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FACTORING THE KAUFMAN–RICKERT INEQUALITY

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ABSTRACT. The Kaufman–Rickert inequality relates the radius of the range of a complex measure to its total variation. This inequality can in a precise sense be factored through a metric space using m-adic unit disks. Some sharp estimates governing geometric properties relating to multilinear Fréchet measures appear as immediate applications.

Consider a (finite) complex measure μ on some measurable space $(\Omega, \mathcal{A})$. Its total variation norm will be denoted $\|\mu\|$ as usual, and the radius of its range will be written

$$\|\mu\|_{(1)} = \sup \big\{ |\mu(E)| \,;\, E \in \mathcal{A} \big\}.$$

The Kaufman–Rickert inequality [KR] then says that

$$\|\mu\| \le \pi \|\mu\|_{(1)},$$

and that this is best possible. Trivially, $\|\mu\|_{(1)} \le \|\mu\|$. In order to anticipate later development, we have the elementary inequality $\|\mu_{\mathbb{R}}\| \le 2\, \|\mu_{\mathbb{R}}\|_{(1)}$ for any signed, finite measure $\mu_{\mathbb{R}}$.

Motivated by an m-adic unit disk, one considers for integers $m \ge 2$ the quantities

$$\|\mu\|_{(m)} = \sup \Big| \sum_{k=0}^{m-1} w_k \mu(E_k) \Big|.$$

Here $\Omega = \cup_{k=0}^{m-1} E_k$ is an $\mathcal{A}$-measurable partition and each w_k is an m-th root of unity, i.e., $w_k^m = 1$ for all k.

Denoting the convex range and m-range of μ, respectively, as

$$Q_1(\mu) = \operatorname{conv} \big\{ \mu(E) \,;\, E \in \mathcal{A} \big\} = \Big\{ \int f d\mu \,;\, 0 \le f \le 1 \Big\},$$

$$Q_m(\mu) = \operatorname{conv} \Big\{ \sum_{k=0}^{m-1} w_k \mu(E_k) \,;\, \Omega = \cup E_k, w_k^m = 1 \Big\} = \Big\{ \int f d\mu \,;\, f \in B_m \Big\},$$

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This piece of researched would certainly not have come into being, should I not have encountered the Kaufman–Rickert inequality during a visit to the Paris Lodron Universität Salzburg. I am truly grateful to Dozent Gerhard Racher in Salzburg for allowing me to visit the Department of Mathematics, and for the mathematics he shared with me.

with partitions and roots of unity as above, and where B_m is the set of $\mathcal{A}$-measurable functions with values in the closed convex hull of the m-th roots of unity. It is then clear that for each $m \geq 1$, the convex set $Q_m(\mu)$ in the complex plane contains the origin and is of radius $\|\mu\|_{(m)}$.

Using the presentation of Kaijser [K], being more detailed than the arguments of Kakutani indicated in [KR], we know that

(1) $Q_1(\mu)$ has perimeter $2\,\|\mu\|$, whereas
(2) $Q_m(\mu)$ has perimeter $2m\sin(\pi/m)\,\|\mu\|$ for $m \geq 2$.

The perimeter of convex sets being monotone with respect to inclusions, we record for $m \geq 2$ the known inequality

$$2m\sin(\pi/m)\,\|\mu\| \leq 2\pi\,\|\mu\|_{(m)}, \quad \text{i.e.,} \quad \|\mu\| \leq \tfrac{\pi/m}{\sin\pi/m}\,\|\mu\|_{(m)}.$$

Here the larger member is the perimeter of a disk that certainly contains $Q_m(\mu)$.

On the other hand, let us consider the particular measure $d\nu(\theta) = e^{i\theta}\,d\theta$ on the interval $[0, 2\pi[$. It is clear that the implied rotation symmetry makes every one of the sets $Q_m(\nu)$ a disk centred at the origin. Hence it follows that $2\pi\,\|\nu\|_{(m)} = 2m\sin(\pi/m)\,\|\nu\|$, which incidentally shows the just derived inequalities to be best possible.

Even though the above argument of Kakutani–Kaijser is complete, it is illuminating to give a second demonstration more in the spirit of Kaufman and Rickert's original calculation for the circular geometry. The details are stunningly effective. Let $\omega = e^{-2\pi i/m}$. Like in [KR] it suffices to study $d\mu = e^{i\theta}\,d|\mu|$, and to calculate

$$\begin{aligned}
\|\mu\|_{(m)} &= \sup\Big|\textstyle\sum_{j=0}^{m-1}\omega_j\,\mu(E_j)\Big| = \sup\Big|\textstyle\sum_{j=0}^{m-1}\omega^j\,\int_{E_j} e^{i\theta}\,d|\mu|(\theta)\Big| \\
&\geq \sup_\lambda\Big|\sum_{j=0}^{m-1}\omega^j\,\int_{\lambda+\frac{2j-1}{m}\pi}^{\lambda+\frac{2j+1}{m}\pi} e^{i(\theta-\lambda)}\,d|\mu|(\theta)\Big| \\
&\geq \sup_\lambda\sum_{j=0}^{m-1}\int_{\lambda+\frac{2j-1}{m}\pi}^{\lambda+\frac{2j+1}{m}\pi}\operatorname{Re} e^{i(\theta-\lambda-2\pi j/m)}\,d|\mu|(\theta) \\
&\geq \int_0^{2\pi}\sum_{j=0}^{m-1}\int_{\lambda+\frac{2j-1}{m}\pi}^{\lambda+\frac{2j+1}{m}\pi}\cos\big(\theta-\lambda-2\pi j/m\big)\,d|\mu|(\theta)\tfrac{d\lambda}{2\pi} \\
&= \int_0^{2\pi}\sum_{j=0}^{m-1}\int_{-\pi/m}^{\pi/m}\cos x\,\tfrac{dx}{2\pi}\,d|\mu|(\theta) = \tfrac{m}{\pi}\,\sin\tfrac{m}{\pi}\,\|\mu\|.
\end{aligned}$$

The indicated intervals of integration are understood as half-open in a consistent manner.

Having followed this kind of reasoning to its completion, we have found that the value $c_m = (\pi/m)/\sin(\pi/m)$ is the optimal value in the factorization to be discussed presently.

Question. Can the Kaufman–Rickert inequality be factored in the sense that for all measures μ,

$$\|\mu\| \leq c_m\,\|\mu\|_{(m)} \leq c_m\,d_m\,\|\mu\|_{(1)},$$

where $c_m d_m = \pi$? If this holds true, the Kaufman–Rickert inequality is said to be m-perfectly factorable. Otherwise, the inequality is perturbedly factorable.

Observe that by necessity (check the measure ν used earlier on) one must have $c_m d_m \geq \pi$ for complex measures. The crux of the matter lies in the possibility of performing the renorming $\|\ \| \to \|\ \|_{(m)} \to \|\ \|_{(1)}$ without loosing optimality, hence the idea of factoring. With an eye to applications, it makes sense to study the same thing with respect to signed measures, where we speak of the constants $c_m^{\mathbb{R}}$ and $d_m^{\mathbb{R}}$, respectively. It is then obvious that $c_m^{\mathbb{R}} d_m^{\mathbb{R}} \geq 2$, and one can speak of real m-perfect factorability. The main result of this paper is contained in the following definition and result.

Definition. Denote for $m \geq 2$ by c_m and d_m the values

$$c_m = \sup_\mu \frac{\|\mu\|}{\|\mu\|_{(m)}}, \qquad d_m = \sup_\mu \frac{\|\mu\|_{(m)}}{\|\mu\|_{(1)}},$$

where μ ranges over all non-trivial complex measures. In case only signed, finite measures are involved, the constants are denoted $c_m^{\mathbb{R}}$ and $d_m^{\mathbb{R}}$.

Main Theorem. i) *The Kaufman–Rickert inequality is real m-perfectly factorable for any m, and in fact*

$$\left\{\begin{matrix} c_m^{\mathbb{R}} = 1 \\ d_m^{\mathbb{R}} = 2 \end{matrix}\right\} \textit{ for even } m, \textit{ whereas } \left\{\begin{matrix} c_m^{\mathbb{R}} = 1/\cos(\pi/2m) \\ d_m^{\mathbb{R}} = 2\cos(\pi/2m) \end{matrix}\right\} \textit{ for odd } m.$$

ii) *For any even m the Kaufman–Rickert inequality is m-perfectly factorable and for these even integers m*

$$c_m = \frac{\pi/m}{\sin(\pi/m)} \quad \textit{and} \quad d_m = m\sin(\pi/m).$$

iii) *For odd orders m, only perturbed factorability obtains, and in fact*

$$c_m = \frac{\pi/m}{\sin(\pi/m)}, \quad \textit{whereas} \quad d_m = \frac{m \sin \pi/m}{\cos(\pi/2m)}.$$

It has already been pointed out that the above value for c_m has been satisfactorily established. The determination of d_m will turn out to be the deepest result in this paper, but still elementary. Let it in passing be noted that the geometric proof of Kakutani–Kaijser for the value of c_m also shows that the corresponding quotient $\|\mu\|/\|\mu\|_{(m)}$ is *strictly* less than c_m as soon as the set $\{e^{i\theta}\,;\, \mu(E) = re^{i\theta}$ for some $r > 0,\ E \in \mathcal{A}\}$ is finite. This is in sharp contrast to the situation for d_m as will be seen later on.

1. The real factorization. The case of real measures can be dealt with in purely geometric terms. Let until further notice $\mu = \mu_+ - \mu_-$ be the Hahn-Jordan decomposition of a signed, finite, non-zero measure on the measurable space $(\Omega, \mathcal{A})$. Fix the integer $m \geq 2$ and write $\omega = e^{2\pi i/m}$.

The measures μ_+ and μ_- being positive, clearly $Q_m(\mu_+)$ is a regular m-gon centred at the origin with one corner in $\mu_+(\Omega)$, whereas $Q_m(-\mu_-)$ has one corner at $-\mu_-(\Omega)$. By the additivity of m-ranges, see [K], $Q_m(\mu)$ is the $2m$-sided polygon centred at the origin and spanned by the points described below. The m-radius follows with no hesitation.

$$Q_m(\mu) = \operatorname{conv}\,\big\{\mu_+(\omega)\omega^j - \mu_-(\Omega)\omega^k\,;\, j,k = 0,\ldots,m-1\big\}$$

$$\|\mu\|_{(m)} = \max_k |\,\mu_+(\Omega) - \mu_-(\Omega)\omega^k\,|.$$

In case m is an even integer it is immediate to deduce the relations

$$\|\mu\|_{(m)} = \mu_+(\Omega) + \mu_-(\Omega) = \|\mu\| \le 2\sup_E |\mu(E)| = 2\|\mu\|_{(1)}.$$

These express exactly what is claimed in the Main theorem, part i) for even m.

Turning to odd m, one first interprets the above computation as

$$\|\mu\|_{(m)} = |\,\mu_+(\Omega) + e^{i\pi/m}\mu_-(\Omega)\,|,$$

from which clearly follows

$$\frac{\|\mu\|_{(m)}}{\|\mu\|} \ge \inf_{0\le\theta\le 1} \big|\,\theta e^{i\pi/2m} + (1-\theta)e^{-i\pi/2m}\big| = \cos\tfrac{\pi}{2m}.$$

This gives an upper bound $c_m^{\mathbb{R}} \le 1/\cos(\pi/2m)$. On the other hand,

$$\frac{\|\mu\|_{(m)}}{\|\mu\|_{(1)}} \le \sup_{0\le\theta\le 1} \frac{|(1-\theta)+\theta e^{i\pi/m}|}{\max\{\theta, 1-\theta\}} = \sup_{0\le x\le 1} |x + e^{i\pi/m}| = 2\cos\tfrac{\pi}{2m},$$

which is an upper bound on $d_m^{\mathbb{R}}$. The product of these two bounds being exactly 2, and since a simple two-point measure has

$$\begin{aligned} 2 = \|\delta_0 - \delta_1\| &\le [\cos\pi/2m]^{-1}\,\|\delta_0 - \delta_1\|_{(m)} \\ &\le 2\cos(\pi/2m)\cdot[\cos\pi/2m]^{-1}\,\|\delta_0 - \delta_1\|_{(1)} = 2, \end{aligned}$$

the above bounds cannot be improved. This completes the proof that in the real-valued case, the Kaufman–Rickert inequality is real m-perfectly factorable for any $m \ge 2$.

2. A relevant extremal problem. Conceptually it seems to be of benefit to reformulate the determination of d_m as a $2m$-dimensional extremal problem, and thus to avoid the large parameter spaces arising from the use of measures.

For a fixed integer $m \ge 2$ consider two classes of vectors, each of whose members has $2m$ components, standardized as $\mathbf{v} = (v_0, \ldots, v_{2m-1})$. To wit,

$$\begin{aligned} \mathcal{V}_m &= \big\{\mathbf{v}\,;\, \{v_k\}_{k=0}^{2m-1} \text{ is permutation of } \{e^{2\pi ik/m}\}_{k=0}^{2m-1}\big\}, \\ \mathcal{U}_m &= \big\{0,1\big\}^{2m}. \end{aligned}$$

Given two vectors $\mathbf{z}$ and $\mathbf{w}$, their product is defined to be $\mathbf{z}\cdot\mathbf{w} = \sum_{k=0}^{2m-1} z_k w_k$.

Lemma 1. *Let* $S_m = \mathbb{C}^{2m} \setminus \{0\}$ *and denote by* β_m *the quantity*

$$\inf_{\mathbf{z}\in S_m} \max_{\mathbf{u}\in\mathcal{U}_m} \min_{\mathbf{v}\in\mathcal{V}_m} \frac{|\,\mathbf{z}\cdot\mathbf{u}\,|}{|\,\mathbf{z}\cdot\mathbf{v}\,|}.$$

Then $d_m = \beta_m^{-1}$.

For the proof we consider any complex, non-zero measure μ on $(\Omega, \mathcal{A})$. There are then a measurable partition $\Omega = E_0 \cup \cdots \cup E_{m-1}$, as well as an $\mathcal{A}$-measurable set F, which when writing $E = F_k \cup F_k'$, $F_k = E_k \cap F$, enjoy the properties

$$\|\mu\|_{(1)} = |\mu(F)| = \big|\sum\nolimits_{k=0}^{m-1} \mu(F_k)\big|, \qquad \|\mu\|_{(m)} = \big|\sum\nolimits_{k=0}^{m-1} \omega^k \mu(E_k)\big|.$$

Consider next the vector

$$\mathbf{w} = \big(\mu(F_0), \mu(F_0'), \dots, \mu(F_{m-1}), \mu(F_{m-1}')\big).$$

By construction the first equality in the following calculation holds.

$$\frac{\|\mu\|_{(1)}}{\|\mu\|_{(m)}} = \max_{\mathbf{u}\in\mathcal{U}_m} \min_{\mathbf{v}\in\mathcal{V}_m} \frac{|\,\mathbf{w}\cdot\mathbf{u}\,|}{|\,\mathbf{w}\cdot\mathbf{v}\,|} \geq \beta_m, \quad \text{i.e.,} \quad \|\mu\|_{(m)} \leq \beta_m^{-1} \|\mu\|_{(1)}.$$

Finally, there is to each $\varepsilon > 0$ some $\mathbf{y} \in S_m$ such that, by definition of the extremal problem,

$$\max_{\mathbf{u}\in\mathcal{U}_m} |\,\mathbf{y}\cdot\mathbf{u}\,| \leq (\beta_m + \varepsilon) \max_{\mathbf{v}\in\mathcal{V}_m} |\,\mathbf{y}\cdot\mathbf{v}\,|.$$

This information suggests a measure on $2m$ points realized as

$$\nu(\{k\}) = y_k, \quad k = 0, \dots, 2m-1.$$

The above property of $\mathbf{y}$ thus translates into

$$\|\nu\|_{(1)} = \max_{\mathbf{u}\in\mathcal{U}_m} |\,\mathbf{y}\cdot\mathbf{u}\,| \leq (\beta_m + \varepsilon) \max_{\mathbf{v}\in\mathcal{V}_m} |\,\mathbf{y}\cdot\mathbf{v}\,| \leq (\beta_m + \varepsilon)\, \|\nu\|_{(m)}.$$

Collecting together, we have therefore (upper bound from the first part)

$$(\beta_m + \varepsilon)^{-1} \leq \frac{\|\nu\|_{(m)}}{\|\nu\|_{(1)}} \leq \beta_m^{-1}.$$

Taking very small values for ε, this completes the verification of the lemma.

Corollary 2. *For even integers* m *the bound* $\beta_m \leq \big[m \sin \frac{\pi}{m}\big]^{-1}$ *obtains, whereas odd orders* m *forces the bound* $\beta_m \leq \cos(\pi/2m)\,\big[m \sin(\pi/m)\big]^{-1}$.

The proof begins with an elementary fact to keep steadily in mind.

Observation. *Let $\{a_n\}_{n=1}^N$ consist of non-zero complex numbers. Any set $E \subseteq \{1,\dots,N\}$ such that $|\sum_{n\in E} a_n| = \max_T |\sum_{n\in T} a_n|$ has the property that for some λ the equality $E = \{n\,;\, \mathrm{Re}\, a_n e^{-i\lambda} \geq 0\}$ holds. In fact, this λ is determined by $\sum_{n\in E} a_n = re^{i\lambda}$ for some $r \geq 0$.*

Suppose E has the maximal property $|\sum_{n\in E} a_n| = \max_T |\sum_{n\in T} a_n|$ and write $\sum_{n\in E} a_n = re^{i\lambda}$, $r \geq 0$. There can be no $m \in E$ with $\mathrm{Re}\, a_m e^{-i\lambda} < 0$, since otherwise $E' = E \setminus \{m\}$ enjoys

$$\big|\sum_{n\in E'} a_n\big| = |re^{i\lambda} - a_m| = |r - a_m e^{-i\lambda}| > r;$$

contradicting the maximality.

On the other hand, any absence $m \notin E$ with $\mathrm{Re}\, a_m e^{-i\lambda} \geq 0$ implies for $E' = E \cup \{m\}$ that by maximality

$$r \geq \big|\sum_{n\in E'} a_n\big| = |re^{i\lambda} + a_m| = |r + \mathrm{Re}\, a_m e^{-i\lambda} + i\,\mathrm{Im}\, a_m e^{-i\lambda}|,$$

whence necessarily $a_m e^{-i\lambda} = 0$, since $\mathrm{Re}\, a_m e^{-i\lambda} \geq 0$. Thus E is described completely by $\{n\,;\, \mathrm{Re}\, a_m e^{-i\lambda} \geq 0\}$ for some λ, and the observation holds true.

For the corollary one considers any $\mathbf{y} \in \mathcal{V}_m$. Clearly $\max_{\mathbf{v}\in\mathcal{V}_m} |\,\mathbf{y}\cdot\mathbf{v}\,| = 2m$. Any closed half-plane pivoting at the origin contains for even $m = 2n$ either $2n$ or $2(n+1)$ numbers appearing as components in $\mathbf{y}$. (They are in a geometric sence consecutive, and in the larger case the opposite numbers cancel in the sum below.) By the observation above

$$\max_{\mathbf{u}\in\mathcal{U}_m} |\,\mathbf{y}\cdot\mathbf{u}\,| = 2\,\big|1 + \cdots + \exp\big(\tfrac{2\pi i}{m}\,(\tfrac{m}{2} - 1)\big)\big| = \frac{2}{\sin(\pi/m)}.$$

Dividing by the observed maximum $2m$ for $\mathbf{v} \in \mathcal{V}_m$, gives $\beta_m \leq \big[m \sin(\pi/m)\big]^{-1}$ and therefore the claim.

For odd orders $m = 2n+1$, the relevant half-plane can contain $2n$ or $2(n+1)$ elements, and bearing $\sum_k y_k = 0$ in mind the observation yields

$$\max_{\mathbf{u}\in\mathcal{U}_m} |\,\mathbf{y}\cdot\mathbf{u}\,| = 2\,\big|1 + \cdots + \exp\big(\tfrac{2\pi i}{m}\,\tfrac{m-3}{2}\big)\big| = \frac{2\cos(\pi/2m)}{\sin \pi/m}.$$

Again dividing by $2m$, an upper bound for β_m appears, which provides the claimed bound for general odd $m \geq 3$, and therefore finishes the corollary.

Proposition 3. *For even orders m the true value is $\beta_m = \big[m \sin\frac{\pi}{m}\big]^{-1}$, and for all odd orders $\beta_m = \frac{\cos(\pi/2m)}{m\sin(\pi/m)}$.*

Proof. By the definition of β_m, matters boil down to providing

$$|\,\mathbf{y}\cdot\mathbf{v}\,| \leq \beta_m^{-1} \max_{T\subseteq Y} \Big|\sum_{x\in T} x\,\Big|$$

for all $\mathbf{y} \in \mathbb{C}^{2m}$ and $\mathbf{v} \in \mathcal{V}_m$. Here Y denotes the multi-set $\{y_0,\dots,y_{2m-1}\}$ derived from $\mathbf{y}$ and which respects possible multiplicity of some element. Write in the sequel $\omega = e^{2\pi i/m}$ and $v_k = \omega^{\rho(k)}$ using any permutation ρ of $\{0,\dots,2m-1\}$ determined by $\mathbf{v} \in \mathcal{V}_m$. The particular choice of ρ does not alter the values $\omega^{\rho(k)}$.

Guided by the proof of corollary 2, even orders $m = 2n$ allow the calculation

$$\begin{aligned}\frac{|\,\mathbf{y}\cdot\mathbf{v}\,|}{\sin\frac{\pi}{m}} &= \left|\left(\sum_{j=0}^{n-1}\omega^j\right)\left(\sum_{k=0}^{2m-1}\omega^{\rho(k)}y_k\right)\right| \\ &= \left|\,\sum_{l=0}^{m-1}\omega^l\left(\sum_{\rho(k)+j\equiv l\ (\mathrm{mod}\ m)} y_k\right)\right| \\ &\le \sum_{l=0}^{m-1}\left|\,\sum\left\{y_k\,;\,\rho(k)+j\equiv l\ (\mathrm{mod}\ m),\ 0\le j\le \tfrac{m}{2}-1\right\}\right| \\ &\le m\max_{\substack{T\subseteq Y\\ |T|=m}}\left|\,\sum T\,\right|\end{aligned}$$

and shows $\beta_m^{-1} \le m\sin\frac{\pi}{m}$. Paired with the result of Corollary 2, the determination of β_m for even orders is complete.

Concerning odd $m = 2n+1$ one records that, again suggested by the proof of the corollary,

$$\begin{aligned}\frac{\cos\frac{\pi}{2m}}{\sin\frac{\pi}{m}}\,|\,\mathbf{y}\cdot\mathbf{v}\,| &= \left|\left(\sum_{j=0}^{n-1}\omega^j\right)\left(\sum_{k=0}^{2m-1}\omega^{\rho(k)}y_k\right)\right| \\ &\le m\max_{\substack{T\subseteq Y\\ |T|=(m-1)/2}}\left|\,\sum T\,\right|,\end{aligned}$$

which establishes

$$|\,\mathbf{y}\cdot\mathbf{v}\,| \le \frac{m\sin\frac{\pi}{m}}{\cos\frac{\pi}{2m}}\max_{\mathbf{u}\in\mathcal{U}_m}|\,\mathbf{y}\cdot\mathbf{u}\,|$$

for every $\mathbf{y}\in\mathbb{C}^{2m}$, every $\mathbf{v}\in\mathcal{V}_m$, and odd order m. Taking maximum over all $\mathbf{v}\in\mathcal{V}_m$ this expresses the required lower bound on β_m, so the determination of said constant follows from the corollary. Thereby the proposition and also the Main Theorem are completely proved.

3. Applications. The investigation that promoted my discovery of the main theorem dealt with multi-linear functionals and in particular Fréchet measures. The reader is urged to consult the commendable monograph by R. Blei [B] in order to get the right perspective. For clarity, I will restrict attention to the discrete Fréchet measure spaces $F_k(\mathbb{N},\ldots,\mathbb{N})$, to be recalled below, but the reader will find little difficulty in handling more general situations.

To begin with, the constants c_m determined in the Main Theorem appear in [B], chapter II, where they are only bounded from above with coarser estimates. These constants are governing interrelations between inequalities of Khintchin, Orlicz, and Littlewood, respectively. Further into the theory these relate also to the Grothendieck inequality, cf. [B]. The present determination of each c_m could potentially produce additional geometric insight in the said situations. The most important new information is now $c_1 = c_2 = \pi/2$, as Blei could repeatedly have applied. In [B], chapter II, c_1 and c_2 are synonymous. From another viewpoint, $c_2 = \pi/2$ governs the best quotient between $\|f\|_{A(R)}$ and $\|f\|_\infty$ when f has Fourier spectrum contained in the Rademacher system.

The Fréchet variation of a complex-valued function β on $\mathbb{N}^k$ is by definition

$$\|\beta\|_{F_k} = \sup\Big\{ \Big\| \sum_{n_j \in S_j} \beta(n_1, \dots, n_k)\, r_{n_1} \otimes \dots \otimes r_{n_k} \Big\|_\infty ;\quad \text{finite sets } S_j \subset \mathbb{N},\ j = 1, \dots, k \Big\}.$$

Generalizing in a natural way, the Fréchet variation of order m is introduced as

$$\|\beta\|_{m,F_k} = \sup\Big\{ \Big\| \sum_{n_j \in S_j} \beta(n_1, \dots, n_k)\, \chi_{n_1} \otimes \dots \otimes \chi_{n_k} \Big\|_\infty ;\quad \text{finite sets } S_j \subset \mathbb{N},\ j = 1, \dots, k \Big\}.$$

Here $\chi_1, \chi_2, \dots$ are the coordinate projections on the compact product space

$$\Omega_m = \big(\mathbb{T}_m\big)^{\mathbb{N}}, \qquad \mathbb{T}_m = \{e^{2\pi i j/m} ; j = 0, \dots, m-1\},$$

and the norm inside the supremum is the one used on $L^\infty\big((\Omega_m)^k\big)$, derived from Haar measure. Thus the original Fréchet variation is the case of $m = 2$, where now $\{r_n\}$ lists the Rademacher functions. The space $F_k = F_k(\mathbb{N}, \dots, \mathbb{N})$ of Fréchet measures consists of all β with $\|\beta\|_{F_k} < \infty$. It is standard to find that $\|\ \|_{F_k}$ and $\|\ \|_{m,F_k}$ are equivalent norms. In addition, $\|\alpha\|_{m,F_1} = \|\alpha\|_{(m)}$ in the previous notation.

A second means of measuring the size of β is needed here. It resembles the radius of the range of a complex measure and can be called the rectangular width of $\beta \in F_k$:

$$[[\beta]]_{F_k} = \sup\Big\{ \Big| \sum_{\substack{n_j \in S_j \\ 1 \le j \le k}} \beta(n_1, \dots, n_k) \Big| ; \text{ finite } S_j \subset \mathbb{N} \Big\}.$$

One way of interpreting general inequalities similar to Kaufman–Rickert's original would be to ask for the existence of finite constants α such that

$$\|\beta\|_{m,F_k} \le \alpha\, [[\beta]]_{F_k}$$

for all β in F_k. The previous work on determining d_m will turn out to answer this fully.

Definition. Denote the minimal α in the previous inequality by $\alpha_{m,k}$.

The quickly established - and well-known - fact that $\alpha_{2,k} \le 4^k$ has been used in [B], section VI:3, and originally in [B2], to develop a theory of extensions of Fréchet measures. The purpose being similar to the extension theory for measures on product algebras to the corresponding product σ-algebras.

Proposition 4. *The exact value of $\alpha_{m,k}$ is $(d_m)^k$.*

Although the exact value $\alpha_{2,k} = 2^k$ is of no obvious additional implications to Blei's extension theory, it should be displayed for the sake of clarity. For all $N \ge 1$,

$$\Big\| \sum_{n_1, \dots, n_k = 1}^{N} \beta(n_1, \dots, n_k) r_{n_1} \otimes \dots \otimes r_{n_k} \Big\|_\infty \le 2^k \sup_{\substack{T_1, \dots, T_k \\ \subseteq \{1, \dots, N\}}} \Big| \sum_{\substack{n_j \in T_j \\ 1 \le j \le k}} \beta(n_1, \dots, n_k) \Big|.$$

Conceivably some geometric content of the Fréchet variation could be extracted from the present, improved information.

The proof of the proposition hinges on a multiplicative structure that is implicit in the F_k-norm and the rectangular width. This is explained in a preparatory result.

Lemma 5. *The quantity* $\alpha_{m,k} = \sup_{\beta\neq 0} \frac{\|\beta\|_{m,F_k}}{[[\beta]]_{F_k}}$ *enjoys* $\alpha_{m,k} = (\alpha_{m,1})^k$.

Consider first the special case where β is an elementary tensor $\beta = \gamma_1 \otimes \cdots \otimes \gamma_k$, with each $\gamma_j \colon \mathbb{N} \to \mathbb{C}$. Clearly

$$\begin{aligned}
\|\gamma_1 \otimes \cdots \otimes \gamma_k\|_{m,F_k} &= \sup_{S_1,\dots,S_k} \Big\| \sum_{n_j\in S_j} \gamma_1(n_1)\dots\gamma_k(n_k)\,\chi_{n_1}\otimes\cdots\otimes\chi_{n_k} \Big\|_\infty \\
&= \|\gamma_1\|_{m,F_1}\dots\|\gamma_k\|_{m,F_1}, \\
[[\gamma_1 \otimes \cdots \otimes \gamma_k]]_{F_k} &= \sup_{S_1,\dots,S_k} \Big| \sum_{n_j\in S_j} \gamma_1(n_1)\dots\gamma_k(n_k) \Big| = [[\gamma_1]]_{F_1}\dots[[\gamma_k]]_{F_1},
\end{aligned}$$

which express

$$\frac{\|\gamma_1 \otimes \cdots \otimes \gamma_k\|_{m,F_k}}{[[\gamma_1 \otimes \cdots \otimes \gamma_k]]_{F_k}} = \prod_{j=1}^{k} \frac{\|\gamma_j\|_{m,F_1}}{[[\gamma_j]]_{F_1}}.$$

Taking supremum for all γ_j implies $\alpha_{m,k} \geq (\alpha_{m,1})^k$. To establish the reverse inequality a shortened form of Blei's original calculation is strong enough to get an inductive argument going.

For fixed, finite $S_1,\dots,S_k \subset \mathbb{N}$ there is in the first stage some $T_1 \subseteq S_1$ and later on also $T_j \subseteq S_j$ for $j \geq 2$ such that for each $(t_1,\dots,t_k) \in (\Omega_m)^k$

$$\begin{aligned}
&\Big| \sum_{n_j\in S_j,\, j\geq 1} \beta(n_1,\dots,n_k)\,\chi_{n_1}(t_1)\dots\chi_{n_k}(t_k) \Big| \\
&\qquad = \Big| \sum_{n_1\in S_1} \Big[\sum_{n_j\in S_j,\, j\geq 2} \beta(n_1,\dots,n_k)\,\chi_{n_2}(t_2)\dots\chi_{n_k}(t_k) \Big] \chi_{n_1}(t_1) \Big| \\
&\qquad \leq \alpha_{m,1} \Big| \sum_{n_1\in T_1} \Big[\sum_{n_j\in S_j,\, j\geq 2} \beta(n_1,\dots,n_k)\,\chi_{n_2}(t_2)\dots\chi_{n_k}(t_k) \Big] \Big| \\
&\qquad \leq \alpha_{m,1} \Big\| \sum_{n_j\in S_j,\, j\geq 2} \big[\textstyle\sum_{n_1\in T_1} \beta(n_1,\dots,n_k) \big]\, \chi_{n_2}\otimes\cdots\otimes\chi_{n_k} \Big\|_\infty \\
&\qquad \leq \alpha_{m,1}\,\alpha_{m,k-1} \Big| \sum_{n_j\in T_j,\, j\geq 2} \textstyle\sum_{n_1\in T_1} \beta(n_1,\dots,n_k) \Big|.
\end{aligned}$$

This calculation clearly says that

$$\|\beta\|_{m,F_k} \leq \alpha_{m,1}\,\alpha_{m,k-1}\,[[\beta]]_{F_k}$$

from which follows

$$\alpha_{k,m} \leq \alpha_{m,1}\,\alpha_{m,k-1} \leq \cdots \leq (\alpha_{m,1})^k.$$

Therefore the lemma has been demonstrated.

Since a quick check at the Main Theorem reveals $\alpha_{m,1} = d_m$, Proposition 4 is an immediate consequence of the lemma. In retrospect, the methods apply unchanged when any real-valued β is considered, so one finds also $\alpha^{\mathbb{R}}_{m,k} = (d^{\mathbb{R}}_m)^k$ in the natural sense.

As a remark, let it be recorded that $\alpha_{m,k}$ is an attained maximum. In fact, bookkeeping in the previous material allows explicit construction of $\gamma_{m,k} \in F_k$, non-zero precisely on $\{1, \dots, 2m\}^k$, such that $\|\gamma_{m,k}\|_{m,F_k} = (d_m)^k\, [[\gamma_{m,k}]]_{F_k}$.

The counterpoint to the above analysis will serve as the first factor in a multilinear Kaufman–Rickert inequality. The standard, less precise result (cf. Lemma IV:2 in [B]) serves to identify F_k as isomorphically a dual Banach space. If the previous application used the knowledge of d_m, the final result focuses on c_m.

Define a measuring quantity on F_k by

$$\left\langle\!\left\langle \beta \right\rangle\!\right\rangle_{F_k} = \sup \Big\{ \Big| \sum_{\substack{n_j \in T_j \\ 1 \le j \le k}} \beta(n_1, \dots, n_k)\, z^{(1)}_{n_1} \dots z^{(k)}_{n_k} \Big| ; \text{ finite } T_j,\ z^{(l)}_j \in \mathbb{C},\ |z^{(l)}_j| \le 1 \Big\}.$$

Observe that on F_1 the identity $\langle\!\langle \alpha \rangle\!\rangle_{F_1} = \|\alpha\|_{\ell^1}$ holds. In fact, $\langle\!\langle\ \rangle\!\rangle_{F_k}$ is the injective tensor norm in $\ell^1 \overset{\vee}{\otimes} \dots \overset{\vee}{\otimes} \ell^1$ with k factors. Thus $\langle\!\langle\ \rangle\!\rangle_{F_k}$ is a norm, weaker than $\|\ \|_{\ell^1(\mathbb{N}^k)}$.

Definition. Let $\gamma_{m,k} = \sup\limits_{\beta \ne 0} \dfrac{\langle\!\langle \beta \rangle\!\rangle_{F_k}}{\|\beta\|_{m,F_k}}$.

Obviously $\gamma_{m,1} = c_m$.

Proposition 6. $\gamma_{m,k} = (c_m)^k$ *for all* $k \ge 1$ *and* $m \ge 2$.

The proof is very similar to the calculation of $\alpha_{m,k}$. It is clear that for $\alpha_j \colon \mathbb{N} \to \mathbb{C}$

$$\left\langle\!\left\langle \alpha_1 \otimes \dots \otimes \alpha_k \right\rangle\!\right\rangle_{F_k} = \left\langle\!\left\langle \alpha_1 \right\rangle\!\right\rangle_{F_1} \dots \left\langle\!\left\langle \alpha_k \right\rangle\!\right\rangle_{F_1},$$

whence

$$\gamma_{m,k} \ge \sup_{\alpha_j} \frac{\langle\!\langle \alpha_1 \otimes \dots \otimes \alpha_k \rangle\!\rangle_{F_k}}{\|\alpha_1 \otimes \dots \otimes \alpha_k\|_{m,F_k}} = \sup_{\alpha_j} \prod_{j=1}^{k} \frac{\|\alpha_j\|_{\ell^1}}{\|\alpha_j\|_{(m)}} = (\gamma_{m,1})^k = (c_m)^k.$$

Consider next a fixed $\beta \in F_k$ and take any finite subsets $T_1, \dots, T_k$ of natural numbers as well as complex numbers $z^{(l)}_j$ of modulus at most one. Then

$$\begin{aligned}
&\Big| \sum_{\substack{n_j \in T_j \\ j \ge 1}} \beta(n_1, \dots, n_k)\, z^{(1)}_{n_1} \dots z^{(k)}_{n_k} \Big| \\
&\qquad = \Big| \sum_{n_1 \in T_1} \Big[\sum_{\substack{n_j \in T_k \\ j \ge 2}} \beta(n_1, \dots, n_k)\, z^{(2)}_{n_2} \dots z^{(k)}_{n_k} \Big]\, z^{(1)}_{n_1} \Big| \\
&\qquad \le \gamma_{m,1} \Big\| \Big[\sum_{\substack{n_j \in T_j \\ j \ge 2}} \beta(n_1, \dots, n_k)\, z^{(2)}_{n_2} \dots z^{(k)}_{n_k} \Big] \Big\|_{m,F_1},
\end{aligned}$$

where the inner quantity is in F_1 as n_1 varies. This latter (m, F_1)-norm is the supremum of the following expression as S_1 and x_1 range through all their values.

$$
\begin{aligned}
&\Big| \sum_{n_1 \in S_1} \Big[\sum_{\substack{n_j \in T_j \\ j \geq 2}} \beta(n_1, \ldots, n_k)\, z^{(2)}_{n_2} \ldots z^{(k)}_{n_k} \Big] \chi_{n_1}(x_1) \Big| \\
&\qquad = \Big| \sum_{\substack{n_j \in T_j \\ j \geq 2}} \Big[\sum_{n_1 \in S_1} \beta(n_1, \ldots, n_k)\, \chi_{n_1}(x_1) \Big] z^{(2)}_{n_2} \ldots z^{(k)}_{n_k} \Big| \\
&\qquad \leq \gamma_{m,k-1} \Big\| \Big[\sum_{n_1 \in S_1} \beta(n_1, \ldots, n_k) \chi_{n_1}(x_1) \Big] \Big\|_{m, F_{k-1}} \qquad \text{(as } n_2, \ldots, n_k \text{ changes)} \\
&\qquad = \gamma_{m,k-1} \sup_{S_j,\, j \geq 2} \Big\| \sum_{\substack{n_j \in S_j \\ j \geq 2}} \Big[\sum_{n_1 \in S_1} \beta(n_1, \ldots, n_k)\, \chi_{n_1}(x_1) \Big] \chi_{n_2} \otimes \cdots \otimes \chi_{n_k} \Big\|_\infty \\
&\qquad \leq \gamma_{m,k-1} \, \|\beta\|_{m, F_k}.
\end{aligned}
$$

Taken together these state

$$
\Big| \sum_{\substack{n_j \in T_j \\ j \geq 1}} \beta(n_1, \ldots, n_k)\, z^{(1)}_{n_1} \ldots z^{(k)}_{n_k} \Big| \leq \gamma_{m,1}\, \gamma_{m,k-1} \, \|\beta\|_{m, F_k}.
$$

Letting $z^{(l)}_j$ and T_j range through all possibilities

$$
\langle\!\langle \beta \rangle\!\rangle_{F_k} \leq \gamma_{m,1}\, \gamma_{m,k-1} \, \|\beta\|_{m, F_k},
$$

whence

$$
\gamma_{m,k} \leq \gamma_{m,1}\, \gamma_{m,k-1} \leq \cdots \leq (\gamma_{m,1})^k = (c_m)^k.
$$

This establishes the claimed result. Looking back, it is clear that the methods involved also allow the conclusion $\gamma^{\mathbb{R}}_{m,k} = (c^{\mathbb{R}}_m)^k$ for the real-valued setting.

To round off, Propositions 4 and 6 express a factoring of a multi-linear Kaufman–Rickert inequality:

$$
\begin{aligned}
[[\beta]]_{F_k} \leq \langle\!\langle \beta \rangle\!\rangle_{F_k} &\leq \left[\frac{\pi/m}{\sin(\pi/m)} \right]^k \|\beta\|_{m, F_k} \\
&\leq (c_m d_m)^k \, [[\beta]]_{F_k} = \begin{cases} \pi^k \, [[\beta]]_{F_k}, & m \text{ even} \\ \left[\dfrac{\pi}{\cos(\pi/2m)} \right]^k [[\beta]]_{F_k}, & m \text{ odd} \end{cases}
\end{aligned}
$$

valid for all $\beta \in F_k$. All these inequalities are individually best possible, the first inequality being trivial.

The classical situation for Fréchet variation, i.e., $m = 2$, deserves to be highlighted:

$$
[[\beta]]_{F_k} \leq \langle\!\langle \beta \rangle\!\rangle_{F_k} \leq \left(\tfrac{\pi}{2}\right)^k \|\beta\|_{F_k} \leq \pi^k \, [[\beta]]_{F_k},
$$

where again no inequality can be improved. The special case $k = 1$ is exactly

$$
\|f\|_{(1)} \leq \|f\|_{\ell^1} \leq \tfrac{\pi}{2} \, \|f\|_{(2)} \leq \pi \, \|f\|_{(1)}, \quad \text{all } f \in \ell^1(\mathbb{N}).
$$

Explicitly writing the norms involved, one has

$$\sup_T \Big| \sum_{n\in T} f(n) \Big| \leq \|f\|_{\ell^1} \leq \tfrac{\pi}{2} \sup_{\varepsilon_n=\pm 1} \Big| \sum_n \varepsilon_n f(n) \Big| \leq \pi \sup_T \Big| \sum_{n\in T} f(n) \Big|,$$

which is the original Kaufman–Rickert inequality in factored form. This should also be recognized as the best possible result for sums of Rademacher functions, visible in the third member.

References

[B] R. Blei, *Analysis in integer and fractional dimension*, Cambridge stud. in adv. math. 71, Cambridge Univ. Press, 2001.

[B2] ———, *An extension theorem concerning Fréchet measures*, Canad. Math. Bull. **38** (1995), no. 3, 278–285.

[K] S. Kaijser, *Representations of tensor algebras as quotients of group algebras*, Arkiv för matematik **10** (1972), no. 1, 107–141.

[KR] R. Kaufman and N. Rickert, *An inequality concerning measures*, Bull. Amer. Math. Soc. **72** (1966), 672–676.

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