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Acknowledgements

This is my thesis for the licentiate degree. It contains four unrelated papers, even though references have for technical reasons been gathered at the end. I would like to thank my teacher Torsten Ekedahl warmly.

Rational Curves on $K3$ Surfaces in Positive Characteristic

To study isolated (smooth) rational curves on $K3$-fibered Calabi-Yau threefolds, as in [3], a major task is to construct a polarized family of $K3$-surfaces, such that the special fiber contains a rational curve C of given degree. This is may be reduced to finding a single $K3$-surface Y that should act as the special fiber. I would like to thank Torsten Ekedahl for suggesting this field of research to me.

We will assume that the base field has positive characteristic. In characteristic zero existence of the desired Y was proved by Oguiso [8], following earlier work of Mori [4]. In this situation one can make use of global period theory and the surjectivity of the period mapping.

Let two positive integers n and d be given. We are supposed to find a $K3$ surface Y, such that NS_Y contains a class H with $H^2 = 2n$, and another class C with $C^2 = -2$ and $C \cdot H = d$. It is furthermore required that H be very ample, and that C should be a smooth rational curve. We proceed by constructing a $K3$-surface X (in characteristic p) with 20-dimensional Néron-Severi group. We embed the rank-two lattice S given by H and C into NS_X, using the work of Nikulin. Thereafter the plethora of curves on X turns hindersome, and we deform it to get Y, such that $NS_Y = S$. On Y then, H will be very ample and C will be smooth. To see that such a deformation of X exists, we must use some variant of local period theory in characteristic p.

As the integral lattice S is of rank two, its discriminant group is generated by two elements (or fewer). We take X as the Kummer surface of $A = E \times E$, where E is an ordinary elliptic curve with complex multiplication. Then NS_X has rank 20. Since X is ordinary, it can be lifted to characteristic zero. Denote the lifting by X'. Then by construction $NS_{X'}$ also has rank 20, and we have a mapping $\varphi : NS_{X'} \to NS_X$. By a reasoning essentially contained in [10], the discriminant group of $NS_{X'}$ is generated by two elements (or fewer). To see this, just note that the transcendental lattice is of rank two. As φ clearly is injective, D_X is a quotient of $D_{X'}$, so it can also be generated by two elements.

Proposition 1 *The given lattice S can be primitively embedded into NS_X.*

Proof: Let an even lattice S be given, and let (s^+, s^-) be its signature, D_S its discriminant group, q_S its discriminant quadratic form and ℓ_S the number of generators of D_S. Introduce analogous notation for M, another even lattice into which we want to embed S primitively. In our case $(s^+, s^-) = (1, 1)$, $\ell_S = 2$, $(m^+, m^-) = (1, 19)$ and $\ell_M \leq 2$. Then there exists an even lattice K with $q_K = -q_S$ and signature (k^+, k^-), where in our case we can take $k^+ = 0$ and $k^- = 8$. Namely, we must have $k^+ - k^- \equiv s^- - s^+ \mod 8$ and $k^+ + k^- > \ell_S$ [6, 1.10.2]. Then put $n^+ = m^+ - k^+ - s^+$ and $n^- = m^- - k^- - s^-$, so that $n^+ - n^- \equiv m^+ - m^- \mod 8$. In our case $n^+ = 1 - 0 - 1 = 0$ and $n^- = 19 - 8 - 1 = 10$. Then by the existence theorem again, there is an even lattice N with $q_N = q_M$ and having signature (n^+, n^-). This holds true, since $n^+ + n^- > \ell_M$.

Now consider the lattice $T = N \times K \times S$. There is an isotropic subgroup H of D_T, given by the graph of the isomorphism $\gamma : D_S \to D_K \subset D_{N\times K}$. This determines, as in [6, 1.5.1], a bigger lattice M' having $q_{M'} = q_N$. Namely, $D_{M'}$ is isometric to the orthogonal complement $H^\perp_{D_T}$ modulo H. But using the fact that S and K are non-degenerate, we see that $H^\perp_{D_T} \cap D_{S\times K}$ is simply H. Therefore $H^\perp_{D_T} = H \times D_N$. The lattice M' that we thus have found is now guaranteed to be isomorphic to the sought-for M, since by [6, 1.13.3] an even non-degenerate indefinite lattice L is uniquely determined by q_L and (l^+, l^-), provided $l^+ + l^- \geq \ell_L + 2$. □

To understand the local moduli space, we use the "period matrices"that arise from the enlarged formal Brauer group of [1]. The relation to moduli of ordinary $K3$ surfaces was explored in [7]. The reason for not using the crystalline theory of [2] is that it uses the p-adic exponential, so it doesn't work for $p = 2$. Let X/A be a lifting to an Artin algebra. The enlarged formal Brauer group $\psi_{X/A}$ is an extension

$$0 \to \mathrm{Br}^\wedge_{X/A} \to \psi_{X/A} \to H^2_{fl}(X/k, \mu_{p^\infty}) \to 0.$$

It is known that such extensions are uniquely given by elements in

$$\mathrm{Hom}_{\mathbf{Z}_p}(H^2_{fl}(X/k, \mathbf{Z}_p(1)), \mathrm{Br}^\wedge_{X/A}(A)).$$

This morphism is constructed as a limit of certain morphisms

$$\text{“}p^r\text{”} : H^2_{fl}(X/k, \mu_{p^r}) \to \mathrm{Br}^\wedge_{X/A}(A).$$

On the other hand, consider the long exact cohomology sequence of

$$0 \to 1 + \mathbf{m}\mathcal{O}_{X/A} \to \mathcal{O}^*_{X/A}/\mathcal{O}^{*p^r}_{X/A} \to \mathcal{O}^*_{X/k}/\mathcal{O}^{*p^r}_{X/k} \to 0.$$

We get a connecting homomorphism

$$\beta_r : H^1(X/k, \mathcal{O}^*_{X/k}/\mathcal{O}^{*p^r}_{X/k}) \to H^2(X/k, 1 + \mathbf{m}\mathcal{O}_{X/A}).$$

The source and target both have other interpretations; we may write

$$\beta_r : H^2_{fl}(X/k, \mu_{p^r}) \to \mathrm{Br}^\wedge_{X/A}(A).$$

By a proposition in [7], we know that "p^r" and β_r are the same maps. Passing to the limit, an invertible sheaf lifts to A exactly when it maps to zero in $\mathrm{Br}^\wedge_{X/A}(A)$, and the locus in the 20 dimensional local moduli space M where it lifts is defined by one non-trivial equation. Then it easily follows that a generic deformation preserving algebraicity of our rank two lattice S will destroy algebraicity of the rest of NS_X. After algebraizing, a typical member Y has $NS_Y = S$.

In fact, identifying M with

$$\mathrm{Hom}_{\mathbf{Z}_p}(H^2_{fl}(X/k, \mathbf{Z}_p(1)), \mathrm{Br}^\wedge_{X/A}(A)),$$

we see that it is a formal torus, and the lifting loci V_L of line bundles L are kernels of characters. So the rank-two lattice S lifts to an 18-dimensional subvariety V. We need to construct a curve in V that is contained in none of the unwanted V_L's. This can be done by induction over NS_X, using the fact that intersections of various V_L's have expected dimension as long as the corresponding line bundles are linearly independent in NS_X.

Proposition 2 *Given positive integers n and d, there exists a $K3$ surface Y over k, such that NS_Y has rank two with intersection form given by*

$$\begin{pmatrix} 2n & d \\ d & -2 \end{pmatrix}.$$

in a basis $\{H, C\}$, where H is very ample and C is a smooth rational curve.

Proof: There remains to prove very ampleness and smoothness. Using the smallness of NS_Y, this follows by arguments contained in [5] and [8]. More precisely, smoothness of C follows by a straightforward lattice theoretic calculation, and the same holds for the fact that H doesn't contract (-2)-curves. Saint-Donat gives (for any p) numerical criteria concerning irreducibilty and freeness from fixed components. These criteria are checked in [5]. Then Saint-Donat proves freeness from fixed points, again for any p. It follows from the theory of varieties of small degree that φ_C must be of degree 1 or 2. Saint-Donat gives criteria for birationality if $p > 2$, and these criteria are checked in [5]. In our case we can exclude a degree two map if $p = 2$, as this would imply unirationality of Y (but Y is an *ordinary* $K3$ surface). Finally, Saint-Donat's argument to show separation of tangent vectors outside (-2)-curves for a birational mapping does not depend on p. $\square$

Equations for some Enriques Surfaces in Characteristic Two

This paper is concerned with the theory of Enriques surfaces in characteristic 2. There are three types: $\boldsymbol{\mu}_2$, $\mathbf{Z}/2$ and $\boldsymbol{\alpha}_2$. The names refer to the structure of their respective Picard schemes. Roughly, $\boldsymbol{\alpha}_2$ surfaces are the most special, and the other two types can degenerate into these.

It is known [13] that an Enriques surface X of type $\mathbf{Z}/2$ contains a regular vector field exactly when it satisfies one of the following (mutually exclusive) conditions:

a) X has a genus one fibering with a double fiber of type $\widetilde{E}_8$. This fibering is then quasi-elliptic.

b) X has a genus one fibering with a double fiber of type $\widetilde{E}_7$. This fibering is then quasi-elliptic.

c) X has a quasi-elliptic fibering with a simple fiber of type $\widetilde{E}_7$; it then also has an elliptic fibering with a nodal double section and a double fiber of $\widetilde{E}_6$ type.

These cases are referred to as types $\widetilde{E}_8$, $\widetilde{E}_7$ and $\widetilde{E}_6$, respectively. By explicit calculation, I show that these surfaces depend on one, two and three moduli, respectively, and we will also see that the implication contained in (c) can be reversed. So we will study genus one fibrations of Enriques surfaces X, such that there is a double section R with self intersection -2, and such that there is a double fiber of type $\widetilde{E}_n$ (or a simple fiber of type $\widetilde{E}_7$). The linear system R represents X as a double cover X' of a ruled surface, where X' is X twice blown up and with the double fiber contracted to a non-rational singularity. We will also investigate surfaces of type $\boldsymbol{\alpha}_2$ while we're at it. Among other things we will get an explicit equation for "the most exceptional Enriques surface"; it is unique, and it looks like this: $z^2 + x^3 + x^8 t + x t^4 = 0$. I would like thank Torsten Ekedahl for suggesting this project to me.

The Equation of a $\mathbf{Z}/2$ Surface with a Nodal Double Section

Let $\pi : X \to \mathbf{P}^1$ be a genus one fibered $\mathbf{Z}/2$ surface, having a given nodal double section R. After blowing up base points to obtain X'', there is a map $\psi : X'' \to Y := \mathbf{P}(\pi_* \mathcal{O}_{X''}(R))$. This map factors through a finite double covering $\varphi : X' \to Y$.

$$\begin{tikzcd}
X' \arrow[dr, "\varphi"] & X'' \arrow[l] \arrow[r] \arrow[d, "\psi"] & X \arrow[d, "\pi"] \\
 & Y \arrow[r] & \mathbf{P}^1
\end{tikzcd}$$

Proposition 3 *The relative system $|R|$ has exactly two base points, occurring at the intersections between R and the two double fibers.*

Proof: The system $|R+nF|$ (F a fiber) is isomorfic locally on $\mathbf{P}^1$, and as $R+nF$ has self-intersection $\gg 0$, Theorem 4.4.1 (p.240) of [12] applies. $\square$

Proposition 4 *Y is isomorphic to the rational ruled surface $\mathbf{F}_2$.*

Proof: After two blowing ups, the self intersection of R is -4. Also, the restriction of φ to R must be a map of degree two, since otherwise π would have R as a section. Therefore R is the pullback of $\varphi(R)$, so we may conclude that $\varphi(R)$ has self intersection -2 (intersection-numbers get doubled on a double covering). $\square$

Write the double covering as $z^2 + zg + f$ with $g \in L$ and $f \in L^2$, where L is a line bundle on Y. Introduce on Y the coordinates x, y, s, t, where x, y are pulled back from $\mathbf{P}^1$ and s, t are vertical. Let $s = 0$ define $\varphi(R)$. Curves on Y are then defined by polynomials that are homogeneous in s, t as well as in t, x, y, where t now has weight 2. Denote a curve that is cut out by a polynomial p by C_p.

Proposition 5 $L \cong \mathcal{O}_Y(2C_t + C_x)$.

Proof: Because of the blow ups $\omega_{X'}$ equals $\mathcal{O}_{X'}(F)$. As $\omega_Y = \mathcal{O}_Y(-2C_s - 4C_x) = \mathcal{O}_Y(-2C_t)$ and $\omega_{X'} = \varphi^*(\omega_Y \otimes L)$, we must have $L \cong \mathcal{O}_Y(2C_t + C_x)$. $\square$

We may assume that f is square free. Namely, put $z \mapsto z + h$, where $h \in L$ is such that the square parts of $h^2 + gh$ and f are equal. Such a polynomial can be found, according to the the following lemma, that I learned from Torsten Ekedahl:

Lemma 6 *Let R_μ be a homogeneous component of a graded polynomial ring (in characteristic p). Let $g \in R_{(p-1)\mu}$ and $u \in R_\mu$. There exists a (non-unique) polynomial $h \in R_\mu$, such that the p'th power part of $h^p + gh$ is u^p.*

Proof: The proof is non-constructive. We will need to consider R_μ as an algebraic group, rather than as a vector space. Define a group scheme endomorphism τ by first sending an $h \in R_\mu$ to the p'th power part of $h^p + gh$ and then dividing all exponents in the resulting polynomial by p. Suppose $\ker(\tau)$ is *not* a finite group scheme. Then there exists [14, sect. 20] a non-constant homomorphism $\sigma\colon \mathbf{G}_a \to R_\mu$, such that composing with τ kills it. Explicitly σ is given by a vector of additive polynomials:

$$x \mapsto (a_{10}x + a_{11}x^p + a_{12}x^{p^2} + \cdots, a_{20}x + \cdots, \ldots).$$

But since τ is the sum of the Frobenius map and a linear transformation, looking at the terms of highest degree that occur in the expression for σ, it is impossible that the compostion $\tau\sigma$ is the zero map. So $\ker(\tau)$ is finite, showing that τ is surjective, which was to be proved. $\square$

As we have determined how sections in L look like, we now know that the covering is given by

$$\begin{gathered} z^2 + z(B_5s^2 + B_3st + B_1t^2) + \\ A_{10}s^4 + A_8s^3t + A_6s^2t^2 + A_4st^3 + A_2t^4 = 0, \end{gathered}$$

where the A_i's and B_i's are forms in x, y of degree i.

When blowing up a rational surface singularity, embedded in three-space, one obtains a conic C in the exceptional plane. If it is reduced, then the singularity was of A_n type. To get the other types, the quadratic piece of the equation that defines the singularity must therefore be a square, say z^2. Next, Lipman [15] considers the cubic piece (modulo z). If the three linear factors are separate, then we have a D_4. If it two of them come together, we get higher D_n types. If all three coincide, then we get some E_n. In this case write the cubic piece as y^3, and introduce a third coordinate x. One proceeds by obtaining conditions on the coeffcients in the original equation, using these coordinates, for the singularity to be of a particular E_n type. The relevant coefficients are first β and γ, in fron of zx^2 and x^4. If these coefficients are general, then we get E_6. If not, we consider ρ and σ, the coefficients of x^3y and x^5. The nature of the first distinguish between E_7 and E_8, whereas the latter tells E_8 apart from non-rational singularities.

Proposition 7 *Any $\mathbf{Z}/2$ surface with a nodal double section can be (birationally) written as*

$$\begin{gathered} z^2 + z\Big((b_1y + b_2x)x^2y^2s^2 + (b_3y + b_4x)xyst\Big)+ \\ (y^4 + x^4)x^3y^3s^4 + (a_1y^2 + a_2xy + a_3x^2)x^3y^3s^3t+ \\ a_4x^3y^3s^2t^2 + a_5x^2y^2st^3 + xyt^4 = 0, \end{gathered}$$

and conversely, these equations define such surfaces (or possibly some non-normal surface).

Proof: Assume first X' is of the given type. Since we are not going to study degenerations into $\boldsymbol{\alpha}_2$ surfaces, we may put the two non-rational singularities at $x = t = 0$ and $y = t = 0$. To get double fibers, x and y must divide f and g. Put $y, s = 1$ to study the non-rational singularity at $x, t = 0$. The equation now looks like

$$\begin{gathered} z^2 + z\Big((\beta_1x + b_1x^2 + b_2x^3 + \beta_2x^4)s^2 + (b_3x + b_4x^2)t\Big) \\ \alpha_1x + \alpha_2x^3 + \alpha_3x^5 + \alpha_4x^7 + \alpha_5x^9+ \\ (\alpha_6x + \alpha_7x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \alpha_8x^6 + \alpha_9x^7)t+ \\ (\alpha_{10}x + a_4x^3 + \alpha_{11}x^5)t^2+ \\ (\alpha_{12}x + a_5x^2 + \alpha_{13}x^3)t^3 + \alpha_{14}xt^4 = 0. \end{gathered}$$

To have a singularity at $x = t = 0$, we must have $\alpha_1 = 0$. To get a non-rational singularity, by Lipman [15] the quadratic part of the polynomial must be a square. Therefore $\beta_1 = \alpha_6 = 0$. The cubic part of f must be a cube, giving $\alpha_7 = \alpha_{10} = 0$. To have X' smooth over C_x, we must have $\alpha_{12} = 0$ and $\alpha_{14} \neq 0$, so put $\alpha_{14} = 1$. If $\alpha_2 = 0$ the genus drops at the first blowing up, and there is then a further rational singularity over C_x. This contradicts Proposition 3, saying that there should be only *one* vertical curve on X'' connecting R to the proper transform of the double fiber. So $\alpha_2 \neq 0$, and we may put it equal to 1 by using the third automorphism of $\mathbf{P}^1$ or $t \mapsto \lambda t$. After repeating everything at the other point, we get the desired polynomial by putting $\alpha_3 = 0$. This is possible by using $t \mapsto t + \lambda sxy$. Doing this, a square is re-introduced into f, namely a linear combination of $x^4y^6s^4$ and $x^6y^4s^4$. This can however easily be removed by $z \mapsto z + \mu_1x^2y^3s^2 + \mu_2x^3y^2s^2$.

Then to check the converse statement, let an equation as above be given. Denote the double cover again by X', and the smooth model by X. To see that it is an Enriques surface (granted X' is normal) we calculate $\chi(X)$ and ω_X. Putting $L = \mathcal{O}_Y(C)$ we get $\chi(X') = \chi(L^{-1}) + \chi(\mathbf{F}_2) = \chi(\mathbf{F}_2) - \chi(C) + \chi(\mathbf{F}_2) = 2\chi(\mathbf{F}_2) - (-C^2 - C \cdot K_Y)/2 = 2 \cdot 1 - (-12 + 10)/2 = 3$. Furthermore, $\omega_{X'} = \varphi^*(\omega_Y \otimes L) = \mathcal{O}_{X'}(F)$. There are two non-rational singularities to resolve, making $\chi(X) \leq 1$. But $\chi(X)$ cannot be *strictly* less than one. Namely, $h^{01}(X) = 0$ since X is nontrivial as a genus one bundle. Therefore the arithmetic genus drops exactly twice during resolution, and we get $\chi(X) = 1$. We also see that exactly minus two half fibers are added to the canonical class, as should be the case. $\square$

The Equation of a $\mathbf{Z}/2$ Surface with an $\widetilde{E}_n + R$ Configuration

We will determine conditions on the parameters, under which the double fiber at $x = 0$ is of $\widetilde{E}_n$ type. Put $s = y = 1$. Blow up at $z = x = t = 0$ by putting $x = tX$, $z = tz'$ and dividing through by t^2, to get

$$z'^2 + z'Xt\Big((b_2X + b_4)Xt + b_1X + b_3\Big) +$$
$$Xt\Big(X^6t^4 + a_3X^4t^3 + (a_4X^2 + a_2X^3 + a_5X + 1)t^2 + a_1X^2t + X^2\Big) = 0.$$

Blow up again at the origin by putting $t = XY$, $z' = XZ$, dividing through by X^2, and finally normalizing by $Z/X \mapsto Z$ to get

$$Z^2 + ZY\Big(b_2X^3Y + b_4X^2Y + b_1X + b_3\Big) +$$
$$Y\Big(X^8Y^4 + a_3X^5Y^3 + a_2X^3Y^2 + a_4X^2Y^2 + a_5XY^2 + a_1XY + Y^2 + 1\Big) = 0.$$

Resolution of rational surface singularities on double coverings is described in the very readable [11, III.7]. We resolve the singularities of the branch locus downstairs, and if the singularity is rational, the multiplicity of the exceptional curve will be 2 or 3. If it is more, we will need to normalize the blown up upper surface. The arithmetic genus of the normalization will be strictly lower in general. The exact amount that the genus can be analysed in terms of resolution data, but we will not need this.

For a general choice of parameters the inverse image of the conductor is an elliptic curve. This, then, is the double fiber. It is obtained by putting $X = 0$. The original curve over C_x, as well as the first exceptional curve, can be blown down. We proceed to study the locus in the parameter space where the double fiber is *not* an elliptic curve, so we need to specialize the parameters further.

We see that the only possible place where the elliptic curve can degenerate (as we vary the parameters) is at $Y = 1$. Put this point over the origin in the X, Y-plane by $Y \mapsto Y + 1$, and thereafter change $Z \mapsto Z + \sqrt{a_4}X + Y$:

$$\begin{gathered}
Z^2 + Z\Big(b_3 + b_3Y + b_1X + b_1XY + b_4X^2 + b_2X^3 + b_4X^2Y^2 + b_2X^3Y^2\Big) + \\
b_3Y + (a_1 + a_5 + \sqrt{a_4}b_3)X + (a_5 + \sqrt{a_4}b_3 + b_1)XY + \sqrt{a_4}b_1X^2 + b_3Y^2 + \\
Y^3 + (b_1 + a_1 + a_5)XY^2 + (b_4 + \sqrt{a_4}b_1 + a_4)X^2Y + (a_2 + \sqrt{a_4}b_4)X^3 + \\
a_5XY^3 + a_4X^2Y^2 + (a_2 + b_2)X^3Y + \sqrt{a_4}b_2X^4 + \\
(b_4 + a_4)X^2Y^3 + (a_2 + \sqrt{a_4}b_4)X^3Y^2 + a_3X^5 + \\
(b_2 + a_2)X^3Y^3 + \sqrt{a_4}b_2X^4Y^2 + \\
X^8 + a_3X^5Y^4 + X^8Y + X^8Y^4 + X^8Y^5 = 0.
\end{gathered}$$

Put $b_3 = a_1 + a_5 = 0$ to get a singularity at all. To have a singularity of type D_n or E_n, the quadratic piece of the equation must be a square. Therefore $b_1 = a_5 = 0$. Writing the equation as $Z^2 + Zp_1(X,Y) + p_2(X,Y)$, we must have the cubic part of p_2 a cube to get an E_n type. Therefore $a_2 + \sqrt{a_4} b_4 = b_4 + a_4 = 0$. Summing up, we have put $a_1, a_5, b_1, b_3 = 0$, $b_4 = a_4 = \alpha^2$ and $a_2 = \alpha^3$, with α as a new parameter. We get then:

$$
\begin{gathered}
Z^2 + Z\Big(\alpha^2 X^2 + b_2 X^3 + \alpha^2 X^2 Y^2 + b_2 X^3 Y^2\Big) + \\
Y^3 + \\
\alpha^2 X^2 Y^2 + (\alpha^3 + b_2) X^3 Y + \alpha b_2 X^4 + \\
a_3 X^5 + \\
(b_2 + \alpha^3) X^3 Y^3 + \alpha b_2 X^4 Y^2 + \\
X^8 + a_3 X^5 Y^4 + X^8 Y + X^8 Y^4 + X^8 Y^5 = 0.
\end{gathered}
$$

Proposition 8 *Any* $\mathbf{Z}/2$ *surface with an* $\widetilde{E}_n + R$ *configuration can be (birationally) written as*

$$
\begin{gathered}
z^2 + z(b_2 x^3 y^2 s^2 + \alpha^2 x^2 y s t) + \\
(y^4 + x^4) x^3 y^y s^4 + (\alpha^3 xy + a_3 x^2) x^3 y^3 s^3 t + \alpha^2 x^3 y^3 s^2 t^2 + x y t^4 = 0,
\end{gathered}
$$

with $\alpha \neq 0$ *or* $\alpha = 0, b_2 \neq 0$ *or* $\alpha = b_2 = 0, a_3 \neq 0$ *according as* $n = 6, 7, 8$. *Conversely, these equations define such surfaces.*

Proof: As in [15], let β, γ denote the coefficients of ZX^2 and X^4, so $\beta = \alpha^2$, $\gamma = \alpha b_2$. For general parameters we have E_6; to get E_7, put $\alpha = 0$, thus making $\beta = \gamma = 0$. Then let ρ, σ denote the coefficients of X^3Y and X^5, so with $\alpha = 0$ we have $\rho = b_2$ and $\sigma = a_3$. Put $b_2 = 0$ to get an E_8. Finally, the singularity is rational as long as $\sigma = a_3 \neq 0$. □

Proposition 9 *With notations as above, a* $\mathbf{Z}/2$ *fibering (with an* $\widetilde{E}_n$ *double fiber and a nodal double section) is quasi-elliptic exactly when* $\alpha = 0$.

Proof: Put $t, y = 1$ in our equation, and differentiate with respect to z and s. We get $D_z = x^2 s(b_2 x + \alpha^2 s)$ and $D_s = x^2(\alpha^2 z + \alpha^3 x^4 s^2 + a_3 x^5 s^2)$. To get a curve of curve singularities, these polynomials must have a common factor, more than just x^2. Therefore $\alpha = 0$, and the cusp curve is $s = 0$. □

Corollary 10 *A genus one fibering on a* $\mathbf{Z}/2$ *surface with a double fiber of* $\widetilde{E}_7$ *or* $\widetilde{E}_8$ *type and with a* -2 *double section is quasi-elliptic, and the* -2 *double section is unique.*

Proof: If there were another -2 double section, it would define a hyperelliptic map such that the cusp curve would not be $s = 0$. □

The Case of $\boldsymbol{\alpha}_2$ Surfaces

The set-up is the same as before. The relative system $|R|$ has exactly one base point, occurring at the intersection between R and the double fiber. It is a double base point, meaning that two blowing ups are needed to resolve it. As before, we get a representation of X as a double cover of $\mathbf{F}_2$. And again the double covering is given by $L \cong \mathcal{O}_Y(2C_t + C_x)$, yielding

$$z^2 + z(B_5 s^2 + B_3 st + B_1 t^2)+$$
$$A_{10}s^4 + A_8 s^3 t + A_6 s^2 t^2 + A_4 st^3 + A_2 t^4 = 0,$$

where the A_i's and B_i's are forms in x, y of degree i.

Proposition 11 *Any $\boldsymbol{\alpha}_2$ surface with a nodal double section can be written birationally as*

$$z^2 + z\Big((Uy^2 + b_1 xy + b_2 x^2)x^3 s^2 + b_3 x^3 st\Big)+$$
$$x^3 y^7 s^4 + (Uy^4 + Wxy^3 + a_1 x^2 y^2 + a_2 x^3 y + a_3 x^4)x^4 s^3 t+$$
$$a_4 x^5 y s^2 t^2 + (Uy + Wx)x^3 st^3 + xyt^4 = 0,$$

and conversely, these equations define such surfaces (or possibly some non-normal surface).

Proof: Assume first X' is of the given type. Put the non-rational singularity at $x, t = 0$. We get the same conditions as before, and some extra ones, as we must now have a *genus 2* singularity. As before, the graph combinatorics shows that the genus must drop at the second blowing up, and locally around $x = 0$, the $\boldsymbol{\alpha}_2$ surface must therefore be a degeneration of a $\mathbf{Z}/2$ surface with an $\widetilde{E}_n$ double fiber. This gives us the following equation so far:

$$z^2 + z\Big((U + b_1 x + b_2 x^2)x^3 + b_3 x^3 t\Big)+$$
$$x^3 + \alpha_1 x^7 + \alpha_2 x^9 + (U + Wx + a_1 x^2 + a_2 x^3 + a_3 x^4)x^4 t+$$
$$a_4 x^5 y t^2 + (U + \alpha_3 x)x^3 t^3 + xt^4 = 0.$$

This time there is no double fiber at $y = 0$. We therefore have more parameters to use. Because of this, we cannot just use Proposition 8 to see that the above equation is the right one. Instead, the proof must be repeated with the new assumptions. This is not done here, as it involves calculations similar to those of Propositions 7 and 8. Note that we have three parameters that must be equal this time. They are denoted U. To get rid of α_1 and α_2, use $y \mapsto y + \lambda x$ and $t \mapsto t + \lambda s x^2$. As before, the newly introduced squares in f can easily be removed. Finally, I have assumed that the zx^3 term is absent, as otherwise the surface is of $\boldsymbol{\mu}_2$ type. The reader may check this. To make the tame $\widetilde{E}_8$ fiber degenerate into a wild elliptic fiber, put $\alpha_3 = W$. The converse statement is proved as before. □

To obtain the equation of an $\boldsymbol{\alpha}_2$ surface with an $\widetilde{E}_n + R$ configuration, blow up twice, normalize and put the singularity at the origin. Then blow up. With our choice of parameters, there will be just one singularity along the new curve; it is a genus one singularity. Blow it up and normalize. There results a supersingular elliptic curve (if the parameters are general). As before, I denote the new coordinates by X and Y, where $X = 0$ defines the elliptic curve and $Y = \infty$ is the intersection with the rest of the graph. Putting $X = 0$ gives

$$Z^2 + ZU + a_4 + b_3 Y + Y^3 = 0.$$

A curve singularity at $Y = \sqrt{b_4}$ appears when $U = 0$. Put this over the origin by $X \mapsto X + \sqrt{b_4}$, and change $Z \mapsto Z + \sqrt{a_4} + \sqrt[4]{b_3}Y$. The coefficient for ZX will be b_3, so to have a singularity of E_n type, we must put $b_3 = 0$, which simplifies a lot. Omitting terms of degree greater than five, we get:

$$Z^2+Z(b_1X^2+b_2X^4)+a_1X+(b_1+W)XY+b_1\sqrt{a_4}X^2+Y^3+a_4X^2Y+a_2X^3+ \\ (b_2+a_1)X^3Y+b_2\sqrt{a_4}X^4+b_1X^3Y^2+b_1\sqrt{a_4}X^4Y+a_3X^5+\cdots=0.$$

By the same reasoning as before, $b_1 = W$ and $a_1, a_4, a_2 = 0$. We then have

$$Z^2+Z(WX^2+b_2X^4)+Y^3+b_2X^3Y+WX^3Y^2+a_3X^5+\cdots=0.$$

Proposition 12 *Any $\boldsymbol{\alpha}_2$ surface with an $\widetilde{E}_n + R$ configuration can be (birationally) written as*

$$z^2+z\Big((Wy+b_2x^2)x^4s^2+\Big)+x^3y^7s^4+(Wy^3+a_3x^3)x^5s^3t+Wx^4st^3+xyt^4=0$$

with $W \neq 0$ or $W = 0, b_2 \neq 0$ or $W = b_2 = 0, a_3 \neq 0$ according to as $n = 6, 7, 8$. Conversely, these equations define such surfaces.

Proof: Use [15] again. □

Finally, we note that our $\boldsymbol{\alpha}_2$ fibering is quasi-elliptic exactly when $W = 0$. Namely, put $t, y = 1$ in the equation, and differentiate with respect to z and s. We get $D_z = x^4s^2(W + b_2x^2)$ and $D_s = x^4(s^2Wx + s^2a_3x^4 + W)$. A curve of cusps appears when $W = 0$; it is given by $s = 0$ as before.

$\mathbf{Z}/2$ Surfaces with a Simple $\widetilde{E}_7$ Fiber in a Quasi-Elliptic Fibering

Since the fibering is quasi-elliptic, it has at least one nodal double section, and as before we study the associated linear system (relative to the fibration). We know X can be written according to Proposition 7. The equation in Proposition 7 does not suit us very well. Instead we use the equation as it looked *before* we put $\alpha_3 = 0$ and $\alpha_2, \alpha_4, \alpha_{14} = 1$ (see the proof of Proposition 7). The following equation, where we have put $s = y = 1$, is of the required type:

$$z^2+kx^3+x^5+(k+1)x^7+\alpha x^3t+\alpha x^5t+\alpha^2x^3t^2+\alpha^3x^2t^3+\beta xt^4=0.$$

The $\widetilde{E}_7$ fiber on X will give rise to an E_7 singularity on X'. We use the automorphisms of $\mathbf{F}_2$ to put the E_7 singularity over the point where $x = 1$ and $t = 0$. We are in fact only interested in exhibiting a three-parameter family with an E_7 singularity at the prescribed spot. To see that our equation defines such a family, change $x \mapsto x + 1$:

$$z^2+\beta t^4(x+1)+\alpha^3t^3(x+1)^2+\alpha^2t^2(x^3+x^2+x+1)+ \\ \alpha tx^2(x^3+x^2+x+1)+(k+1)(x^7+x^6)+k(x^5+x^4)+x^3+x^2=0.$$

Theorem 13 *If a $\mathbf{Z}/2$ surface has an elliptic fibration with a nodal double section and an $\widetilde{E}_6$ double fiber, then it has a quasi-elliptic fibration with an $\widetilde{E}_7$ simple fiber.*

Proof: The cubic part of the equation is $(\alpha t)^3 + (\alpha t)^2x + \alpha tx^2 + x^3$. As it is a cube, we have a singularity of type E_n. As the fibering is quasi-elliptic, it is not E_6. As the coefficient of $tx^3 \neq 0$, we have an E_7. In prescribing the locus where three singularities should lie we have used up all automorphisms of $\mathbf{F}_2$. Our family is therefore genuinely three-dimensional. As there likewise is a three-dimensional family of $\mathbf{Z}/2$ surfaces with an $\widetilde{E}_6$ double fiber and a nodal double section, the statement follows, since we know by [13] that the first family is contained in the second. And we also know by the above that the second family is irreducible. $\square$

Triangulation of Infinitesimal Cubes

I think that explicit formulae for triangulation of cubes and vice versa should get more attention. Among other things, there may be applications to combinatorics or number theory. Here is a very simple construction that works only infinitesimally.

Consider a topological space X and let $\triangle_n(X)$ and $\boxdot_n(X)$ denote the sets of singular simplices and pointed cubes in it. Then we have real-valued functions on these sets denoted $\triangle^n(X)$ and $\boxdot^n(X)$. We often omit X from notation.

There are cubical differentials

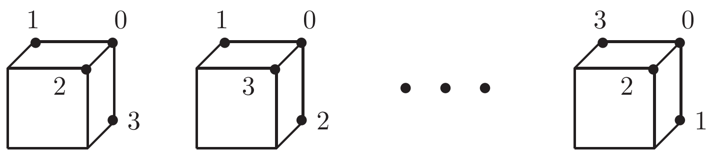
$$d_i^\epsilon \colon \boxdot_n \to \boxdot_{n-1}\,, \tag{1}$$

where $1 \leq i \leq n$ and $\epsilon = \pm 1$. They are given by including I^{n-1} in I^n at the level of the point, or *minus* that level, depending on ϵ. If we let degeneracies put the point at 1 along the new coordinate, then the usual $\square_*$ is the cubical subobject of $\boxdot_*$ where the point lies in the $(1, \ldots, 1)$ corner.

Let $\xi \colon * \to I^n \to X$ be a pointed cube in X. Let the point have coordinates $z^0 = (z_1, \ldots, z_n)$. Then we get a simplex in X by mapping the i'th vertex to $z^i = (z_1, \ldots, -z_i, \ldots, z_n)$. Then we extend linearly. More generally we also consider the simplex which has the i'th vertex at $z^{\sigma(i)}$ for some $\sigma \in \Sigma$. We denote this by $\lambda_\sigma(\xi)$. We get mappings $\lambda_\sigma \colon \boxdot_n \to \triangle_n$. We put

$$\lambda = (-1)^n \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \lambda^\sigma\,, \tag{2}$$

where λ^σ are the adjoint mappings.



(3)

Our basic claim is that λ is a chain mapping. Once we know that, we can apply acyclic models to see that it is a chain homotopy equivalence. Unfortunately our "basic claim" is (obviously) false as stated. Let us go down to the infinitesimal level where we may hope for cancellations to save us. Our reason for using *pointed* cubes is now clear: the corners of the cube disappears infinitesimally, so we must supply a point of our own.

Assume that everything is differentiable from now on. We change slightly the meaning of $\triangle_n$. It will denote the direct product of $n+1$ copies of X, so "simplex" will mean point configuration. It is almost the same infinitesimally near the diagonal in $\triangle_n$, which is where we will work. To feel more at home we switch to the language of algebraic geometry, although we could just as well stick to smooth manifolds. Let's work over an algebraically closed field, just to fix ideas. Assume for safety that the characteristic isn't 2. There is no longer any need to assume that X is smooth, but that is the most interesting case. We may for simplicity let it be affine since everything is so very local anyway.

We need a new I^n. The *microcube* I^n is the direct product of copies of $\operatorname{Spec} k[x]/x^2$. We let $\boxdot_n$ denote the "space" of pointed microcubes in X. It is a non-reduced variety of finite dimension. The presence of a point in I^n gives rise to a natural mapping $\boxdot_n \to X$. Moreover, as already noted, there are mappings $\lambda_\sigma \colon \boxdot_n \to \triangle_n$. We let $\boxdot^n$ and $\triangle^n$ denote the *full* function algebras, and we let $\widetilde{\boxdot}^n$ and $\widetilde{\triangle}^n$ denote the functions that vanish on degenerate cubes and simplices.

Theorem 14 *The diagram*

$$\begin{tikzcd} \triangle^{n-1} \arrow[r, "d"] \arrow[d, "\lambda"] & \triangle^n \arrow[d, "\lambda"] \\ \boxdot^{n-1} \arrow[r, "d"] & \boxdot^n \end{tikzcd} \tag{4}$$

commutes (it is not very natural in this context to query whether it induces an isomorphism in homology).

Proof: The proof is a direct computation, which I omit. $\square$

Let us turn to the easier claim that

$$\begin{tikzcd} \widetilde{\triangle}^{n-1} \arrow[r, "d"] \arrow[d, "\lambda"] & \widetilde{\triangle}^n \arrow[d, "\lambda"] \\ \widetilde{\boxdot}^{n-1} \arrow[r, "d"] & \widetilde{\boxdot}^n \end{tikzcd} \tag{5}$$

commutes. Elements $\triangle^n$ are tensors. For a $b \in \triangle^0$, we write

$$b_i = 1 \otimes \cdots \otimes b \otimes \cdots \otimes 1 \ \in \triangle^n \,, \tag{6}$$

where the insertion is at the i'th place. We let a^i denote the i'th coordinate function on I^n. There is no risk of confusion with the i'th power, since $(a^i)^2 = 0$.

Following [16] we define a subscheme $\triangleright_n$ of $\triangle_n$. As i, j varies from 0 to n and b, b' varies over $\triangle^0$, let E^n be generated by expressions like

$$(b_i - b_j)(b'_i - b'_j) \tag{7}$$

or equivalently, by expressions like

$$(b_i - b_0)(b_j - b_0)\,. \tag{8}$$

This is an ideal in $\triangle^n$ and therefore define a subscheme of $\triangle_n$. We have a corresponding $\widetilde{\triangleright}^n$, which now can be easily described. It is the ideal in $\triangleright^n$ that is generated by expressions like

$$(b^1_1 - b^1_0) \cdots (b^p_p - b^p_0)\,. \tag{9}$$

Put $X = I^n$, as we may just as well do. Let us note the important fact that

$$\lambda^\sigma(b_i) = \pm b\,, \tag{10}$$

with the minus when $a^{\sigma(i)}$ divides b. This implies that $\lambda^\sigma(b_i - b_0)$ is divisible by $a^{\sigma(i)}$. Moreover

$$\lambda^\sigma(a_i^{\sigma(i)} - a_0^{\sigma(i)}) = -2a^{\sigma(i)} \,. \tag{11}$$

Theorem 15 $\lambda(E^n) = 0$.

Proof: Consider first a generator of type $\psi = (b_i - b_0)(b_j - b_0)$, where $i \neq j$. Let $\sigma' = \sigma\sigma_{ij}$, where σ_{ij} is the (ij) transposition. Then $\lambda^\sigma(\psi) = \lambda^{\sigma'}(\psi)$, but as $|\sigma'| = |\sigma| + 1$, they are given different signs by formula (2). We finish the argument by having σ run through the even permutations. The same reasoning works for $\psi = (b_i - b_j)(b'_i - b'_j)$ as well. The final case is $\psi = (b_i - b_0)(b'_i - b'_0)$. We have seen that $\lambda^\sigma(b_i - b_0)$ and $\lambda^\sigma(b'_i - b'_0)$ are divisible by $a^{\sigma(i)}$. Now we use that $(a^{\sigma(i)})^2 = 0$. □

The advantage of knowing this is that we may restrict attention to the ideal (9). In a certain sense one may say that the image of $\boxdot_n$ in $\triangle_n$ under the adjoint of λ is contained in $\triangleright_n$, but that is abuse of language since λ is not a ring mapping. In fact, we may regard Theorem 15 as the main theorem of this paper. Once we know this, we may switch to the convenient language of differential forms. One sees that elements of $\widetilde{\triangleright}^n$ may be decomposed into expressions

$$b_0^0(b_1^1 - b_0^1)\cdots(b_n^n - b_0^n) \,, \tag{12}$$

which we identify with $b^0 db^1 \wedge \cdots \wedge db^n$ (cf. [16]). By the paucity of differential forms on I^n, we may restrict attention to

$$\varphi^i = a^i \, da^1 \wedge \cdots \wedge \widehat{da^i} \wedge \cdots \wedge da^n \,, \tag{13}$$
$$\varphi = da^1 \wedge \cdots \wedge da^n \tag{14}$$

when we consider $\widetilde{\triangleright}^{n-1}$ and $\widetilde{\triangleright}^n$.

Theorem 16 *Diagram (5) commutes.*

Proof: We compute $d\varphi^i = (-1)^{i+1}\varphi$ and

$$\lambda^\sigma(\varphi) = (-1)^n(-2)^n a^1 \cdots a^n \,, \tag{15}$$

according to (11) and (2). So in all we get

$$\lambda^\sigma(d\varphi^i) = (-1)^{i+1}2^n \, a^1 \cdots a^n \,. \tag{16}$$

Next we compute

$$\lambda^\sigma(\varphi^i) = 2^{n-1}a^1 \cdots a^n \,, \tag{17}$$

which we give the following meaning. It is a function of points on I^n, where a point determine a subcube $I^{n-1} \hookrightarrow I^n$, the i'th insertion. Therefore it is also a function of singular $(n-1)$-cubes in I^n. The cubical differential then gives the same sign as in (16) and we also get another 2 from the way the cubical differential works. So we are done. □

A Holomorphic Analog of Cubical Subdivision

Cubes have been less popular among topologists than simplices. For good reasons maybe, but in a wider framework of "cubical objects" and "simplicial objects" we are sometimes forced to choose cubes. In complex algebraic geometry polydisks make up good "cubes", the boundary being parametrized over S^1 instead of S^0.

To make this rigorous we should set up a theory somewhat along the following lines. We consider holomorphic mappings of closed polydisks D^n into an analytic manifold X. Let Q_n be the "infinite-dimensional manifold" that parametrizes such things. Let C^n be the corresponding ring of smooth complex-valued test functions (without growth restrictions, say). Let C_n be the distributions that we thus get. All of these things are "S^1-cubical objects" (or cocubical).

A S^1*-cubical object* is a contravariant functor from the "polydisk category" where objects are the standard polydisks and differentials d^i_t are insertions $D^{n-1} \hookrightarrow D^n$ in the evident way, where $1 \leq i \leq n$ and $t \in \partial D$. Degeneracies s^i are given by projections onto factors, so they don't depend on t. For an S^1-cocubical complex vector space C^* there is a cochain complex given by:

$$d^i = c \cdot \int_{\partial D} d^i_z \, dz \qquad \text{and} \qquad d = \sum_i (-1)^{i+1} \, d^i \,. \tag{18}$$

The constant c may be chosen freely. The natural choices are

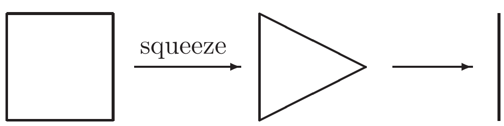
$$c = 1 \qquad \text{or} \qquad c = \frac{1}{2\pi i} \,. \tag{19}$$

I will use the latter option. Note that a similar choice must be made in the S^0-cocubical case, where the natural choice is $c = \frac{1}{2}$ beside $c = 1$.

Given a singular n-dimensional polydisk, we may pull back smooth forms of type $(0, n)$ from X and integrate against the standard $(n, 0)$-form on D^n. One checks (using the Stokes formula) that this is compatible with the $\bar{\partial}$-operator, so we get a chain mapping $C_* \to A_{0,*}$ to the compactly supported currents on X. Here I write $A_{0,n}$ for currents that are functionals on forms on type $(0, n)$. To check that we get an isomorphism in homology we need a subdivision operator.

We will actually subdivide down *to the infinitesimal level*: we will, as it were, partition the unit disk into its tangent vectors. This seems to be the only sensible option, as opposed to the case of euclidean cubes where dichotomization works well.

By some straightforward combinatorics one reduces subdivision of D^n to the D^1 case. This in turn is achieved by mapping $D^2 \to D^1$ in a suitable way. In the euclidean case it looks like



(20)

But in our situation we must use a whole *family* of mappings, each one looking somewhat like this:

(21)

The mapping $D^2 \to D^1 \subset \mathbf{C}$ for some fixed ε and r.

Image of the locus where $z_1 = \pm 1$ or $z_2 = \pm 1$.

The formula for these mappings is

$$\begin{tikzcd} U \times D^2 \arrow[d, "(r+\varepsilon z_1)z_2"] \\ D^1 \subset \mathbf{C} \end{tikzcd} , \tag{22}$$

where z_1 and z_2 are variables on D^2 and r and ε are variables on the parameter space $U \cong \mathbf{C}^2$. To get a chain we must specify a distribution ξ on U. It is

$$\xi(g) = \frac{-1}{2\pi i} \int_\Delta \frac{\partial g}{\partial \bar{\varepsilon}}\, dr \wedge d\bar{r}, \tag{23}$$

where $\Delta \subset U_0$ is the locus where $|r| < 1$ and $\varepsilon = 0$. There remains to verify that $d_1\xi = id$, where id denotes the unit point measure at the identity mapping $D^1 \to D^1$. We must impose the further axiom on test functions:

$$f(s(\tau)) = sf(\tau)\,, \tag{24}$$

where s denotes rotation by $s \in \partial D$ of a factor in D^q and τ is a singular polydisk seen as a point in Q_q. See [21] for the analogous, now obsolete, axiom in algebraic topology.

The image of our family under $Q_2(D^1) \times \partial D \to Q_1(D^1)$ depends on a one-dimensional parameter space, which it is convenient to identify with $\mathbf{C}$ (with ρ as coordinate):

$$\begin{tikzcd} U \times \partial D \times D^1 \arrow[rr, "(r+\varepsilon z_1, z_2)"] \arrow[dr, "(u+\varepsilon z_1)z_2"'] & & \mathbf{C} \times D^1 \arrow[dl, "\rho z_2"] \\ & D^1 \subset \mathbf{C} & \end{tikzcd} . \tag{25}$$

Let $f \in C^1(D^1)$ and consider its restriction to our one-dimensional space. The variable r identifies this space with U_0:

$$\begin{tikzcd} U \times \partial D \arrow[r, "r+\varepsilon z_1"] & \mathbf{C} \arrow[d, "f"] \\ U_0 \times 1 \arrow[u, hook] \arrow[ur, "r"] \arrow[r, "\tilde{f}"] & \mathbf{C} \end{tikzcd} . \tag{26}$$

We write $\tilde{f}$ for f pulled back to U_0, so $f(r) = \tilde{f}$.

Computing $d_1\xi(f)$ means doing three things: pulling back f to $U \times \partial D$, integrating over ∂D, and applying ξ. The first step consists in turning attention to the function $f(r + \varepsilon z_1)$.

Theorem 17 *$d_1\xi(f) = f(id)$, where id now denotes the identity $D^1 \to D^1$.*

Proof: Integrating over z_1 and applying $\partial/\partial\bar{\varepsilon}$ we obtain

$$\frac{1}{2\pi i}\int_{\partial D} \frac{\partial f(r + \varepsilon z_1)}{\partial \bar{\varepsilon}}\, dz_1 = \frac{1}{2\pi i}\int_{\partial D} \frac{\partial f}{\partial \bar{z}}(r + \varepsilon z_1)\, \bar{z}_1\, dz_1\,, \tag{27}$$

which when $\varepsilon = 0$ becomes

$$\frac{\partial f}{\partial \bar{z}}(r) = \frac{\partial \tilde{f}}{\partial \bar{r}}\,. \tag{28}$$

Then we should sum over Δ:

$$\frac{-1}{2\pi i}\int_{\Delta} \frac{\partial \tilde{f}}{\partial \bar{r}}\, dr \wedge d\bar{r} = \frac{1}{2\pi i}\int_{\partial\Delta} \tilde{f}\, dr\,. \tag{29}$$

By axiom (24) the restriction of $\tilde{f}$ to $\partial\Delta$ equals $\tilde{f}(1)\bar{r}$, so we obtain $\tilde{f}(1)$, or equivalently $f(1)$, so f gets evaluated at the identity endomorphism of D^1. $\square$

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