg$(\mathfrak{m}^l)$ and let $r(X, Y) = (b_j - b_i)X + (a_i - a_j)Y + l[a_i(b_i - b_j) + b_i(a_j - a_i)] = 0$ the straight line in $\mathbb{R}^2$ connecting (la_i, lb_i) and (la_j, lb_j). It is straightforward to prove that for every k such that $0 \leq k < l$, the integer point $((l-k)a_i, (l-k)b_i) + (ka_j, kb_j) = ((l-k)a_i + ka_j, (l-k)b_i + kb_j)$ is in the straight line in $\mathbb{R}^2$ with equation $r(X, Y)$.

Let $\mathfrak{m} = (x_1^{a_1}x_2^{b_1}, \ldots, x_1^{a_r}x_2^{b_r})k[x_1, x_2]$ and $x^\gamma \in \overline{\mathfrak{m}^l} \setminus \mathfrak{m}^l$ (hence $l \geq 2$) with $\gamma = (\gamma_1, \gamma_2)$. By Proposition 3.2, γ is an integer point in the convex hull of the union of the set $b + \mathbb{N}^d$, where b is an exponent of an element in $\mathfrak{m}^l$. Hence, by Corollary 4.9, there exists $(la_i, lb_i)(la_{i+1}, lb_{i+1})$ spg$(\mathfrak{m}^l)$ such that (γ_1, γ_2) is not under the straight line in $\mathbb{R}^2$ connecting (la_i, lb_i) and (la_{i+1}, lb_{i+1}) and such that $la_i < \gamma_1 < la_{i+1}$.

So γ is in the triangle in $\mathbb{R}^2$ with vertices $(la_i, lb_i), (la_{i+1}, lb_{i+1}), (la_{i+1}, lb_i)$ (we note that γ cannot be out of the triangle since $x^\gamma \notin \mathfrak{m}^l$ and $x_1^{la_i}x_2^{lb_i}$, $x_1^{la_{i+1}}x_2^{lb_{i+1}}$ are generators for $\mathfrak{m}^l$ as a $k[x]$-module by the proof of Lemma 4.8).

Finally, since γ is in the triangle in $\mathbb{R}^2$ with vertices $(la_i, lb_i), (la_{i+1}, lb_{i+1})$, (la_{i+1}, lb_i) and since (cf. Remark 4.10), for every k with $1 \leq k < l$, $((l-k)a_i, (l-k)b_i) + (ka_j, kb_j)$ is in the straight line connecting (la_i, lb_i) and (la_{i+1}, lb_{i+1}), we get

$$\gamma \text{ is in the triangle in } \mathbb{R}^2 \text{ with vertices}$$
$$(la_i, lb_i), ((l-k)a_i, (l-k)b_i) + (ka_j, kb_j), ((l-k)a_i + ka_j, lb_i)$$

or

$$\gamma \text{ is in the triangle in } \mathbb{R}^2 \text{ with vertices}$$
$$((l-k)a_i, (l-k)b_i) + (ka_j, kb_j), (la_j, lb_j), (la_j, (l-k)b_i + kb_j).$$

Theorem 4.11. *Let $\mathfrak{m} = (x_1^{a_1}x_2^{b_1}, \ldots, x_1^{a_r}x_2^{b_r})k[x_1, x_2]$ and suppose there exists $t \geq 2$ such that $\mathfrak{m}^t = \overline{\mathfrak{m}^t}$. Then $\mathfrak{m}^l = \overline{\mathfrak{m}^l}$ for every $l \geq t$.*

Proof. It is enough to prove that if $\mathfrak{m}^t = \overline{\mathfrak{m}^t}$ with $t \geq 2$, then $\mathfrak{m}^{t+1} = \overline{\mathfrak{m}^{t+1}}$.

Suppose $x^\gamma \in \overline{\mathfrak{m}^{t+1}} \setminus \mathfrak{m}^{t+1}$ and let $l = t+1$ (hence $l \geq 3$). By what is written above, for a fixed k with $1 \leq k < l$ we have either

$$\gamma \text{ is in the triangle in } \mathbb{R}^2 \text{ with vertices}$$
$$(la_i, lb_i), ((l-k)a_i, (l-k)b_i) + (ka_j, kb_j), ((l-k)a_i + ka_j, lb_i)$$

or

$$\gamma \text{ is in the triangle in } \mathbb{R}^2 \text{ with vertices}$$
$$((l-k)a_i, (l-k)b_i) + (ka_j, kb_j), (la_j, lb_j), (la_j, (l-k)b_i + kb_j).$$

Let us consider the first case. Hence

$$(\gamma_1, \gamma_2) = \lambda_1(la_i, lb_i) + \lambda_2((l-k)a_i + ka_j, (l-k)b_i + kb_j) + \lambda_3((l-k)a_i + ka_j, lb_i),$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1,\ \lambda_i \in \mathbb{Q}_{\geq 0}.$$

Let $\delta = (\delta_1, \delta_2) = (\gamma_1, \gamma_2) - (a_i, b_i)$. Since $x^\gamma \notin \mathfrak{m}^l$, necessary $x^\delta \notin \mathfrak{m}^{l-1}$. But

$$\delta = (\delta_1, \delta_2) = \lambda_1((l-1)a_i, (l-1)b_i) + \lambda_2((l-k-1)a_i + ka_j, (l-k-1)b_i + kb_j) +$$

$$\lambda_3((l-k-1)a_i + ka_j, (l-1)b_i),\ \lambda_1 + \lambda_2 + \lambda_3 = 1,\ \lambda_i \in \mathbb{Q}_{\geq 0},$$

hence

δ is in the triangle in $\mathbb{R}^2$ with vertices $((l-1)a_i,(l-1)b_i),((l-k-1)a_i,(l-k-1)b_i)+(ka_j,kb_j),((l-k-1)a_i+ka_j,(l-1)b_i)$,

that is δ is an integer point in the convex hull of the union of the set $b+\mathbb{N}^d$, where b is an exponent of an element in $\mathfrak{m}^{l-1}$. By $x^\delta \in R$ and Proposition 3.2, we get $x^\delta \in \overline{\mathfrak{m}^{l-1}} = \mathfrak{m}^{l-1}$. Absurd.

Similarly we get the proof for the other case.

The statement of Theorem 4.11 is, in general, not true for other kind of rings R as in the Introduction (cf. Remark 4.33).

Remark 4.12. We note that it is not true in general that if $\mathfrak{m}^t = \overline{\mathfrak{m}^t}$, then $\mathfrak{m}^l = \overline{\mathfrak{m}^l}$ for some $l < t$. Indeed, let $n \geq 1$ and $R = k+(x_1^n, x_1^{n-1}x_2, x_1x_2^{n-1}, x_2^n)k[x_1,x_2]$. In Remark 4.20 we show that $\overline{\mathfrak{m}^k} = \mathfrak{m}^k$ if and only if $k \geq n-2$.

As corollary to Theorem 4.11 and by $\mathfrak{m} = \overline{\mathfrak{m}}$, we get a criterion for $\mathfrak{m}$ to be normal.

Corollary 4.13. *The graded maximal ideal* $\mathfrak{m}$ *is normal if and only if* $\mathfrak{m}^2 = \overline{\mathfrak{m}^2}$.

Example 4.14. Let $R = k + (x_2^8, x_1x_2^6, x_1^2x_2^5, x_1^4x_2^4, x_1^8x_2^2, x_1^9x_2, x_1^{11})k[x_1,x_2]$ as in Figure 4. By Proposition 3.2,

$$\mathfrak{m}^2 = (x_2^{16}, x_1x_2^{14}, x_1^2x_2^{12}, x_1^3x_2^{11}, x_1^4x_2^{10}, x_1^6x_2^9, x_1^8x_2^8, x_1^{10}x_2^7, x_1^{11}x_2^6, x_1^{13}x_2^5, x_1^{15}x_2^4, x_1^{17}x_2^3,$$

$$x_1^{18}x_2^2, x_1^{20}x_2, x_1^{22})k[x_1,x_2] = \overline{\mathfrak{m}^2}$$

and, by Corollary 4.13, we get that $\mathfrak{m}$ is normal.

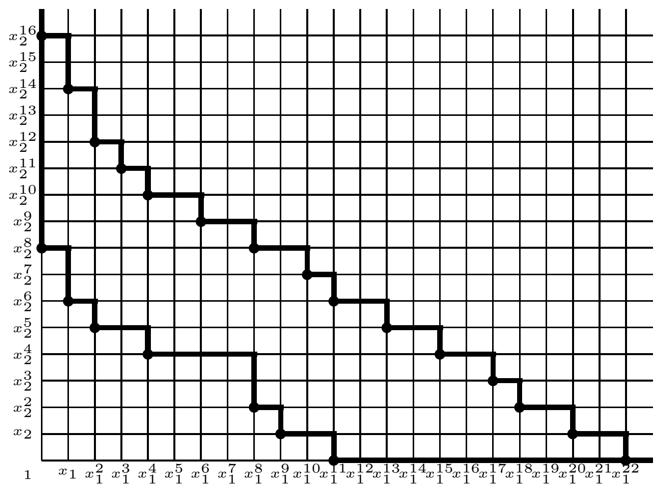

FIGURE 4

We note that in the Example 4.14, $x_1^7x_2^3 \notin R$, while $(7,3)$ is an integer point in the convex hull of the union of the set $b+\mathbb{N}^d$, where b is an exponent of an element in $\mathfrak{m}$.

By the proof of Theorem 4.11, we get the following corollary.

Corollary 4.15. *If for every integer point a in the convex hull of the union of the set $b+\mathbb{N}^d$, where b is an exponent of an element in $\mathfrak{m}$, we get $x^a \in \mathfrak{m}$, then $\mathfrak{m}$ is normal.*

Example 4.16. Let be $k+(x_2^7, x_1^2x_2^6, x_1^3x_2^5, x_1^7x_2^4, x_1^{11}x_2^3, x_1^{14}x_2^2, x_1^{18}x_2, x_1^{22})k[x_1,x_2]$. Since for every integer point a in the convex hull of the union of the set $b+\mathbb{N}^d$, where b is an exponent of an element in $\mathfrak{m} = (x_2^7, x_1^2x_2^6, x_1^3x_2^5, x_1^7x_2^4, x_1^{11}x_2^3, x_1^{14}x_2^2, x_1^{18}x_2, x_1^{22})k[x_1,x_2]$, we get $x^a \in \mathfrak{m}$, then, by Corollary 4.15, $\mathfrak{m}$ is normal.

4.1.1 A class of examples

Let $\mathfrak{n} = (x_1, x_2)$ be the graded maximal ideal of the polynomial ring $k[x_1,x_2] = k[x]$. We look for rings $R = k + (x^{\delta_1}, \dots, x^{\delta_t})k[x]$ with graded maximal ideal $\mathfrak{m} = (x^{\delta_1}, \dots, x^{\delta_t})k[x]$ such that $\mathfrak{m}^2 = \mathfrak{n}^{2n}$ $(n \geq 1)$. Indeed by Proposition 3.2 and by Corollary 4.13, we have that $\mathfrak{m}$ is normal.

Let $a_1 = \min\{\alpha_1 \neq 0 \mid x_1^{\alpha_1} \in \mathfrak{m}\}$ and $b_2 = \min\{\beta_2 \neq 0 \mid x_2^{\beta_2} \in \mathfrak{m}\}$ as at the beginning of Subsection 3.1. By $\mathfrak{m}^2 = \mathfrak{n}^{2n} = (x_1,x_2)^{2n}$, we necessarily have that $a_1 = n = b_2$ and that, if $x^r = x_1^{r_1}x_2^{r_2}$ is a generator for $\mathfrak{m}$ as a $k[x]$-module, then $r_1 + r_2 \geq n$. Furthermore if $r_1 + r_2 > n$, then this generator is uninteresting in our discussion as if $(x^{\delta_1}, \dots, x^{\delta_t})^2k[x] = (x_1,x_2)^{2n}$ then $\mathfrak{a} := (\{x^{\delta_1}, \dots, x^{\delta_t}\} \setminus \{x^r\})^2k[x]$ is equal to $(x_1,x_2)^{2n}$. Indeed if there exists $x_1^{c_1}x_2^{c_2} \in \mathfrak{m}^2 \setminus \mathfrak{a}$ with $x_1^{c_1}x_2^{c_2}$ generator for $\mathfrak{m}^2$ as a $k[x]$-module, then $x_1^{c_1}x_2^{c_2} = (x_1^{r_1}x_2^{r_2})(x_1^{b_1}x_2^{b_2})$. This is absurd as $2n = c_1 + c_2 = r_1 + r_2 + b_1 + b_2 > 2n$. Hence we can assume $r_1 + r_2 = n$

Finally by Proposition 4.1 and Remark 4.2, $x_1^{n-1}x_2, x_1x_2^{n-1} \in R$. Moreover, since we can suppose $r_1 + r_2 = n$ whenever $x^r = x_1^{r_1}x_2^{r_2}$ is a generator for $\mathfrak{m}$ as a $k[x]$-module, we have that $x_1^{n-1}x_2, x_1x_2^{n-1}$ are generators of $\mathfrak{m}$ as a $k[x]$-module.

By what is written above we can translate the problem to a merely combinatorial problem just considering the powers of the x_2's in the generators of $\mathfrak{m}$ as a $k[x]$-module. Indeed we look for a class of sets X with $\{0,1,n-1,n\} \subseteq X \subseteq \{0,1,\dots,n\}$ such that $2X := X + X = \{0,1,\dots,2n\}$.

From now on, given two integers a and b with $a \leq b$, we denote the set of integers between a and b (included) by $[a,b]$.

Proposition 4.17. *Let $X = \{0,1,\dots,h_1-1,h_1,h_2,\dots,h_z = n-h_1, n-h_1+1,\dots,n\}$ with $h_1 \geq 1$ and $h_{i+1} - h_i \leq h_1 + 1$ for every $i \in [1, z-1]$ (*). Then $2X = [0,2n]$.*

Proof. We show that for every $x \in [0,2n]$, there exist $x_1, x_2 \in X$ such that $x_1 + x_2 = x$.

If $x \in [0,h_1]$, then $x_1 = x$ and $x_2 = 0$.

If $x \in [h_1, h_z]$, then there exists i such that $h_i \leq x \leq h_{i+1}$. If $x = h_i$ or $x = h_{i+1}$, then $x_1 = x$ and $x_2 = 0$. Suppose hence that $h_i < x < h_{i+1}$, that is

$h_i+1 \le x \le h_{i+1}-1$. By (*), $h_{i+1}-1 \le h_i+h_1$, hence $h_i+1 \le x \le h_i+h_1$. So $x = h_i+h$ with $0 \le h \le h_1$ and we can assume $x_1 = h_i$ and $x_2 = h$.

If $x \in [h_z, n]$, then $x_1 = x$ and $x_2 = 0$.

If $x \in [n, n+h_1]$, then $x = n+h$ with $0 \le h \le h_1$. Hence $x_1 = n$ and $x_2 = h$.

If $x \in [n+h_1, n+h_z]$, then there exists i such that $n+h_i \le x \le n+h_{i+1}$. If $x = n+h_i$, then $x_1 = n$ and $x_2 = h_i$. Suppose hence that $n+h_i < x \le n+h_{i+1}$. So $n+h_i-h_{i+1} < x-h_{i+1} \le n$. By (*), $n-h_1-1 \le n+h_i-h_{i+1}$ and this implies $n-h_1-1 < x-h_{i+1} \le n$. Hence $x-h_{i+1} \in X$ and we can assume $x_1 = h_{i+1}$ and $x_2 = x-h_{i+1}$.

Finally if $x \in [n+h_z, 2n] = [2n-h_1, 2n]$, then $x = 2n-h_1+h$ with $0 \le h \le h_1$. Since $n-h_1 \le n-h_1+h \le n$, then $n-h_1-h \in X$. Hence $x_1 = n-h_1+h$ and $x_2 = n$.

Corollary 4.18. *The graded maximal ideal $\mathfrak{m}$ of $R = k+(x_1^n, x_1^{n-1}x_2, \dots, x_1^{n-h_1}x_2^{h_1}, x_1^{n-h_2}x_2^{h_2}, \dots, x_1^{n-h_z}x_2^{h_z}, x_1^{h_1-1}x_2^{n-h_1+1}, \dots, x_2^n)k[x_1,x_2]$ with $h_1 \ge 1$ and $h_{i+1}-h_i \le h_1+1$ for every $i \in [1, z-1]$ is normal.*

We now generalize Proposition 4.17. Indeed we look for rings R with graded maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^k = \mathfrak{n}^{kn}$, $(n \ge 1)$. By Proposition 3.2 and by Theorem 4.11, this implies $\mathfrak{m}^l = \overline{\mathfrak{m}^l}$ for every $l \ge k$.

As above we have that $a_1 = n = b_2$, that $x_1^{n-1}x_2, x_1x_2^{n-1} \in R$ are generators of $\mathfrak{m}$ as a $k[x]$-module and that if $x_1^{r_1}x_2^{r_2}$ is a generator for $\mathfrak{m}$ as a $k[x]$-module, then we can assume $r_1+r_2 = n$. Hence we look for a class of sets X with $\{0,1,n-1,n\} \subseteq X \subseteq [0,n]$ such that $kX = [0,kn]$.

Proposition 4.19. *Let X be a set with $\{0,1,n-1,n\} \subseteq X \subseteq [0,n]$ and such that $pX = [0,pn]$. Then for every $q \ge p$, $qX = [0,qn]$.*

Proof. It is enough to prove the proposition for $q = p+1$. By $[0,(p+1)n] = \{\{0\}+pX\} \cup \{\{n\}+pX\} \subseteq (p+1)X$, we get the proof.

Remark 4.20. We note that if $X = \{0,1,n-1,n\}$, then $kX = [0,kn]$ if and only if $k \ge n-2$. Indeed for every $i = 0, \dots, n-2$, $[i(n-1)+(n-i-2)\cdot 0, in+(n-i-2)\cdot 1] = [in-i, (i+1)n-(i+2)] \subseteq (n-2)X$. Hence $[0,(n-2)n] = \bigcup_{i=0}^{n-2}[in-i,(i+1)n-(i+2)] \subseteq (n-2)X$. Moreover $n-2 \notin \alpha X$ when $\alpha \le n-3$.

In particular, if $k \ge n-2$, then $kX = [0,kn]$ for every set X such that $\{0,1,n-1,n\} \subseteq X \subseteq [0,n]$. Hence if $\mathfrak{m}$ is the graded maximal ideal of R for which $x_1^n, x_1^{n-1}x_2, x_1x_2^{n-1}, x_2^n$ are part of a set of generators for $\mathfrak{m}$ as a $k[x]$-module, then $\overline{\mathfrak{m}^k} = \mathfrak{m}^k$ for every $k \ge n-2$.

As a particular case of Proposition 4.19 we get that the class of set X as in Proposition 4.17, satisfies $kX = [0,kn]$ for every $k \ge 2$.

Now we find a class of sets X such that $kX = [0,kn]$ but, in general, $(k-1)X \ne [0,(k-1)n]$. To this aim we generalize the class X of Proposition 4.17.

Proposition 4.21. *Let* $X = \{0, 1, \dots, h_1 - 1, h_1, h_2, \dots, h_z = n - h_1, n - h_1 - 1, \dots, n\}$ *such that* $h_1 \geq 2$ *and* $h_{i+1} - h_i \leq h_1 + k - 1$ *for every* $i \in [1, z-1]$ *(**). Then* $kX = [0, kn]$.

Proof. The case $k = 2$ is Proposition 4.17. We suppose hence $k \geq 3$.

Let $Y := \{0, 1, \dots, h_1, n - h_1, \dots, n\} \subseteq X$. So $kY = \{0, \dots, kh_1, n - h_1, \dots, n + (k-1)h_1, 2(n-h_1), \dots, 2n + (k-2)h_1, \dots, k(n-h_1), \dots, kn\} \subseteq kX$. To get the proof we need to cover all the holes in kX between $in + (k-i)h_1$ and $(i+1)(n - h_1)$ for every $0 \leq i \leq k - 1$.

We start covering all the holes in kX between kh_1 and $n - h_1$. We note that $(k-1)Y = \{0, \dots, (k-1)h_1, n - h_1, \dots, n + (k-2)h_1, 2(n-h_1), \dots, 2n + (k-3)h_1, \dots, (k-1)(n-h_1), \dots, (k-1)n\} \subseteq (k-1)X$.

For every $l \in [1, z-1]$ we have $\{h_l\} + [0, (k-1)h_1] = [h_l, h_l + (k-1)h_1] \subseteq kX$. By (**) and by $h_1 \geq 2$ and $k \geq 3$, we get $h_{l+1} \leq h_l + (k-1)h_1$. Hence $\{h_1, h_2, \dots, h_z\} + [0, (k-1)h_1] = \{h_1, h_1 + 1, \dots, h_z + (k-1)h_1\} = [h_1, h_z + (k-1)h_1] \subseteq kX$.

By $k \geq 3$ and $h_z = n - h_1$, we get $h_1 \leq kh_1$ and $n - h_1 \leq h_z + (k-1)h_1$, that is $[kh_1, n - h_1] \subseteq [h_1, h_z + (k-1)h_1]$ and we have covered all the holes in kX between kh_1 and $n - h_1$.

Making exactly the same sort of calculus as above you can check that $\{h_1, h_2, \dots, h_z\} + [i(n-h_1), in + (k-i-1)h_1]$ covers all the holes in kX between $in + (k-i)h_1$ and $(i+1)(n-h_1)$ for every $1 \leq i \leq k-1$. Hence $[0, kn] = kX$.

Corollary 4.22. *Let* $R = k + (x_1^n, x_1^{n-1}x_2, \dots, x_1^{n-h_1}x_2^{h_1}, x_1^{n-h_2}x_2^{h_2}, \dots, x_1^{n-h_z}x_2^{h_z}, x_1^{h_1-1}x_2^{n-h_1+1}, \dots, x_2^n)k[x_1, x_2]$ *with* $h_1 \geq 2$*,* $h_{i+1} - h_i \leq h_1 + k - 1$ *for every* $i \in [1, z-1]$*. Then for every* $l \geq k$*, the* l*-th power of the graded maximal ideal* $\mathfrak{m}$ *of* R *is integrally closed.*

Remark 4.23. We note that h_1 in Proposition 4.21 must be greater than 1. Indeed if $X = \{0, 1, 4, 7, 9, 10\}$ (hence $n = 10$, $k = 3$ and $h_1 = 1$), then $22, 25 \notin 3X$.

Remark 4.24. If X is a set in the class as in Proposition 4.21, then we know $kX = [0, kn]$. Anyway in general $(k-1)X \neq [0, (k-1)n]$ (in particular the converse to Proposition 4.19 does not hold). Indeed let $X = \{0, 1, 2, 6, 10, 12, 13, 14\}$ (hence $n = 14$, $h_1 = 2$ and $k = 3$). By Proposition 4.21, $3X = [0, 3n] = [0, 42]$. Anyway $5, 9, 17 \notin 2X$.

Let $V = [0, n]$ and $k \geq 2$ and let us consider the collection Δ_k of subset of V such that $F \in \Delta_k$ if and only if $0, 1, n-1, n \notin F$ and $k(V \setminus F) = [0, kn]$. Since $F \in \Delta_k$ whenever $F \subseteq G$ for some $G \in \Delta_k$ and since $\{i\} \in \Delta_k$ for every $i \in V \setminus \{0, 1, n-1, n\}$, then Δ_k is a (finite) simplicial complex on $V \setminus \{0, 1, n-1, n\}$.

We first find a lower and an upper bound for $\dim \Delta_2 := \sup\{\dim(F) \mid F \in \Delta_2\}$, where $\dim(F) := |F| - 1$.

Theorem 4.25. $\dim \Delta_2 \leq n - \lceil \frac{-1+\sqrt{16n+9}}{2} \rceil$*. Furthermore, if* $n \geq 4$ *then* $n - \sqrt{8n} + 1 \leq \dim \Delta_2$.

Proof. Let F be a face in Δ_2 with $|V \setminus F| = m$. As F is a face, $|2(V \setminus F)| = |[0, 2n]| = 2n + 1$. Since there are no more than $\frac{(m-1)m}{2}$ sums of mutually different numbers from X along with no more than m sums of equal numbers, we get that $2n+1 \leq \frac{m(m+1)}{2}$, hence $m \geq \lceil \frac{-1+\sqrt{16n+9}}{2} \rceil$. So $\dim \Delta_2 \leq ((n+1) - \lceil \frac{-1+\sqrt{16n+9}}{2} \rceil) - 1 = n - \lceil \frac{-1+\sqrt{16n+9}}{2} \rceil$.

Let now $n \geq 4$ and let us consider $X = \{0, 1, \dots, h_1 - 1, h_1, h_2, \dots, h_z = n - h_1, n - h_1 + 1, \dots, n\}$ with $h_1 \geq 1$, $h_{i+1} - h_i = h_1 + 1$ for every $i \in [1, z-2]$ (note that $z-2 \geq 1$ as $n \geq 4$) and $h_z - h_{z-1} \leq h_1 + 1$. By Proposition 4.17, $2X = [0, 2n]$. Since in X we exclude $2(h_1+1) + \lfloor \frac{n-2(h_1+1)}{h_1+1} \rfloor = 2(h_1+1) + \lfloor \frac{n}{h_1+1} \rfloor - 2$ integers from $[0, n]$, we get $n - 2(h_1+1) - \lfloor \frac{n}{h_1+1} \rfloor + 2$ number of vertices in a maximal face of Δ_2.

Let us consider $f(h_1) := n - 2(h_1+1) - \frac{n}{h_1+1} + 2$ as a function of h_1. The derivative $f'(h_1) = 0$ if and only if $h_1 = \sqrt{\frac{n}{2}} - 1$. Since $f(\sqrt{\frac{n}{2}} - 1) = n - \sqrt{8n} + 2$ and $-\lfloor \frac{n}{h_1+1} \rfloor \geq -\frac{n}{h_1+1} \geq -\lfloor \frac{n}{h_1+1} \rfloor - 1$, we have $\dim \Delta_2 \geq n - \sqrt{8n} + 1$.

Remark 4.26. We note that since, for $n \gg 0$, $n - \sqrt{8n} + 1 \approx n - \sqrt{8n}$ and $n - \lceil \frac{-1+\sqrt{16n+9}}{2} \rceil \approx n - \sqrt{4n}$, then the class of sets X as in the second part of the proof of Theorem 4.25 is a almost extremal class of examples.

Now we generalize part of Theorem 4.25 finding a lower bound for Δ_k.

Proposition 4.27. *Let $n \geq 4$. Then* $\dim \Delta_k \geq n - 2(\sqrt{\frac{n}{2} + k - 2} - k) - \frac{n+2k-4}{\sqrt{\frac{n}{2}+k-2}} - 3$

Proof. Let $X = \{0, 1, \dots, h_1 - 1, h_1, h_2, \dots, h_z = n - h_1, n - h_1 + 1, \dots, n\}$ with $h_1 \geq 2$, $h_{i+1} - h_i = h_1 + k - 1$ for every $i \in [1, z-2]$ and $h_z - h_{z-1} \leq h_1 + k - 1$. By Proposition 4.21, $kX = [0, kn]$. Since in X we exclude $2(h_1+1) + \lfloor \frac{n-2(h_1+1)}{h_1+k-1} \rfloor$ integers from $[0, n]$, we get $n - 2(h_1+1) - \lfloor \frac{n-2(h_1+1)}{h_1+k-1} \rfloor$ number of vertices in a maximal face of Δ_k.

Let us consider $f(h_1) := n - 2(h_1+1) - \frac{n-2(h_1+1)}{h_1+k-1}$ as a function of h_1. Using the same argument as in Theorem 4.25, we get the proof.

We note that for $n = 1, 2, 3$ then $\Delta_k = \{\emptyset\}$ for every $k \geq 2$; for $n = 4$, $\Delta_k = \{\emptyset, \{2\}\}$ for every $k \geq 2$; for $n = 5$, $\Delta_2 = \{\emptyset, \{2\}, \{3\}\}$ and $\Delta_k = \{\emptyset, \{2, 3\}\}$ for every $k \geq 3$; for $n = 6$, $\Delta_2 = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 4\}\}$, $\Delta_3 = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, $\Delta_k = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}$ for every $k \geq 4$.

Our next aim is to prove that if $k \geq 3$, then Δ_k is connected for every n and that Δ_2 is connected for every $n \neq 5, 6$. We need to show that if $n \geq 7$ then for every $a, b \in V \setminus \{0, 1, n-1, n\}$ there exist $\gamma_1, \dots, \gamma_m \in V \setminus \{0, 1, n-1, n\}$ such that $\{a, \gamma_1\}$,$\{\gamma_1, \gamma_2\}$, ... ,$\{\gamma_m, b\}$ are in Δ_k for every $k \geq 2$.

Lemma 4.28. *If $F \in \Delta_i$, then $F \in \Delta_l$ for every $l \geq i$.*

Proof. This follows by definition of face of Δ_i and by Proposition 4.19.

Lemma 4.29. *Let $n \geq 7$ and $a, b \in V \setminus \{0, 1, n-1, n\}$ with $a < b$. Then $\{a, b\} \in \Delta_k$, with $k \geq 3$, and $\{a, b\} \in \Delta_2$ if and only if the pair (a, b) is different from $(2, 3)$ and from $(n-3, n-2)$.*

Proof. Let (a, b) be different from $(2, 3)$ and $(n-3, n-2)$ (this is possible as $n \geq 7$). We need to prove that $\{a, b\}$ is a face in Δ_2, that is $2(V \setminus \{a, b\}) = [0, 2n]$. Let $x \in [0, 2n]$ and suppose first that $x \leq n$.

If $x \neq a, b$, then $x = x + 0 \in 2(V \setminus \{a, b\})$. If $x = a$, then $x = (x-1) + 1 \in 2(V \setminus \{a, b\})$. Finally if $x = b$, then $x = (b-1) + 1 \in 2(V \setminus \{a, b\})$ if $a \neq b-1$ and $x = (b-2) + 2 \in 2(V \setminus \{a, b\})$ if $a = b-1$.

Suppose now $n < x \leq 2n-4$. Since $x - n \leq n-4$, we get $n > x-n$, $n-1 > x-n+1$ and $n-2 \geq x-n+2$. Hence we can write x in at least three different ways as a sum of two natural number less or equal to n. Precisely $x = n + (x-n)$, $x = (n-1) + (x-n+1)$ and $x = (n-2) + (x-n+2)$. This implies that in at least one of the sums above the two summands are both different from a and b. Thus $x \in 2(V \setminus \{a, b\})$.

Suppose now $2n-3 \leq x \leq 2n$. Since $\{n-2, n-1, n\} \subseteq V \setminus \{a, b\}$ or $\{n-3, n-1, n\} \subseteq V \setminus \{a, b\}$, then $x \in 2(V \setminus \{a, b\})$. Hence $\{a, b\} \in \Delta_2$.

To get the proof we need to show that $\{2, 3\}$ and $\{n-3, n-2\}$ are not in Δ_2. Indeed if $X = [0, n] \setminus \{2, 3\}$ then $[0, 2n] \setminus 2X = \{3\}$ and if $X = [0, n] \setminus \{n-3, n-2\}$ then $[0, 2n] \setminus 2X = \{2n-3\}$.

Let now $k \geq 3$. By the first part of the proof and by Lemma 4.28 we know that if $a, b \in V \setminus \{0, 1, n-1, n\}$ with (a, b) different from $(2, 3)$ and from $(n-3, n-2)$, then $\{a, b\} \in \Delta_k$. Let $V \setminus \{2, 3\} = \{0, 1, 4, 5, 6, \ldots, n\}$. By $3\{0, 1, 4, 5, 6\} = [0, 12]$, by $3[4, n] = [12, 3n]$ and by Lemma 4.28, we have $\{2, 3\} \in \Delta_k$. Finally let $V \setminus \{n-3, n-2\} = \{0, 1, \ldots, n-4, n-1, n\}$. By $3[0, n-4] = [0, 3n-12]$, by $3\{n-5, n-4, n-1, n\} = [3n-12, 3n]$ and by Lemma 4.28, we have $\{n-3, n-2\} \in \Delta_k$.

Theorem 4.30. *If $k \geq 3$, then Δ_k is connected for every n. Δ_2 is connected for every $n \neq 5, 6$.*

Proof. By what is written before Lemma 4.28, we need to show that if $n \geq 7$, then Δ_k is connected for every $k \geq 2$. The case $k \geq 3$ is immediate by Lemma 4.29.

Let $k = 2$. Given $a, b \in V \setminus \{0, 1, n-1, n\}$. If $(a, b) \neq (2, 3), (n-3, n-2)$, then, by Lemma 4.29, $\{a, b\} \in \Delta_2$. Otherwise, since $n \geq 7$, there exists $\gamma \in V \setminus \{0, 1, a, b, n-1, n-2\}$. By Lemma 4.29, $\{a, \gamma\}$ and $\{\gamma, b\}$ are in Δ_2.

4.2 The second case

Let us denote the set of all power products in the indeterminates $x_1, x_2, \ldots,$ x_d of degree i in $k[x_1, x_2, \ldots, x_d]$, that is $\{x_1^i, x_1^{i-1}x_2, \ldots, x_d^i\}$, by F_i. Let $g_1, g_2, \ldots, g_n$ be positive natural numbers with $g_1 < g_2 < \cdots < g_n$ and $\gcd(g_1, g_2,$ $\ldots, g_n) = 1$.

In this second case we study the integral closure of powers of the graded maximal ideal of $R = k[F_{g_1}, F_{g_2}, \ldots, F_{g_n}]$.

We define the k-algebra homomorphism

$$\psi : k[x_1, x_2, \ldots, x_d] \longrightarrow k[t]$$

with $\psi(f(x_1, \ldots, x_d)) = f(t, \ldots, t)$.

We also define the k-algebra homomorphism

$$\phi : k[t] \longrightarrow k[x_1, x_2, \ldots, x_d]$$

with $\phi(l(t)) = l(x_1)$.

Theorem 4.31. *Let $R = k[F_{g_1}, F_{g_2}, \ldots, F_{g_n}]$ and $T = k[t^{g_1}, t^{g_2}, \ldots, t^{g_n}]$. Let $\mathfrak{m} = (F_{g_1}, F_{g_2}, \ldots, F_{g_n})$ and $M = (t^{g_1}, t^{g_2}, \ldots, t^{g_n})$ denote the graded maximal ideal of R and T respectively. Then for every natural number a, $\overline{\mathfrak{m}^a} \setminus \mathfrak{m}^a \neq \emptyset$ if and only if $\overline{M^a} \setminus M^a \neq \emptyset$.*

Proof. It is straightforward to verify that, since $\gcd(g_1, g_2, \ldots, g_n) = 1$, the complements to $\mathbb{N}^d$ and $\mathbb{N}$ respectively of the set of exponents of all power products in R and in T are finite.

Suppose there exists $z \in \overline{\mathfrak{m}^a} \setminus \mathfrak{m}^a$ for some a. By Theorem 3.1, we can suppose that z is a monomial in $k[x_1, x_2, \ldots, x_d]$, that is $z = x_{i_1}^{j_1} \cdots x_{i_p}^{j_p}$, $1 \leq i_1 < \cdots < i_p \leq d$.

We note that $\psi(z) = t^{j_1} \cdots t^{j_p} \notin M^a$ (since $z \notin \mathfrak{m}^a$). We will show that $\psi(z) \in \overline{M^a}$.

By $z \in \overline{\mathfrak{m}^a}$, there exist $c_1, c_2, \ldots, c_m$ with $c_i \in (\mathfrak{m}^a)^i$ such that $z^m + c_1 z^{m-1} + \cdots + c_m = 0$. Hence

$$0 = \psi(z^m + c_1 z^{m-1} + \cdots + c_m) = \psi(z^m) + \psi(c_1)\psi(z^{m-1}) + \cdots + \psi(c_m) =$$

$$\psi(z)^m + \psi(c_1)\psi(z)^{m-1} + \cdots + \psi(c_m).$$

Since $c_i \in (\mathfrak{m}^a)^i$, we get $\psi(c_i) \in (M^a)^i$. Thus $\psi(z) \in \overline{M^a}$.

Suppose now $\overline{M^a} \setminus M^a \neq \emptyset$ for some a. By Theorem 3.1 we can suppose $t^b \in \overline{M^a} \setminus M^a$ for some b. Let us consider x_1^b. Clearly $x_1^b \notin \mathfrak{m}^a$ (if not $\psi(x_1^b) = t^b \in M^a$). We will show that $x_1^b \in \overline{\mathfrak{m}^a}$.

By $t^b \in \overline{M^a}$ there exist $c_1, c_2, \ldots, c_m$ with $c_i \in (M^a)^i$ such that $(t^b)^m + c_1(t^b)^{m-1} + \cdots + c_m = 0$. Hence

$$0 = \phi((t^b)^m + c_1(t^b)^{m-1} + \cdots + c_m) = \phi((t^b)^m) + \phi(c_1)\phi((t^b)^{m-1}) + \cdots + \phi(c_m) =$$

$$(x_1^b)^m + \phi(c_1)(x_1^b)^{m-1} + \cdots + \phi(c_m).$$

Since $\phi(c_i) \in (\mathfrak{m}^a)^i$, we get $x_1^a \in \overline{\mathfrak{m}^a}$.

Corollary 4.32. *Let $(R, \mathfrak{m})$ and (T, M) be rings as in Theorem 4.31. Then $\mathfrak{m}$ is normal if and only if M is normal.*

A study on normal graded maximal ideal for rings $k[t^{g_1}, t^{g_2}, \ldots, t^{g_n}]$ with $\gcd(g_1, g_2, \ldots, g_n) = 1$ can be found in [1].

Remark 4.33. Let $R = k[F_6, F_7, F_{11}]$. Then, using Theorem 4.31, it is easy to check that $\mathfrak{m}^2 = \overline{\mathfrak{m}^2}$, but $x_1^{22} \in \overline{\mathfrak{m}^3} \setminus \mathfrak{m}^3$. Hence the statement of Theorem 4.11 is not true for this kind of rings.

Corollary 4.34. *Let R be a ring as above with graded maximal ideal $\mathfrak{m} = (F_{g_1}, F_{g_2}, \dots, F_{g_n})$.*

(i) If $\mathfrak{m}$ is normal, then $g_2 = g_1 + 1$ and $g_n < 2g_1$.

(ii) $\mathfrak{m}$ is normal if and only if $\overline{\mathfrak{m}^{a+1}} = \overline{\mathfrak{m}^a}\mathfrak{m}$ for every $a \geq 0$.

(iii) Let $\mathfrak{m}_i$ denote the maximal ideal of $k[F_{g_1}, F_{g_2}, \dots, F_{g_i}]$; if $\mathfrak{m}_i$ is not normal for some $i < n$, then $\mathfrak{m}_n (= \mathfrak{m})$ is not normal.

Proof. (i) This follows by Corollary 4.32 and [1, Proposition 3.1]).
(ii) Use Corollary 4.32 and [1, Theorem 3.5].
(iii) By Corollary 4.32 and [1, Theorem 3.14].

In the 3-generated case ($n = 3$) we can give a concrete characterization for normal graded maximal ideal $\mathfrak{m}$. Suppose $g_2 = g_1 + 1$ and $g_3 < 2g_1$ (if one of these two conditions is not satisfied, then $\mathfrak{m}$ is not normal by (i) of Corollary 4.34) and let α be the unique integer such that $(\alpha - 1)g_3 < \alpha g_1$ and $\alpha g_3 \geq (\alpha + 1)g_1$.

Corollary 4.35. *Let R, g_2 and g_3 as above. Then $\mathfrak{m}$ is normal if and only if $\alpha g_3 \leq (\alpha + 1)g_2$.*

Proof. This follows by Corollary 4.32 and [1, Theorem 3.25].

Example 4.36. Let

$$k[F_{10}, F_{11}, F_{17}] = k[x_1^{10}, x_1^9 x_2, \dots, x_d^{10}, x_1^{11}, x_1^{10} x_2, \dots, x_d^{11}, x_1^{17}, x_1^{16} x_2, \dots, x_d^{17}].$$

Then $\alpha = 2$ and, by Corollary 4.35, we get $\mathfrak{m} = (F_{10}, F_{11}, F_{17})$ is not normal. Using the same argument as above it is easy to check that in $k[F_{10}, F_{11}, F_{16}]$, $\mathfrak{m} = (F_{10}, F_{11}, F_{17})$ is normal.

References

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