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# Krein's formula and perturbation theory

P. Kurasov and S.T. Kuroda

Abstract. Krein's formula and its modification are discussed from the viewpoint that they describe all selfadjoint operators in relation to a given unperturbed operator. A direct proof of Krein's formula is also given for the case when the restricted operator is not necessarily densely defined and possibly has infinite deficiency indices.

## 1. Introduction

In the present paper we shall take another look at Krein's formula in the theory of selfadjoint extensions of a symmetric operator. Krein's formula describes the relation between the resolvents of two selfadjoint extensions $H_0$ and $H$ of one symmetric operator $H_{00}$. Usually, one fixes $H_{00}$ and regards Krein's formula as a formula describing an arbitrary extension of $H_{00}$ in relation to a particular extension $H_0$. If we change the viewpoint and allow $H_{00}$ to vary, $H_0$ being fixed, then Krein's formula can be regarded as a formula describing all selfadjoint operators $H$ in relation to a given selfadjoint operator $H_0$. In this respect Krein's formula may be considered as a formula in the perturbation theory. In fact in the the present paper we prove a counterpart of Krein's formula in the operator theory and reexamine Krein's formula in the extension theory from the operator view point.

The original Krein's formula was derived by M.Krein and M.Naimark for the case where $H_{00}$ has deficiency indices $(1,1)$ [**7, 8, 9, 12**]. It was generalized later for arbitrary (including infinite) deficiency indices by S.N.Saakjan [**14**]. See also recent paper [**4**] where Saakjan's result is discussed. We shall write down below what we consider the most general form of Krein's formula in the extension situation. (We do not deal with the so-called non-orthogonal extensions.)

Let $H_0$ be a selfadjoint extension of a closed symmetric operator $H_{00}$. Let $M = [(H_{00}-i)\mathrm{D}\,(H_{00})]^{\perp}$ be the deficiency subspace of $H_{00}$ at $i$ and let $P_M$ be the orthogonal projection onto $M$. ($\mathrm{D}\,(A)$ stands for the domain of $A$.) Then, Krein's formula can be written as

$$\frac{1}{H-z} = \frac{1}{H_0-z} - \frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z}, \tag{1.1}$$

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where $Q(z)$ is given by

$$Q(z) = P_M \frac{1 + zH_0}{H_0 - z} P_M, \tag{1.2}$$

for $z \in \rho(H_0)$, and $\gamma$ is a selfadjoint operator in $M$. The operator $Q(z)$ is a generalization of Krein's $Q$-function. In the original papers by M.Krein [**7, 8, 9**] the $Q$-function was defined up to a certain real parameter. We find it more convenient to fix the $Q$-operator using (1.2). Krein's formula is valid for $H$ such that $H$ and $H_0$ are relatively prime extensions of $H_{00}$ and under an additional condition on $\gamma$, which we call the admissibility condition. But we shall not dwell on these details here. We only note that for $z = i$ (1.1) takes a simple form that

$$\frac{1}{H - i} = \frac{1}{H_0 - i} - \frac{H_0 + i}{H_0 - i} \frac{1}{\gamma + i} P_M. \tag{1.3}$$

We now change the viewpoint. We fix $H_0$ and consider an arbitrary selfadjoint operator $H$. Let us denote by $H \wedge H_0$ the maximal common restriction of the operators $H$ and $H_0$, i.e. the restriction to the domain

$$\mathrm{D}\,(H \wedge H_0) := \{u \in \mathrm{D}\,(H) \cap \mathrm{D}(H_0) \,|\, Hu = H_0 u\}. \tag{1.4}$$

Then the operators $H$ and $H_0$ are two relatively prime extensions of the Hermitian operator $H \wedge H_0$ and (1.1) applies with $M = [(H_0 - i)\mathrm{D}\,(H \wedge H_0)]^\perp$. We also see that with varying $H$ all subspaces $M$ will appear. Thus, we expect that with a fixed $H_0$ and with varying $M$ and $\gamma$ (1.1) or (1.3) will describe all selfadjoint operators $H$. This is a perturbation theoretical aspect of Krein's formula.

In Section 2 we shall in fact prove that, given $H_0$, relation (1.3) gives a bijective correspondence between the set of all selfadjoint operators $H$ and the set of all pairs $\{M, \gamma\}$ of a closed subspace $M$ and a selfadjoint operator $\gamma$ in $M$ satisfying the condition

$$\mathrm{Ker} \left\{ \frac{1}{H_0 + i} - \frac{1}{\gamma + i} P_M \right\} = 0. \tag{1.5}$$

We call condition (1.5) the admissibility condition. This will be done in Theorem 1. (1.3) relates the resolvents of $H$ and $H_0$ only at $z = i$. In Section 3 we shall prove formula (1.1) for general $z$ (Theorem 2) and also discuss the relation to the extension theory. In this way we recapture the original Krein's formula in the extension situation (Theorem 3). It might be said that our approach would provide a transparent proof of Krein's formula.

It should be noted that Krein's formula (1.1) itself does not provide direct estimate on the difference of the resolvents. This is related to the fact that the operator $\gamma$ appearing in (1.1) cannot be considered as a perturbation operator because the difference of the resolvent decreases if the norm of $\gamma$ increases (see Theorem 6). In this connection we shall derive a modification of Krein's formula. Like in (1.3) $H$ is described by a closed subspace $L$ and a selfadjoint operator $\beta$ in $L$. The modified formula reads

$$\frac{1}{H - i} = \frac{1}{H_0 - i} + \frac{H_0 + i}{H_0 - i} \left( i + \frac{1}{\beta + i} P_M \right). \tag{1.6}$$

With this $\beta$ we can estimate the difference of the resolvents (see (5.1)).

In order to deal with (1.3) and (1.6) in one stroke we introduce a parameter $\theta \in [0, 2\pi)$ and consider in Theorem 1 a family of correspondence $\{M, \gamma\} \leftrightarrow H(M, \gamma; \theta)$

distinguished by the value of $\theta$. $\theta = 0$ and $\theta = \pi$ corresponds to (1.3) and (1.6) respectively.

Some comments on the perturbation theory are due. In [**10, 11**] H.Nagatani and the second author proved a resolvent formula (for general $z$ ) which relates all selfadjoint operators to a given $H_0$. Originally, the formula was written in the framework of what is called "$\mathcal{H}_{-2}$ resolvent construction" in [**10**]. There, the results were expressed using the language of a scale of Hilbert spaces associated to $H_0$. The same results can be presented using only operators in the original Hilbert space. In this form, given $H_0$, the results of [**10, 11**] give a bijective correspondence between the set of all selfadjoint operators $H$ in $\mathcal{H}$ and the set of all normal operators $\tau$ satisfying certain additional conditions. The correspondence is given by

$$\frac{1}{H-z} - \frac{1}{H_0-z} = \frac{H_0+i}{H_0-z}\frac{1}{1+(z-i)\tau\frac{H_0+i}{H_0-z}}\tau\frac{H_0-i}{H_0-z}. \tag{1.7}$$

In particular, with $z = i$ (1.7) becomes

$$\frac{1}{H-i} - \frac{1}{H_0-i} = \frac{H_0+i}{H_0-i}\tau. \tag{1.8}$$

The similarity between (1.3) (or (1.6)) and (1.8) is evident. Thus, by taking $\tau$ properly, it is expected that results obtained in the $\mathcal{H}_{-2}$ perturbation theory can be converted to $\{M,\gamma\}$ situation. In particular, in the case of (1.3) we can recapture (1.1) from (1.7) and in the case of (1.6) we can derive a resolvent expression for general $z$ from (1.7) (see (4.4) ). Unfortunately, however, (4.4) has a rather complicated appearance.

**Notations.** Throughout the present paper we shall use the following notations.

We shall work in a fixed Hilbert space $\mathcal{H}$. For brevity of the exposition we put

$$\mathcal{C}_{sa}(\mathcal{H}) = \{\text{the set of all selfadjoint operators in } \mathcal{H}\}, \tag{1.9}$$

$$\mathcal{M} = \{\text{the set of all closed subspaces of } \mathcal{H}\}. \tag{1.10}$$

For $M \in \mathcal{M}$ we denote by $M^\perp$ the orthogonal complement of $M$ and by $P_M$ the orthogonal projection on $M$.

For a closed operator $A$ in $\mathcal{H}$ the resolvent set of $A$ is denoted by $\rho(A)$. As we already did we denote the resolvent and related operators by the fractions: $(H-z)^{-1} = \dfrac{1}{H-z}$; $(H-w)(H-z)^{-1} = \dfrac{H-w}{H-z}$. We also note that the following simple relation is rather useful in our discussion:

$$\frac{H+i}{H-i} = 1 + \frac{2i}{H-i}. \tag{1.11}$$

## 2. General correspondence

In this section we fix $\theta \in [0, 2\pi)$ and let $H_0 \in \mathcal{C}_{sa}(\mathcal{H})$. For $M \in \mathcal{M}$ and $\gamma \in \mathcal{C}_{sa}(M)$ we introduce the condition

$$\mathrm{Ker}\left\{1 - \frac{2i}{\gamma+i}P_M - e^{i\theta}\frac{H_0-i}{H_0+i}\right\} = \{0\}, \tag{2.1}$$

and call it *admissibility condition*. We also call a pair $\{M,\gamma\}$ satisfying condition (2.1) an *admissible pair*. See Section 6 where the admissibility condition is studied in the case $\theta = 0$.

The main result in this section is the following theorem.

THEOREM 1. *Let $\theta$ and $H_0$ be as above. Then, for any $H \in \mathcal{C}_{sa}(\mathcal{H})$ their exists an admissible pair $\{M, \gamma\}$ such that the following equivalent relations* (2.2)–(2.5) *hold*

$$\frac{H+i}{H-i} = e^{-i\theta}\frac{H_0+i}{H_0-i}\left(1-\frac{2i}{\gamma+i}P_M\right) \tag{2.2}$$

$$= e^{-i\theta}\frac{H_0+i}{H_0-i}\left(P_{M^\perp}+\frac{\gamma-i}{\gamma+i}P_M\right), \tag{2.3}$$

$$\frac{1}{H-i}-\frac{1}{H_0-i} = \frac{H_0+i}{H_0-i}\left(\frac{e^{-i\theta}-1}{2i}-\frac{e^{-i\theta}}{\gamma+i}P_M\right) \tag{2.4}$$

$$= \frac{H_0+i}{H_0-i}\left(\frac{e^{-i\theta}-1}{2i}P_{M^\perp}+\frac{1}{2i}\left[e^{-i\theta}\frac{\gamma-i}{\gamma+i}-1\right]P_M\right). \tag{2.5}$$

*The correspondence $H \longleftrightarrow \{M, \gamma\}$ is a bijection between $\mathcal{C}_{sa}(\mathcal{H})$ and the set of all admissible pairs $\{M, \gamma\}$.*

*Remark.* With $\theta = 0$ formula (2.4) coincides with (1.3), and with $\theta = \pi$ it is (1.6) ($\beta = \gamma$).

*Proof.* (i) The equivalence of (2.2) and (2.3) (or (2.4) and (2.5)) is a result of simple manipulations. To see that (2.2) is equivalent to (2.4) we use (1.11) on the left hand side of (2.2) and see that (2.2) is equivalent to

$$\begin{aligned}\frac{2i}{H-i} &= e^{-i\theta}\frac{H_0+i}{H_0-i}\left(1-\frac{2i}{\gamma+i}P_M\right)-1\\ &= \frac{H_0+i}{H_0-i}-1+e^{-i\theta}\frac{H_0+i}{H_0-i}\left(1-e^{i\theta}-\frac{2i}{\gamma+i}P_M\right).\end{aligned}$$

Applying (1.11) to $H_0$ and dividing by $2i$, we see that this is equivalent to (2.4).

(ii) Given $H \in \mathcal{C}_{sa}(\mathcal{H})$, we put

$$U = e^{i\theta}\frac{H_0-i}{H_0+i}\;\frac{H+i}{H-i}, \tag{2.6}$$

which is the ratio of the Cayley transforms of $H_0$ and $H$ multiplied by $e^{i\theta}$. $U$ is a unitary operators in $\mathcal{H}$. Let $K = \{\psi \in \mathcal{H} \,|\, U\psi = \psi\}$ be the eigenspace of $U$ corresponding to the eigenvalue 1 and put $M = K^\perp$. Then, $K$ and $M$ reduce $U$ and the part $U|_M$ of $U$ in $M$ is a unitary operator in $M$ which does not have 1 as an eigenvalue. Hence, by the theory of Cayley transforms there exists a unique $\gamma \in \mathcal{C}_{sa}(M)$ such that $U|_M = \dfrac{\gamma-i}{\gamma+i}$. Equation (2.3) follows from this observation and (2.6).

It is not difficult to see that $M$ and $\gamma$ given above satisfy (2.1). Indeed, the operator on the left hand side of (2.2) does not have 1 as an eigenvalue. Hence, by multiplying the right hand side of (2.2) by $e^{i\theta}\dfrac{H_0-i}{H_0+i}$, we see that

$$\operatorname{Ker}\left\{1-\frac{2i}{\gamma+i}P_M-e^{i\theta}\frac{H_0-i}{H_0+i}\right\} = \{0\}. \tag{2.7}$$

This is equivalent to (2.1).

(iii) The uniqueness of the pair $\{M, \gamma\}$ satisfying (2.3) is verified as follows. Let $M'$ and $\gamma'$ satisfy (2.3) with $M$ and $\gamma$ replaced by $M'$ and $\gamma'$, respectively. Then, it is clear that $M'^\perp \subset K = \{\psi \in \mathcal{H} \,|\, U\psi = \psi\}$. If $M'^\perp$ does not exhaust $K$, then there exists a non-zero $\psi \in M'$ such that $U\psi = \psi$. Furthermore, (2.3) implies that $U|_{M'} = \dfrac{\gamma' - i}{\gamma' + i}$. This means that $\dfrac{\gamma' - i}{\gamma' + i}\psi = \psi$, $\psi \neq 0$, which is impossible because $\gamma'$ is selfadjoint in $M'$. Hence, $M' = M$. Then, the Cayley transform of $\gamma$ and $\gamma'$ are both equal to $U|_M$. Hence, $\gamma' = \gamma$.

(iv) Conversely, given a pair $\{M, \gamma\} \in \mathcal{M}$, the right hand side of (2.3) defines a unitary operator in $\mathcal{H}$. As is mentioned above, the admissibility condition (2.1) is equivalent to (2.7), which in turn implies that the operator on the right hand side of (2.2) does not have 1 as an eigenvalue. Hence, there exist $H \in \mathcal{C}_{sa}(\mathcal{H})$ such that (2.3) holds. This establishes that the correspondence $H \longmapsto \{M, \gamma\}$ is onto. ∎

Example 1. *Among all correspondences indexed by $\theta$ the cases $\theta = 0$ and $\theta = \pi$ are particularly interesting. We reproduce formulas* (2.1), (2.2)–(2.5) *for these cases.*

(i) *The case $\theta = 0$. The admissibility condition* (2.1) *takes the form*

$$\operatorname{Ker}\left\{\frac{1}{H_0 + i} - \frac{1}{\gamma + i}P_M\right\} = \{0\} \tag{2.8}$$

*and equations* (2.2)–(2.5) *read*

$$\frac{H + i}{H - i} = \frac{H_0 + i}{H_0 - i}\left(1 - \frac{2i}{\gamma + i}P_M\right) = \frac{H_0 + i}{H_0 - i}\left(P_{M^\perp} + \frac{\gamma - i}{\gamma + i}P_M\right), \tag{2.9}$$

$$\frac{1}{H - i} - \frac{1}{H_0 - i} = -\frac{H_0 + i}{H_0 - i}\,\frac{1}{\gamma + i}P_M. \tag{2.10}$$

(ii) *The case $\theta = \pi$. The admissibility condition* (2.1) *takes the form*

$$\operatorname{Ker}\left\{\frac{H_0}{H_0 + i} - \frac{i}{\gamma + i}P_M\right\} = \{0\} \tag{2.11}$$

*and equations* (2.2)–(2.5) *read*

$$\frac{H + i}{H - i} = -\frac{H_0 + i}{H_0 - i}\left(1 - \frac{2i}{\gamma + i}P_M\right) = -\frac{H_0 + i}{H_0 - i}\left(P_{M^\perp} + \frac{\gamma - i}{\gamma + i}P_M\right), \tag{2.12}$$

$$\frac{1}{H - i} - \frac{1}{H_0 - i} = \frac{H_0 + i}{H_0 - i}\left(i + \frac{1}{\gamma + i}P_M\right) = \frac{H_0 + i}{H_0 - i}\left(iP_{M^\perp} + i\frac{\gamma}{\gamma + i}P_M\right). \tag{2.13}$$

*We just mention that in the case that $\theta = 0$ $H_0$ itself corresponds to the pair* $(\{0\}, 0)$, *where* $0$ *is the zero operator with the domain* $\{0\}$, *while in the case that $\theta = \pi$ $H_0$ corresponds to the pair* $(\mathcal{H}, 0)$, *where* $0$ *is the zero operator with the domain $\mathcal{H}$.*

Sometimes it is convenient to show the dependence of $H$ on $\{M, \gamma\}$ explicitly. For this purpose we use $H(M, \gamma; \theta)$ to denote the operator determined by $\{M, \gamma\}$ in the correspondence with $\theta$. When we use the notation $H(M, \gamma; \theta)$, it is tacitly assumed that the pair $\{M, \gamma\}$ satisfy the admissibility condition (2.1).

## 3. Krein's formula

**3.1. The resolvent formula.** In this section we are concerned with the case $\theta = 0$. In the previous section we established a bijective correspondence $H \leftrightarrow \{M, \gamma\}$ by using only the resolvent at $z = i$. The next problem is to derive (1.1) at general $z$. We say that a bounded operator $A$ in $M$ is boundedly invertible in $M$ if $A$ has an everywhere defined bounded inverse.

THEOREM 2. *Let $H_0$ be fixed. Then for any selfadjoint operator $H$ exists unique admissible pair $\{M, \gamma\}$, such that the resolvent of $H$ is given by*

$$\frac{1}{H(M,\gamma;0)-z} = \frac{1}{H_0-z} - \frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z}, \quad z \in \rho(H)\cap\rho(H_0), \tag{3.1}$$

*where $Q(z) = P_M \frac{1+zH_0}{H_0-z} P_M$ for $z \in \rho(H_0)$. Here $z \in \rho(H)\cap\rho(H_0)$ if and only if the operator $\gamma + Q(z)$ in $M$ is boundedly invertible.*

*Proof.* Consider any complex $z$, $\Im z \neq 0$. Let us denote by $R(z)$ the right hand side of formula (3.1)

$$R(z) := \frac{1}{H_0-z} - \frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z}.$$

Let us prove that $R(z)$ is a resolvent of a certain selfadjoint operator. Following [**11**] direct proof of the theorem can be carried out using the following two propositions.

PROPOSITION 1. *The family $R(z)$ satisfies the resolvent equation*

$$R(z) - R(w) = (z-w)R(z)R(w). \tag{3.2}$$

*Proof.* Observe first that

$$Q(z) - Q(w) = P_M(H_0-i)\left(\frac{1}{H_0-z} - \frac{1}{H_0-w}\right)(H_0+i)P_M, \tag{3.3}$$

since the operator $Q(z)$ can be written in the form:

$$Q(z) = P_M(H_0-i)\left(\frac{1}{H_0-z} - \frac{H_0}{H_0^2+1}\right)(H_0+i)P_M.$$

It follows that

$$\begin{aligned} &\frac{1}{\gamma+Q(z)} - \frac{1}{\gamma+Q(w)} \\ &= \frac{1}{\gamma+Q(z)}P_M(H_0-i)\left(\frac{1}{H_0-w} - \frac{1}{H_0-z}\right)(H_0+i)P_M\frac{1}{\gamma+Q(w)}. \end{aligned} \tag{3.4}$$

Then formula (3.2) can be proven directly using the resolvent identity for the operator $H_0$

$$(z-w)\frac{1}{H_0-z}\frac{1}{H_0-w} = \frac{1}{H_0-z} - \frac{1}{H_0-w}$$

and (3.4) as follows

$$
\begin{aligned}
& (z-w)R(z)R(w) - R(z) + R(w) \\
= \;& (z-w)\frac{1}{H_0-z}\frac{1}{H_0-w} - (z-w)\frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z}\frac{1}{H_0-w} \\
& -(z-w)\frac{1}{H_0-z}\frac{H_0+i}{H_0-w}\frac{1}{\gamma+Q(w)}P_M\frac{H_0-i}{H_0-w} \\
& +(z-w)\frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z}\frac{H_0+i}{H_0-w}\frac{1}{\gamma+Q(w)}P_M\frac{H_0-i}{H_0-w} \\
& -\frac{1}{H_0-z} + \frac{H_0+i}{H_0-z}\frac{1}{\gamma+Q(z)}P_M\frac{H_0-i}{H_0-z} \\
& +\frac{1}{H_0-w} - \frac{H_0+i}{H_0-w}\frac{1}{\gamma+Q(w)}P_M\frac{H_0-i}{H_0-w} \\
= \;& \frac{H_0+i}{H_0-z}\left\{\frac{1}{\gamma+Q(z)} - \frac{1}{\gamma+Q(w)}\right. \\
& \left.+\frac{1}{\gamma+Q(z)}P_M(H_0-i)\left(\frac{1}{H_0-z} - \frac{1}{H_0-w}\right)(H_0+i)P_M\frac{1}{\gamma+Q(w)}\right\}P_M \\
& \frac{H_0-i}{H_0-w} \\
= \;& 0.
\end{aligned}
$$

This proposition implies in particular that the kernel of the operator $R(z)$ does not depend on $z$.

Proposition 2. *The kernel of the "resolvent" operator $R(z)$ is trivial if and only if*

$$
\operatorname{Ker}\left(\frac{1}{H_0+i} - \frac{1}{\gamma+i}P_M\right) = \{0\}. \tag{3.5}
$$

*Proof.* Since the kernel of $R(z)$ is independent of $z$, consider point $z=i$. Then

$$
R(i) = \frac{H_0+i}{H_0-i}\left(\frac{1}{H_0+i} - \frac{1}{\gamma-i}P_M\right)
$$

implies that the kernel of the operator $R(i)$ coincides with the kernel of the operator $\frac{1}{H_0+i} - \frac{1}{\gamma+i}P_M$. The proposition is proven.

Condition (3.5) coincides with the admissibility condition (2.8) and therefore is satisfied for the pair $\{M,\gamma\}$. It follows that $R(z)$ is a resolvent of a certain selfadjoint operator in $\mathcal{H}$. Considering the point $z=i$ we conclude that this operator coincides with the operator $H(M,\gamma;0)$ determined by Theorem 1. Therefore (3.1) must hold. It has been already proven that the correspondence $H \leftrightarrow \{M,\gamma\}$ is a bijection.

Formula (3.1) established for all non real $z \notin \mathbf{R}$ can be written as follows

$$-\frac{H_0 - z}{H_0 + i}\left(\frac{1}{H(M,\gamma;0) - z} - \frac{1}{H_0 - z}\right)\frac{H_0 - z}{H_0 - i} = \frac{1}{\gamma + Q(z)} P_M. \tag{3.6}$$

It follows that, provided $z \in \rho(H_0)$, the operator $\gamma + Q(z)$ is boundedly invertible whenever the operator $H - z$ is boundedly invertible, i.e. for all $z \in \rho(H) \cap \rho(H_0)$. The theorem is proven. ∎

We remark that the theorem can also be verified as a consequence of results in the $\mathcal{H}_{-2}$-theory given in [**10, 11**] (see Section 4). Formula (3.1) is obtained using operator analysis and to prove it no knowledge of the extension theory is needed. This is the main difference between formula (3.1) and classical Krein's formula appearing in the extension theory.

**3.2. Krein's formula.** In the case $\theta = 0$ the approach described above is nicely related to the extension theory. We shall explain it and rewrite Theorems 1 and 2 in the situation of extension theory. Two selfadjoint extensions $H$ and $H_0$ of a closed Hermitian operator $H_{00}$ are called relatively prime if the operator $H_{00}$ coincides with the maximal common restriction of the operators $H$ and $H_0$. Using (1.4) the extensions $H$ and $H_0$ can be shown to be relatively prime if

$$H_{00} = H \wedge H_0. \tag{3.7}$$

We make a simple observation that

$$\mathrm{Ker}\left(\frac{1}{H - i} - \frac{1}{H_0 - i}\right) = (H_0 - i)\mathrm{D}\,(H \wedge H_0). \tag{3.8}$$

If $H = H(M,\gamma;0)$ in (3.8), then the left hand side of (3.8) is equal to $M^\perp$ by (2.10). This and (3.8) imply that

$$M = \left[(H_0 - i)\mathrm{D}\,(H(M,\gamma;0) \wedge H_0)\right]^\perp, \tag{3.9}$$

which in turn implies

$$\mathrm{D}\,(H(M,\gamma;0) \wedge H_0) = \frac{1}{H_0 - i} M^\perp. \tag{3.10}$$

(3.10) says that the maximal common restriction of $H(M,\gamma;0) \wedge H_0$ is determined only by $M$ independently of $\gamma$. (This occurs only in the case $\theta = 0$.) Putting in another way, we can say that given a closed subspace $M$ all relatively prime selfadjoint extensions of $H_0|_{\frac{1}{H_0 - i}M^\perp}$ are given by $H(M,\gamma;0)$.

Following F.Riesz and M.Krasnosel'skii we are going to call an operator $A$ in $\mathcal{H}$ *Hermitian* if

$$\langle Af, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in \mathrm{D}\,(A)$ without assuming that $\mathrm{D}\,(A)$ is dense in $\mathcal{H}$. The conditions for the existence of selfadjoint extensions of Hermitian operators have been studied in detail by M.Krasnosel'skii [**6**].

On the basis of these observations we can reformulate Theorems 1 and 2 as follows.

THEOREM 3. *Let $H_{00}$ be a closed Hermitian operator in a Hilbert space $\mathcal{H}$ and $H_0$ be a selfadjoint extension of $H_{00}$. Then the following* (i) *and* (ii) *hold:*
(i) *Between the set of all selfadjoint extensions $H$ of $H_{00}$ which are relatively prime to $H_0$ and the set of all selfadjoint operators $\gamma$ in the deficiency subspace $M = [(H_{00} - i)\mathrm{D}\,(H_{00})]^\perp$ which satisfy the admissibility condition* (2.8)*, there exists a*

*bijective correspondence determined by* (2.10). *Precisely, the correspondence* $\gamma \to H$ *is given by* $H = H([(H_{00} - i)\mathrm{D}(H_{00})]^{\perp}, \gamma; 0)$.
(ii) *For* $z \in \rho(H_0)$, *let* $Q(z)$ *be defined by* (1.2) *with* $M = [(H_{00} - i)\mathrm{D}(H_{00})]^{\perp}$. *Then,* $z \in \rho(H)$ *if and only if* $\gamma + Q(z)$ *is boundedly invertible in* $M$ *and the following formula holds*

(3.11)
$$\frac{1}{H([(H_{00}-i)\mathrm{D}(H_{00})]^{\perp},\gamma;0) - z} - \frac{1}{H_0 - z} = -\frac{H_0 + i}{H_0 - z}\frac{1}{\gamma + Q(z)}P_{[(H_{00}-i)\mathrm{D}(H_{00})]^{\perp}}\frac{H_0 - i}{H_0 - z}.$$

Formula (3.11) is Krein's formula. In the case where the operator $H_{00}$ is densely defined this formula has been derived by S.N.Saakjan [**14**]. See recent paper [**4**] for a comprehensive study of this problem and extensive reference list.

The bounded operator $Q(z)$ appearing in (1.1) depends analytically on $z \notin \mathbf{R}$ and has positive imaginary part in $\Im z > 0$. Really

$$\Im Q(z) = \Im z\, P_M \frac{H_0^2 + 1}{(H_0 - \Re z)^2 + \Im z^2} P_M \geq 0.$$

## 4. Application of $\mathcal{H}_{-2}$ perturbation theory

As is reviewed in the introduction results in [**11**] say that resolvent formula (1.7) gives a bijective correspondence between all selfadjoint operators $H$ and all bounded normal operators such that

$$\text{(4.1)} \qquad \operatorname{Ker}\left(\frac{1}{H_0 + i} - \tau\right) = \{0\}, \ \ \sigma(\tau) \subset \left\{ z \,\middle|\, |z + \frac{i}{2}| = \frac{1}{2} \right\},$$

where $\sigma(\tau)$ is the spectrum of $\tau$. It was also shown that $z \in \rho(H_0) \cap \rho(H)$ if and only if the operator $B(z) := 1 + (z - i)\tau\dfrac{H_0 + i}{H_0 - z}$ is boundedly invertible in $\mathcal{H}$.

Let us first consider the case that $\theta = 0$. Comparing (1.8) with (2.10) $\tau$ corresponding to $H(M, \gamma; 0)$ is given by

$$\text{(4.2)} \qquad \tau = -\frac{1}{\gamma + i}P_M.$$

In this case $B(z) = 1 - \dfrac{z - i}{\gamma + i}P_M\dfrac{H_0 + i}{H_0 - z}$. Then, if $B(z)$ is boundedly invertible in $\mathcal{H}$, then it maps $M$ onto $M$ and hence the restriction of $B(z)$ to $M$ is an operator in $M$ and it is boundedly invertible in $M$. It is then a simple matter to derive (3.1) from (1.7).

Next, let us consider the case that $\theta = \pi$. In this case we have

$$\text{(4.3)} \qquad \tau = i + \frac{1}{\gamma + i}P_M.$$

Then, $B(z)$ does not map $M$ into $M$. For this reason we cannot make much simplifications. All we can do now is to write down (1.7) with the above $\tau$ and call it the resolvent formula in the case of $\theta = \pi$:

$$\text{(4.4)} \qquad \begin{aligned} &\frac{1}{H(M, \gamma; \pi) - z} - \frac{1}{H - z} \\ &= \frac{H_0 + i}{H_0 - z}\frac{1}{1 + (z - i)\left(i + \frac{1}{\gamma+i}P_M\right)\frac{H_0+i}{H_0-z}}\left(i + \frac{1}{\gamma + i}P_M\right)\frac{H_0 - i}{H_0 - z}. \end{aligned}$$

The admissibility conditions for $\{M, \gamma\}$ and (4.1) are equivalent to each other. It would not be necessary to discuss this in detail.

## 5. Resolvent estimates

Parameterization of the selfadjoint operators as perturbations of a given selfadjoint operator $H_0$ using the pair $\{M, \gamma\}$ in the case $\theta = \pi$ leads to efficient estimates of the difference between the resolvents of the perturbed and unperturbed operators.

THEOREM 4. *Let $H_0 \in \mathcal{C}_{sa}(\mathcal{H})$ and $\theta = \pi$. Then the difference between the resolvents of the unperturbed operator $H_0$ and the perturbed operator $H = H(M, \gamma; \pi)$ determined by the admissible pair $\{M, \gamma\}$ can be estimated as follows*

$$\parallel \frac{1}{H-i} - \frac{1}{H_0 - i} \parallel_{\mathcal{H}} \leq \begin{cases} 1, & \text{if} \quad M \neq \mathcal{H}; \\ \min\{\parallel \gamma \parallel, 1\}, & \text{if} \quad M = \mathcal{H}. \end{cases} \tag{5.1}$$

*Proof.* The difference between the resolvents at point $i$ is given by (2.13)

$$\frac{1}{H-i} - \frac{1}{H_0 - i} = \frac{H_0 + i}{H_0 - i} \left( i P_{M^\perp} + i \frac{\gamma}{\gamma - i} P_M \right).$$

To estimate the norm of the operator

$$i P_{M^\perp} + i \frac{\gamma}{\gamma - i} P_M$$

we note first that this sum is orthogonal. Therefore the norm of this operator is equal to the maximum of the norms of the summands. The norm of the operator $i \frac{\gamma}{\gamma - i} P_M$ can be estimated by $\min\{\parallel \gamma \parallel, 1\}$. The orthogonal projector $P_{M^\perp}$ has norm 1 if $M \neq \mathcal{H}$. We conclude that the norm of the operator sum is equal to 1 if the subspace $M^\perp$ is not trivial and to $\min\{\parallel \gamma \parallel, 1\}$ in the opposite case. The norm of the Cayley transform $\frac{H_0 + i}{H_0 - i}$ is equal to 1 and formula (5.1) is proven. ∎

The following statement is an easy corollary of the last theorem. It can also be proven directly using estimates involving Cayley transform.

COROLLARY 1. *Let $H$ and $H_0$ be two arbitrary selfadjoint operators in the Hilbert space $\mathcal{H}$. Then the difference between the resolvents at point $i$ satisfies the estimate*

$$\parallel \frac{1}{H-i} - \frac{1}{H_0 - i} \parallel \leq 1. \tag{5.2}$$

*Proof.* The following estimates prove the corollary without using Theorem 4 but rather formula (1.11)

$$\parallel \frac{1}{H-i} - \frac{1}{H_0 - i} \parallel = \frac{1}{2} \parallel \frac{H+i}{H-i} - \frac{H_0 + i}{H_0 - i} \parallel \leq 1.$$

∎

EXAMPLE 2. *Let the original operator $H_0$ be equal to zero $H_0 = 0$ with the domain $\mathrm{D}(H_0) = \mathcal{H}$. Then the pair $\{M, \gamma\}$ is admissible only if the subspace $M$ coincides with $\mathcal{H}$. Then any operator $\gamma$ is admissible. The formula* (2.13) *reads as follows*

$$\frac{1}{H-i} = \frac{1}{-\gamma - i}.$$

*The perturbation operator $\gamma$ coincides with the operator $-H$ in this case.*

## 6. Some remarks

In this section the general correspondence is discussed in the case $\theta = 0$. The following theorem describes when does the admissibility condition (2.8) is fulfilled for all selfadjoint operators $\gamma$ provided that the closed subspace $M$ is fixed.

THEOREM 5. *Let $H_0$ be a selfadjoint operator and $M$ be any closed subspace in $\mathcal{H}$. Let $H(M,\gamma;0)$ be the selfadjoint operator parameterized by the pair $\{M,\gamma\}$. Then the following* (i) *and* (ii) *hold:*
(i) *If $M \cap \mathrm{D}\,(H_0) = \{0\}$, then the domain $\mathrm{D}\,(H(M,\gamma;0) \wedge H_0)$ is dense and any selfadjoint operator $\gamma$ in $M$ is admissible.*
(ii) *If $M \cap \mathrm{D}\,(H_0) \neq \{0\}$, then the domain $\mathrm{D}\,(H(M,\gamma;0) \wedge H_0)$ is not dense and there exist selfadjoint operator $\gamma$ in $M$, which is not admissible.*

*Proof.* Note first that formula (3.9) implies that the subspace $M$ is exactly the deficiency subspace for the maximal common restriction of the operators $H_0$ and $H(M,\gamma;0)$. Therefore the maximal common restriction $H_{00} = H(M,\gamma;0) \wedge H_0$ does not depend on the operator $\gamma$.

Suppose that $\varphi \in M \cap \mathrm{D}\,(H_0)$, $\parallel \varphi \parallel = 1$. Then every vector from $\mathrm{D}\,(H_{00})$ is orthogonal to the vector $(H_0 + i)\varphi \in \mathcal{H}$, and it follows that the operator $H_{00}$ is not densely defined. Then any selfadjoint operator $\gamma$ in $M$, which maps $\varphi \mapsto P_M H_0 \varphi$ is not admissible.

Suppose that $M \cap \mathrm{D}\,(H_0) = \{0\}$. It follows that any selfadjoint operator $\gamma$ in $M$ is admissible, since the ranges of the operators $\frac{1}{H_0+i}$ and $\frac{1}{\gamma+i}P_M$ belong to $\mathrm{D}\,(H_0)$ and $M$ respectively and the equality

$$\left(\frac{1}{H_0 + i} - \frac{1}{\gamma + i}\right) f = 0$$

implies that $f$ belongs to the kernel of the resolvent $\frac{1}{H_0+i}$ and therefore is equal to zero. Suppose that the operator $H_{00}$ is not densely defined. It follows that there exist $f \in \mathcal{H}$, such that $f \perp \mathrm{D}\,(H_{00})$. Then the vector $\varphi = \frac{1}{H_0+i} f$ belongs both to $M$ and $\mathrm{D}\,(H_0)$. Contradiction proves that no such vector $f$ exists and the operator $H_{00}$ is densely defined. The theorem is proven. ∎

The necessity of the admissibility condition was discovered first by M.Krasnosel'skii [**6**] during the studies of selfadjoint extensions of Hermitian not densely defined operators. It was proven that the admissibility condition is not needed if the restricted operator is densely defined.

Consider two extreme cases $M = \{0\}$ and $M = \mathcal{H}$. When $M = \{0\}$, the only selfadjoint operator in $M$ is the zero operator and the admissibility condition (2.8) is satisfied. We have $H(\{0\}, 0; 0) = H_0$. Next let $M = \mathcal{H}$. A pair $\{\mathcal{H}, \gamma\}$ satisfies (2.8) if and only if $\mathrm{D}\,(H_0 \wedge \gamma) = \{0\}$. In particular if $\mathrm{D}\,(H_0) \cap \mathrm{D}\,(\gamma) = \{0\}$, then $(\mathcal{H}, \gamma)$ is an admissible pair.

We mentioned that in the case of $\theta = 0$ the difference of the resolvent decreases when the norm of $\gamma$ increases. For example we have the following theorem

THEOREM 6. *Let $\gamma \in \mathcal{C}_{sa}(\mathcal{H})$ be such that $\mathrm{D}\,(\gamma) \cap \mathrm{D}\,(H_0) = \{0\}$ and zero is not an eigenvalue of $\gamma$. Then, $(\mathcal{H}, t\gamma)$ satisfies* (2.8) *for any real $t \neq 0$. Furthermore, as $t \to \pm\infty$ $H(\mathcal{H}, t\gamma; 0)$ converges to $H_0$ in the sense of strong resolvent convergence.*

*Proof.* Consider any element $f$ of the Hilbert space. Then the difference of the resolvents can be written using the spectral measure $\mu_f(\lambda)$ for the operator $\gamma$ and the element $P_M f$ as follows

$$\| \left( \frac{1}{H-i} - \frac{1}{H_0 - i} \right) f \|^2 = \int_{\mathbf{R}} |\frac{1}{t\lambda + i}|^2 d\mu_f(\lambda) \to_{t \to \pm\infty} 0,$$

since point zero is not an eigenvalue of the operator $\gamma$. The theorem is proven. ∎

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## References

[1] S.Albeverio, F.Gesztesy, R.Høegh-Krohn and H.Holden, Solvable models in quantum mechanics, Springer, 1988.

[2] S.Albeverio and P.Kurasov, Singular perturbations of differential operators, London Mathematical Society Lecture Notes vol. 271, Cambridge Univ. Press, Cambridge, 2000.

[3] F.A.Berezin and L.D.Faddeev, Remark on the Schrödinger equation with singular potential, *Dokl. Akad. Nauk SSSR*, **137**, 1011–1014, 1961.

[4] F. Gesztesy, K. A. Makarov, E. Tsekanovskii, An addendum to Krein's formula, *J. Math. Anal. Appl.*, **222** (1998), 594–606.

[5] A. Kiselev and B. Simon, Rank-one perturbations at infinite coupling, J. Funct. Anal., **128**, 1995, 245-252.

[6] M.Krasnosel'skii, On selfadjoint extensions of Hermitian operators, *Ukrain. Mat. Zurnal*, **1**, 21-38, 1949.

[7] M.Krein, On Hermitian operators whose deficiency indices are 1, *Comptes Rendue (Doklady) Acad. Sci. URSS (N.S.)*, **43**, 131-134, 1944.

[8] M.Krein, On Hermitian operators whose deficiency indices are equal to one. II, *Comptes Rendue (Doklady) Acad. Sci. URSS (N.S.)*, **44**, 131-134, 1944.

[9] M.Krein, On a remarkable class of Hermitian operators, *Comptes Rendue (Doklady) Acad. Sci. URSS (N.S.)*, **44**, 175-179, 1944.

[10] S.T.Kuroda, H.Nagatani, $\mathcal{H}_{-2}$-construction and some applications, in Mathematical Results in Quantum Mechanics, J.Dittrich, P.Exner, M.Tartar, ed., Operator Theory: Advances and Applications, Vol. 108, Birkhäuser Verlag, Basel, 1999.

[11] S.T.Kuroda, H.Nagatani, $\mathcal{H}_{-2}$-construction of general type and its applications to point interactions, to appear.

[12] M.Naimark, On spectral functions of a symmetric operator, *Bull. Acad. Sciences URSS*, **7**, 285-296, 1943.

[13] B.Pavlov, The theory of extensions and explicitly solvable models, *Uspekhi Mat. Nauk*, **42**, 99-131, 1987.

[14] S.N.Saakjan, Theory of resolvents of a symmetric operator with infinite defect numbers, *Akad. Nauk. Armjan. SSR Dokl.*, **41**, 193-198, 1965.

[15] B.Simon, Spectral analysis of rank one perturbations and applications, in: Mathematical quantum theory. II. Schrödinger operators (Vancouver, BC, 1993), 109–149, *CRM Proc. Lecture Notes*, **8**, AMS, Providence, RI, 1995.

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