ISSN: 1401-5617

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Research Reports in Mathematics
Number 8, 2001

Department of Mathematics
Stockholm University

Electronic versions of this document are available at
http://www.matematik.su.se/reports/2001/8

Date of publication: April 3, 2001
2000 Mathematics Subject Classification: Primary 32A05, Secondary 14H45, 26B25, 52B20.

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Polynomial amoebas and convexity

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Abstract

The amoeba of a polynomial f in n complex variables is defined to be the image of the hypersurface $f^{-1}(0)$ under the mapping $\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$. Amoebas were introduced by Gelfand, Kapranov and Zelevinsky, and have also been studied by Forsberg, Passare and Tsikh. An application to the topology of real algebraic curves has been found by Mikhalkin. In this thesis a special convex function N_f, which we call the Ronkin function, is applied to the study of amoebas. Using this function, two kinds of results are obtained. First, the Ronkin function provides a connection between the amoeba and the Newton polytope and makes precise a rather striking sense of duality between these objects which was noticed by Forsberg in his doctoral thesis. Second, we find that the Monge-Ampère measure of N_f has interesting properties. Using this measure, we obtain an estimate on the area of the amoeba in terms of the Newton polytope for polynomials in two variables. It turns out that the amoebas with maximal area correspond to so-called Harnack curves. We also study the number of connected components of the amoeba complement, or equivalently, the number of convergent Laurent series expansions of the rational function $1/f$.

Contents

Acknowledgements. I would like to thank my advisor Mikael Passare, with whom I have had the pleasure to write a joint paper on the subject of this thesis. His support and encouragement have been invaluable. I also thank Grigory Mikhalkin for an enjoyable and instructive collaboration. Many thanks also to everyone at the Department of Mathematics at Stockholm University for providing a pleasant and stimulating atmosphere.

1 Introduction

1.1 Background

Consider the polynomial $f(z) = z^2 + z - 2 = (z-1)(z+2)$. Suppose we are interested in expanding the rational function $1/f(z)$ in a Laurent series, that is, an infinite linear combination of Laurent monomials z^k where k is an integer. It is not difficult to find such a Laurent series expansion: By the geometric series $(1-\zeta)^{-1} = 1 + \zeta + \zeta^2 + \dots$ we have

$$\frac{1}{f(z)} = \frac{1}{-2} \cdot \frac{1}{1 - \frac{z^2+z}{2}} = -\frac{1}{2} \sum_{j=0}^{+\infty} \left(\frac{z^2+z}{2} \right)^j .$$

When the terms in the sum on the right are expanded, we see that only terms where $k/2 \leq j \leq k$ contain the monomial z^k. So by expanding all terms and collecting the finitely many monomials with the same exponent k, we obtain a series $\sum_{k=0}^{+\infty} a_k z^k$. It is not difficult to show that this series converges when $|z| < 1$, and is then equal to $1/f(z)$.

Another expansion is obtained from the computation

$$\frac{1}{f(z)} = \frac{1}{z^2} \cdot \frac{1}{1 - \frac{2-z}{z^2}} = \frac{1}{z^2} \sum_{j=0}^{+\infty} \left(\frac{2}{z^2} - \frac{1}{z} \right)^j .$$

This time we obtain, after expanding the terms in the sum, a series containing only negative powers of z. Some computations show that this series converges when $|z| > 2$ and is then equal to $1/f(z)$.

Are there any other Laurent series expansions of $1/f(z)$? Observing that the two expansions we have found so far were obtained by dividing out the constant term and the z^2-term respectively, and then using a geomeric series, we try the same trick with the z-term:

$$\frac{1}{f(z)} = \frac{1}{z} \cdot \frac{1}{1 - \frac{2-z^2}{z}} = \frac{1}{z} \sum_{j=0}^{+\infty} \left(\frac{2}{z} - z \right)^j .$$

This time we run into a difficulty which was not encountered in the previous two computations. When the terms in the sum are expanded, we find that, for example, a nonzero constant term is present whenever j is even. In order to know whether the sum represents a Laurent series (let alone a convergent Laurent series) we would have to check whether the sum of all these constant terms converges, and similarly for all other powers of z.

A better understanding can be gained by regarding the problem from a geometric point of view. The function $1/f(z)$ has two poles at $z = 1$ and $z = -2$. Hence it is holomorphic in the disc $|z| < 1$ and in the unbounded annulus $|z| > 2$, and this is precisely where the first two series we computed converge. It is also holomorphic in the annulus $1 < |z| < 2$, and by a familiar theorem in the theory of holomorphic functions, it follows that $1/f(z)$ is represented there by a Laurent series $f(z) = \sum_{k=-\infty}^{+\infty} a_k z^k$ where

$$a_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{z^{-k-1}\, dz}{f(z)}$$

and $1 < r < 2$.

From the geometric picture we thus see immediately that there are precisely three convergent Laurent series expansions of $1/f(z)$. Moreover, we have seen how two of these can be computed explicitly, to whatever degree we like, by a kind of geometric series trick. More generally, if $f(z)$ is any polynomial in one variable, then the number of convergent Laurent series expansions of $1/f(z)$ is equal to one plus the maximal number of nonzero roots of f having mutually distinct absolute values.

Let us now consider polynomials in several variables. Such a polynomial can be written $f(z) = \sum_{\alpha \in A} c_\alpha z^\alpha$. Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$ are multiorders, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and c_α are arbitrary complex numbers. The summation takes place over a finite subset A of the lattice $\mathbf{Z}^n$. (Actually, if f is a polynomial, all the components α_j must be positive, but it is just as natural here to allow f to be a Laurent polynomial where the variables may be raised to negative powers.) Singling out one of the points in A, say β, we try to write the series

$$\frac{1}{f(z)} = \frac{1}{c_\beta z^\beta} \cdot \frac{1}{1 + \sum_{\alpha \in A'} \frac{c_\alpha}{c_\beta} z^{\alpha-\beta}} = \frac{1}{c_\beta z^\beta} \sum_{j=0}^{+\infty} \left(- \sum_{\alpha \in A'} \frac{c_\alpha}{c_\beta} z^{\alpha - \beta} \right)^j \tag{1}$$

where $A' = A \setminus \{\beta\}$. How can we know if the sum on the right, when each term is expanded, can be rearranged as a Laurent series? Let us say that the sum (1) is well behaved if every monomial z^ν occurs only in the expansions of a finite number of terms

$$\left(- \sum_{\alpha \in A'} \frac{c_\alpha}{c_\beta} z^{\alpha - \beta} \right)^j.$$

The problem of deciding whether the sum (1) is well behaved can easily be understood from a geometric consideration. Imagine that we plot the points in A in Euclidean space. The convex hull of these points is called the Newton polytope of f. For example, if $f(z_1, z_2) = 1 + z_1^5 + 80 z_1^2 z_2 + z_1^3 z_2 + 40 z_1^3 z_2^2 + z_1^3 z_2^4$, then $A = \{(0,0), (5,0), (2,1), (3,1), (3,2), (3,4)\}$ and the Newton polytope of f is the triangle shown on the left in Figure 1. The significance of the Newton polytope here is that the sum (1) is well behaved if and only if β is a vertex of the Newton polytope. In our example, the sum is well behaved if β is one of the points $(0,0), (5,0)$ or $(3,4)$, which are corners of the triangle, but not if β is one of the other three points. If f is a polynomial in one variable, the Newton polytope is just a segment, and the vertices are its endpoints. These correspond to the monomials of lowest and highest degree in f respectively. Our first two computations succeded, because there our β was one of these endpoints, but the third computation ran into trouble (and would in fact have failed if we had carried on) since β was a point inside the segment.

Let us now carry over the other way of understanding Laurent series expansions to polynomials of several variables. The multidimensional analogue of an annulus is a circular domain $\{z \in \mathbf{C}^n; (|z_1|, \ldots, |z_n|) \in E\}$ where E is an open connected subset of $\mathbf{R}^n_{>0}$. If a function is holomorphic in such a circular domain, then it can be represented there by a convergent Laurent series $\sum_{\alpha \in \mathbf{Z}^n} a_\alpha z^\alpha$. For various reasons, it is more convenient to describe a circular domain in the following way. Let $\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|)$, so that Log is a mapping from $(\mathbf{C} \setminus \{0\})^n$ to $\mathbf{R}^n$. Then a circular domain can be written in the form $\mathrm{Log}^{-1}(E) = \{z; \mathrm{Log}(z) \in E\}$ where E is an open connected set in $\mathbf{R}^n$. Notice that if one of the coordinates, say z_1, is replaced by z_1^{-1}, which is quite natural when we are dealing with Laurent series, the set E will simply be reflected in the plane $x_1 = 0$. Another nice property of this convention is that we need only consider sets E which are convex. In fact, it can be shown that if E is connected, then every function which is holomorphic in $\mathrm{Log}^{-1}(E)$ can automatically be extended to $\mathrm{Log}^{-1}(\mathrm{conv}\, E)$ where $\mathrm{conv}\, E$ denotes the convex hull of E.

When f is a polynomial in more than one variable, the singularities of $1/f$, which must be avoided by $\mathrm{Log}^{-1}(E)$, are not discrete but spread out along the surface of complex dimension $n-1$ where $f(z) = 0$. The image of this surface under the mapping Log is called the amoeba of f. Each connected component E of the complement of the amoeba corresponds to a circular domain $\mathrm{Log}^{-1}(E)$ where $f(z)$ is never zero and hence $1/f(z)$ is holomorphic. We conclude that there is a one-to-one correspondence between the connected components of the complement of the amoeba and the convergent Laurent series expansions of $1/f(z)$.

Figure 1 shows the amoeba of the polynomial $f(z_1, z_2) = 1 + z_1^5 + 80z_1^2z_2 + z_1^3z_2 + 40z_1^3z_2^2 + z_1^3z_2^4$. The three "tentacles" of the amoeba extend outside the picture to infinity. There are five clearly visible complement components of the amoeba in the picture. Two of these are bounded, while the remaining three are infinitely large. These three unbounded regions are where the three well behaved Laurent series which we found by the geometric series computation converge.

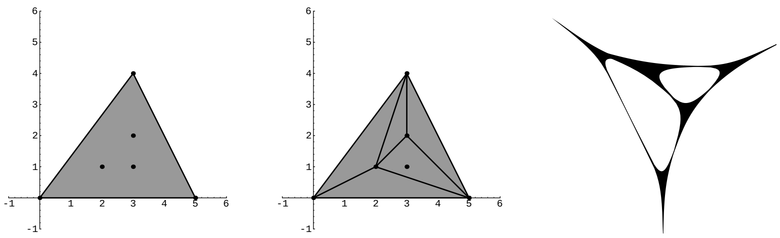

Figure 1: Newton polytope, triangulated Newton polytope and amoeba of the polynomial $f(z) = 1 + z_1^5 + 80z_1^2z_2 + z_1^3z_2 + 40z_1^3z_2^2 + z_1^3z_2^4$

Notice that there seems to be a kind of duality between the amoeba and the Newton polytope. This feeling becomes even stronger if the Newton polytope is triangulated by drawing lines as in the second picture. A more precise statement is that the amoeba looks like a thickened graph, whose edges are perpendicular to certain edges in the triangulation of the Newton polytope.

It is almost obvious from looking at these pictures that each unbounded component of the amoeba complement is associated to a vertex of the Newton polytope. One might also surmise, that the two bounded components belong to the points $(2, 1)$ and $(3, 2)$ in the Newton polytope, corresponding to the terms $80z_1^2z_2$ and $40z_1^3z_2^2$ in $f(z)$. This is actually the case, in a sense made precise by the concept of the order of a complement component introduced by Forsberg, Passare and Tsikh. Notice that there seems to be no complement component corresponding to the term $z_1^3 z_2$. It is tempting to attribute this to the fact that its coefficient is so much smaller than the coefficients of $z_1^2 z_2$ and $z_1^3 z_2^2$. There is some truth to this statement; a connection between the size of a coefficient and the existence and size of a corresponding complement component does exist. However, the connection is weaker than one might initially be lead to hope (see Example 7 in section 9).

1.2 Outline of the thesis

There are three main problems with which this thesis is concerned. These are treated in sections 4, 5 and 6. The treatment is based on the papers [21], [22] and [27], where many of the results of the thesis can also be found.

In section 2 we give the definitions of the objects we will be dealing with, together with a discussion of the complex torus which is the space in which these objects live. Section 3 outlines some results, to the most part taken from [8], [11] and [26], which are of fundamental importance in the following treatment. We also introduce certain functions, which are very nearly hypergeometric in the GKZ sense, which turn out to have several interesting connections with amoebas.

After these foundations have been established, we explore in section 4 the duality between the amoeba and the Newton polytope. Certain aspects of this duality were found in [7], [8] and [11]. We obtain, in section 4, another manifestation of this duality by exploiting a special convex function N_f associated to the polynomial f. The idea of using this function in the study of amoebas comes from the paper [26] It is defined by letting $N_f(x)$ be the average of $\log|f(z)|$ as z runs through $\mathrm{Log}^{-1}(x)$. This function encodes information both about the amoeba (Theorem 1) and the Newton polytope (Theorem 2).

In section 5 we consider the problem of finding the number of complement components of the amoeba of a given polynomial. Lower and upper bounds on this number, in terms of the Newton polytope, were found in [11] and [8]. We show that these bounds are sharp and also give some partial ansers to certain related problems.

Finally, in section 6 we study the convexity of the function N_f by means of the Monge-Ampère operator, and find several relations between this function and the hypersurface $f^{-1}(0)$. One rather remarkable consequence is that for polynomials of two variables, the area of the amoeba is no greater than π^2 times the area of the Newton polytope. Moreover, as was discovered jointly with Mikhalkin, the polynomials for which the amoeba has maximal area are those defining so-called Harnack curves which arise in real algebraic geometry. The connection with real algebraic curves is outlined in section 7.

A possible generalization of the concepts treated in the thesis is outlined in section 8. Section 9 contains some explicit examples, and in section 10 we present a few open problems.

2 The complex torus

In this section we will define several objects related to a Laurent polynomial which will be our main interest in later sections. First we shall discuss the space where all these objects live, the complex torus.

The complex torus could simply be defined as the product space $\mathbf{C}_*^n$, where $\mathbf{C}_* = \mathbf{C} \setminus \{0\}$ is the multiplicative group of the complex field and n is a positive integer. However, it is more natural to state the definitions in coordinate free manner. Let therefore L be an n-dimensional lattice and $L^* = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ its dual. The complex torus associated to L is the abelian group $L_{\mathbf{C}_*} = L \otimes \mathbf{C}_*$ (where the tensor product is taken over $\mathbf{Z}$). By choosing a basis for L, we obtain an isomorphism between $L_{\mathbf{C}_*}$ and $\mathbf{C}_*^n$. In particular, we see that $L_{\mathbf{C}_*}$ has the structure of a complex manifold as well as an abelian group. Wherever it is convenient, we shall assume that such a basis has been chosen so that expressions may be written in terms of the coordinate functions on $\mathbf{C}_*^n$. However, we shall here establish notations which make it possible to avoid a particular choice of basis most of the time.

The homomorphism $\zeta \mapsto \log|\zeta|$ from $\mathbf{C}_*$ to $\mathbf{R}$ determines a mapping $\mathrm{Log} = \mathrm{id} \otimes \log|\cdot| : L_{\mathbf{C}_*} \to L_{\mathbf{R}} = L \otimes \mathbf{R}$. If a basis is chosen for L, this mapping can be written explicitly $\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|)$.

Since $\mathrm{Log}^{-1}(0)$ is a compact subgroup of $L_{\mathbf{C}_*}$, it has a unique Haar measure η_0 which is translation invariant and has total mass 1. If $x \in L_{\mathbf{R}}$ and $z \in \mathrm{Log}^{-1}(x)$, then multiplication by z maps $\mathrm{Log}^{-1}(0)$ to $\mathrm{Log}^{-1}(x)$. Any two such mappings differ only by a translation in $\mathrm{Log}^{-1}(0)$ and hence the direct images of η_0 under all such mappings coincide and therefore define a measure η_x on $\mathrm{Log}^{-1}(x)$. We shall drop the subscript and denote by η the probability measure η_x on any fiber $\mathrm{Log}^{-1}(x)$. The measure η can be computed by integrating the differential form

$$\frac{1}{(2\pi i)^n} \cdot \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n}.$$

Every $\alpha \in L^*$ determines a function $\alpha \otimes \mathrm{id} : L_{\mathbf{C}_*} \to \mathbf{Z} \otimes \mathbf{C}_* = \mathbf{C}_*$. This function is usually denoted $z \mapsto z^\alpha$ and called a Laurent monomial. Similarly, every $a \in L$ gives rise to a function $\zeta \mapsto \zeta^a := a \otimes \zeta$ from $\mathbf{C}_*$ to $L_{\mathbf{C}_*}$. If $e_1, \ldots, e_n$ is a basis for L, then an isomorphism between $L_{\mathbf{C}_*}$ and $\mathbf{C}_*^n$ is given explicitly by $z \mapsto (z^{e_1}, \ldots, z^{e_n})$. A finite linear combination of Laurent monomials is called a Laurent polynomial. If f is a Laurent polynomial, we will let f_α denote the coefficient for z^α in f. Hence, $f(z) = \sum_{\alpha \in L^*} f_\alpha z^\alpha$. The set of all Laurent polynomials clearly is a $\mathbf{C}$-algebra, isomorphic to the group algebra $\mathbf{C}[L^*]$.

A Laurent series is a (formal) linear combination $f(z) = \sum_{\alpha \in L^*} f_\alpha z^\alpha$ where there may be infinitely many nonzero terms in the sum. The Laurent series is said to converge at z if the sum is absolutely convergent. Its domain of convergence is the largest open set in which it converges at every point, and f is called convergent if its domain of convergence is nonempty. If f is a convergent Laurent series, it is well known that its domain of convergence is of the form $\mathrm{Log}^{-1}(\Omega)$ where $\Omega \subset L_{\mathbf{R}}$ is a convex open set. A convergent Laurent series defines a holomorphic function on its domain of convergence. Conversely, if $f(z)$ is a holomorphic function defined in $\mathrm{Log}^{-1}(\Omega)$ where Ω is open and connected, then $f(z)$ is represented by a Laurent series whose domain of convergence contains $\operatorname{Log}^{-1}(\Omega)$. The coefficients of the Laurent series are given by

$$f_\alpha = \int_{\operatorname{Log}^{-1}(x)} z^{-\alpha} f(z)\, d\eta(z)$$

for any $x \in \Omega$. If f and g are convergent Laurent series, their sum $f+g$ need not be convergent and their product need not even be defined. If, however, their domains of convergence have nonempty intersection, then $f+g$ and $f \cdot g$ are both holomorphic functions in the intersection of the domains of convergence of f and g, and hence define convergent Laurent series.

Definition 1 (Gelfand, Kapranov, Zelevinsky [11]). *If f is a Laurent polynomial or a convergent Laurent series, then the amoeba of f, denoted $\mathcal{A}_f$, is the set* $\operatorname{Log}(f^{-1}(0))$.

If the domain of convergence of f is $\operatorname{Log}^{-1}(\Omega)$, the amoeba of f should be considered as a subset of Ω. We will write $\mathcal{A}_f^c = \Omega \smallsetminus \mathcal{A}_f$ for the complement of the amoeba in Ω.

Definition 2. *If f is a Laurent series, P_f will denote the convex hull in $L^*_{\mathbf{R}}$ of the set $\{\alpha \in L^*; f_\alpha \neq 0\}$. If f is a Laurent polynomial, P_f is called the Newton polytope of f.*

Definition 3 (Ronkin [26]). *If f is a Laurent polynomial (or a Laurent series converging in* $\operatorname{Log}^{-1}(\Omega)$*), the function N_f is defined in $L_{\mathbf{R}}$ (or in Ω) by*

$$N_f(x) = \int_{\operatorname{Log}^{-1}(x)} \log|f(z)|\, d\eta(z). \tag{2}$$

We will call N_f the Ronkin function of f.

A linear mapping $T : L \to M$ between two lattices, not necessarily of the same dimension, induces in an obvious way mappings $T^* : \mathbf{C}[M^*] \to \mathbf{C}[L^*]$, $T_{\mathbf{R}} : L_{\mathbf{R}} \to M_{\mathbf{R}}$ and $T_{\mathbf{C}_*} : L_{\mathbf{C}_*} \to M_{\mathbf{C}_*}$. It is easily verified that

$$T^* f(z) = f(T_{\mathbf{C}_*} z) \tag{3}$$

$$T_{\mathbf{R}}(\operatorname{Log} z) = \operatorname{Log}(T_{\mathbf{C}_*} z) \tag{4}$$

for any $z \in L_{\mathbf{C}_*}$ and $f \in \mathbf{C}[M^*]$. If moreover $T^* : M^* \to L^*$ is injective, then both $T_{\mathbf{R}}$ and $T_{\mathbf{C}_*}$ are surjective. In this case,

$$T_{\mathbf{C}_*} \eta_L = \eta_M \tag{5}$$

where η_L and η_M denote the Haar measures on the fibres of $\operatorname{Log} : L_{\mathbf{C}_*} \to L_{\mathbf{R}}$ and $\operatorname{Log} : M_{\mathbf{C}_*} \to M_{\mathbf{R}}$ respectively.

Let $a \in L$. The mapping $\zeta \mapsto \zeta^a$ from $\mathbf{C}_*$ to $L_{\mathbf{C}_*}$ induces a homomorphism of homology groups $H_1(\mathbf{C}_*, \mathbf{Z}) \to H_1(L_{\mathbf{C}_*}, \mathbf{Z})$. The generator of $H_1(\mathbf{C}_*, \mathbf{Z})$ represented by the unit circle with its usual orientation, is mapped by this homomorphism to an element in $H_1(L_{\mathbf{C}_*}, \mathbf{Z})$ which we will denote $\rho_1(a)$. It is not difficult to see that $\rho_1 : L \to H_1(L_{\mathbf{C}_*}, \mathbf{Z})$ is a homomorphism. Similarly, any $\alpha \in L^*$ defines an element $\rho^1(\alpha)$ in the cohomology group $H^1(L_{\mathbf{C}_*}, \mathbf{Z})$

via the mapping $z \mapsto z^\alpha$. In de Rham cohomology, $\rho^1(\alpha)$ is represented by the differential form $(2\pi i)^{-1} z^{-\alpha}\, dz^\alpha = (2\pi i)^{-1}\, d\log z^\alpha$. For every pair $a \in L, \alpha \in L^*$, it is easy to see that $\langle \rho^1(\alpha), \rho_1(a)\rangle = \langle \alpha, a\rangle$. Since $H_1(L_{\mathbf{C}_*}, \mathbf{Z})$ and $H^1(L_{\mathbf{C}_*}, \mathbf{Z})$ are free abelian groups of rank n, it follows that $\rho_1 : L \to H_1(L_{\mathbf{C}_*}, \mathbf{Z})$ and $\rho^1 : L^* \to H^1(L_{\mathbf{C}_*}, \mathbf{Z})$ are in fact isomorphisms.

For any $k = 1, \ldots, n$ there is a Künneth map

$$H_1(L_{\mathbf{C}_*}, \mathbf{Z})^{\otimes k} \to H_k((L_{\mathbf{C}_*})^k, \mathbf{Z}).$$

Together with the homomorphism induced in homology by the multiplication map $(L_{\mathbf{C}_*})^k \to L_{\mathbf{C}_*}$ this defines a homomorphism $H_1(L_{\mathbf{C}_*}, \mathbf{Z})^{\otimes k} \to H_k(L_{\mathbf{C}_*}, \mathbf{Z})$ which is easily seen to be alternating. Hence, composition with ρ_1 defines a homomorphism $\rho_k : \bigwedge^k L \to H_k(L_{\mathbf{C}_*}, \mathbf{Z})$. Similarly, the multiplication in the cohomology ring $H^*(L_{\mathbf{C}_*}, \mathbf{Z}) = \bigoplus H^k(L_{\mathbf{C}_*}, \mathbf{Z})$ together with ρ^1 determines a homomorphism $\rho^k : \bigwedge^k L^* \to H^k(L_{\mathbf{C}_*}, \mathbf{Z})$. In de Rham cohomology, $\rho^k(\alpha_1 \wedge \ldots \wedge \alpha_k)$ is represented by the differential form $(2\pi i)^{-k} z^{-\alpha_1 - \ldots - \alpha_k}\, dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_k}$. The mappings ρ_k and ρ^k are all isomorpisms, and satisfy $\langle \rho^k(\alpha), \rho_k(a)\rangle = \langle \alpha, a\rangle$ for all $a \in \bigwedge^k L, \alpha \in \bigwedge^k L^*$.

If E is a contractible subset of $L_{\mathbf{R}}$, then there are canonical isomorphisms $H_k(L_{\mathbf{C}_*}, \mathbf{Z}) \cong H_k(\mathrm{Log}^{-1}(E), \mathbf{Z})$ and $H^k(L_{\mathbf{C}_*}, \mathbf{Z}) \cong H^k(\mathrm{Log}^{-1}(E), \mathbf{Z})$, and we may therefore identify these homology and cohomology groups for any E.

Let f be a Laurent polynomial or a convergent Laurent series and let E be a connected component of $\mathcal{A}_f^c$. We shall see that any such component is convex, so in particular, it is contractible. Then f defines a mapping from $\mathrm{Log}^{-1}(E)$ to $\mathbf{C}_*$, and hence a homomorphism $f^* : H^1(\mathbf{C}_*, \mathbf{Z}) \to H^1(\mathrm{Log}^{-1}(E), \mathbf{Z})$. Let ω be the generator of $H^1(\mathbf{C}_*, \mathbf{Z})$ represented by the differential form $(2\pi i\zeta)^{-1}\, d\zeta$.

Definition 4 (Forsberg, Passare, Tsikh [8]). *The vector $\alpha = {\rho^1}^{-1}(f^*\omega)$ is called the order of the complement component E.*

An alternative definition is that the order is the unique $\alpha \in L^*$ such that the mappings $z \mapsto f(z)$ and $z \mapsto z^\alpha$ from $\mathrm{Log}^{-1}(E)$ to $\mathbf{C}_*$ are homotopic. The following relations are useful for computations with orders. If α is the order of the complement component E and $a \in L$, then

$$\langle \alpha, a\rangle = \langle f^*\omega, \rho_1(a)\rangle = \frac{1}{2\pi i}\int_{|\zeta|=1} d\log f(\zeta^a z).$$

where z is any point in $\mathrm{Log}^{-1}(E)$. In particular the coordinates of α with respect to some basis of L^* are given by

$$\begin{aligned} \alpha_j &= \frac{1}{2\pi i}\int_{|\zeta|=1} d\log f(z_1, \ldots, \zeta z_j, \ldots, z_n), \quad z \in \mathrm{Log}^{-1}(E) \\ &= \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial f/\partial z_j}{f(z)}\, d\eta(z), \quad x \in E. \end{aligned} \tag{6}$$

If A is a subset of L^*, we will let $\mathbf{C}^A$ denote the set of all Laurent polynomials of the form $f(z) = \sum_{\alpha \in A} f_\alpha z^\alpha$. Usually A will be a finite set. Thus $\mathbf{C}^A$ is a complex vector space whose points are Laurent polynomials. There is a natural choice of coordinates on this space, namely the coefficients f_α. The space $\mathbf{C}^A$ can be treated just as any other complex manifold. For example, we will consider holomorphic functions defined on $\mathbf{C}^A$. The fact that f is used both to denote a point in $\mathbf{C}^A$ and a function on $L_{\mathbf{C}_*}$ is not likely to cause much confusion.

This is a natural place to establish some terminology and notations concerning toric varieties which will be needed in sections 7 and 8. The following paragraphs are not needed for the main part of the thesis. For an extensive treatment of toric varieties, we refer to [6] and [4].

Let σ be a cone in $L^*_{\mathbf{R}}$ generated by finitely many vectors in L^*. Then $\mathbf{C}[\sigma \cap L^*]$ is the subalgebra of $\mathbf{C}[L^*]$ generated by all monomials z^α with $\alpha \in \sigma \cap L^*$. The affine toric variety associated to σ is defined to be the maximal spectrum of $\mathbf{C}[\sigma \cap L^*]$, and will be denoted X_σ. This variety can be given a concrete realization as follows. Take a set of generators $\alpha_1, \ldots, \alpha_k$ for σ and consider the mapping $\phi : L_{\mathbf{C}_*} \to \mathbf{C}^k$ defined by $\phi(z) = (z^{\alpha_1}, \ldots, z^{\alpha_k})$. Then X_σ is isomorphic to the closure of $\phi(L_{\mathbf{C}_*})$ in $\mathbf{C}^k$ (with the metric or Zariski topology, whichever one prefers). If $\sigma \subset \tau$ are cones, then there is a natural injective mapping $X_\tau \to X_\sigma$.

A general toric variety is constructed by gluing together affine toric varieties. To keep track of the operations it is useful to introduce the notion of a fan.

A fan Σ in a real vector space V is a finite collection of polyhedral cones $\sigma \subset V$, such that (i) if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau \in \Sigma$ and (ii) if $\sigma \in \Sigma$ and τ is any cone contained in σ, then $\tau \in \Sigma$ precisely if τ is a face of σ. A fan is said to be complete if the union of all its cones is the entire vector space in which it lives. If C is a cone in V, then the dual of C is defined to be the cone $C^\vee = \{\xi \in V^*; \langle \xi, x\rangle \leq 0, \forall x \in C\}$. If C is generated by finitely many vectors from a lattice in V, then its dual is generated by finitely many vectors of the dual lattice

Suppose now that Σ is a fan in $L_{\mathbf{R}}$ whose cones are generated by finitely many lattice vectors. For every $\sigma \in \Sigma$, consider the affine toric variety $X_{\sigma^\vee}$. From the disjoint union of all these varieties we form a new variety by identifying $X_{\tau^\vee}$ with its image in $X_{\sigma^\vee}$ whenever $\tau \subset \sigma$ are in Σ. The variety so obtained is called a toric variety and will be denoted X_Σ. The affine toric variety arising from the zero cone $\sigma = \{0\}$ deserves special attention. The dual of σ is the whole space $L^*_{\mathbf{R}}$ and its associated affine toric variety $X_{\sigma^\vee}$ is isomorphic to the complex torus $L_{\mathbf{C}_*}$. This space is contained as an open dense subset in X_Σ. The toric variety X_Σ is compact if and only if Σ is complete.

Now let P be a polytope in $L^*_{\mathbf{R}}$ with vertices in $L^*_{\mathbf{Q}}$. If F is a face of P, then the normal cone to P at F is defined to be the cone $\mathrm{nc}(F, P) = \{x; \langle \xi - \eta, x\rangle \leq 0, \forall \xi \in P, \eta \in F\}$. The cone dual to $\mathrm{nc}(F, P)$ is $\mathrm{cone}(F, P) = \{t(\xi - \eta); t \geq 0, \xi \in P, \eta \in F\}$, which can be thought of as the cone of all vectors pointing from F into P. The collection of all normal cones, as F runs over all faces of P, is a complete fan Σ, and hence defines a compact toric variety $X_P = X_\Sigma$. If F is any face of P then we write $X(F) = X_{\mathrm{cone}(F,P)}$ and let $V(F)$ denote the closure of $X(F) \setminus \cup_{G \supset F} X(G)$ where the union is taken over all faces G properly containing F. This system of subvarieties of X_P looks like the polytope P, in the sense that $\dim V(F) = \dim F$ and $V(F) \subset V(G)$ precisely if $F \subset G$. We remark that there is a mapping, known as the moment map, from X_P to P, taking $V(F)$ onto F for every face F of P. If f is a Laurent polynomial whose Newton polytope is P, then f defines a closed hypersurface $V \subset X_P$. The image of V under the moment map was called the compactified amoeba in [11].

Two familiar examples of toric varieties are obtained by taking the polytope P to be the standard simplex $\mathrm{conv}(0, e_1, \ldots, e_n)$ or the unit cube $[0,1]^n$ in $\mathbf{R}^n$. In the first case, X_P is the projective space $\mathbf{P}^n$. For example, if $n = 2$, then P is a triangle and X_P is the projective plane. When F is a side of the triangle P, $V(F)$ is one of the coordinate axes or the line at infinity. In the second case, X_P is the product $\mathbf{P}^1 \times \ldots \times \mathbf{P}^1$ of n copies of the Riemann sphere.

3 Foundations

This section contains a collection of results which are fundamental for the study of polynomials from the amoeba point of view. A major part of the material has appeared earlier in [7], [8], [11] and [26]. However, it seems that much is gained by studying the interplay between different objects, namely the Newton polytope, the amoeba and the convex function N_f, which have not been considered all at once in these earlier works. For this reason, a selection of previously known results are presented here in an attempt to carry out a unified approach to the subject. One new aspect in this presentation is that certain results are generalized from Laurent polynomials to arbitrary convergent Laurent series. Although a fairly straightforward generalization, it has not been carried out earlier.

As our starting point we take Theorem 1, which is essentially due to Ronkin [26], although it is stated here in slightly different terminology. The other main results are Theorem 2, Theorem 3 which is due to Forsberg, Passare and Tsikh [8], and Theorem 4 which was the main motivation for Gelfand, Kapranov and Zelevinsky to introduce amoebas in [11]. The functions Φ_α, which are very similar to GKZ-hypergeometric functions, appeared first in [22].

3.1 Some classical results

Theorem 1 (Ronkin [26]). *Let f be a Laurent polynomial or a convergent Laurent series. Then the following holds.*

(i) The function N_f is convex.

(ii) N_f is affine linear in an open connected set E precisely if E is contained in $\mathcal{A}_f^c$.

(iii) If E is a connected component of $\mathcal{A}_f^c$, then $\operatorname{grad} N_f|_E$ is equal to the order of E.

Proof. The convexity of N_f follows from the fact that $\log|f|$ is plurisubharmonic. Indeed, $N_f(\operatorname{Log} z)$ is a superposition of plurisubharmonic functions, and is therefore itself plurisubharmonic. This is easily seen to be equivalent to convexity of N_f. (See also [25], Corollary 1 on p. 84.) If $E \subset \mathcal{A}_f^c$ then $\log|f|$ is actually pluriharmonic in $\operatorname{Log}^{-1}(E)$, and it follows that N_f is affine linear in

E. Conversely, if N_f is affine linear in E, then $N_f(\operatorname{Log} z)$ is pluriharmonic in $\operatorname{Log}^{-1}(E)$ which implies that $\log|f|$ must be pluriharmonic in $\operatorname{Log}^{-1}(E)$. But then $f(z) \neq 0$ there, so $E \subset \mathcal{A}_f^c$. Differentiation with respect to x_j under the integral sign in the definition (2) of N_f yields

$$\begin{aligned}\frac{\partial N_f}{\partial x_j} &= \frac{\partial}{\partial x_j} \int_{\operatorname{Log}^{-1}(0)} \log|f(z)|\, d\eta(z) \\ &= \operatorname{Re} \int_{\operatorname{Log}^{-1}(x)} \frac{z_j \partial f/\partial z_j}{f(z)}\, d\eta(z)\end{aligned}$$

which is precisely the real part of the second integral in (6). However, the integral (6) is always real valued so $\partial N_f/\partial x_j$ is the jth component of the order. Hence $\operatorname{grad} N_f$ is equal to the order of the complement component. ■

An immediate consequence is the following corollaries which were proved by other methods in [11] and [8].

Corollary 1. *Every connected component of $\mathcal{A}_f^c$ is a convex open set.*

Proof. It is clear that $\mathcal{A}_f$ is closed, hence every complement component is open. If E is a connected component of $\mathcal{A}_f^c$ of order α, then $N_f(x) = c + \langle \alpha, x\rangle$ in E by Theorem 1. By convexity of N_f it follows that $N_f(x) \geq c + \langle \alpha, x\rangle$ for all x. Hence the set K consisting of all x such that $N_f(x) = c + \langle \alpha, x\rangle$ is convex and the interior of K does not intersect $\mathcal{A}_f$. It follows that $E = \operatorname{int} K$ is convex. ■

Corollary 2. *Different components of $\mathcal{A}_f^c$ have different orders.*

Proof. If E is a complement component of order α, then there exists a constant c such that $N_f(x) \geq c + \langle \alpha, x\rangle$ with equality precisely in the closure of E. If E' is another component of order α, and c' the corresponding constant, then it follows that $c = c'$, and then that $E = E'$. ■

Hence, if $\mathcal{A}_f^c$ has a component of order α, this component is uniquely determined by α.

Definition 5. *When $\mathcal{A}_f^c$ has a component of order α, this component will be denoted E_α or $E_\alpha(f)$. If $\mathcal{A}_f^c$ has no component of order α, we set $E_\alpha = \varnothing$.*

A simple and useful criterion for determining the order of a complement component is that a dominating term in the Laurent polynomial determines the order of a complement component. To make this precise we introduce the following notation. If f is a convergent Laurent series whose domain of convergence is $\operatorname{Log}^{-1}(\Omega)$, and $\alpha \in L^*$ we write

$$m_\alpha(f;x) = \frac{\sum_{\beta\neq\alpha} |f_\beta| \exp\langle \beta, x\rangle}{|f_\alpha| \exp\langle \alpha, x\rangle}$$

and

$$m_\alpha(f) = \inf_{x\in\Omega} m_\alpha(f;x).$$

Note that $\Omega = \operatorname{int}\{x; m_\alpha(f;x) < +\infty\}$ for any α with $f_\alpha \neq 0$. The following lemma is proved in [8].

Lemma 1. *Let $f(z)$ be a Laurent polynomial or a convergent Laurent series and suppose that $m_\alpha(f;x) < 1$ for some $\alpha \in L^*$ and $x \in \Omega$. Then $x \in \mathcal{A}_f^c$ and the complement component containing x has order α.*

Proof. By the triangle inequality,

$$|f(z)| \geq |f_\alpha| \exp\langle \alpha, x\rangle - \sum_{\beta \neq \alpha} |f_\beta| \exp\langle \beta, x\rangle > 0$$

for all $z \in \operatorname{Log}^{-1}(x)$, so $x \in \mathcal{A}_f^c$. Suppose the component containing x has order β. By equation (6) and the argument principle,

$$\begin{aligned}
\beta_j &= \frac{1}{2\pi i} \int_{|\zeta|=1} d \log f(e^{x_1}, \ldots, \zeta e^{x_j}, \ldots, e^{x_n}) \\
&= \frac{1}{2\pi i} \int_{|\zeta|=1} d \log \left(\sum_\nu f_\nu \exp\langle \nu, x\rangle \zeta^{\nu_j} \right) \\
&= \frac{1}{2\pi i} \int_{|\zeta|=1} d \log (f_\alpha \exp\langle \alpha, x\rangle \zeta^{\alpha_j}) = \alpha_j
\end{aligned}$$

so $\alpha = \beta$. ■

Corollary 3. *If α is a vertex of P_f, then $m_\alpha(f) = 0$, hence $\mathcal{A}_f^c$ has a component of order α.*

Proof. Since α is a vertex of P_f, there exists a $y \in L_{\mathbf{R}}$ so that $\langle \alpha - \beta, y\rangle > 0$ for all $\beta \neq \alpha$ with $f_\beta \neq 0$. Take any $x \in \Omega$, where $\operatorname{Log}^{-1}(\Omega)$ is the domain of convergence of f and a positive number ϵ. Then $m_\alpha(f;x) < +\infty$, so there is a finite set $A \subset L^*$ such that $\sum_{\beta \notin A} |f_\beta| \exp\langle \beta, x\rangle / |f_\alpha| \exp\langle \alpha, x\rangle < \epsilon$. Now, for any $\beta \in L^* \cap P_f$, $\exp\langle \beta - \alpha, x + ty\rangle$ is decreasing as a function of t and has the limit 0 when $t \to +\infty$. If $t > 0$ it follows that

$$\begin{aligned}
m_\alpha(f; x+ty) &= \sum_{\beta \in A \smallsetminus \{\alpha\}} \frac{|f_\beta| \exp\langle \beta, x+ty\rangle}{|f_\alpha| \exp\langle \alpha, x+ty\rangle} + \sum_{\beta \notin A} \frac{|f_\beta| \exp\langle \beta, x+ty\rangle}{|f_\alpha| \exp\langle \alpha, x+ty\rangle} \\
&\leq \sum_{\beta \in A \smallsetminus \{\alpha\}} \frac{|f_\beta| \exp\langle \beta, x+ty\rangle}{|f_\alpha| \exp\langle \alpha, x+ty\rangle} + \epsilon \\
&\leq 2\epsilon \quad \text{for large } t.
\end{aligned}$$

Since clearly $x + ty \in \Omega$ for all $t \geq 0$, it follows that $m_\alpha(f) = 0$ as required. ■

If u is a convex function defined in a domain $\Omega \subset L_{\mathbf{R}}$, then by the gradient of u at $x_0 \in \Omega$ we will mean the set

$$\operatorname{grad} u(x_0) = \{\xi \in L_{\mathbf{R}}^*; u(x) - u(x_0) \geq \langle \xi, x - x_0\rangle, \forall x \in \Omega\}. \tag{7}$$

When u is differentiable at x_0, $\operatorname{grad} u(x_0)$ consists of a single point which is just the usual gradient of u. By the gradient image of a set $E \subset \Omega$, we will mean the set

$$\operatorname{grad} u(E) = \bigcup_{x \in E} \operatorname{grad} u(x). \tag{8}$$

Theorem 2. *If f is a Laurent series converging in* $\operatorname{Log}^{-1}(\Omega)$, *then* $\operatorname{grad} N_f(\Omega) \subset P_f$. *If f is a Laurent polynomial, then moreover* $\operatorname{relint} P_f \subset \operatorname{grad} N_f(L_{\mathbf{R}})$.

Proof. Let $\xi \in \operatorname{grad} N_f(x)$ for some $x \in \Omega$. Assume that $a \in L$ and $k \in \mathbf{Z}$ are such that $\langle \alpha, a\rangle \geq k$ for all $\alpha \in L^*$ with $f_\alpha \neq 0$. If $z \in \operatorname{Log}^{-1}(x)$, then $\zeta^{-k} f(\zeta^a z)$ is holomorphic as a function of ζ in the unit disc. Applying the maximum principle to this function and taking $\zeta = e^{-t}$ it follows that

$$\sup_{z \in \operatorname{Log}^{-1}(x-ta)} e^{kt}|f(z)| \leq \sup_{z \in \operatorname{Log}^{-1}(x)} |f(z)|$$

hence

$$N_f(x - ta) = \int_{\operatorname{Log}^{-1}(x-ta)} \log|f(z)|\, d\eta(z) \leq -kt + \sup_{\operatorname{Log}^{-1}(x)} \log|f(z)|.$$

It follows that

$$\langle \xi, -ta\rangle \leq N_f(x - ta) - N_f(x) \leq -kt + \sup_{\operatorname{Log}^{-1}(x)} \log|f(z)| - N_f(x).$$

Letting $t \to +\infty$ it follows that $\langle \xi, a\rangle \geq k$, hence $\xi \in P_f$.

Suppose now that f is a Laurent polynomial and that $\xi \in \operatorname{relint} P_f$. Assume first that P_f is n-dimensional so that $\xi \in \operatorname{int} P_f$. If α is a vertex of P_f, then $\mathcal{A}_f^c$ has a component of order α by Corollary 3, so it follows from Theorem 1 that $N_f(x) \geq C + \langle \alpha, x\rangle$ for some constant C. Here we may assume that the constant C is the same for all vertices. It follows that

$$N_f(x) - \langle \xi, x\rangle \geq C + \max_{\alpha \in \operatorname{vert} P_f} \langle \alpha - \xi, x\rangle \to +\infty \quad \text{when } x \to \infty.$$

Hence $N_f(x) - \langle \xi, x\rangle$ has a minimum at some x_0, and then $\xi \in \operatorname{grad} N_f(x_0)$. The case when $\dim P_f < n$ is handled similarly. The only difference is that $N_f(x) - \langle \xi, x\rangle$ is constant along subspaces orthogonal to P_f in this case. ■

Theorem 3 (Forsberg, Passare, Tsikh [8]). *The mapping which takes a connected component of $\mathcal{A}_f^c$ to its order is an injection from the set of complement components to $P_f \cap L^*$.*

Proof. By definition, the order of any complement component is in L^*. That the order of any complement component is in P_f follows from Theorem 1 and Theorem 2. The injectivity is just Corollary 2. ■

The following estimates were obtained by Gelfand, Kapranov and Zelevinsky [11] and Forsberg, Passare and Tsikh [8].

Corollary 4. *The number of components of $\mathcal{A}_f^c$ is at least equal to the number of vertices of P_f and at most equal to the number of lattice points in P_f.*

There is much more to be said about the relation between the geometry of a complement component and its order. For example, there is a strong connection between the size of a complement component and the location of its order in the Newton polytope. Thus, a component is bounded precisely if its order is in the interior of the Newton polytope, while the largest components (in a certain sense) are those which correspond to vertices of the Newton polytope. We refer to [7], [8] and [11] for further results on these matters.

3.2 A class of almost hypergeometric functions

Definition 6. *If $E_\alpha(f)$ is nonempty, define*

$$\begin{aligned}\Phi_\alpha(f) &= \int_{\mathrm{Log}^{-1}(x)} \log(f(z)/z^\alpha)\,d\eta(z), \quad x \in E_\alpha \\ &= \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(x)} \frac{\log(f(z)/z^\alpha)\,dz_1\wedge\ldots\wedge dz_n}{z_1\ldots z_n}.\end{aligned} \tag{9}$$

Notice that the the function $\log(f(z)/z^\alpha)$ has a globally defined holomorphic branch in $\mathrm{Log}^{-1}(E_\alpha)$. Hence the integral (9) defines a holomorphic function in the coefficients of f with values in $\mathbf{C}/2\pi i\mathbf{Z}$. The second integral shows that the definition is independent of the choice of $x \in E_\alpha$. In fact, the integration may be performed over any cycle γ homologous to $\mathrm{Log}^{-1}(x)$ in $L_{\mathbf{C}_*} \smallsetminus f^{-1}(0)$ on which $\log(f(z)/z^\alpha)$ has a holomorphic branch. This can be used to define an analytic continuation of Φ_α to all $f \in \mathbf{C}^A$ whose principal A-determinant (see [11]) is nonzero, where $A \subset L^*$ is a fixed finite set.

These functions Φ_α are interesting in several ways. An immediate observation is that $N_f(x) \geq \langle\alpha, x\rangle + \mathrm{Re}\,\Phi_\alpha(f)$ with equality when $x \in E_\alpha$. It follows that

$$N_f(x) \geq \max\left(\mathrm{Re}\,\Phi_\alpha(f) + \langle\alpha, x\rangle\right)$$

with equality in the closure of $\mathcal{A}_f^c$. The maximum is taken over all α such that $E_\alpha(f)$ is nonempty. This approximation of the function N_f will be used in section 4 to construct a polyhedral complex approximating the amoeba. We note also that when $m_\alpha(f) < 1$ we have the estimate

$$|\Phi_\alpha(f) - \log f_\alpha| \leq -\log(1 - m_\alpha(f)) \tag{10}$$

which follows immediately from the definition.

The following result was alluded to in the introduction. After giving the precise statement we shall show that all the coefficients in a Laurent series expansion of $1/f$ can be expressed in terms of the functions Φ_α.

Theorem 4 (Gelfand, Kapranov, Zelevinsky [11]). *If f is a Laurent polynomial, then the convergent Laurent series g such that $fg = 1$ are in bijective correspondence with the connected components of $\mathcal{A}_f^c$.*

Proof. If E is a complement component of $\mathcal{A}_f$, then $1/f$ is a holomorphic function in $\mathrm{Log}^{-1}(E)$. This function is represented by a convergent Laurent series g, and evidently $fg = 1$. Conversely, if g is a convergent Laurent series, converging say in a domain $\mathrm{Log}^{-1}(E)$, with $fg = 1$, then the holomorphic function defined by g in $\mathrm{Log}^{-1}(E)$ is equal to $1/f(z)$. It follows that $f(z) \neq 0$ in $\mathrm{Log}^{-1}(E)$ which means that $E \subset \mathcal{A}_f^c$. If E' is the complement component containing E, then g is equal to the unique Laurent series expansion of $1/f(z)$ in $\mathrm{Log}^{-1}(E')$. This proves the theorem. ∎

Theorem 5. *Assume $E_\alpha(f)$ is nonempty, and define $c_\nu = \partial\Phi_\alpha/\partial f_\nu(f)$ for all $\nu \in L^*$. Then*

$$\frac{1}{f(z)} = \sum_{\nu \in L^*} c_\nu z^{-\nu},$$

the series converging in $\mathrm{Log}^{-1}(E_\alpha)$.

Proof. Differentiation under the integral sign in (9) yields

$$\frac{\partial \Phi_\alpha}{\partial f_\nu} = \int_{\mathrm{Log}^{-1}(x)} \frac{z^\nu \, d\eta(z)}{f(z)}.$$

This is precisely the coefficient for $z^{-\nu}$ in a Laurent series expansion of $1/f$ converging in a neighbourhood of $\mathrm{Log}^{-1}(x)$. ■

The following theorem shows that the functions Φ_α satisfy a system of differential equations which is very similar to a so-called A-hypergeometric system (see [10]). In fact, the only way the following equations fail to be of A-hypergeometric type is that the right-hand sides of equations (14) and (15) are nonzero.

Theorem 6 ([22]). *Let* $f(z) = \sum_{\nu \in A} f_\nu z^\nu$ *be a general Laurent polynomial in* $\mathbf{C}^A$ *where* A *is a fixed finite subset of* L^**. Then the holomorphic functions* Φ_α *have the power series expansion*

$$\Phi_\alpha(f) = \log f_\alpha + \sum_{k \in K_\alpha} \frac{(-k_\alpha - 1)!}{\prod_{\beta \neq \alpha} k_\beta!} (-1)^{k_\alpha - 1} f^k, \tag{11}$$

where $f^k = \prod_{\nu \in A} f_\nu^{k_\nu}$ *and*

$$K_\alpha = \{k \in \mathbf{Z}^A; k_\alpha < 0, k_\beta \geq 0 \text{ if } \beta \neq \alpha, \sum_\nu k_\nu = 0, \sum_\nu \nu k_\nu = 0\}. \tag{12}$$

The domain of convergence of the series is the set of all f *with* $m_\alpha(f) < 1$*. Moreover,* Φ_α *satisfies the differential equations*

$$(\partial^u - \partial^v)\Phi_\alpha = 0 \quad \text{if} \quad \sum_\nu (u_\nu - v_\nu) = 0 \quad \text{and} \quad \sum_\nu \nu(u_\nu - v_\nu) = 0 \tag{13}$$

and

$$\sum_\nu f_\nu \partial_\nu \Phi_\alpha = 1 \tag{14}$$

$$\sum_\nu \nu f_\nu \partial_\nu \Phi_\alpha = \alpha. \tag{15}$$

where $\partial_\nu = \partial / \partial f_\nu$ *and* $\partial^u = \prod \partial_\nu^{u_\nu}$ *when* $u \in \mathbf{Z}^A_{\geq 0}$.

Remark. Notice that the power series in (11) only involves those coefficients of f which belong to the smallest face of P_f containing α. In particular, if α is a vertex of P_f, then $\Phi_\alpha(f) = \log f_\alpha$.

Proof. Use the power series expansion of the logarithm function to write

$$\log(f(z)/z^\alpha) = \log f_\alpha + \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \left(\sum_{\beta \neq \alpha} \frac{f_\beta z^\beta}{f_\alpha z^\alpha} \right)^m.$$

Now

$$\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\left(\sum_{\beta\neq\alpha}\frac{f_\beta z^\beta}{f_\alpha z^\alpha}\right)^m = \sum_{m\geq 1}\sum_{\Sigma k_\beta = m}\frac{(-1)^{m-1}}{m}\frac{m!}{\prod k_\beta!}\frac{\prod f_\beta^{k_\beta} z^{k_\beta \beta}}{f_\alpha^m z^{m\alpha}}$$
$$= \sum_{L_\alpha}(-1)^{k_\alpha - 1}\frac{(-k_\alpha - 1)!}{\prod k_\beta!} f^k z^{\Sigma k_\nu \nu}.$$

Here all sums and products indexed by β are taken over $\beta \in A \setminus \{\alpha\}$ while ν ranges over all of A and $L_\alpha = \{k \in \mathbf{Z}^A; k_\alpha < 0, k_\beta \geq 0, \sum k_\nu = 0\}$. The constant terms in this expression, considered as monomials in the z variables, are precisely those corresponding to the set K_α, and we have proved (11).

Next we compute the domain of convergence of the power series. Let

$$\theta_k = \frac{(-k_\alpha - 1)!}{\prod_{\beta\neq\alpha} k_\beta!}(-1)^{k_\alpha - 1} f^k$$

be the term corresponding to $k \in K_\alpha$. Using Stirling's formula $\log m! = m\log m - m + O(\log m)$ and the relations $\sum k_\nu = 0, \sum \nu k_\nu = 0$ we find that

$$\log|\theta_k| = -k_\alpha \log(-k_\alpha) + k_\alpha - \sum_\beta (k_\beta \log k_\beta - k_\beta) + \sum_\nu k_\nu \log|f_\nu| + O(\log|k_\alpha|)$$
$$= -k_\alpha \sum_\beta \frac{k_\beta}{-k_\alpha}\log\frac{-k_\alpha}{k_\beta} + \sum_\nu k_\nu \log(|f_\nu|\exp\langle \nu, x\rangle) + O(\log|k_\alpha|)$$
$$= -k_\alpha \sum_\beta \frac{k_\beta}{-k_\alpha}\log\frac{-k_\alpha |f_\beta|\exp\langle\beta, x\rangle}{k_\beta |f_\alpha|\exp\langle\alpha, x\rangle} + O(\log|k_\alpha|)$$

for any $x \in L_{\mathbf{R}}$. Assume now that $m_\alpha(f) < 1$ and take x so that $m_\alpha(f;x) < 1$. It follows then from Jensen's inequality that

$$\log|\theta_k| = -k_\alpha \sum_\beta \frac{k_\beta}{-k_\alpha}\log\frac{-k_\alpha |f_\beta|\exp\langle\beta, x\rangle}{k_\beta |f_\alpha|\exp\langle\alpha, x\rangle} + O(\log|k_\alpha|)$$
$$\leq -k_\alpha \log\sum_\beta \frac{|f_\beta|\exp\langle\beta, x\rangle}{|f_\alpha|\exp\langle\alpha, x\rangle} + O(\log|k_\alpha|)$$
$$= -k_\alpha \log m_\alpha(f;x) + O(\log|k_\alpha|).$$

Since the number of terms with a given k_α increases polynomially with k_α, it follows that the series $\sum_{k\in K_\alpha}\theta_k$ is absolutely convergent.

To prove the converse, we may assume that α is in the relative interior of P_f. Otherwise we simply replace f with its truncation to the smallest face containing α. This operation leaves both $m_\alpha(f)$ and the series (11) unchanged. Then $m_\alpha(f;x)$ attains its minimal value. Take a point x where this minimum is achieved, and note that

$$\operatorname{grad} m_\alpha(f;x) = \frac{1}{|f_\alpha|\exp\langle\alpha, x\rangle}\sum |f_\beta|\exp\langle\beta, x\rangle(\beta - \alpha) = 0.$$

Setting $\phi_\beta = |f_\beta|\exp\langle\beta, x\rangle$ and $\phi_\alpha = -\sum\phi_\beta$, this means that $\sum\phi_\nu = 0$ and $\sum\nu\phi_\nu = 0$, so that the vector ϕ belongs to the cone generated by K_α. Since K_α

is a semigroup, there is a constant C such that for any $t > 0$ there is a $k \in K_\alpha$ with $|k - t\phi| < C$. For the corresponding term θ_k we then have

$$\begin{aligned}
\log|\theta_k| &= -k_\alpha \sum_\beta \frac{k_\beta}{-k_\alpha} \log \frac{-k_\alpha |f_\beta| \exp\langle\beta, x\rangle}{k_\beta |f_\alpha| \exp\langle\alpha, x\rangle} + O(\log t) \\
&= -t\phi_\alpha \sum_\beta \frac{k_\beta}{k_\alpha} \log \frac{-t\phi_\alpha \phi_\beta}{\phi_\beta |f_\alpha| \exp\langle\alpha, x\rangle} + O(\log t) \\
&= -t\phi_\alpha \log m_\alpha(f; x) + O(\log t).
\end{aligned}$$

If the series is to converge, these terms must remain bounded as $t \to +\infty$, which implies that $m_\alpha(f; x) \le 1$. Since the domain of convergence is by definition an open set, it follows that it is defined by the inequality $m_\alpha(f) < 1$.

To verify the differential equations, we differentate under the sign of integration defining Φ_α. By a simple computation,

$$\partial^u \log(f(z)/z^\alpha) = -\left(\sum u_\nu - 1\right)!\, z^{\Sigma u_\nu \nu}(-f(z))^{-\Sigma u_\nu}$$

which depends only on $\sum u_\nu$ and $\sum u_\nu \nu$. Also,

$$\sum f_\nu \partial_\nu \log(f(z)/z^\alpha) = \sum f_\nu z^\nu / f(z) = 1.$$

This verifies (13) and (14). Finally,

$$\sum \nu_j f_\nu \partial_\nu \log(f(z)/z^\alpha) = \sum \nu_j f_\nu z^\nu / f(z) = \frac{z_j \partial f / \partial z_j}{f(z)}.$$

Comparing this to the second integral in (6) proves the relation (15). ■

3.3 Functorial properties

For convenience, we record here how certain changes of coordinates affect the amoeba and the Ronkin function.

Theorem 7. *Let $T : L \to M$ be a linear mapping between two lattices such that $T^* : M^* \to L^*$ is injective. Let $f \in \mathbf{C}[M]$ be a Laurent polynomial. Then $T_{\mathbf{R}}^{-1}(\mathcal{A}_f) = \mathcal{A}_{T^*f}$, $T_{\mathbf{R}}^{-1}(E_\alpha(f)) = E_{T^*\alpha}(T^*f)$, $N_f(T_{\mathbf{R}}x) = N_{T^*f}(x)$ and $\Phi_\alpha(f) = \Phi_{T^*\alpha}(T^*f)$.*

Proof. Notice that $T_{\mathbf{R}}$ and $T_{\mathbf{C}_*}$ are surjective since T^* is injective. From the relation (3) it follows that $(T^*f)^{-1}(0) = T_{\mathbf{C}_*}^{-1}(f^{-1}(0))$. Applying Log to both sides of this equality and using (4), it follows that $\mathcal{A}_{T^*f} = T_{\mathbf{R}}^{-1}(\mathcal{A}_f)$. In particular, $T_{\mathbf{R}}^{-1}(E_\alpha(f))$ is a connected component of $\mathcal{A}_{T^*f}^c$. That the order of this component is $T^*\alpha$ can be seen directly from the definition. An alternative is to use the gradient of the Ronkin function. From (5) it follows that

$$\begin{aligned}
N_f(T_{\mathbf{R}}x) &= \int_{\mathrm{Log}^{-1}(T_{\mathbf{R}}x)} \log|f(w)|\, d\eta_M(w) \\
&= \int_{\mathrm{Log}^{-1}(x)} \log|f(T_{\mathbf{C}_*}z)|\, d\eta_L(z) \\
&= \int_{\mathrm{Log}^{-1}(x)} \log|T^*f(z)|\, d\eta_L(z) \\
&= N_{T^*f}(x).
\end{aligned}$$

Finally, if $x \in E_{T^*\alpha}(T^*f)$, then $T_{\mathbf{R}}x \in E_\alpha(f)$ and it follows that

$$\begin{aligned}
\Phi_\alpha(f) &= \int_{\mathrm{Log}^{-1}(T_{\mathbf{R}}x)} \log(f(w)/w^\alpha)\, d\eta_M(w) \\
&= \int_{\mathrm{Log}^{-1}(x)} \log(f(T_{\mathbf{C}_*}z)/(T_{\mathbf{C}_*}z)^\alpha)\, d\eta_L(z) \\
&= \int_{\mathrm{Log}^{-1}(x)} \log(T^*f(z)/z^{T^*\alpha})\, d\eta_L(z) \\
&= \Phi_{T^*\alpha}(T^*f). \quad \blacksquare
\end{aligned}$$

4 Patchworking amoebas

In this section, amoebas are compared to certain polyhedral subdivisions of the space $L_{\mathbf{R}}$. It is shown that every amoeba can be approximated by a polyhedral complex of a special kind (Theorem 8) and conversely, all such polyhedral complexes can be approximated by amoebas if rescalings are allowed (Theorem 9). The construction is reminiscent of the patchworking technique invented by Viro for constructing real algebraic curves with prescribed topology, hence the title. The ideas in this section are inspired by the computer generated pictures in [7, section 5], and provide an explanation for the empirical observations made there.

4.1 Dual polyhedral subdivisions

Definition 7. *Let K be a polyhedron in a real vector space V (possibly all of V). By a polyhedral subdivision of K we will mean a finite collection Σ of nonempty polyhedra whose union is K and satisfying the following properties.*

(i) If $\sigma, \tau \in \Sigma$ and $\sigma \cap \tau$ is nonempty, then $\sigma \cap \tau \in \Sigma$.

(ii) If $\sigma \in \Sigma$ and $\tau \subset \sigma$ is a polyhedron, then $\tau \in \Sigma$ precisely if τ is a face of σ.

If σ is a polyhedron, and τ is a face of σ, we shall denote by $\mathrm{cone}(\tau, \sigma) = \{t(x-y); t \geq 0, x \in \sigma, y \in \tau\}$ the cone of vectors pointing from τ into σ. If C is a closed convex cone in V, its dual is defined to be the cone $C^\vee = \{\xi \in V^*; \langle \xi, x\rangle \leq 0, \forall x \in C\}$. It is well known that $C^{\vee\vee} = C$. We shall therefore say that two cones C_1, C_2 are dual if $C_1^\vee = C_2$ or equivalently $C_1 = C_2^\vee$.

Definition 8. *Let Σ, Σ' be polyhedral subdivisions of polyhedra $K \subset V, K' \subset V^*$ respectively. We shall say that Σ and Σ' are dual if there is a bijection $\sigma \mapsto \sigma^*$ from Σ to Σ' such that $\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$ and whenever this is the case, $\mathrm{cone}(\tau, \sigma)$ and $\mathrm{cone}(\sigma^*, \tau^*)$ are dual.*

Next we discuss a particular method of constructing dual polyhedral subdivisions. Let K be a polyhedron in V and let s be a piecewise linear function on K, by which we here mean that s is the maximum of finitely many linear functions. The Legendre transform of s,

$$\tilde{s}(\xi) = \sup_{x \in K}(\langle \xi, x\rangle - s(x))$$

is also a piecewise linear function on the polyhedron K' consisting of all points ξ with $\tilde{s}(\xi) < +\infty$. Define

$$S(\xi, x) = s(x) + \tilde{s}(\xi) - \langle \xi, x \rangle.$$

Let Σ be the collection of all sets of the form $\{x \in K; S(\xi, x) = 0\}$ for some $\xi \in K'$ and let Σ' consist of all sets of the form $\{\xi \in K'; S(\xi, x) = 0\}$ for some $x \in K$.

Proposition 1. *With notations as in the preceding paragraph, Σ and Σ' are dual polyhedral subdivisions of K and K'.*

Proof. If E is any subset of K, define $E^* = \{\xi \in K'; S(\xi, x) = 0, \forall x \in E\}$ and define E^* similarly if $E \subset K'$. Now it is clear that $E \subset F \Rightarrow F^* \subset E^*$, $E \subset E^{**}$ and $E^* = E^{***}$. The main points in the proof are to show that if $\sigma \subset K$ is nonempty then $\sigma \in \Sigma \Leftrightarrow \sigma^{**} = \sigma$ and that if $\sigma \in \Sigma$ and τ is a nonempty subset of σ, then $\operatorname{cone}(\sigma^*, \tau^*) = \{\xi; \langle \xi, x \rangle \leq \langle \xi, y \rangle, \forall x \in \sigma, y \in \tau\}$. From this it follows that $x \in \{x\}^{**} \in \Sigma$ for any $x \in K$, so the union of all $\sigma \in \Sigma$ is equal to K and that if $\sigma, \tau \in \Sigma$, then $(\sigma \cap \tau)^{**} \subset (\sigma^* \cup \tau^*)^* \subset \sigma^{**} \cap \tau^{**} = \sigma \cap \tau$ so that $\sigma \cap \tau \in \Sigma$. It follows also that the mapping $\sigma \mapsto \sigma^*$ is an inclusion reversing bijection from Σ to Σ'. Moreover, if $\sigma \in \Sigma$ and τ is a nonempty subset of σ then it follows that τ^{**} is the smallest face of σ containing τ. This shows that Σ is a polyhedral subdivision. Finally, it is easy to see that $\{\xi; \langle \xi, x \rangle \leq \langle \xi, y \rangle, \forall x \in \sigma, y \in \tau\}$ is dual to $\operatorname{cone}(\tau, \sigma)$ and this shows that Σ and Σ' are dual subdivisions. ■

Remark. Polyhedral subdivisions which can be obtained from a piecewise linear convex function are called coherent and play an important role in the theory of discriminants. It is known that not all subdivisions are coherent. Proposition 1 shows that every coherent subdivision of a polyhedron $K \subset V$ is dual to a subdivision of a polyhedron $K' \subset V^*$. Conversely, it is easy to see that if Σ is a subdivision which is dual to some subdivision Σ', then Σ is coherent.

4.2 The spine of an amoeba

Let now f be a Laurent polynomial and let A be the set of all $\alpha \in L^*$ such that $\mathcal{A}_f^c$ has a component of order α. Take

$$s(x) = \max_{\alpha \in A} (\operatorname{Re} \Phi_\alpha(f) + \langle \alpha, x \rangle) \tag{16}$$

and take K to be all of $L_{\mathbf{R}}$. The Legendre transform $\tilde{s}$ is then finite precisely on the convex hull of A which is equal to the Newton polytope of f. Let Σ and Σ' be the polyhedral subdivisions constructed from these functions and let $\mathcal{S}_f$ be the union of all polyhedra in Σ whose dimension is smaller than n. This last set will be called the spine of $\mathcal{A}_f$ for reasons which are obvious from the following theorem.

Theorem 8 ([22]). *Let f be a Laurent polynomial. Then Σ' is a polyhedral subdivision of the Newton polytope of f, and Σ is a dual subdivision of $L_{\mathbf{R}}$ such that $E_\alpha = \{\alpha\}^* \cap \mathcal{A}_f^c$ whenever $\mathcal{A}_f^c$ has a component of order α. Moreover, $\mathcal{S}_f$ is a strong deformation retract of $\mathcal{A}_f$.*

Remark. Other choices of the function s may also produce a polyhedral subdivision compatible with the amoeba in the sense of Theorem 8. For example, the proof of Theorem 12 shows that, under the assumptions of that Theorem, the constants $\operatorname{Re}\Phi_\alpha(f)$ in (16) may be replaced by $\log|f_\alpha|$.

Proof. Since $\operatorname{Re}\Phi_\alpha(f) + \langle \alpha, x\rangle \le N_f(x)$ with equality precisely in the closure of E_α and $\tilde{s}(\alpha) = -\operatorname{Re}\Phi_\alpha(f)$ it follows immediately from the definition that $E_\alpha = \{\alpha\}^* \cap \mathcal{A}_f^c$. Now let σ^* be a polyhedron in $\mathcal{S}_f$ where $\sigma \in \Sigma'$ with $\dim \sigma \ge 1$. Take two points $\alpha, \beta \in \sigma \cap A$. Then it follows that $\sigma^* \cap \mathcal{A}_f^c \subset E_\alpha \cap E_\beta = \varnothing$, which proves that $\mathcal{S}_f \subset \mathcal{A}_f$. Since every connected component of the complement of $\mathcal{S}_f$ is a convex polyhedron $\{\alpha\}^*$ which contains exactly one nonempty component E_α of $\mathcal{A}_f^c$ it follows easily that $\mathcal{S}_f$ is a strong deformation retract of $\mathcal{A}_f$. ■

In most cases it is of course much easier to compute $\Phi_\alpha(f)$ for all $\alpha \in A$ than to compute the exact shape of the amoeba. Hence, the spine provides an approximation to the amoeba which is much more convenient to compute explicitly. It should be noted, however, that the spine cannot be used to find the number of complement components of the amoeba, since the set A of orders of the complement components is needed in the construction of the spine.

We now reverse the operation and construct amoebas approximating a prescribed polyhedral subdivision. Let A be a finite subset of L^* and c_α be an arbitrary real number for every $\alpha \in A$. Let

$$s(x) = \max_{\alpha \in A} (c_\alpha + \langle \alpha, x\rangle),$$

let Σ and Σ' be the polyhedral subdivision of $L_{\mathbf{R}}$ and $\operatorname{conv} A$ determined by s and let $\mathcal{S}$ be the union of all polyhedra in Σ of dimension smaller than n. Also, let $A' \subset A$ be the set of vertices of the subdivision Σ'. Let $f^t \in \mathbf{C}^A$ be a family of Laurent polynomials such that $\log|f^t_\alpha| = tc_\alpha$ for all $\alpha \in A$. If $\operatorname{dist}(E, F)$ denotes the Hausdorff distance between two sets E, F in L_R (with respect to any norm), then we have the following result.

Theorem 9. *With notations as in the preceding paragraph,* $\operatorname{dist}(t^{-1}\mathcal{A}_{f^t}, \mathcal{S}) \to 0$ *and* $\operatorname{dist}(t^{-1}E_\alpha(f^t), \{\alpha\}^*) \to 0$ *for all* $\alpha \in A'$ *when* $t \to +\infty$.

Remark. Note that the complements of these amoebas will be dominated by the components whose orders are vertices of the dual subdivision Σ' of the Newton polytope. However, we are not asserting that these will be the only components in the complement of the amoeba. In general, there will be components which have other orders as well.

Proof. For every $\alpha \in A'$ and $\delta > 0$, let

$$F_\alpha^\delta = \{x; c_\alpha + \langle \alpha, x\rangle - \delta \ge c_\beta + \langle \beta, x\rangle, \forall \beta \ne \alpha\}$$

and

$$G_\alpha^\delta = \{x; c_\alpha + \langle \alpha, x\rangle + \delta \ge c_\beta + \langle \beta, x\rangle, \forall \beta \ne \alpha\}.$$

Then $F_\alpha^\delta \subset \{\alpha\}^* \subset G_\alpha^\delta$ and F_α^δ and G_α^δ converge to $\{\alpha\}^*$ when $\delta \to 0$. If N is the cardinality of A and $x \in F_\alpha^\delta$, then it is easy to see that $m_\alpha(f^t; tx) \le Ne^{-t\delta}$.

It follows from Lemma 1 that $F_\alpha^\delta \subset t^{-1}E_\alpha(f^t)$ for sufficiently large t. Moreover, it follows that $m_\alpha(f^t) \to 0$ when $t \to +\infty$, hence by (10), $t^{-1}\operatorname{Re}\Phi_\alpha(f^t) \to c_\alpha$ for all $\alpha \in A'$. Let $c_\alpha^t = t^{-1}\operatorname{Re}\Phi_\alpha(f^t)$ and $s^t(x) = \max_{\alpha\in A'}(c_\alpha^t + \langle\alpha, x\rangle)$. For sufficiently large t it follows that $t^{-1}E_\alpha(f^t) \subset \{x; c_\alpha^t + \langle\alpha, x\rangle = s^t(x)\} \subset G_\alpha^\delta$. Hence $\operatorname{dist}(t^{-1}E_\alpha(f^t), \{\alpha\}^*) \to 0$. Since $t^{-1}\mathcal{A}_{f^t} \subset \mathbf{R}^n \smallsetminus \cup F_\alpha^\delta$ and every line segment with endpoints in F_α^δ and F_β^δ where $\alpha \neq \beta$ intersects $t^{-1}\mathcal{A}_{f^t}$ when t is sufficiently large, it follows that $\operatorname{dist}(t^{-1}\mathcal{A}_{f^t}, \mathcal{S}) \to 0$. ■

5 The complement components of an amoeba

Gelfand, Kapranov and Zelevinsky posed in [11] the following problem: Given a polynomial f, find all connected components of $\mathcal{A}_f^c$. At that time, the order of a complement component had not been defined. In view of this concept and Theorem 3, it is natural to reformulate the problem as follows: Given a polynomial f, find all $\alpha \in P_f \cap L^*$ such that $\mathcal{A}_f^c$ has a component of order α.

The present section is concerned with variations of this problem. We give some partial answers to certain questions relating to the original problem. In this context, we should also mention the work of Sadykov [28] on the amoebas of certain special functions.

We introduce the sets U_α^A of all Laurent polynomials $f \in \mathbf{C}^A$ such that $\mathcal{A}_f^c$ has a component of order α. The fact that these sets are semialgebraic means that there exists, at lest in principle, a procedure for determining whether $\mathcal{A}_f^c$, for a given f, has a component of order α. Next we pose the following problem: When is U_α^A nonempty? Theorem 11 gives one necessary and one sufficient condition, although the gap between them is, for most sets A, very large. Theorem 12 gives the complete answer for certain simple sets A, which include cases where it is nontrivial to compute amoebas explicitly.

In the previous section it was shown that by choosing certain coefficients of a polynomial f appropriately, the corresponding complement components can be made very large. In the opposite direction, we show now that the coefficients may be chosen in such a way that certain prescribed complement components vanish altogether. This shows that the set of lattice points occuring as orders of complement components of $\mathcal{A}_f^c$, where the Newton polytope P_f is given, is subject only to those restrictions imposed by Corollary 3 and Theorem 3. In particular, the estimates on the number of complement components given in Corollary 4 are sharp.

Finally, we prove a statement concerning the topology of the sets U_α^A.

Definition 9. *If $A \subset L^*$ and $\alpha \in L^*$, let U_α^A denote the set of all $f \in \mathbf{C}^A$ such that $E_\alpha(f) \neq \varnothing$.*

Theorem 10. *All the sets U_α^A are open and semialgebraic.*

Proof. In the product space $\mathbf{C}^A \times L_{\mathbf{C}_*}$, consider the algebraic surface $V = \{(f, z); f(z) = 0\}$. By the Tarski-Seidenberg theorem, V is mapped onto a semialgebraic set by the mapping $\phi : (f, z) \mapsto (f, (|z_1|^2, \dots, |z_n|^2))$. Now the set $\{(f, x) \in \mathbf{C}^A \times \mathbf{R}_{>0}^n; \frac{1}{2}\operatorname{Log} x \in E_\alpha(f)\}$ consists of certain connected components of $\mathbf{C}^A \times \mathbf{R}_{>0}^n \smallsetminus \phi(V)$, and hence is semialgebraic. Therefore its projection on $\mathbf{C}^A$, which is precisely U_α^A, is also semialgebraic. ■

Let us next turn to the following problem. For which α is U^A_α nonempty? Let $\operatorname{aff} A$ denote the affine lattice generated by A. Then we have

Theorem 11. *A necessary condition for U^A_α to be nonempty is that $\alpha \in \operatorname{conv} A \cap \operatorname{aff} A$.*

A sufficient condition for U^A_α to be nonempty is that there exists a line l such that $\alpha \in \operatorname{conv}(A \cap l) \cap \operatorname{aff}(A \cap l)$.

Proof. If $f \in U^A_\alpha$ it follows from Theorem 3 that $\alpha \in P_f \subset \operatorname{conv} A$. Moreover, we may assume that $\operatorname{aff} A$ contains 0, and then it follows from Theorem 7 with T^* the inclusion mapping $\operatorname{aff} A \to L^*$ that $\alpha \in \operatorname{aff} A$. This proves the first part.

Suppose now that l is a line with $\alpha \in \operatorname{aff}(A \cap l) \cap \operatorname{conv}(A \cap l)$. Without loss of generality we may assume that $A \subset l$. In fact, we may assume that $A \subset \mathbf{Z}$ with $\operatorname{aff} A = \mathbf{Z}$ and that the largest element in A is N and the smallest element 0. Consider polynomials of the form $f(z) = z^N - 1 + \epsilon e^{i\theta} g(z)$ where $g(z) = \sum g_\alpha z^\alpha$ and the sum is taken over $A \setminus \{0, N\}$. Then the zeros of f are given by $\omega^j(1 - \epsilon e^{i\theta} g(\omega^j)/N) + o(\epsilon), j = 0, \ldots, N-1$ where $\omega = e^{2\pi i/N}$. For generic choices of the coefficients g_α, all the numbers $g(\omega^j)$ are distinct, hence for suitable θ, $\operatorname{Re}(e^{i\theta} g(\omega^j))$ are all distinct. It follows that for sufficiently small ϵ, all the zeros of f have different absolute values, so $\mathcal{A}^c_f$ has components of all orders between 0 and N. ■

Theorem 12. *Suppose that $A \subset L^*$ has no more than $2n$ points and that no $k+2$ of these lie in an affine k-dimensional subspace for $k = 1, \ldots, n-1$. If $f \in \mathbf{C}^A$ has a component of order α, then $\alpha \in A$. In other words, U^A_α is nonempty if and only if $\alpha \in A$.*

Proof. Let $f \in \mathbf{C}^A$ and take a point $x \in \mathcal{A}^c_f$. Without loss of generality, assume that $x = 0$. Write $f(z) = \sum_{j=1}^{2n} c_j z^{\alpha_j}$ with $|c_1| \geq |c_2| \geq \ldots \geq |c_{2n}|$. Suppose the complement component containing x has order α. We will show that $\alpha = \alpha_1$. There is no loss of generality in assuming that $\alpha_1 = 0$ and that $c_1 \geq 0$, otherwise we divide f by $c_1 z^{\alpha_1}$. For each $k = 2, \ldots, n+1$ choose $a_k \in L$ orthogonal to $\alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}$ (a_k is uniquely determined up to scalar multiplication) and a point $z_k \in \operatorname{Log}^{-1}(0)$ such that $c_j z_k^{\alpha_j} \geq 0$ for $j = 2, \ldots, n+1, j \neq k$. Since $\operatorname{Re} f(\zeta^{a_k} z_k) \geq |c_1| + \ldots + |c_{k-1}| + |c_{k+1}| + \ldots + |c_{n+1}| - |c_k| - |c_{n+2}| - \ldots - |c_{2n}| \geq 0$ whenever $|\zeta| = 1$ it follows that

$$\langle \alpha, a_k \rangle = \frac{1}{2\pi i} \int_{|\zeta|=1} d \log f(\zeta^{a_k} z_k) = 0.$$

Since $a_2, \ldots, a_{n+1}$ is a basis for $L_{\mathbf{R}}$ it follows that $\alpha = 0 = \alpha_1$ as required. ■

Theorem 13. *Let $f \in \mathbf{C}^A$ and let $B \subset A$. Then there exists a polynomial $g \in \mathbf{C}^B$ such that $E_\alpha(f+g) = \varnothing$ for all $\alpha \in B$.*

Proof. We may assume that $P_f = \operatorname{conv} A$. Suppose B contains some vertices of P_f and let $B_0 = B \cap \operatorname{vert} P_f$. We begin by taking $g \in \mathbf{C}^{B_0}$ so that $g_\alpha = -f_\alpha$ for all $\alpha \in B_0$. Then P_{f+g} does not contain any point of B_0. If $B \setminus B_0$ contains vertices of P_{f+g} the procedure can be repeated. Hence we can assume from the beginning that B does not contain any vertices of P_f. This implies that

$m_\alpha(f+g)$ is continuous as a function of $g \in \mathbf{C}^B$ (with values in $\mathbf{R} \cup \{+\infty\}$) for all α. Assume also that $f_\alpha = 0$ for all $\alpha \in B$. Let

$$X_\alpha = \{g \in \mathbf{C}^B; E_\alpha(f+g) = \varnothing\}$$

and

$$Y_\alpha = \{g \in \mathbf{C}^B; m_\alpha(f+g) > 1/2\}$$

for all $\alpha \in B$. By Lemma 1, Y_α is an open neighbourhood of X_α. Let V_α be any open neighbourhood of X_α in Y_α and let ϕ_α be smooth functions defined in $\mathbf{C}^B$ which satisfy $\phi_\alpha(g) = \exp(-\Phi_\alpha(f+g))$ outside V_α. Then ϕ_α is holomorphic outside V_α and by (10)

$$|g_\alpha \phi_\alpha(g) - 1| \leq C m_\alpha(f+g), \quad g \notin Y_\alpha$$

where C is a constant (we may take $C = 2e$). Consider the differential form

$$\omega = \bigwedge_{\alpha \in B} (\bar\partial \phi_\alpha \wedge dg_\alpha) = \bigwedge_{\alpha \in B} (d\phi_\alpha \wedge dg_\alpha).$$

Notice that ω has its support in $\overline{\cap V_\alpha}$. Hence, if it can be shown that $\omega \neq 0$, it will follow that $\cap V_\alpha \neq \varnothing$. Since V_α were arbitrarily small neighbourhoods of X_α and X_α are closed this implies that $\cap X_\alpha \neq \varnothing$, which is precisely what we want to prove.

To prove that $\omega \neq 0$, we will evaluate its integral over $\mathbf{C}^B$. Order the elements in $B = \{\alpha_1, \ldots, \alpha_k\}$ and let $dg = dg_{\alpha_1} \wedge \ldots \wedge dg_{\alpha_k}$ and $\phi = \phi_{\alpha_1} \ldots \phi_{\alpha_k}$.

Lemma 2. *Let $D \subset \mathbf{C}^B$ be a polydisc centered at the origin whose distinguished boundary $\partial' D$ does not meet $\cup Y_\alpha$, and let ψ be holomorphic in a neighbourhood of $\overline{D}$. Then*

$$\int_D \psi\omega = \int_{\partial' D} \psi\phi \, dg.$$

Proof. Use induction on the number of elements in B. If $k = 1$ this is a simple application of Stokes' theorem. In the general case let $B' = B \setminus \{\alpha_1\}$, $\omega' = \bigwedge_{\alpha \in B'} (\bar\partial\phi_\alpha \wedge dg_\alpha)$, and write $D = D_1 \times D'$ with $D_1 \subset \mathbf{C}$ and $D' \subset \mathbf{C}^{B'}$. Since m_α is decreasing as a function of $|g_\alpha|$ and increasing as a function of $|g_\beta|$ for any $\beta \neq \alpha$, it follows that $\partial D_1 \times D'$ does not intersect Y_{α_1} and $D_1 \times \partial D'$ does not intersect $\cap_{\alpha \in B'} Y_\alpha$. Hence ϕ_{α_1} is holomorphic on $\partial D_1 \times D'$ and $\omega' = 0$ on $D_1 \times \partial D'$. By Stokes' theorem and the inductive hypothesis it follows that

$$\begin{aligned} \int_D \psi\omega &= \int_{\partial D} \psi\phi_{\alpha_1} \, dg_{\alpha_1} \wedge \omega' = \int_{\partial D_1} dg_{\alpha_1} \int_{D'} \psi\phi_{\alpha_1}\omega' \\ &= \int_{\partial D_1} dg_{\alpha_1} \int_{\partial' D'} \psi\phi \, dg' = \int_{\partial' D} \psi\phi \, dg. \quad \blacksquare \end{aligned}$$

Returning to the proof of the theorem, let $s : L_{\mathbf{R}} \to \mathbf{R}$ be a strictly concave function which is positive on A. Let $D(t) \subset \mathbf{C}^B$ be the polydisc $\{g \in \mathbf{C}^B; |g_\alpha| < e^{ts(\alpha)}\}$. For large t, we then have $ts(\beta) > \log|f_\beta|$. For every $\alpha \in A$ there is some $x_\alpha \in L_{\mathbf{R}}$ such that $s(\alpha) + \langle \alpha, x_\alpha \rangle > s(\beta) + \langle \beta, x_\alpha \rangle$ for all $\beta \neq \alpha$. It follows that $m_\alpha(f+g) \leq m_\alpha(f+g; tx_\alpha) \to 0$ uniformly for $g \in \partial' D(t)$ as $t \to +\infty$ for all $\alpha \in B$. In particular, the hypothesis of Lemma 2 is satisfied for large t and $\psi \equiv 1$. It follows that

$$\int_{\mathbf{C}^A} \omega = \lim_{t\to+\infty} \int_{\partial' D(t)} \phi \, dg = \lim_{t\to+\infty} \int_{\partial' D(t)} \frac{dg}{g_{\alpha_1}\cdots g_{\alpha_k}} = (2\pi i)^k,$$

so that certainly $\omega \neq 0$. This completes the proof. ■

Corollary 5 ([27]). *Let $f \in \mathbf{C}^A$ and let B, C be disjoint subsets of A. Then there exists a polynomial $g \in \mathbf{C}^{B\cup C}$ such that $E_\alpha(f+g) \neq \varnothing$ for all $\alpha \in B$ and $E_\alpha(f+g) = \varnothing$ for all $\alpha \in C$.*

Proof. Assume that $f_\alpha = 0$ for all $\alpha \in B \cup C$. Let N be the number of elements in A and take a strictly concave function s which is positive on A. For every $\alpha \in A$, take a point $x_\alpha \in L_{\mathbf{R}}$ such that $s(\alpha) + \langle \alpha, x_\alpha\rangle > s(\beta) + \langle \beta, x_\alpha\rangle$ for all $\beta \in A \setminus \{\alpha\}$. After multiplying s by a large constant and modifying x_α accordingly, we may assume that

$$s(\alpha) + \langle \alpha, x_\alpha\rangle > s(\beta) + \langle \beta x_\alpha\rangle + \log N \tag{17}$$

and that $s(\alpha) > \log|f_\alpha|$ for all $\alpha \in A$. Now take a polynomial $g^1 \in \mathbf{C}^B$ such that $\log|g_\beta| = s(\beta)$ for all $\beta \in B$. By Theorem 13 there is a polynomial $g^2 \in \mathbf{C}^C$ such that $E_\gamma(f + g^1 + g^2) = \varnothing$ for every $\gamma \in C$. Write $h = f + g^1 + g^2$ and take an $\alpha \in A$ with $\log|h_\alpha| - s(\alpha)$ maximal. Then it follows from (17) that $m_\alpha(h, x_\alpha) < 1$, so that $E_\alpha(h) \neq \varnothing$. Hence $\alpha \notin C$, but then by the construction of s and g^1, $\log|h_\alpha| - s(\alpha) \leq 0$. Since $\log|h_\beta| = s(\beta)$ for all $\beta \in B$, it follows that $E_\beta(h) \neq \varnothing$ for all such β. Therefore $g = g^1 + g^2$ has the required properties. ■

Corollary 6. *If P is a lattice polytope in $L^*_{\mathbf{R}}$ and A is a subset of $P \cap L^*$ containing all vertices of P, then there is a Laurent polynomial f, with $P_f = P$ such that $E_\alpha(f)$ is nonempty precisely if $\alpha \in A$. In particular, the estimates in Corollary 4 are sharp.*

One might also try to prove statements about the topology of the sets U^A_α. Since U^A_α is invariant under multiplication by nonzero scalars, it is reasonable to consider the projective sets $\tilde{U}^A_\alpha = \{[f]; f \in U^A_\alpha\} \subset \mathbf{PC}^A$. The following result is most conveniently stated in the projective setting.

Theorem 14 ([27]). *The intersection of the complement of $\tilde{U}^A_\alpha$ with any complex line in $\mathbf{PC}^A$ is nonempty and connected.*

Remark. This is indeed a very special statement. However, it is just about all that can be said about the topology of the intersection of a general $\tilde{U}^A_\alpha$ with a general complex line. For instance, the intersection of $\tilde{U}^A_\alpha$ with a complex line can have any number of connected components, as Example 5 in section 9 shows. It seems to be an open question whether U^A_α always is connected.

Proof. For any $f \in \mathbf{C}^A$, let

$$\begin{aligned} u_\alpha(f) &= \inf_{x\in L_{\mathbf{R}}} (N_f(x) - \langle \alpha, x\rangle) \\ &= \operatorname{Re}\Phi_\alpha(f) \quad \text{if } f \in U^A_\alpha. \end{aligned}$$

These are plurisubharmonic as a consequence of Kiselman's minimum principle (see [16]) and pluriharmonic in U_α^A.

Let l be a complex line in $\mathbf{PC}^A$. If l is contained in the complement of $\tilde{U}_\alpha^A$ there is nothing to prove. Otherwise, take two points on l represented by Laurent polynomials f and g with $g \in U_\alpha^A$. Let $K = \{t \in \mathbf{C}; f + tg \notin U_\alpha^A\}$ and write $\Phi(t) = \Phi_\alpha(f + tg)$ for $t \in \mathbf{C} \setminus K$ and $u(t) = u_\alpha(f + tg)$. We want to show that K is connected. Suppose K' is a closed and relatively open subset of K. Let ω be a bounded open set with smooth boundary such that $K' = K \cap \omega$ and define

$$N(K') = \frac{1}{2\pi} \int_{\partial\omega} d \operatorname{Im} \Phi.$$

Then $N(K')$ is an integer, and if K', K'' are disjoint, $N(K' \cup K'') = N(K') + N(K'')$. We intend to show that $N(K') \geq 1$ whenever K' is nonempty and that $N(K) = 1$. From this it follows that K is nonempty and connected.

First, since u is subharmonic

$$\frac{1}{2\pi} \int_{\partial\omega} d \operatorname{Im} \Phi = \int_{\partial\omega} d^c u = \int_\omega dd^c u \geq 0$$

where $d^c = (\partial - \bar{\partial})/2\pi i$. Equality occurs if and only if u is harmonic in ω. In this case, Φ can be continued analytically across ω. For the same reason, $\Phi_\alpha(f + tg + sz^\nu)$ has an analytic continuation to $t \in \omega$ for all $\nu \in L^*$ and sufficiently small s. Set

$$c_\nu(t) = \frac{\partial}{\partial s} \Phi_\alpha(f + tg + sz^\nu)\big|_{s=0}$$

and consider the Laurent series $\sum_{\nu \in L^*} c_\nu(t) z^{-\nu}$. When $t \in \partial\omega$ it follows from Theorem 5 that this is a Laurent series expansion of $1/(f + tg)$ which converges in $\operatorname{Log}^{-1}(E_\alpha(f + tg))$. It follows from the maximum principle that the series is convergent for all $t \in \omega$. This implies that $\mathcal{A}^c_{f+tg}$ has a component of order α for all $t \in \omega$, which means that $K' = \varnothing$.

Hence $N(K') \geq 1$ if K' is nonempty. From the fact that

$$\Phi(t) = \log t + \Phi_\alpha(f/t + g) = \log t + \Phi_\alpha(g) + O(|t|^{-1})$$

looks asymptotically like $\log t$ when $t \to \infty$ it follows that $N(K) = 1$. This completes the proof. ■

6 Convexity of the Ronkin function

According to Theorem 1, the function N_f is affine linear precisely in the complement of the amoeba of f. This section is concerned with the question of how much N_f deviates from being linear at points in the amoeba.

If u is a smooth convex function, its Hessian

$$\operatorname{Hess}(u) = \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)$$

is a positive definite matrix which in a certain sense measures how convex u is. The trace and determinant of the Hessian matrix are respectively the Laplace and the Monge-Ampère operator. We will see that the Monge-Ampère operator has useful properties for studying convexity of the function N_f.

After a brief discussion of the Monge-Ampère operator we define the Monge-Ampère measure μ_f of the function N_f. One motivation for choosing to work with the Monge-Ampère operator is Theorem 15, which relates the total mass of μ_f to the Newton polytope P_f. Next we relate the measure μ_f to local properties of the hypersurface $f^{-1}(0)$. Finally, the Monge-Ampère measure is used to derive an estimate on the area of amoebas in the two dimensional case. It turns out, that the amoebas with maximal area, for a give Newton polytope, correspond to polynomials defining so-called Harnack curves which arise in real algebraic geometry.

6.1 The Monge-Ampère operator

Let Ω be a domain in $\mathbf{R}^n$. The Monge-Ampère operator is defined on smooth convex functions in Ω as the determinant of the Hessian matrix. More precisely,

$$\mathrm{M}\, u = \det \mathrm{Hess}(u) \cdot \lambda \tag{18}$$

is called the Monge-Ampère measure of u, where λ denotes Lebesgue measure. The reason for defining $\mathrm{M}\, u$ as a measure is that the definition can then be extended to all convex functions without any requirements on smoothness. For an arbitrary convex function u and a Borel set E,

$$\mathrm{M}\, u(E) = \lambda(\operatorname{grad} u(E))$$

where $\operatorname{grad} u(E)$ is defined by (7) and (8). It can be shown, although it is not entirely obvious, that this defines a positive Borel measure $\mathrm{M}\, u$ for any convex function u. Moreover, M is a continuous operator from the space of convex functions with the topology of uniform convergence on compact sets, to the space of measures with the weak topology. A good reference for the Monge-Ampère operator is [24].

When $n > 1$, the Monge-Ampère operator is not linear. However, it can be turned into a multilinear operator, taking n convex functions as arguments. The construction is described in Proposition 2. This multilinear operator will be called the mixed Monge-Ampère operator in analogy with the term mixed volume in the theory of convex bodies.

Let A be a real $n \times n$ matrix whose entries are considered as indeterminates. The determinant of A is then a homogeneous polynomial of degree n. From this it follows that

$$\det(A_1, \ldots, A_n) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \ldots < j_k \le n} (-1)^{n-k} \det(A_{j_1} + \ldots + A_{j_k}). \tag{19}$$

defines a symmetric multilinear form on the space of all $n \times n$ matrices, with the property that $\det A = \det(A, \ldots, A)$. It is known that when $A_1, \ldots, A_n$ are positive definite (in particular, symmetric) $\det(A_1, \ldots, A_n)$ is positive. In this case it actually holds that (see [1])

$$\det(A_1, \ldots, A_n) \ge (\det A_1 \ldots \det A_n)^{1/n}. \tag{20}$$

From this it is easy to deduce

Proposition 2. *There exists a unique operator* $\widetilde{\mathrm{M}}$ *taking* n *convex functions as arguments with the properties that* $\widetilde{\mathrm{M}}(u_1, \dots, u_n)$ *is a positive measure depending multilinearly and symmetrically on* $u_1, \dots, u_n$ *and* $\widetilde{\mathrm{M}}(u, \dots, u) = \mathrm{M}\, u$ *for every* u.

Proof. If there exists an operator with the required properties, it is easy to show that it must satisfy

$$\widetilde{\mathrm{M}}(u_1, \dots, u_n) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \le j_1 < \ldots < j_k \le n} (-1)^{n-k}\, \mathrm{M}(u_{j_1} + \ldots + u_{j_k}). \tag{21}$$

Hence uniqueness is established. To prove existence, we define $\widetilde{\mathrm{M}}$ by the formula (21), and prove that it has the required properties.

If $u_1, \dots, u_n$ are two times differentiable and we set $A_j = \mathrm{Hess}(u_j)$, it follows that $\widetilde{\mathrm{M}}(u_1, \dots, u_n) = \det(A_1, \dots, A_n) \cdot \lambda$ is a positive measure depending multilinearly and symmetrically on $u_1, \dots, u_n$ and that $\widetilde{\mathrm{M}}(u, \dots, u) = \mathrm{M}(u)$. The general case follows by approximating u_j with smooth functions and passing to the limit. ■

When $u_1, \dots, u_n$ are convex functions there is in general no geometric interpretation of $\widetilde{\mathrm{M}}(u_1, \dots, u_n)(E)$ in terms of the gradient images $\operatorname{grad} u_j(E)$. However, if $E = \mathbf{R}^n$ we have the following

Proposition 3. *For a convex function* u *defined in* $\mathbf{R}^n$, *let* K_u *be the closure of* $\operatorname{grad} u(\mathbf{R}^n)$. *Then* K_u *is convex, and if it is also compact, then the total mass of* $\mathrm{M}\, u$ *equals* $\mathrm{Vol}(K_u)$. *If* $K_{u_1}, \dots, K_{u_n}$ *are all compact, then the total mass of* $\widetilde{\mathrm{M}}(u_1, \dots, u_n)$ *is equal to the mixed volume* $\mathrm{Vol}(K_{u_1}, \dots, K_{u_n})$.

Proof. Assume without loss of generality that $u(0) = 0$. For $t \ge 1$ define $u_t(x) = u(tx)/t$. Then u_t is an increasing family of convex functions and $u_\infty(x) = \sup u_t(x)$ is a positively homogeneous convex function with values in $\mathbf{R} \cup \{+\infty\}$. Let $K'_u = \{\xi; \langle \xi, x\rangle \le u_\infty(x), \forall x \in \mathbf{R}^n\}$. It is clear that K'_u is a closed convex set. If $\xi \in \operatorname{grad} u(\mathbf{R}^n)$, then $u(x) \ge c + \langle \xi, x\rangle$ for some constant c, hence $u_t(x) \ge c/t + \langle \xi, x\rangle$ and $u_\infty(x) \ge \langle \xi, x\rangle$. It follows that $\operatorname{grad} u(\mathbf{R}^n) \subset K'_u$. On the other hand, if $\xi \in \operatorname{int} K'_u$, then $u_\infty(x) - \langle \xi, x\rangle \ge c|x|$ for some constant $c > 0$. It follows that for sufficiently large t, $u_t(x) - \langle \xi, x\rangle > 0$ when $x \in \partial B$, where B is the unit ball, so $\xi \in \operatorname{grad} u_t(B) \subset \operatorname{grad} u(\mathbf{R}^n)$. Similarly, if $\xi \in \operatorname{relint} K'_u$ it follows that $\xi \in \operatorname{grad} u(\mathbf{R}^n)$. Hence $\operatorname{relint} K'_u \subset \operatorname{grad} u(\mathbf{R}^n) \subset K'_u$, and it follows that $K_u = K'_u$ is convex. It follows also that the total mass of $\mathrm{M}\, u$ is equal to $\mathrm{Vol}(K_u)$. Moreover, it is clear that $(u + v)_\infty = u_\infty + v_\infty$ for all convex functions u and v, and if K_u and K_v are compact this implies that $K_{u+v} = K_u + K_v$. Hence $\widetilde{\mathrm{M}}(u_1, \dots, u_n)$ and $\mathrm{Vol}(K_{u_1}, \dots, K_{u_n})$ both depend multilinearly on $u_1, \dots, u_n$. Since we have already shown that the total mass of $\widetilde{\mathrm{M}}(u, \dots, u) = \mathrm{M}\, u$ is equal to $\mathrm{Vol}(K_u, \dots, K_u) = \mathrm{Vol}(K_u)$ it follows that the total mass of $\widetilde{\mathrm{M}}(u_1, \dots, u_n)$ is equal to $\mathrm{Vol}(K_{u_1}, \dots, K_{u_n})$. ■

There is also a complex version of the Monge-Ampère operator. If U is a smooth plurisubharmonic function defined in $\mathbf{C}^n$, then integration of the differential form $(dd^c U)^n$ (where $d^c = (\partial - \bar\partial)/2\pi i$) defines the Monge-Ampère measure of U. The multilinear version of the complex Monge-Ampère measure is defined by integrating the form $dd^cU_1 \wedge \ldots \wedge dd^cU_n$ where $U_1, \ldots, U_n$ are smooth plurisubharmonic functions. The complex Monge-Ampère operator can be extended to a continuous operator from the space of continuous plurisubharmonic functions with the topology of uniform convergence on compact sets, to the space of measures with the weak topology (see [2]).

If u is a convex function defined in $\mathbf{R}^n$, we may define a plurisubharmonic function in $\mathbf{C}_*^n$ by

$$U(z) = u(\operatorname{Log} z). \tag{22}$$

Similarly, if U is a plurisubharmonic function defined in $\mathbf{C}_*^n$, then

$$u(x) = \int_{\operatorname{Log}^{-1}(x)} U(z)\, d\eta(z) \tag{23}$$

is a convex function defined in $\mathbf{R}^n$. The real and complex Monge-Ampère operators are related by the following properties.

Proposition 4. *If $u_1, \ldots, u_n$ are convex functions and $U_j(z)$ are defined by* (22)*, then*

$$\int_E \widetilde{\mathrm{M}}(u_1, \ldots, u_n) = \frac{1}{n!} \int_{\operatorname{Log}^{-1}(E)} dd^cU_1 \wedge \ldots \wedge dd^cU_n. \tag{24}$$

for any Borel set E. If $U_1, \ldots, U_n$ are continuous plurisubharmonic functions and $u_j(x)$ are defined by (23)*, then*

$$\int_E \widetilde{\mathrm{M}}(u_1, \ldots, u_n) = \frac{1}{n!} \int_{\mathbf{T}^{n^2}} \int_{\operatorname{Log}^{-1}(E)} dd^cU_1(t^{(1)}z) \wedge \ldots \wedge dd^cU_n(t^{(n)}z) d\eta'(t) \tag{25}$$

where $\mathbf{T}^{n^2}$ denotes the real n^2-dimensional torus $\{t = (t_j^{(k)}); t_j^{(k)} \in \mathbf{C}, |t_j^{(k)}| = 1, j, k = 1, \ldots, n\}$ with the normalized Haar measure η', and $t^{(k)} = (t_1^{(k)}, \ldots, t_n^{(k)})$ acts on $\mathbf{C}_*^n$ by componentwise multiplication.

Proof. We first prove (24) in the case where $u_1 = \ldots = u_n = u$ is a smooth function (see also [23]). Let $U(z) = u(\operatorname{Log} z)$. Since $dd^cU = i\partial\bar{\partial}U/\pi$ we have

$$(dd^cU)^n = n! \left(\frac{i}{\pi}\right)^n \det\left(\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}\right) dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.$$

Moreover, since

$$\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k} = \frac{1}{4 z_j \bar{z}_k} \frac{\partial^2 u}{\partial x_j \partial x_k}$$

it follows that

$$\det\left(\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}\right) = \frac{1}{4^n |z_1|^2 \ldots |z_n|^2} \det\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right).$$

Writing $z_j = \exp(x_j + iy_j)$, we have $dz_j \wedge d\bar{z}_j = -2i|z_j|^2\, dx_j \wedge dy_j$ and it follows that

$$(dd^c U)^n = \frac{n!}{(2\pi)^n} \det\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n.$$

Hence

$$\begin{aligned}&\int_{\mathrm{Log}^{-1}(E)} (dd^c U)^n \\ &\quad = \frac{n!}{(2\pi)^n} \int_E \det\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) dx_1 \wedge \ldots \wedge dx_n \int_{0<y_j<2\pi} dy_1 \wedge \ldots \wedge dy_n \\ &\quad = n! \int_E \mathrm{M}\, u.\end{aligned}$$

The proof of (24) is complete in this special case. The case with arbitrary convex u is obtained by a limiting procedure. Recall that both the real and the complex Monge-Ampère operators can be extended in a continuous way to all convex and continuous plurisubharmonic functions respectively. Finally, the result is extended to arbitrary convex functions $u_1, \ldots, u_n$ by observing that both sides of (24) depend multilinearly and symmetrically on $u_1, \ldots, u_n$.

To prove (25), let U_j be continuous plurisubharmonic functions and let u_j be defined by (23). Also, write $\tilde{U}_j(z) = u_j(\mathrm{Log}\, z)$. Then it follows by reversing the order of integration that

$$\begin{aligned}&\int_{\mathbf{T}^{n^2}} \int_{\mathrm{Log}^{-1}(E)} dd^c U_1(t^{(1)} z) \wedge \ldots \wedge dd^c U_n(t^{(n)} z) d\eta'(t) \\ &\quad = \int_{\mathrm{Log}^{-1}(E)} dd^c \tilde{U}_1 \wedge \ldots \wedge dd^c \tilde{U}_n = n! \int_E \widetilde{\mathrm{M}}(u_1, \ldots, u_n). \quad \blacksquare\end{aligned}$$

6.2 Monge-Ampère measures on amoebas

Throughout this section, Ω denotes a convex domain in $\mathbf{R}^n$ and $f, f_1, f_2, \ldots$ denote holomorphic functions, all defined in $\mathrm{Log}^{-1}(\Omega)$ unless specified otherwise.

Definition 10 ([22]). *Define μ_f to be the Monge-Ampère measure $\mathrm{M}\, N_f$, and $\mu_{f_1,\ldots,f_n}$ to be the mixed Monge-Ampère measure $\widetilde{\mathrm{M}}(N_{f_1}, \ldots, N_{f_n})$.*

Theorem 15. *The measure μ_f has its support in $\mathcal{A}_f$, and $\mu_{f_1,\ldots,f_n}$ has its support in $\mathcal{A}_{f_1} \cap \ldots \cap \mathcal{A}_{f_n}$. If $f, f_1, \ldots, f_n$ are Laurent polynomials, then the total mass of μ_f equals $\mathrm{Vol}(P_f)$ and the total mass of $\mu_{f_1,\ldots,f_n}$ equals the mixed volume $\mathrm{Vol}(P_{f_1}, \ldots, P_{f_n})$.*

Proof. Since N_f is affine linear outside $\mathcal{A}_f$ it follows from the definition of the Monge-Ampère operator that $\mathrm{supp}\, \mu_f \subset \mathcal{A}_f$ and that $\mathrm{supp}\, \mu_{f_1,\ldots,f_n} \subset \mathcal{A}_{f_1} \cap \ldots \cap \mathcal{A}_{f_n}$. The statement about the total mass follows directly from Proposition 3 and Theorem 2. $\blacksquare$

Theorem 16 ([22]). *If E is a Borel set in Ω, then $\mu_{f_1,\ldots,f_n}$ equals the average number of solutions in $\mathrm{Log}^{-1}(E)$ to the system of equations*

$$f_j(t_1^{(j)} z_1, \ldots, t_n^{(j)} z_n) = 0 \quad j = 1, \ldots, n \tag{26}$$

as $t = (t_k^{(j)})$ ranges over the torus $\mathbf{T}^{n^2}$.

In the proof we will need the following lemma.

Lemma 3. *Let V be an analytic set in a neighbourhood of a compact set $D_1 \times D_2 \subset \mathbf{C}^k \times \mathbf{C}^m$. Then there exists a constant C such that $V \cap \{z\} \times D_2$ either has positive dimension or has at most C points for every $z \in D_1$.*

Proof. Take any point, say $(0,0)$ in $D_1 \times D_2$. It suffices to prove the statement with D_1 and D_2 replaced by arbitrarily small neighbourhoods of $(0,0)$. Use induction on m. If $\dim(V \cap \{0\} \times D_2) < m$ then $V \cap D_1 \times D_2$ can be properly projected to $D_1 \times D_2' \subset \mathbf{C}^k \times \mathbf{C}^{m-1}$, and this way the problem is reduced to a smaller m. So we may assume that the statement is already proved when $\dim(V \cap \{0\} \times D_2) < m$. Assume therefore that $V \supset \{0\} \times D_2$ and let $f_j(z,w) = \sum f_{j,\alpha}(z) w^\alpha$ be defining functions for V near the origin. Consider the ideal generated by all $f_{j,\alpha}$ in the ring of germs of holomorphic functions, and let $g_1, \ldots, g_r$ be a finite subset of the $f_{j,\alpha}$ generating the same ideal. Write $f_{j,\alpha} = \sum \phi_{j,\alpha,l} g_l$. Since for every l there is some j, α with $g_l = f_{j,\alpha}$ we may assume that the corresponding $\phi_{j,\alpha,l} \equiv 1$. Now consider the analytic set W in $\mathbf{C}^k \times \mathbf{C}^r \times \mathbf{C}^m$ defined by the functions $h_j(z,\zeta,w) = \sum \phi_{j,\alpha,l}(z) \zeta_l w^\alpha$. If $\zeta \neq 0$, then some $\zeta_l \neq 0$ and if j, α are such that $\phi_{j,\alpha,l} \equiv 1$, it follows that the function $w \mapsto h_j(z,\zeta,w)$ is not identically 0 for any z. Hence $\dim(W \cap \{z\} \times \{\zeta\} \times D_2) < m$ whenever $\zeta \neq 0$. By the inductive hypothesis we may then assume that $W \cap \{z\} \times \{\zeta\} \times D_2$ either has positive dimension or has no more than C points for all $z \in D_1$ and all ζ with $|\zeta| = 1$. But since $h_j(z, t\zeta, w) = t h_j(z,\zeta,w)$, the same estimate holds for all $\zeta \neq 0$. By taking $\zeta_l = g_l(z)$, it follows that $V \cap \{z\} \times D_2$ either has positive dimension or has no more than C points for any $z \in D_1$. ■

Proof of Theorem 16. If U_j are smooth plurisubharmonic functions which converge to $\log|f_j|$, then u_j defined by (23) converge to N_{f_j}. By the continuity of the real Monge-Ampère operator this implies that $\tilde{M}(u_1, \ldots, u_n)$ converges to $\tilde{M}(N_{f_1}, \ldots, N_{f_n})$ in the weak topology. Also $dd^c U_1(t^{(1)}z) \wedge \ldots \wedge dd^c U_n(t^{(n)}z)$ converges weakly to the sum of point masses at the solutions of $f_1(t^{(1)}z) = \ldots = f_n(t^{(n)}z) = 0$. Hence the theorem follows by using (25) and passing to the limit if we only show that

$$\int_{\mathrm{Log}^{-1}(E)} dd^c U_1(t^{(1)}z) \wedge \ldots \wedge dd^c U_n(t^{(n)}z)$$

remains uniformly bounded as $U_j \to \log|f_j|$ for almost all $t \in \mathbf{T}^{n^2}$. Here we may assume that E is compact and that U_j is of the form $U_j = \psi(\log|f_j|)$ where ψ is a convex function, constant near $-\infty$, which will converge to the identity function. Let $f_t(z) = (f_1(t^{(1)}z), \ldots, f_n(t^{(n)}z))$. Then $f_t(z)$ is a holomorphic function in z and t defined for z in a neighbourhood of $\mathrm{Log}^{-1}(E)$ and t in a complex neighbourhood of $\mathbf{T}^{n^2}$. By Lemma 3 there exists a constant C such that the number of solutions z in $\mathrm{Log}^{-1}(E)$ to the equation $f_t(z) = w$ is bounded above by C for almost all $t \in \mathbf{T}^{n^2}$ and $w \in \mathbf{C}^n$. Since $\omega = dd^c\psi(\log|w_1|) \wedge \ldots \wedge dd^c\psi(\log|w_n|)$ induces a positive measure on $\mathbf{C}^n$ with total mass 1, it follows that

$$0 \leq \int_{\mathrm{Log}^{-1}(E)} f_t^* \omega \leq C$$

for almost all t, and this completes the proof. ■

Theorem 16 can be thought of as a local analog of Bernstein's theorem relating the number of solutions to a system of polynomial equations to the mixed volume of the Newton polytopes. In fact, Bernsteins result can rather easily be derived from Theorem 16.

Corollary 7 (Bernstein's theorem [3]). *Let $P_1, \ldots, P_n$ be integer polytopes in $\mathbf{R}^n$ and let $f_1, \ldots, f_n$ be generic Laurent polynomials subject to the restriction that $P_{f_j} = P_j$. Then the number of solutions in $\mathbf{C}_*^n$ of the system of equations $f_1(z) = \ldots = f_n(z) = 0$ is equal to $n! \operatorname{Vol}(P_1, \ldots, P_n)$.*

Proof. We assume it is known that the number of solutions is equal to some constant N when $(f_1, \ldots, f_n)$ is outside some subvariety in the space of n-tuples of polynomials with Newton polytopes $P_1, \ldots, P_n$. If $f_1(z) = \ldots = f_n(z) = 0$ has the generic number of solutions, then the same is true for $f_1(t^{(1)}z) = \ldots = f_n(t^{(n)}z) = 0$ for almost all $t \in \mathbf{T}^{n^2}$. It follows from Theorem 15 and Theorem 16 that $N = n! \operatorname{Vol}(P_1, \ldots, P_n)$. ■

Theorem 17. *If $u_1, \ldots, u_{n-1}$ are convex functions defined in Ω and $U_j(z) = u_j(\operatorname{Log} z)$, then*

$$n! \int_E \widetilde{\mathrm{M}}(u_1, \ldots, u_{n-1}, N_f) = \int_{\operatorname{Log}^{-1}(E) \cap f^{-1}(0)} dd^c U_1 \wedge \ldots \wedge dd^c U_{n-1}$$

for any Borel set $E \subset \Omega$.

Proof. Since $dd^c \log |f|$ is equal to the current of integration along $f^{-1}(0)$ it follows from (24) that

$$\begin{aligned} n! \int_E \widetilde{\mathrm{M}}(u_1, \ldots, u_{n-1}, N_f) &= \int_{\operatorname{Log}^{-1}(E)} dd^c U_1 \wedge \ldots \wedge dd^c U_{n-1} \wedge dd^c N_f(\operatorname{Log} z) \\ &= \int_{\operatorname{Log}^{-1}(E)} dd^c U_1 \wedge \ldots \wedge dd^c U_{n-1} \wedge dd^c \log |f| \\ &= \int_{\operatorname{Log}^{-1}(E) \cap f^{-1}(0)} dd^c U_1 \wedge \ldots \wedge dd^c U_{n-1}. \quad ■ \end{aligned}$$

Corollary 8 ([22]). *If Δ denotes the Laplace operator and E is any Borel set in Ω, then*

$$(n-1)! \int_E \Delta N_f = \int_{\operatorname{Log}^{-1}(E) \cap f^{-1}(0)} \omega^{n-1}$$

where $\omega = (|z_1|^{-2} d\bar{z}_1 \wedge dz_1 + \ldots + |z_n|^{-2} d\bar{z}_n \wedge dz_n)/2\pi i$. Hence $(n-1)! \Delta N_f$ is the direct image of $\omega|_{f^{-1}(0)}$ (considered as a measure) under the mapping Log.

Proof. This follows immediately from Theorem 17 since

$$\Delta N_f = n \, \widetilde{\mathrm{M}}(|x|^2, \ldots, |x|^2, N_f)$$

and $\omega = dd^c |\operatorname{Log} z|^2$. ■

Finally, we shall give a formula for μ_f in the two-dimensional case which has some interesting consequences. Let Ω be a convex domain in $\mathbf{R}^2$ and let f be a holomorphic function in Ω. Let F be the set of critical values of the mapping $\mathrm{Log} : f^{-1}(0) \longrightarrow \mathbf{R}^2$, and take a small open set V in $\mathcal{A}_f \setminus F$. Let k be the cardinality of $f^{-1}(0) \cap \mathrm{Log}^{-1}(x)$ for $x \in V$. Then there exist smooth functions ϕ_j, ψ_j, $j = 1, \dots, k$ defined in V such that $x \mapsto (\exp(x_1 + i\phi_j(x)), \exp(x_2 + i\psi_j(x)))$ are local inverses of Log, i.e. $f^{-1}(0) \cap \mathrm{Log}^{-1}(V) = \cup_{j=1}^k \{(\exp(x_1 + i\phi_j(x)), \exp(x_2 + i\psi_j(x))); x = (x_1, x_2) \in V\}$.

Theorem 18 ([22]). *With notations as in the preceding paragraph, the Hessian of N_f is given in V by the formula*

$$\mathrm{Hess}\, N_f = \frac{1}{2\pi} \sum_{j=1}^{k} \pm \begin{pmatrix} \partial\psi_j/\partial x_1 & \partial\psi_j/\partial x_2 \\ -\partial\phi_j/\partial x_1 & -\partial\phi_j/\partial x_2 \end{pmatrix}. \tag{27}$$

The summands in the right hand side are positive definite matrices with determinant equal to 1.

Proof. Differentiating the integral (2) defining N_f with respect to x_1 we obtain

$$\begin{aligned} \frac{\partial N_f}{\partial x_1} &= \mathrm{Re}\, \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(x)} \frac{\partial f/\partial z_1\, dz_1\, dz_2}{f(z) z_2} \\ &= \frac{1}{2\pi i} \int_{\log|z_2| = x_2} n(f(\cdot, z_2), x_1) \frac{dz_2}{z_2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} n(f(\cdot, e^{x_2 + iy_2}), x_1) dy_2. \end{aligned}$$

If f is a Laurent polynomial, $n(f(\cdot, z_2), x_1)$ is the number of zeros minus the number of poles of the function $z_1 \mapsto f(z_1, z_2)$ inside the disc $\{\log|z_1| < x_1\}$. In general, $n(f(\cdot, z_2), x_1)$ is an integer valued function such that $n(f(\cdot, z_2), x_1) - n(f(\cdot, z_2), x_1')$ is equal to the number of zeros of $z_1 \mapsto f(z_1, z_2)$ in the annulus $\{x_1' < \log|z_1| < x_1\}$ when $x_1' < x_1$. Hence, the integrand in the last integral is a piecewise constant function with jumps of magnitude 1 at $y_2 = \psi_j(x)$. It follows that the gradient of $\partial N_f/\partial x_1$ is given by a sum of terms $\pm(2\pi)^{-1}\, \mathrm{grad}\, \psi_j$. This proves the first row of the identity (27), up to sign changes. The correct sign of each term can be found by observing that $n(f(\cdot, e^{x_2+iy_2}), x_1)$ is increasing as a function of x_1, hence all the terms contributing to $\partial^2 N_f/\partial x_1^2$ should be positive. A similar computation involving $\partial N_f/\partial x_2$ proves the second row. However, we have not yet shown that the choices of signs in the two rows are consistent.

We shall now prove that all the terms on the right hand side of (27) are symmetric, positive definite matrices with deteminant equal to 1. Take a point x and an index j. Differentiating the expression $f(e^{x_1+i\phi_j(x)}, e^{x_2+i\psi_j(x)}) = 0$ with respect to x_1 and x_2 yields the equations

$$\begin{aligned} z_1 \frac{\partial f}{\partial z_1} \left(1 + i\frac{\partial \phi_j}{\partial x_1}\right) + z_2 \frac{\partial f}{\partial z_2} i \frac{\partial \psi_j}{\partial x_1} &= 0 \\ z_1 \frac{\partial f}{\partial z_1} i \frac{\partial \phi_j}{\partial x_2} + z_2 \frac{\partial f}{\partial z_2} \left(1 + i\frac{\partial \psi_j}{\partial x_2}\right) &= 0. \end{aligned}$$

Writing $a = z_1 \partial f/\partial z_1$, $b = z_2 \partial f/\partial z_2$, these equations have the solution

$$\begin{pmatrix} \partial\psi_j/\partial x_1 & \partial\psi_j/\partial x_2 \\ -\partial\phi_j/\partial x_1 & -\partial\phi_j/\partial x_2 \end{pmatrix} = \frac{1}{\mathrm{Im}(\bar{a}b)} \begin{pmatrix} |a|^2 & \mathrm{Re}(\bar{a}b) \\ \mathrm{Re}(\bar{a}b) & |b|^2 \end{pmatrix}.$$

This matrix clearly has determinant 1. Changing the sign if $\mathrm{Im}(\bar{a}b) < 0$ we also have that the diagonal elements are positive, so the matrix is positive definite. Since we have already observed that the diagonal elements in the right hand side of (27) must be positive, it follows that these matrices are positive definite with determinant equal to 1. ■

Corollary 9. *If f is a Laurent polynomial in two variables, then $\mu_f \geq \pi^{-2}\lambda|_{\mathcal{A}_f}$ where λ denotes Lebesgue measure.*

Proof. Since the set F of critical values of $\mathrm{Log} : f^{-1}(0) \to \mathbf{R}^2$ is a null set for Lebesgue measure it suffices to prove the inequality in the complement of this set. If A_1, A_2 are 2×2 positive definite matrices, then it follows from the inequality (20) that $\det(A_1 + A_2) = \det A_1 + \det A_2 + 2\det(A_1, A_2) \geq \det A_1 + \det A_2 + 2\sqrt{\det A_1 \det A_2}$. Repeated use of this inequality leads to $\sqrt{\det(A_1 + \ldots + A_k)} \geq \sqrt{\det A_1} + \ldots + \sqrt{\det A_k}$. Applying this inequality to the sum (27), which contains at least two terms for every $x \in \mathcal{A}_f \setminus F$, yields the result. ■

Theorem 19 ([22], [21]). *If f is a Laurent polynomial in two variables, then $\mathrm{Area}(\mathcal{A}_f) \leq \pi^2 \mathrm{Area}(P_f)$. When P_f has positive area, equality holds precisely if $\mathrm{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in at most two points for every $x \in \mathbf{R}^2$ and there exist constants $a, b_1, b_2 \in \mathbf{C}_*$ such that $af(b_1z_1, b_2z_2)$ is a polynomial with real coefficients.*

Proof. The inequality $\mathrm{Area}(\mathcal{A}_f) \leq \pi^2 \mathrm{Area}(P_f)$ follows immediately from Corollary 9 and Theorem 15. Notice that equality holds if and only if $\mu_f = \pi^{-2}\lambda|_{\mathcal{A}_f}$.

Suppose that $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ has at most two points for all $x \in \mathbf{R}^2$ and that $af(b_1z_1, b_2z_2)$ has real coefficients. Without loss of generality we may assume that $a = b_1 = b_2 = 1$. Then the sum (27) contains precisely two terms for all $x \in \mathcal{A}_f$ and since complex conjugation of the coordinates leaves $f^{-1}(0)$ unchanged, we have $\phi_2 = -\phi_1, \psi_2 = -\psi_1$. This means that the two terms are actually equal, and it follows that $\det \mathrm{Hess}(N_f) = \pi^{-2}$ and hence $\mu_f = \pi^{-2}\lambda$ in $\mathcal{A}_f \setminus F$. Now F is a null set with respect to Lebesgue measure, so if we can show that $\mu_f(F) = 0$ it will follow that $\mu_f = \pi^{-2}\lambda|_{\mathcal{A}_f}$, which is what we want to prove.

Let $\tilde{F}$ be a real algebraic curve containing the set of critical values of the mapping $f^{-1}(0) \to \mathbf{R}^2 : (z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$. Consider the product space $\mathbf{C}_*^2 \times \mathbf{T}^2$ and let $\pi_1 : \mathbf{C}_*^2 \times \mathbf{T}^2 \to \mathbf{R}^2$ and $\pi_2 : \mathbf{C}_*^2 \times \mathbf{T}^2 \to \mathbf{T}^2$ be defined by $\pi_1(z,t) = (|z_1|^2, |z_2|^2)$ and $\pi_2(z,t) = t$. Let $C = \pi_1^{-1}(\tilde{F}) \cap \{f(z_1, z_2) = f(t_1z_1, t_2z_2) = 0\}$. Since $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ is a finite set for every $x \in \mathbf{R}^2$, it follows that the projection $\pi_1 : C \to \tilde{F}$ has finite fibers, hence C is a real curve. It follows that $\pi_2(C)$ is a null set in $\mathbf{T}^2$, which means that the system of equations $f(z_1, z_2) = f(t_1z_1, t_2z_2) = 0$ has no solutions in $\mathrm{Log}^{-1}(F)$ for almost all $t \in \mathbf{T}^2$. It follows from Theorem 16 that $\mu_f(F) = 0$ as required.

Suppose now conversely that $\mu_f = \pi^{-2}\lambda|_{\mathcal{A}_f}$. First we show that f is irreducible. If K, L are compact convex sets in $\mathbf{R}^2$, then it follows from the monotonicity properties of mixed volumes that $\operatorname{Area}(K+L) \geq \operatorname{Area}(K) + \operatorname{Area}(L)$ with strict inequality unless either K or L is a point or K and L are two parallel segments. If $f = gh$ is a nontrivial factorization of f we therefore have

$$\begin{aligned}\operatorname{Area}(\mathcal{A}_f) &\leq \operatorname{Area}(\mathcal{A}_g) + \operatorname{Area}(\mathcal{A}_h) \leq \pi^2(\operatorname{Area}(P_g) + \operatorname{Area}(P_h)) \\ &< \pi^2 \operatorname{Area}(P_f)\end{aligned}$$

contradicting the assumption that $\operatorname{Area}(\mathcal{A}_f) = \pi^2 \operatorname{Area}(P_f)$.

It follows from Theorem 18 that $\operatorname{Log}^{-1}(0) \cap f^{-1}(0)$ has at most two points for all x outside F and that the two terms in the sum (27) are equal. After a change of coordinates $(z_1, z_2) \mapsto (z_1/b_1, z_2/b_2)$ we may then assume that $\phi_2 = -\phi_1, \psi_2 = -\psi_1$ in a neighbourhood of some point $x \in \mathcal{A}_f \setminus F$ (such points exist by the assumption that $\operatorname{Area}(\mathcal{A}_f) = \pi^2 \operatorname{Area}(P_f) > 0$). But this means that $\overline{f(\bar{z}_1, \bar{z}_2)}$ vanishes on an open subset of $f^{-1}(0)$. Since f is irreducible it follows that $f(z_1, z_2)$ and $\overline{f(\bar{z}_1, \bar{z}_2)}$ are equal up to a constant multiple. Hence $af(z_1, z_2)$ has real coefficients for a suitable constant a.

It remains to be shown that $\operatorname{Log}^{-1}(x_0) \cap f^{-1}(0)$ has at most two points for all $x_0 \in F$. To do this, we consider two cases. Consider a discrete point of $\operatorname{Log}^{-1}(x_0) \cap f^{-1}(0)$ and a small neighbourhood U of it in $f^{-1}(0)$. Now, either $\operatorname{Log}(U)$ contains a neighbourhood of x_0, or there is an open half plane H with x_0 on its boundary such that $\operatorname{Log}^{-1}(x) \cap U$ has two points for all $x \in H$ near x_0. If there are three discrete points in $\operatorname{Log}^{-1}(x_0)$ this implies that one can find x outside F such that $\operatorname{Log}^{-1}(x) \cap f^{-1}(0)$ contains more than two points, a contradiction. If instead $\operatorname{Log}^{-1}(x_0) \cap f^{-1}(0)$ contains a real curve, then it must be of the form $z^\alpha = c$ for some $\alpha \in \mathbf{Z}^2$ and $c \in \mathbf{C}$, otherwise μ_f would have a point mass at x_0 by Theorem 16. But then f has a factor $z^\alpha - c$, which is impossible since $\operatorname{Area}(P_f)$ is positive and f is irreducible. ■

7 Amoebas and real algebraic geometry

A relation between amoebas and real algebraic geometry has recently been discovered by Mikhalkin, and who used amoebas to obtain results about the topology of real algebraic curves (see [20]). In this section we briefly outline the background to these results.

The study of the topology of real algebraic curves can be traced back to the paper [12] by Harnack from 1876. The topology of real curves is also the subject of Hilberts sixteenth problem. A more recent development in the theory is the patchworking technique of Viro.

Throughout this section, P will denote a lattice polygon in $\mathbf{R}^2$, and $F_1, \ldots, F_k$ denote the edges of P, numbered in cyclic order. Let g be the number of lattice points in the interior of P, and let the number of lattice points on F_j be $d_j + 1$. The polygon P detemines a toric variety X_P which is a compactification of the complex torus $\mathbf{C}_*^2$. The closure of $\mathbf{R}_*^2$ in X_P is a real toric variety, which we denote $\mathbf{R}X_P$. (See section 2 for the notations being used for toric varieties.)

Let f be a Laurent polynomial with real coefficients whose Newton polytope is P, and let $V_f = f^{-1}(0)$ be the hypersurface defined by f in $\mathbf{C}_*^2$. Also let $\overline{V}_f$ be the closure of V_f in X_P and let $\mathbf{R}V_f$ and $\mathbf{R}\overline{V}_f$ be the intersection of $\mathbf{R}X_P$ with V_f and $\overline{V}_f$ respectively.

Theorem 20 (Harnack [12], Khovanskii [15]). *The genus of $\overline{V}_f$ is equal to g and the number of intersection points between $\overline{V}_f$ and $V(F_j)$ is equal to d_j. Moreover, $\mathbf{R}\overline{V}_f$ has at most $g+1$ connected components.*

Definition 11 (see [14], [20], [21]). *A Laurent polynomial f is said to define a Harnack curve if the following conditions hold.*

(i) $\mathbf{R}\overline{V}_f$ *consists of $g+1$ connected components.*

(ii) One of these components can be divided into k consecutive arcs $\gamma_1, \dots, \gamma_k$ such that γ_j intersects $V(F_j)$ in d_j points and γ_j does not intersect $V(F_l)$ if $j \neq l$.

(iii) None of the other components of $\mathbf{R}\overline{V}_f$ *intersects $V(F_j)$, $j = 1, \dots, k$.*

Let A denote the set of lattice points in P and let c_α be a real number for every $\alpha \in A$. Set

$$s(x) = \max_{\alpha \in A} (c_\alpha + \langle \alpha, x \rangle)$$

and let Σ' be the subdivision of P obtained by the recipe in section 4. Assume that the numbers c_α have been chosen so that the 2-dimensional polygons in Σ' are all triangles and that all points in A appear as vertices in Σ'.

Let f^t be a family of polynomials in $\mathbf{R}^A$ such that $\log|f^t_\alpha| = tc_\alpha$ and f^t_α is negative if $\alpha \in 2\mathbf{Z}^2$ and positive otherwise.

Theorem 21 (Harnack [12], Itenberg, Viro [14], Mikhalkin [20]). *For sufficiently large t, f^t defines a Harnack curve.*

Theorem 22 (Mikhalkin [21]). *A nonsingular real polynomial f defines a Harnack curve if and only if* $\mathrm{Log}^{-1}(x)$ *intersects V_f in at most two points for all $x \in \mathbf{R}^2$. In this case,* $\mathbf{R}V_f = V_f \cap \mathrm{Log}^{-1}(\partial \mathcal{A}_f)$.

Combining this with Theorem 19 we obtain

Corollary 10 ([21]). *A nonsingular real polynomial f defines a Harnack curve if and only if* $\mathrm{Area}(\mathcal{A}_f) = \pi^2 \,\mathrm{Area}(P_f)$.

Since Harnack curves exist for every polytope P by Theorem 21 we also have

Corollary 11 ([21]). *The inequality* $\mathrm{Area}(\mathcal{A}_f) \leq \pi^2 \,\mathrm{Area}(P_f)$ *in Theorem 19 is sharp.*

8 Amoebas of varieties of codimension greater than 1

In this section we discuss a possible generalization of the material in previous sections. The ideas presented are rather tentative, and we do not prove any results. The central problem will be how the definitions of the main objects we have studied might be carried over to a more general situation where the hypersurface $f^{-1}(0)$ is replaced by an arbitrary algebraic variety.

Let V be an algebraic variety in $L_{\mathbf{C}_*}$. We assume that V is of pure dimension k and let $r = n - k$ be its codimension. By the amoeba of V we shall mean the set $\mathcal{A}_V = \mathrm{Log}(V)$.

8.1 The amoeba complement

If f is a Laurent polynomial, and $V = f^{-1}(0)$ is a hypersurface, the amoeba complement $\mathcal{A}_V^c$ consists of a number of connected components. Each such component is convex, and in particular contractible. Therefore, the topology of $\mathcal{A}_V^c$ is determined completely by the number of connected components.

The concept of the order of a complement component plays an important role in the study of amoebas. Notice that the order of a complement component cannot be defined only in terms of the hypersurface V. Indeed, if f is multiplied by an arbitrary Laurent monomial z^ν, then V is not changed, but the order of each complement component is translated by the vector ν. On the other hand, the difference between the orders of two components does not change, so this difference may be possible to define in terms of V alone. This can be done as follows. Let E_α and E_β be two components of $\mathcal{A}_V^c$ whose orders are α and β (with respect to the defining polynomial f). Take arbitrary points $z_\alpha \in \operatorname{Log}^{-1}(E_\alpha$ and $z_\beta \in \operatorname{Log}^{-1}(E_\beta)$ and let $a \in L$. Let C_α and C_β be the oriented curves parametrized by $\zeta^a z_\alpha$ and $\zeta^a z_\beta$ where ζ runs along the unit circle $\mathbf{T}$ in the counterclockwise direction. If B is an oriented surface in $L_{\mathbf{C}_*}$ with boundary $C_\alpha - C_\beta$, then $\langle \alpha - \beta, a\rangle$ is equal to the number of intersection points (counted with signs) between B and V.

We now generalize this construction to the case where the codimension r of V is greater than 1. In this case the connected components of $\mathcal{A}_V^c$ are no longer convex (in general) so $\mathcal{A}_V^c$ may have nontrivial homology groups. We will focus here on $H_{r-1}(\mathcal{A}_V^c, \mathbf{Z})$, and define a homomorphism $\operatorname{ord} : H_{r-1}(\mathcal{A}_V^c, \mathbf{Z}) \to \bigwedge^r L^*$, which seems to be a natural generalization of the order of complement components. Let $c \in H_{r-1}(\mathcal{A}_V^c, \mathbf{Z})$ and $a \in \bigwedge^r L$. Since $\operatorname{Log}^{-1}(\mathcal{A}_V^c)$ is homeomorphic (in a canonical way) to $\operatorname{Log}^{-1}(0) \times \mathcal{A}_V^c$, $\rho_r(a) \otimes c$ defines a homology class in $H_{2r-1}(\operatorname{Log}^{-1}(\mathcal{A}_V^c), \mathbf{Z})$. Let C be a $(2r-1)$-cycle representing this homology class. Since c is null homologous in $L_{\mathbf{R}}$, there exists a $2r$-chain B in $L_{\mathbf{C}_*}$ whose boundary is C. The number of intersection points between B and V depends only on a and c and not on the choices made in selecting C and B. Moreover, the dependence on a and c is bilinear. Hence $\operatorname{ord}(c)$ may be defined as the unique element in $\bigwedge^r L^*$ such that $\langle \operatorname{ord}(c), a\rangle$ is equal to the number of intersection points between B and V for all $a \in \bigwedge^r L$.

8.2 The Ronkin function

If we try to define the function N_f entirely in terms of $V = f^{-1}(0)$ we face the same problem as with the orders of complement components; when f is multiplied by a monomial the function N_f is changed but V remains the same. In order to rescue at least some fragment of the Ronkin function we note that with respect to the Monge-Ampère operator these changes are of no significance. Theorem 17 gives an explicit formula for $\widetilde{\mathrm{M}}(u_1, \ldots, u_{n-1}, N_f)$ in terms of V where $u_1, \ldots, u_{n-1}$ are arbitrary convex functions; if $U_j(z) = u_j(\operatorname{Log} z)$, then

$$\widetilde{\mathrm{M}}(u_1, \ldots, u_{n-1}, N_f)(E) = \frac{1}{n!} \int_{\operatorname{Log}^{-1}(E)\cap V} dd^c U_1 \wedge \ldots \wedge dd^c U_{n-1}.$$

This can easily be generalized to the case where $k = \dim V$ is arbitrary. If we define

$$\widetilde{\mathrm{M}}_V(u_1, \dots, u_k)(E) = \frac{1}{n!} \int_{\mathrm{Log}^{-1}(E) \cap V} dd^c U_1 \wedge \dots \wedge dd^c U_k,$$

then $\widetilde{\mathrm{M}}_V(u_1, \dots, u_k)$ is a positive measure with support on $\mathcal{A}_V$ depending multilinearly on $u_1, \dots, u_k$. The interpretation of $\mu_{f_1,\dots,f_n}$ given in Theorem 16 is also open to generalizations.

8.3 The Newton polytope

Finally, we propose an approach to generalized Newton polytopes. We suggest that the Newton polytope of a variety of arbitrary dimension should be found in the polytope algebra discovered by McMullen. Since many readers are probably not familiar with this intriguing structure, we give a brief description here, refering to [18] and [19] for details. A relation between the polytope algebra and toric varieties which seems to be of interest in this context has been found by Fulton and Sturmfels [9].

The polytope algebra Π is an abelian group generated by all polytopes in a real vector space, which in our case will be $L_{\mathbf{R}}$. If P is a polytope, its class in Π is denoted $[P]$. The generators are subject to the relations $[P+x] = [P]$ for any translation vector x (translation invariance) and $[P] + [Q] = [P \cup Q] + [P \cap Q]$ whenever P, Q and $P \cup Q$ are polytopes (the valuation property). The polytope algebra is the universal group for functions which are translation invariant and satisfy the valuation property: If ϕ is a function from the set of all polytopes into an abelian group which satisfies $\phi(P + x) = \phi(P)$ and $\phi(P) + \phi(Q) = \phi(P \cup Q) + \phi(P \cap Q)$, then ϕ can be extended in a unique way to a group homomorphism on Π.

A multiplication is defined on Π by the rule $[P] \cdot [Q] = [P + Q]$, where $P + Q$ denotes the Minkowski (or vector) sum of P and Q. This operation makes Π into a commutative ring.

There is a direct sum decomposition $\Pi = \bigoplus_{r=0}^n \Xi_r$ where each Ξ_r is an abelian group. With respect to the multiplication in Π, $\Xi_r \cdot \Xi_s = \Xi_{r+s}$ (here $\Xi_r = 0$ if $r > n$). Moreover, each Ξ_r with $r \geq 1$ is in a natural way a real vector space. Loosely speaking, Ξ_r carries information about the r-dimensional faces of a polytope for $r = 1, \dots, n$.

The component Ξ_0 is isomorphic to $\mathbf{Z}$ and is generated by the class of a single point, while $\Xi_n \cong \mathbf{R}$. The Ξ_n-component of a polytope is proportional to its volume.

The subgroup $Z_1 = \bigoplus_{r=1}^n \Xi_r$ is a nilpotent ideal in Π. For any $p \in Z_1$ one may therefore define

$$\log(1 + p) = \sum_{j \geq 1} (-1)^{j-1} \frac{p^j}{j}$$

and

$$\exp p = \sum_{j \geq 0} \frac{p^j}{j!}.$$

The functions log and exp are inverses of each other and satisfy $\exp(p_1 + p_2) = \exp p_1 \cdot \exp p_2$ and $\log((1+p_1)(1+p_2)) = \log(1+p_1) + \log(1+p_2)$. For any polytope P, $[P] - 1$ belongs to Z_1 and $\log[P]$ is the Ξ_1-component of $[P]$. The set $\{\log[P]; P \text{ a polytope}\}$, is a convex cone in Ξ_1.

If P is any polytope, we write $h(P,x) = \sup_{\xi \in P} \langle \xi, x \rangle$ and $P_x = \{\xi \in P; \langle \xi, x \rangle = h(P,x)\}$.

If P is a polytope, then $\Pi(P)$ denotes the subalgebra of Π generated by all polytopes Q which are Minkowski summands of P, that is $P = tQ + R$ for some $t > 0$ and some polytope R. Also, $\Xi_r(P) = \Xi_r \cap \Pi(P)$. If Q is a Minkowski summand of P, there is a mapping $F \mapsto Q_F$ from the faces of P to the faces of Q; if $F = P_x$ then $Q_F = Q_x$. For any face F, Q_F is a Minkowski summand of F and the mapping $Q \mapsto [Q_F] \in \Pi(F)$ is translation invariant and satisfies the valuation property, hence it induces a homomorphism $\Pi(P) \to \Pi(F)$, which we will denote $p \mapsto p_F$.

Suppose now that P is a lattice polytope. For any face F of P, let L_F^* denote the sublattice of L^* which lies in the linear subspace of $L_{\mathbf{R}}^*$ parallel to F, and let L_F be the dual lattice of L_F^*, which is isomorphic to a quotient of L. If G is a face of P and F is a facet of G, the set $\{a \in L_G; \langle \xi - \eta, a \rangle \leq 0, \forall \xi \in G, \eta \in F\}$ is a semigroup isomorphic to $\mathbf{Z}_{\geq 0}$. Let x_{FG} denote the generator of this semigroup; it is a kind of outer normal to the facet F of G. A real valued function w on the set of all r-dimensional faces of P is called an r-weight. An r-weight w is called a Minkowski weight if it satisfies the Minkowski relation

$$\sum_{F \subset G} w(F) x_{FG} = 0$$

for every $(r+1)$-dimensional face G of P, where the sum is taken over all facets F of G. (We have deviated sligtly here from McMullens treatment, since he uses an inner product on the vector space to define the outer unit normal to a facet of a polytope, while we use a lattice for this purpose.)

Now let Q be a Minkowski summand of P. For every r-dimensional face F of P, the lattice L_F^* determines a volume form on subspaces parallel to F. Let $w(F)$ be the volume of Q_F with respect this volume form. It can then be shown that w is a Minkowski weight. Similarly, any $p \in \Pi(P)$ determines a Minkowski r-wieght on P for any $r = 0, \ldots, n$. Moreover, if P is a simple polytope (that is, exactly n facets meet at every vertex of P), then McMullen has shown that the set of r-weights on P is isomorphic, via the above construction, to $\Xi_r(P)$.

We are now ready to define the Newton polytope of a variety V. Let P be a simple polytope, and assume that the toric variety X_P is nonsingular. Let V be a k-dimensional subvariety of X_P which intersects $V(F)$ transversely for every face F of P. Let $r = n - k$ be the codimension of V and define an r-weight on P by letting $w(F)$ be the number of intersection points between V and $V(F)$. Then w is a Minkowski weight, which can be seen as follows. If G is an $(r+1)$-dimensional face of P and $\alpha \in L_G^*$, then z^α defines a rational function on $V(G)$ whose divisor is $-\sum_{F \subset G} \langle \alpha, x_{FG} \rangle V(F)$. Hence $\sum_{F \subset G} \langle \alpha, x_{FG} \rangle w(F)$ is equal to the number of poles minus the number of zeros of z^α on $V \cap V(G)$, which must be 0. Since this is true for all $\alpha \in L_G^*$, it follows that w satisfies the Minkowski relations. Hence w can be identified with an element in $\Xi_r(P)$, which can be thought of as a generalized Newton polytope. If f is a Laurent polynomial whose Newton polytope is a Minkowski summand of P, the construction may be applied to the closure of $f^{-1}(0)$ in X_P. The generalized Newton polytope obtained in this case is $\log[P_f]$, from which P_f can be reconstructed up to translations.

9 Examples of amoebas

To calculate explicitly the amoeba and Ronkin function of a given polynomial is very messy in all but the simplest cases. To motivate and exemplify the theory in the previous sections we give here some examples of polynomials for which certain calculations can be carried out without too much effort.

Example 1. First we consider a polynomial in one variable $f(z) = f_0 + f_1 z + \ldots + f_{m-1}z^{m-1} + z^m = (z+a_0)\ldots(z+a_m)$ where it is assumed that $|a_1| \leq \ldots \leq |a_m|$. The amoeba of f is then the discrete point set $\{\log|a_1|, \ldots, \log|a_m|\}$. A typical complement component of $\mathcal{A}_f$ is an interval $(\log|a_\alpha|, \log|a_{\alpha+1}|)$ for some $\alpha = 1, \ldots, m-1$. The order of this component is α. In addition, there are the two unbounded components $(-\infty, \log|a_1|)$ and $(\log|a_m|, +\infty)$, whose orders are 0 and m respectively.

Suppose $\mathcal{A}_f^c$ has a component of order α and let x be a point in that component. Then it follows that

$$\begin{aligned}\Phi_\alpha(f) &= \int_{\mathrm{Log}^{-1}(x)} \log\frac{f(z)}{z^\alpha}\, d\eta(z)\\ &= \sum_{j=1}^{\alpha}\int_{\log|z|=x} \log\frac{z+a_j}{z}\frac{dz}{2\pi i z} + \sum_{j=\alpha+1}^{m}\int_{\log|z|=x}\log(z+a_j)\frac{dz}{2\pi i z}\\ &= \sum_{j=\alpha+1}^{m}\log a_j\\ &= \log(a_{\alpha+1}\ldots a_m).\end{aligned}$$

Note that Φ_α has a branched analytic continuation to all polynomials without multiple roots. The branches of this continuation correspond to various $(m-\alpha)$-element subsets of $\{1, \ldots, m\}$. It is amusing to note that the sum of all branches of $\exp\Phi_\alpha(f)$ is equal to f_α.

Recall that for any Laurent polynomial f we have $N_f(x) \geq \max_\alpha(\operatorname{Re}\Phi_\alpha(f) + \langle\alpha, x\rangle)$ with equality in the closure of $\mathcal{A}_f^c$. If f is a polynomial in one variable, then $\mathcal{A}_f^c$ is dense in $\mathbf{R}$ and so we have

$$N_f(x) = \max_\alpha(\log|a_{\alpha+1}\ldots a_m| + \langle\alpha, x\rangle).$$

This is just a different formulation of the classical Jensen formula

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\, d\theta = \log|f(0)| + \sum_{j=1}^{\alpha}\log\left(\frac{r}{|a_\alpha|}\right)$$

where α is the largest index such that $|a_\alpha| < r$.

Example 2. Let us write up defining equations and inequalities for the set U_α^A in the simplest nontrivial case, namely for quadratic polynomials in one variable. Let $f(z) = f_0 + f_1 z + f_2 z^2$. The zeros of f are

$$\frac{-f_1 \pm \sqrt{f_1^2 - 4f_0 f_2}}{2f_2}$$

and these have the same modulus precisely if f_1 and $i\sqrt{f_1^2 - 4f_0f_2}$ are linearly dependent over $\mathbf{R}$. This happens precisely if f_1^2 and $4f_0f_2 - f_1^2$ are positive multiples of each other, or equivalently

$$\bar{f}_1^2(4f_0f_2 - f_1^2) \geq 0.$$

Hence $f \in U_1^{\{0,1,2\}}$ precisely if this relation is not satisfied.

Example 3. If $A \subset L^*$ is affinely independent, and $f \in \mathbf{C}^A$, then $\mathcal{A}_f = \{x; m_\alpha(f;x) \geq 1, \forall \alpha \in A\}$. When $A = \{0, e_1, \ldots, e_n\}$ where $e_1, \ldots, e_n$ is the standard basis for $\mathbf{Z}^n$, $\mathcal{A}_f$ is called a hyperplane amoeba. When f is a product of such linear factors, $\mathcal{A}_f$ is called an arrangement of hyperplane amoebas. Arrangements of hyperplane amoebas were studied extensively in [8].

Example 4. Next we consider polynomials in two variables of the form $f(z) = a + z_1 + z_2 + z_1 z_2$, assuming to begin with that a is an arbitrary complex constant. It can then be shown that the amoeba of f is the set of points satisfying

$$\begin{aligned} |a|^4 - 2|a|^2 e^{2x_1} + e^{4x_1} - 2|a|^2 e^{2x_2} - (2 - 8\operatorname{Re} a + 2|a|^2) e^{2x_1+2x_2} \\ - 2e^{4x_1+2x_2} + e^{4x_2} - 2e^{2x_1+4x_2} + e^{4x_1+4x_2} \leq 0. \end{aligned} \tag{28}$$

We now specialize to the case where a is real. It turns out that the amoeba looks rather different depending on the sign of a. Consider first the case $a < 0$. The inequality (28) can then be written

$$\begin{aligned} (e^{x_1+x_2} - e^{x_1} - e^{x_2} - |a|)(e^{x_1+x_2} - e^{x_1} + e^{x_2} + |a|) \\ \times (e^{x_1+x_2} + e^{x_1} - e^{x_2} + |a|)(e^{x_1+x_2} + e^{x_1} + e^{x_2} - |a|) \leq 0. \end{aligned}$$

Each of the factors vanishes on the boundary of one of the complement components of the amoeba. Moreover, $\operatorname{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in at most two points for every x so $\mu_f = \pi^{-2}\lambda|_{\mathcal{A}_f}$ and the area of $\mathcal{A}_f$ is π^2 by Theorem 19.

If $a > 0$, there is a similar factorization

$$\begin{aligned} (e^{x_1+x_2} - e^{x_1} - e^{x_2} + |a|)(e^{x_1+x_2} - e^{x_1} + e^{x_2} - |a|) \\ \times (e^{x_1+x_2} + e^{x_1} - e^{x_2} - |a|)(e^{x_1+x_2} + e^{x_1} + e^{x_2} + |a|) \leq 0. \end{aligned}$$

Notice that the fourth factor is always positive, while the first factor vanishes on the boundaries of two complements components (those of orders $(0,0)$ and $(1,1)$ if $a < 1$ and those of orders $(0,1)$ and $(1,0)$ if $a > 1$). The remaining factors define two curves, each of which constitutes part of the boundary of two different complement components. The curves intersect at their common point of inflection $(\log a/2, \log a/2)$. Moreover, $\operatorname{Log}^{-1}(x)$ intersects $f^{-1}(0)$ in

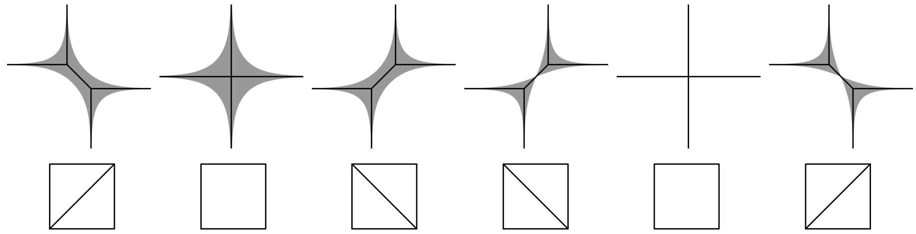

Figure 2: Amoebas of the polynomial $f(z) = a + z_1 + z_2 + z_1 z_2$ for $a = -5, -1, -1/5, 1/5, 1$ and 5 together with their spines and dual subdivisions of the Newton polytope.

at most two points except when $x = (\log a/2, \log a/2)$. For this special x, $\mathrm{Log}^{-1}(x) \cap f^{-1}(0)$ is a real curve. It follows from Theorem 16 and Theorem 18 that the measure μ_f is equal to $\pi^{-2}\lambda|_{\mathcal{A}_f}$ plus a point mass at $(\log a/2, \log a/2)$. The size of the point mass can be computed explicitly by means of Theorem 16, and one finds that it is

$$\mu_a = \frac{16}{\pi^2} \int_0^{\pi/2} \arcsin(a^{\pm 1/2} \cos t)\, dt.$$

where the positive sign is chosen in the exponent if $a < 1$ and the negative sign otherwise. Consequently, the area of the amoeba is $\pi^2(1 - \mu_a)$. The area of teh amoeba can also be computed directly.

If $a = 1$, then $f(z) = (z_1 + 1)(z_2 + 1)$ and the amoeba is the union of two lines.

Example 5. Consider Laurent polynomials in one variable of the form $f(z) = g(z + z^{-1}) - a$, where g is an arbitrary polynomial and a is a constant. We shall determine the set of a for which $\mathcal{A}_f^c$ has a component of order 0. Note that $f(z^{-1}) = f(z)$, hence the amoeba of f is symmetric with respect to reflection in the origin. In particular, if $\mathcal{A}_f^c$ has a component of order 0, then it is mapped onto itself by this reflection. Hence a complement component of order 0, if it exists, must contain the origin. Conversely, if a complement component of order α contains the origin, then reflection in the origin maps it to a complement component of order $-\alpha$. Since these components have nonempty intersection, it follows that $\alpha = 0$. We conclude that $\mathcal{A}_f^c$ has a component of order 0 if and only if $0 \in \mathcal{A}_f^c$. Now, $0 \in \mathcal{A}_f$ means that f has a zero on the unit circle. Since $z \mapsto z + z^{-1}$ maps the unit circle onto the interval $[-2, 2]$, it follows that $\mathcal{A}_f^c$ has a component of order 0 precisely if $a \notin g([-2, 2])$. Keeping g fixed and letting a vary, we see that the set $\{a; E_0(f) \neq \varnothing\}$ may have any (finite) number of connected components. The complement of this set is of course connected, in accordance with Theorem 14.

Example 6. Another class of polynomials on which certain computations can easily be carried out explicitly is polynomials of the form $f(z) = 1 + z_1^{n+1} + \ldots + z_n^{n+1} + az_1 \ldots z_n$ where a is an arbitrary complex constant. The Newton polytope of f is a simplex with precisely one lattice point, namely $(1, \ldots, 1)$, in its interior. It follows from Theorem 12 or Theorem 11 that the only lattice points in P_f which can occur as orders of complement components, are the vertices and the interior point $(1, \dots, 1)$. Now, if α is a vertex of P_f there is always a complement component of order α. We shall compute the set of a for which $\mathcal{A}_f^c$ has a component of order $(1, \dots, 1)$.

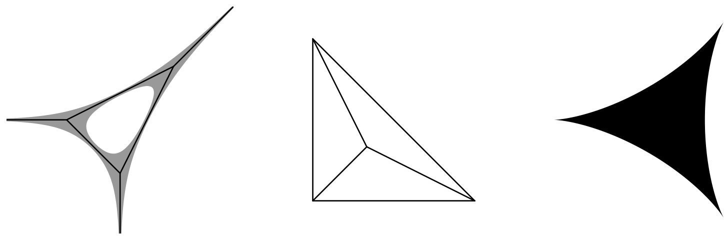

Figure 3: Amoeba and triangulated Newton polytope of the polynomial $f(z) = 1 + z_1^3 + z_2^3 + az_1z_2$ for $a = -6$ and the set of a for which $E_{(1,1)}(f)$ is empty.

For reasons of symmetry, as in the previous example, it can be shown that a component of order $(1, \dots, 1)$ must necessarily contain the origin, and conversely, if a complement component contains the origin, then its order must be $(1, \dots, 1)$. It is also clear that $0 \in \mathcal{A}_f$ precisely if $-a$ belongs to the set $K_n = \{t_0 + \ldots + t_n; |t_0| = \ldots = |t_n| = t_0 \ldots t_n = 1\} \subset \mathbf{C}$. This set is contained in the closed disc of radius $n+1$ and contains the disc of radius $n-1$ centered at the origin. The boundary of K_n has $n+1$ cusps; the corresponding values of a give rise to polynomials defining singular hypersurfaces, and are branching points for $\Phi_{(1,\dots,1)}(f)$. Finally, we note that the power series expansion of $\Phi_{(1,\dots,1)}(f)$ computed in Theorem 6 takes the simple form

$$\Phi_{(1,\dots,1)}(f) = \log a - \sum_{k \geq 1} \frac{((n+1)k-1)!}{(k!)^{n+1}}(-a)^{(n+1)k}.$$

Example 7. If $f(z) = \sum_{\alpha \in A} f_\alpha z^\alpha$ is a Laurent polynomial where A has no more than $2n$ elements, and these are in sufficiently general position, then by the proof of Theorem 12, the order of a complement component of $\mathcal{A}_f$ is determined by the dominating term in f. The assumption about general position implies in particular, that no three points in A are collinear. The following example shows that the situation becomes quite different if A is allowed to have three collinear points.

Let f be a Laurent polynomial of the form $f(z) = a_0 + \sum_{\alpha \in A} a_\alpha(z^\alpha - z^{-\alpha})$. Here we assume that A is a finite set not containing the origin, and that all the coefficients a_α except a_0 are real. For reasons of symmetry, a complement component of $\mathcal{A}_f$ containing the origin must have order 0, and conversely, a component of order 0 must contain the origin. Now, if $z \in \mathrm{Log}^{-1}(0)$, then $z^\alpha + z^{-\alpha}$ is real for any α. It follows that $\mathcal{A}_f^c$ has a complement component of order 0 as soon as a_0 is not real, no matter how small it is.

10 Some open problems

Here are a few seemingly interesting and open problems related to the subject of this thesis. After each problem we give some comments including elementary observations, known results of a similar nature and some guesses about solutions.

Problem 1. *Let $A \subset \mathbf{Z}^n$ be a finite set and let $\alpha \in \mathbf{Z}^n$ be a point. Find a necessary and sufficient condition for the existence of a Laurent polynomial $f \in \mathbf{C}^A$ with $E_\alpha(f) \neq \varnothing$.*

Theorem 11 gives one necessary condition and one sufficient condition. However, the gap between the two conditions is usually very large. For certain simple sets A the complete answer is given by Theorem 12. A rather wild guess would be that the second condition in Theorem 11 is also a necessary condition.

Problem 2. *Given an integer polytope $P \subset \mathbf{Z}^2$, what is the minimal area of the amoeba $\mathcal{A}_f$ given that $P_f = P$?*

This problem was suggested by Oleg Viro and communicated to the author by Grigory Mikhalkin. If P is a zonotope, that is the Minkowski sum of line segments, then f can be taken to be a product of binomials and the area of the amoeba is zero in this case. If P is a triangle whose area is $1/2$, the area of $\mathcal{A}_f$ will always be $\pi^2/2$, whereas if P is an arbitrary triangle and the only nonzero coefficients in f are those corresponding to the vertices of P, then $\operatorname{Area}(\mathcal{A}_f) = \pi^2/(4\operatorname{Area}(P))$. In view of this it seems reasonable to conjecture that $\operatorname{Area}(\mathcal{A}_f) \geq c/\operatorname{Area}(P_f)$ for some constant $c > 0$ unless P_f is a zonotope. The constant c can be no greater than $\pi^2/4$. However, this estimate is probably not sharp for most polytopes P.

Problem 3. *Let $f_0, \ldots, f_m$ be Laurent polynomials. Classify all convergent fractional Laurent series g satisfying the equation $f_0 + f_1 g + \ldots + f_m g^m = 0$.*

A fractional Laurent series is an infinite linear combination of fractional Laurent monomials z^α, where $\alpha \in L_1^*$ and $L_1 \subset L$ is a sublattice of the same rank as L.

If $m = 1$ and $f_0 = -1$, Theorem 4 associates every such g with a connected component of $\mathcal{A}_{f_1}^c$, and each such component is associated with a lattice point in P_{f_1}. Hence there is a bijective correspondence between the convergent Laurent series g and a subset of $L^* \cap P_{f_1}$.

As demonstrated in the introduction, the Laurent series associated with a vertex of P_{f_1} are most easily computed. The analogous series g for the equation $f_0 + f_1 g + \ldots + f_m g^m = 0$ were classified by McDonald in [17]. There it was shown that they correspond in a natural way to certain edges of $P_f \subset L_R^* \times \mathbf{R}$, where $f(z,t) = f_0(z) + f_1(z)t + \ldots + f_m(z)t^m$. It would be nice to find a similar generalization of the non-vertex lattice points.

11 List of notations

Symbol	Explanation	Defined on page
aff A	Affine lattice generatd by A	
$\mathcal{A}_f$	Amoeba of f	8
$\mathcal{A}_f^c$	Complement of amoeba	8
Area	Area in $\mathbf{R}^2$	
$C^\vee$	Dual of cone C	10, 19
$\mathbf{C}_*$	$\mathbf{C} \smallsetminus \{0\}$	
$\mathbf{C}^A$	The space of Laurent polynomials $\sum_{\alpha \in A} f_\alpha z^\alpha$	10
cone(F, P)	Cone of vectors from F into P	10, 19
conv	Convex hull	
d^c	$(\partial - \bar{\partial})/2\pi i$	28
E_α	Complement component of order α	12
η	Haar measure	7
Hess	Hessian matrix	26
int	Interior of a set	
λ	Lebesgue measure	
Log		7
M	Monge-Ampère operator	27
$\widetilde{\mathrm{M}}$	Mixed Monge-Ampère operator	28
$m_\alpha(f)$	$\inf_x\, m_\alpha(f;x)$	12
$m_\alpha(f;x)$	$\sum_{\beta \neq \alpha} \lvert f_\beta / f_\alpha \rvert \exp\langle \beta - \alpha, x\rangle$	12
μ_f	$\mathrm{M}\, N_f$	30
$\mu_{f_1,\dots,f_n}$	$\widetilde{\mathrm{M}}(N_{f_1}, \dots, N_{f_n})$	30
N_f	Ronkin function of f	8
nc(F, P)	Normal cone to P at F	10
P_f	Newton polytope of f	8
Φ_α		15
ρ_k, ρ^k		8
relint	Relative interior of a convex set	
$\mathcal{S}_f$	Spine of $\mathcal{A}_f$	20
$\mathbf{T}$	Unit circle in $\mathbf{C}$	
$\mathbf{T}^n$	$\mathbf{T} \times \ldots \times \mathbf{T}$ (n factors)	
U_α^A	$\{f \in \mathbf{C}^A; E_\alpha(f) \neq \varnothing\}$	22
vert P	Set of vertices of P	
Vol	Volume in $\mathbf{R}^n$	
X_P, X_Σ	Toric varieties	10

References

[1] A.D. Aleksandrov: Zur Theorie von konvexen Körpern IV: Die gemischten Diskriminanten und die gemischten Volumina, *Matem. Sb. SSSR* **3** (1938), 227 - 251.

[2] Eric Bedford, Alan Taylor: The Dirichlet problem for a complex Monge-Ampère equation, *Invent. Math.* **37** (1976), 1 - 44.

[3] David Bernstein: The number of roots of a system of equations, *Functional Anal. Appl.* **9** (1975), 183 - 185.

[4] V. I. Danilov: The geometry of toric varieties, *Russian Math. Surveys*, **33**:2 (1978), 97 - 154.

[5] Johannes Duistermaat, Wilberd van der Kallen: Constant terms in powers of a Laurent polynomial, *Indag. Math.* **9** (1998), 221 - 231.

[6] Günther Ewald: *Combinatorial convexity and algebraic geometry*, Springer-Verlag, New York, 1996.

[7] Mikael Forsberg: *Amoebas and Laurent series*, Doctoral thesis, KTH Stockholm, 1998.

[8] Mikael Forsberg, Mikael Passare, August Tsikh: Laurent determinants and arrangements of hyperplane amoebas, *Adv. in Math.* **151** (2000), 45 - 70.

[9] William Fulton, Bernd Sturmfels: Intersection theory on toric varieties, *Topology* **36** no. 2 (1997), 335 - 353.

[10] Israel Gelfand, Mikhail Kapranov, Andrei Zelevinsky: Generalized Euler integrals and A-hypergeometric functions, *Adv. in Math.* **84** (1990), 255 - 271.

[11] Israel Gelfand, Mikhail Kapranov, Andrei Zelevinsky: *Discriminants, resultants and multidimensional determinants,* Birkhäuser, Boston, 1994.

[12] Axel Harnack, *Über Vieltheiligkeit der ebenen algebraischen Curven,* Math. Ann. **10** (1876), 189-199.

[13] Lars Hörmander: *Notions of convexity,* Birkhäuser, Boston, 1994.

[14] Ilia Itenberg, Oleg Viro: Patchworking algebraic curves disproves the Ragsdale conjecture, *Math. Intelligencer* **18** (1996), no. 4, 19 - 28.

[15] A. G. Khovanskii: Newton polyhedra and the genus of complete intersections, *Functional Anal. Appl.* **12** (1978), 38 - 46.

[16] Christer Kiselman: The partial Legendre transformation for plurisubharmonic functions, *Invent. Math.* **49** (1978) 137-148.

[17] John McDonald: Fiber polytopes and fractional power series, *J. Pure Appl. Algebra*, **104** (1995), 213 - 233.

[18] Peter McMullen: The polytope algebra, *Adv. in Math.*, **78** (1989), 76 - 130.

[19] Peter McMullen: Separation in the polytope algebra, *Beiträge Algebra Geometrie*, **34** (1993), 15 - 30.

[20] Grigory Mikhalkin: Real algebraic curves, moment map and amoebas, *Ann. of Math.*, **151** (2000),no. 1, 309 - 326.

[21] Grigory Mikhalkin, Hans Rullgård: Amoebas of maximal area, Preprint: http://xxx.lanl.gov/abs/math.CV/0010087 (To appear in *International Mathematical Research Notices*).

[22] Mikael Passare, Hans Rullgård, Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope. Preprint: http://www.matematik.su.se/reports/2000/10

[23] Alexander Rashkovskii: Indicators for plurisubharmonic functions of logarithmic growth. Preprint: http://xxx.lanl.gov/abs/math/9911240

[24] Jeffrey Rauch, Alan Taylor: The Dirichlet problem for the multidimensional Monge-Ampère equation, *Rocky Mountain J. Math.* **7** (1977), 345 - 364.

[25] Lev Ronkin: *Introduction to the theory of entire functions of several variables,* Translations of mathematical monographs, AMS, Providence, 1974.

[26] Lev Ronkin: *On zeros of almost periodic functions generated by holomorphic functions in a multicircular domain*, To appear in "Complex Analysis in Modern Mathematics", Fazis, Moscow, 2000, pp. 243-256.

[27] Hans Rullgård: Stratification des espaces de polynômes de Laurent et structure de leurs amibes, *C. R. Acad. Sci.* 2000, 355 - 358.

[28] Timur Sadykov: Hypergeometric systems of differential equations and amoebas of rational functions, Preprint: Potsdam 1999.

[29] Oleg Viro: Real algebraic curves: Constructions with controlled topology, *Leningrad Math. J.*, **1** (1990), 1059 - 1134.