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Vincenzo Micale*

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1 Introduction

Let R a Noetherian ring with $K \subset R \subseteq K[[t]]$, K a field of characteristic zero, $\overline{R} = K[[t]]$ and the conductor $\mathfrak{C} = (R : K[[t]])$ different from zero. The above conditions on R imply that R is a one-dimensional Noetherian local domain. Note that if $x \in (t) \setminus (t^2)$, then $K[[t]] = K[[x]]$. This means that $x = ut$ for some unit u of $K[[t]]$ or equivalently that we have $t = x(a_0 + a_1x + a_2x^2 + \cdots)$ or $x = t(b_0 + b_1t + b_2t^2 + \cdots)$ with $a_0b_0 = 1$. We shall without loss of generality always assume that $a_0 = b_0 = 1$.
If $v : K((t))^* \to \mathbb{Z}$ is the natural valuation for $K((t))$, that is $v\left(\sum_{h=i}^{\infty} r_h t^h\right) = i$, with $i \in \mathbb{Z}$ and $r_i \neq 0$, then $v(R) = S$ is a numerical semigroup and $v(\overline{R}) = \mathbb{N}$.
An early paper on the connection between semigroups and one-dimensional local domains is [A]. This connection has since been studied in e.g. [H-K] and an extensive study on numerical semigroups and their applications to integral domains is in [B-D-F].
Let be $S = \langle g_1, \ldots, g_n \rangle$ with $g_1, \ldots, g_n$ a minimal set of generators. Without loss of generality we may assume that $\gcd(g_1, \ldots, g_n) = 1$. By $K[[t^S]]$ we mean $K[[t^{g_1}, \ldots, t^{g_n}]]$. A ring R is called a *semigroup ring* if $R = K[[x^S]]$ for some $x \in (t) \setminus (t^2)$. In general if S is fixed and we consider all the rings R with $v(R) = S$, is not true that all these rings are semigroup rings.
In [P-S] the notion of monomial semigroup has been introduced.
We call a semigroup S in $\mathbb{N}$ a *monomial semigroup* if each subring R as above with $v(R) = S$, is a semigroup ring.
In [P-S, Theorem 3.10] is given a theoretical description and a concrete classification of the monomial semigroups, however the proof is not completely correct.

1.1 Description of the content

We now make a closer description of the content of this paper. In Section 2, we recall some known results about the numerical semigroups and we introduce $v(R)$, the value semigroup associated to a ring R. In Section 3 we give the definition of m-critical number and we use it (cf. Theorem 3.11) for a correct

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proof of [P-S, Theorem 3.10]. Moreover we introduce an invariant $P(S)$ of S and we find a bound for $P(S)$, (cf. Theorem 3.17). We have $P(S) = 0$ if and only if S is a monomial semigroup. In Section 4 we give (cf. Example 4.3) a concrete classification of the numerical semigroups with $\text{crit}(S) = 1$ and we give an example of a numerical semigroup with $P(S) = 1$ and $\text{crit}(S) > 1$.

2 Preliminaries

Let $\mathbb{N}$ denote the natural numbers. A subsemigroup S of $(\mathbb{N}, +)$ with $0 \in S$ is called a *numerical semigroup.* Each semigroup S has a natural partial ordering $\leq_S$ where for two elements s and t in S we have $s \leq_S t$ if there is a $u \in S$ such that $t = s + u$. The set $\{g_i\}$ of the minimal elements in $S \setminus \{0\}$ in this ordering is called a *a minimal set of generators* for S. In fact all elements of S are linear combination with non-negative integers coefficients of minimal elements. The set $\{g_i\}$ of minimal generators is finite since for any $s \in S$, $s \neq 0$, we have $g_i \not\equiv g_j \pmod{s}$. The same argument shows that the number of minimal generators is at most $\min\{s \in S \mid s \neq 0\}$. We denote the semigroup generated by $g_1, g_2, \ldots, g_n$ by $\langle g_1, g_2, \ldots, g_n \rangle$. Since the semigroup $\langle g_1, g_2, \ldots, g_n \rangle$ is isomorphic to $\langle dg_1, dg_2, \ldots, dg_n \rangle$ for any $d \in \mathbb{N} \setminus \{0\}$, we assume, in the sequel, that $\gcd(g_1, g_2, \ldots, g_n) = 1$. This is easily seen to be equivalent to $|\mathbb{N} \setminus S| < \infty$.
For a semigroup S we denote $g(S) := \max\{x \in \mathbb{Z} \mid x \notin S\}$. This number is often called the *Frobenius number* of S.
For a semigroup S we denote by $g - S$ the set of numbers $\{g(S) - s \mid s \in S\}$. Clearly we have $S \cap (g - S) = \emptyset$.
The semigroup S is called *symmetric* if $S \cup (g - S) = \mathbb{Z}$. There are several alternative descriptions of the concept of symmetric semigroup (cf. [F-G-H, Lemma 1.1]). It is classically known (cf. [S]) that $S = \langle g_1, g_2 \rangle$ is a symmetric semigroup.
Since $|\mathbb{N} \setminus S| < \infty$, there exists in S elements s such that the set $\{s, s+1, \longrightarrow\} \subseteq S$ (where the symbol "$\longrightarrow$" means that all subsequent natural numbers belong to the set). We call the *conductor* of S, the minimal of such elements s and denote it with c. Clearly, from the definition of Frobenius number, we have $c = g(S)+1$.
Throughout the rest of the paper we will assume $R \subseteq K[[t]]$ be a Noetherian domain with K field of characteristic zero, the conductor of R in $K[[t]]$, that is the greatest ideal of R and $K[[t]]$, be different from zero, $K \subset R$ and $K[[t]]$ as integral closure.
We call $v(R) := \{v(r) \mid r \in R\}$ the *value semigroup associated* to R. It is clear from the definition of t-adic valuation that if $S = \langle g_1, g_2, \ldots, g_n \rangle$ is any nonzero numerical semigroup, then every semigroup ring $K[[t^S]]$ has as valuation the semigroup S. However not every R of our type is a semigroup ring, e.g. $R = K[[t^4, t^6 + t^9, t^{11}]]$ has $v(R) = \langle 4, 6, 11 \rangle$ but, as we will show, R is not a semigroup ring.

3 The main theorems

Throughout the rest of the paper we will assume that $g_1 < g_2 < \cdots < g_n$ is a minimal system of generators for S and that $\gcd(g_1, \ldots, g_n) = 1$; moreover we let g_s denote the greatest generator of S less than the conductor.

The following are easy to see:

$$\text{if } [a, a + g_1 - 1] \subseteq S, \text{ then } [a, \infty) \subseteq S, \text{ i.e. } a \geq c \tag{3.1}$$

$$\text{if } g \in [g_1, g_1 + g_2 - 1] \cap S \setminus g_1\mathbb{N}, \text{ then } g = g_i \text{ for some } 1 \leq i \leq n \tag{3.2}$$

We say that a natural number k is a *critical number* for g_i if $g_i + k \notin S$. In general, we call k an *m-critical number* if it is critical number for m generators of S.

In [P-S, Theorem 3.10] is given a theoretic and a concrete description of monomial semigroups. However, the proof of the theorem of characterization of monomial semigroup is not completely correct. We will give a correct proof of the theorem, giving a more intuitive theoretical description of the monomial semigroups. To this purpose we prove the following lemma in which the condition (i), present in [P-S, Theorem 3.10], is replaced by other more evident conditions.

Lemma 3.1. *Let $S = \langle g_1, \ldots, g_n \rangle$ be a numerical semigroup. Then the following are equivalent:*

(i) If $x \in \mathbb{N} \setminus S$ and $c(x) := \min\{n \in \mathbb{N} \mid [n, \infty) \subseteq S \cup (x + S)\}$, then $S \cap (x + S) \subseteq [c(x), \infty)$.

(ii) For every $k \geq 0$ and for every (i, j), with $i \neq j$ and $i, j = 1, \ldots, n$, we have that $g_i + k \in S$ or $g_j + k \in S$.

(iii) Every integer $k \geq 0$ is a critical number for at most one generator of the semigroup.

(iv) If a and b are in S with $a > b$ and $a - b \notin S$, then $a + k \in S$ or $b + k \in S$ for every integer $k \geq 0$.

Proof. (i) $\Rightarrow$ (iv): Let be a and b in S with $b < a$ and such that $a - b = x \notin S$. We have to prove that $a + k \in S$ or that $b + k \in S$ for every integer $k \geq 0$. Since $a \in S \cap (S + x)$, we have $c(x) \leq a$. Hence $a + k \in S \cup (x + S)$ for every integer $k \geq 0$.

(iv) $\Rightarrow$ (i): Let be $y \in S \cap (x + S)$. We have $y \geq c(x) \Leftrightarrow y + k \in S \cup (x + S)$ for every integer $k \geq 0 \Leftrightarrow y + k \in S$ or $(y - x) + k \in S$ and the last statement is true because $y - (y - x) = x \notin S$.

(ii) $\Rightarrow$ (iv): Let a and b be in S, with $a > b$, such that $a - b \notin S$. Suppose there exists an integer $k \geq 0$ such that $a + k \notin S$ and $b + k \notin S$. Since a and b are in S, they are combination of generators of S. But if they are combination of more than one generator or if they are multiple of different generators, then by (ii) we have that $a+k$ or $b+k$ are in S, that is a contradition. Hence we only consider the case $a = \alpha g_i$ and $b = \beta g_i$. But in this case $a - b$ is in S. Absurd.

(iv) $\Rightarrow$ (ii): Trivial, since g_i and g_j are minimal generators.

(ii) $\Leftrightarrow$(iii): Trivial from definition of critical number.

Lemma 3.2. *Let $S = \langle g_1, \ldots, g_n \rangle$ be a numerical semigroup. If the conditions of Lemma 3.1 are true, then $g_i + g_j + k \in S$ for every integer $k \geq 0$ and for every (i, j), with $i \neq j$ and $i, j = 1, \ldots, n$. (i.e. $g_i + g_j \geq c$, where c is the conductor of S)*

Proof. It is enough to prove the lemma for $i = 1$ and $j = 2$. If there exists an integer $k > 0$ such that $g_1 + g_2 + k \notin S$ then $g_1 + k \notin S$ and $g_2 + k \notin S$. Hence we have a contradition to (ii) of 3.1.

From now on we denote by K_i the set of critical numbers of g_i. It is for us an important set and we use it many times in the paper.

Remark 3.3. Let K_i be as above, where $i = 1, \ldots, s$. If $S = \langle g_1, \ldots, g_n \rangle$ is a numerical semigroup, with g_s the generator above, then it easy to see that every ring R with $v(R) = S$, has a unique canonical representation of this sort:

$$R = K[[f_1, \ldots, f_s, t^r \mid r \geq c]]$$

where $f_i = t^{g_i} + \sum_{k_{i_j} \in K_i} a_j t^{g_i + k_{i_j}}$. In fact if there exists another representation, say

$$R = K[[f'_1, \ldots, f'_s, t^r \mid r \geq c]]$$

where $f'_i = t^{g_i} + \sum_{k_{i_j} \in K_i} b_j t^{g_i + k_{i_j}}$, we would have $f_i - f'_i = \sum_{k_{i_j} \in K_i} (a_i - b_i) t^{g_i + k_{i_j}} \in R$ contradicting $g_i + k_{i_j} \notin S$.

Remark 3.4. If $S = \langle g_1, \ldots, g_n \rangle$ and k is a m-critical number with $m \geq 2$ and g_i is generator which has k as critical number, it is not always true that $R = K[[t^{g_i} + \alpha t^{g_i + k}, t^{g_j} \mid j \in \{1, \ldots, n\} \setminus \{i\}]]$ has associated the semigroup S.

Example 3.5. Consider $S = \langle 4, 6, 15, 17 \rangle$ and let $R = K[[t^4 + t^5, t^6, t^{15}, t^{17}]]$. We have that $(t^4 + t^5)^3 - (t^6)^2 = 3t^{13} + 3t^{14} + t^{15} \in R$, but 13 is not in S.

This is the mistake in the proof of [P-S, Theorem 3.10]. We prove now that the statement becomes true for a right choose of k and g_i. The following lemma is generalization of Lemma 3.2.

Lemma 3.6. *Let $S = \langle g_1, \ldots, g_n \rangle$ be a numerical semigroup with conductor c. Consider the sets, possibly empty, $K_{i,j}$ of critical numbers for both g_i and g_j and let $h_{i,j}$ be the greatest element in $K_{i,j}$, where $h_{i,j} = -1$ if $K_{i,j} = \emptyset$. Then $g_i + g_j + h_{i,j} \geq c - 1$.*

Proof. By the choice of $h_{i,j}$ we have $g_i + h_{i,j} + x \in S$ or $g_j + h_{i,j} + x \in S$ for every $x \geq 1$, so $g_i + g_j + h_{i,j} + x \in S$, hence $g_i + g_j + h_{i,j} + 1 \geq c$ that is $g_i + g_j + h_{i,j} \geq c - 1$.

Lemma 3.7. *Let k be the greatest m-critical number of S with $m \geq 2$ and let g_i the greatest generator of S which has k as critical number. Then if $R = K[[t^{g_i} + t^{g_i+k}, t^{g_j} \mid j \in \{1, \ldots, n\} \setminus \{i\}]]$, we have $v(R) = S$.*

Proof. The only possibility to get a value outside S is to have in R an element $y = [\prod_{j \neq i}(t^{g_j})^{m_j}](t^{g_i} + t^{g_i+k})^n - \prod_{j \neq i}(t^{g_j})^{n_j}$ where $\sum_{j \neq i} g_j m_j + ng_i + k = \sum_{j \neq i} n_j g_j + k \notin S$. By factoring out common factors, we may assume that $n_j = 0$ if $m_j \neq 0$. Now $y = t^{(\sum_{j \neq i} g_j m_j)+ng_i+k} + \cdots$. We know by Lemma 3.6 that $g_i + g_j + k \geq c - 1$ if k is the greatest critical number for g_i and g_j and $i \neq j$. Thus we have to consider only four cases (we suppose that $g_j < g_i$ and $g_t < g_r$):

- $g_i + g_j + k = g_r + g_t + k = c - 1$ with $r, t \neq i, j$
- $g_i + g_j + k = dg_r + k = c - 1$ with $r \neq i, j$ $d > 1$
- $ng_i + k = g_r + g_t + k = c - 1$ with $r, t \neq i$ and $n > 1$
- $ng_i + k = dg_r + k \notin S$ with $i \neq r$ and $n, d > 1$.

Consider the first case. We get $g_i + k = g_t + g_r + k - g_j$. We have $g_i + k \notin S$ and $g_r + k - g_j > k$. Since $g_r + k = g_j + (g_r + k - g_j) \notin S$ and $g_i + k = g_t + (g_r + k - g_j)$, we get that $g_r + k - g_j$ is critical for both g_j and g_t, which is a contradiction to the fact that k is the largest m-critical number for some $m > 1$.
Consider the second case. We have $g_i + k = g_r + (d-1)g_r + k - g_j$. We get that $(d-1)g_r + k + g_j$ is critical for g_r and $(d-1)g_r - g_j + k > k$, so $(d-1)g_r - g_j + k$ cannot be critical for g_j, thus $g_j + (d-1)g_r - g_j + k = (d-1)g_r + k \in S$ and $g_r + (d-1)g_r + k = dg_r + k \in S$, a contradiction.
Consider the third case. Then we have $g_r + k = ng_i + k - g_t$. Since $(n-1)g_i + k - g_t > k$, and $(n-1)g_i + k - g_t$ is critical for g_i, it cannot be critical for g_t, so $g_t + (n-1)g_i + k + g_t = (n-1)g_i + k \in S$, so $g_i + (n-1)g_i + k = ng_i + k \in S$, a contradiction.
Consider now the last particular case. By Lemma 3.6 we have $g_i + g_r + k = c - 1$, but $g_i + g_r + k < g_i + g_i + k$, hence $ng_i + k \in S$. We conclude that for every y as above, $v(y) \in S$, that is $v(R) = S$.

We recall that a ring R is called a semigroup ring if $R = K[[x^S]]$ for some $x \in (t) \setminus (t^2)$. Let $S = \langle g_1, g_2, \ldots, g_n \rangle$ be a numerical semigroup. We call a polynomial $f(t) = \sum a_i t^i \in K[[t]]$ an *S-polynomial* in t if $a_i \neq 0$ implies $i \in S$.

Lemma 3.8. *$K[[f_1(t), f_2(t), \ldots]] = K[[t^S]]$ if and only if all the $f_i(t)$ are S-polynomial in t.*

The proof is trivial.

Lemma 3.9. *Let $S = \langle g_1, \dots, g_n \rangle$ be a numerical semigroup and let r an integer that is not a critical number in S. If $f_i(x)$ is an S-polynomial, then $f_i'(y) = f_i(y(1 - c_r y^r))$ is an S-polynomial.*

Proof. Since $f_i(x)$ is an S-polynomial in x, we have $f_i(x) = x^{s_1} + d_{s_2} x^{s_2} + \cdots$, with $s_i \in S$. We can restrict to a monomial in $f_i(x)$. Let $s \in S$, then $x^s = y^s(1 - c_r y^r)^s = y^s + \sum_{i \geq 1} d_i y^{s+ir}$, hence for the definition of critical number and S-polynomial, we have the proof.

Lemma 3.10. *Let k be an m-critical number of S with $m > 1$ and let g_i a generator of S which has k for critical number. If $R = K[[f_i(t) = t^{g_i} + t^{g_i+k}, f_j(t) = t^{g_j} \mid j \in \{1, \dots, n\} \setminus \{i\}]]$, then R is not a semigroup ring.*

Proof. Suppose that R is a semigroup ring. Then there exists an $x \in (t) \setminus (t)^2$, that is $t = x(1 + a_r x^r + \cdots)$, such that $R = K[[x^S]]$. We know that $R = K[[f_1'(x), f_2'(x), \dots]]$, where, by Lemma 3.8, $f_i'(x) = x^{g_i} + \cdots$ is an S-polynomial for every i. If $r > k$, then we get a contradiction by Lemma 3.8. In fact $f_i'(x) = x^{g_i} + x^{g_i+k} + \cdots$.
If $r = k$ and g_j is a generator, different from g_i, which has k for critical number, we get a contradiction by Lemma 3.8. In fact $f_j'(x) = x^{g_j} + g_j a_r x^{g_j+r} + \cdots$.
Thus $r < k$. Then r is not a critical number. In fact if r is a critical number for g_d, then we get a contradiction by Lemma 3.8 since $f_d' = x^{g_d} + g_d a_r x^{g_d+r} + \cdots$. So r is not a critical number. We choose x such that $K[[f_1'(x), f_2'(x), \dots]] = K[[x^S]]$ and $t - x \in (t)^{r+1}$ with r as big as possible. Let y such that $x = y(1 - a_r y^r)$ (it easy to see that such y exists). Then $t = x(1 + a_r x^r + \cdots) = y(1 - a_r y^r)(1 + a_r y^r(1 - a_r y^r)^r + \cdots) = y(1 + a_b y^b + \cdots)$ with $b > r$. By Lemma 3.9 we have $K[[f_1''(y), f_2''(y), \dots]] = K[[y^S]]$ and we get a contradiction by the definition of r.

Now we are ready to give a new version of [P-S, Theorem 3.10].

Theorem 3.11. *Let $S = \langle g_1, \dots, g_n \rangle$ be a numerical semigroup. Then the following are equivalent:*

- *(i) S is a monomial semigroup.*
- *(ii) S is a semigroup from the following list:*
  - *(1) S is such that the only elements below the conductor are multiples of g_1,*
  - *(2) $x \notin S$ only for one $x > g_1$,*
  - *(3) The only elements greater than g_1 that are not in S are $g_1 + 1$ and $2g_1 + 1$ and $g_1 \geq 3$.*
- *(iii) S satisfies the conditions of the Lemma 3.1*

Proof. (i) $\Rightarrow$ (iii): Suppose (iii) is not true. Hence there exist integers that are m-critical numbers with $m \geq 2$. Let k be the greatest of these integers and let g_i the greatest generator which has k for critical number. We have to show that there exists a ring R with associated semigroup S, which is not a semigroup ring. Let $R = K[[t^{g_i} + t^{g_i+k}, t^{g_j} \mid j \in \{1, \dots, n\} \setminus \{i\}]]$. By Lemma 3.7 we have $v(R) = S$ and by Lemma 3.10, R is not a semigroup ring.

(iii) $\Rightarrow$ (i): Let $S = \langle g_1, \dots, g_n \rangle$ be a numerical semigroup, and let $K_i = \{k \geq 0 \mid g_i + k \notin S\} = \{k_{i_1}, \dots, k_{i_{n_i}}\}$ for $i = 1, \dots, s$.

By Remark 3.3 we know that every ring R associated with S has a unique canonical representation of this sort:

$$R = K[[f_1, \dots, f_s, t^r \mid r \geq c]]$$

where $f_i = t^{g_i} + \sum_{k_{i_j} \in K_i} a_{i_j} t^{g_i + k_{i_j}}$. Let $K = \cup K_i$. By (iii) this is a disjoint union of K_i. Let k_{i_j} be the minimal element of K such that a_{i_j} is different from zero. By $t = x(1 - (a_{i_j}/g_i)x^{k_{i_j}})$ with $x \in (t) \setminus (t)^2$ we have $R = K[[f_1'(x), \dots, f_s'(x), x^r \mid r \geq c]]$ where $f_i'(x) = x^{g_i} + \sum_{h_{i_j} \in K_i} b_{i_j} x^{g_i + h_{i_j}}$ and where $\min\{h_{i_j} \mid b_{i_j} \neq 0\} > \min\{k_{i_j} \mid a_{i_j} \neq 0\}$. Since $|K| < \infty$, proceeding in the same way, we get R is a semigroup ring.

(ii) $\Rightarrow$ (iii): This is an easy case by case check.

(iii) $\Rightarrow$ (ii): Suppose that S satisfies (iii). If $g_2 > 2g_1$ then $\{1, \dots, g_1 - 1\}$ are critical numbers for g_1, hence $[g_2, g_2 + g_1 - 1] \subseteq S$ and $g_2 \geq c$ follows from (3.1). In this case S is of the type (1). Suppose next that $g_2 < 2g_1$. If $g_1 = 2$, then $S = \langle 2, 3 \rangle$ is of the type (1).

Otherwise $g_1 \geq 3$. Suppose first $g_1 + 1 = g_2$. If $g_2 \geq c$, then S is of the type (1). Otherwise $g_2 < c - 1$. So there exists a critical number k with $2 \leq k < g_1 - 1$ for g_1 with $[g_1, g_1 + k - 1] \subseteq S$ hence $[2g_1, 2g_1 + 2k - 2] \subseteq S$. So $[g_1 + k + 1, 2g_1 + 2k - 2] \in S$ and S is of the type (2) by (3.1). It remains now to consider the case $g_2 = g_1 + b$, with $b > 1$. If $b \geq 3$ we get that 1 and 2 are critical numbers for g_1, then $[g_2, g_2 + g_1 - 1] \subseteq S$ and S is of the type (1) by (3.1).

Now $b = 2$. Since 1 is a critical number for g_1, we have $[g_2, 2g_1] \subseteq S$. If $2g_1 + 1 \in S$, then $[g_2, \infty) \subseteq S$ by (3.1), so S is of the type (1). If $2g_1 + 1 \notin S$, then S is of the type (3) by (3.1). In fact $3g_1 + 1 = (2g_1 - 1) + g_2$, hence $[2g_1 + 2, 3g_1 + 1] \in S$.

Remark 3.12. In the notation of the paper by Pfister and Steenbrink we have that our semigroups S of type (1) are their class $S_{m,s,b} := \{im \mid i = 0, 1, \dots, n\} \cup [sm + b, \infty)$ with $1 \leq b < m, s \geq 1$, our semigroups of type (2) are theirs $S_{m,r} := \{0\} \cup [m, m + r - 1] \cup [m + r + 1, \infty)$ with $2 \leq r \leq m - 1$ and our semigroups of type (3) are theirs $S_m := \{0, m\} \cup [m + 2, 2m] \cup [2m + 2, \infty)$ with $m \geq 3$.

Now we show some sufficient conditions for $v(R) = S$.

Theorem 3.13. *If for every set $\{g_{i_1}, \dots, g_{i_e}\}$ of generators with the same critical number k, $\min\{(g_{i_j} + S) \cap \langle g_{i_1}, \dots, g_{i_{j-1}}, g_{i_{j+1}}, \dots, g_{i_e} \rangle\} \geq c$ then $v(R) = S$.*

Proof. In fact the only possibility to have $S \subset v(R)$ is that $g_{i_j} + \sum a_r g_r + k = \sum_{r \neq i_j} b_r g_r + k \notin S$.

Corollary 3.14. *If for every pair (g_i, g_j) with the same critical number k, $g_i + g_j + k \geq c$ then $v(R) = S$.*

Corollary 3.15. *Let $S = \langle g_1, g_2 \rangle$. Then $v(R) = S$.*

Proof. In fact the only way to have $S \subset v(R)$ is that there exist a, b and k with $a > b$ and k critical number for g_1 and g_2, such that $ag_1 = bg_2$, $ag_1 + k \notin S$ and $a \geq 3$. Since $\gcd(g_1, g_2) = 1$, we have $a = ng_2$ and $b = ng_1$. Hence $ag_1 + k$ is in S for every $k \geq 0$, since the conductor of S is $g_1 g_2 - g_1 - g_2 + 1$.

Corollary 3.16. *If g_1 and g_2 are the only elements of S below the conductor and $\gcd(g_1, g_2) = 1$, then $v(R) = S$.*

Proof. Use the same argument of the Corollary 3.15, knowing that, in this case, the conductor is less than or equal to $g_1 g_2 - g_1 - g_2 + 1$.

Our aim is to find presentations of the rings of our class, which are as easy as possible. We have seen that a ring associated to a monomial semigroup is a semigroup ring. In general, if R is such that $v(R) = S$ and has a presentation $R = K[[f_1, \dots, f_s, t^r \mid r \geq c]]$ with $f_i = t^{g_i} + \sum_{k_{i_j} \in K_i} a_{i_j} t^{g_i + k_{i_j}}$, where K_i is the set of critical numbers for g_i, we want minimize the number of nonzero a_{i_j}. We denote with $P_x(R)$ the minimal number of these coefficients in the canonical form of R with $x \in (t) \setminus (t^2)$.
We define $P(R) := \min\{P_x(R) \mid x \in (t) \setminus (t^2)\}$.
We denote by $P(S) := \max\{P_R \mid v(R) = S\}$. We note that $P(S) = 0$ if and only if S is monomial. Let $K = \cup K_i$, and let $m_k = \mathrm{card}\{g_j \mid k \in K_j\}$.
We call $\mathrm{crit}(S) = \sum_{k \in K}(m_k - 1)$.

The following theorem shows that $P(S) \leq \mathrm{crit}(S)$.

Theorem 3.17. *Let R be a ring with semigroup S. Then R has a presentation $R = K[[f_1, \dots, f_s, t^r \mid r \geq c]]$, $f_i = t^{g_i} + \sum_{k_{i_j} \in K_i} a_{i_j} t^{g_i + k_{i_j}}$, with at most crit$(S)$ non zero coefficients a_{i_j}.*

Proof. Let $K = \{k_1, \dots, k_r\}$ be the set of critical numbers of S, with $k_1 < \dots < k_r$. Let g_j be one of the generators which have k_1 as critical number. By $t = x(1 - (d/g_j)x^{k_1})$, where d is the coefficient of $t^{g_j + k_1}$ in $f = t^{g_j} + dt^{g_j + k_1} + \dots$ and $x \in (t) \setminus (t)^2$, we have that the coefficient of $x^{g_j + k_1}$ in $f'(x)$ becomes equal to zero. Repeating for $k_2, k_3, \dots, k_r$ in order, we have the proof.

Corollary 3.18. *Let S a semigroup, if $\mathrm{crit}(S) = 1$, then $P(S) = 1$*

Proof. It follows immediately by Theorems 3.11 and 3.17.

4 Classification of the numerical semigroups with $\mathrm{crit}(S) = 1$

In this section we show a method to get a classification of semigroups with $\mathrm{crit}(S) = 1$ and we use it to produce many examples of semigroups with $P(S) = 1$. We will give also an example of a numerical semigroup with $P(S) = 1$ but $\mathrm{crit}(S) > 1$.

Lemma 4.1. *Let S be a semigroup and let S' be the semigroup obtained from S adding $g(S)$, the Frobenius number of S. then* $\mathrm{crit}(S') \leq \mathrm{crit}(S)$.

Proof. Follows from the definition of $\mathrm{crit}(S)$.

Remark 4.2. By Lemma 4.1, we have a method for a concrete classification of semigroups S with $\mathrm{crit}(S) = 1$, knowing, by Theorem 3.11, the classification of all monomial semigroups S. Some of the semigroups S with $\mathrm{crit}(S) = 1$ are obtained deleting a generator g from a monomial semigroup S', such that $g(S') + 1 \leq g \leq g(S') + g_1$. We will call this "deleting a large generator". In this way we find all the semigroups with $\mathrm{crit}(S) = 1$ for which, adding their number of Frobenius, the semigroups S' just obtained is monomial. Some others semigroups with $\mathrm{crit}(S) = 1$ are obtained deleting a generator g from all semigroups S' with $\mathrm{crit}(S') = 1$ just obtained, such that $g(S') + 1 \leq g \leq g(S') + g_1$. In this way we find all the semigroups S with $\mathrm{crit}(S) = 1$ for which, adding their number of Frobenius, the semigroups S' just obtained have $\mathrm{crit}(S') = 1$. And so on for every semigroup with $\mathrm{crit}(S') = 1$ just obtained. We will show that after a finite number of steps, we will find all the semigroups with $\mathrm{crit}(S) = 1$.

Now we use the method above.

Example 4.3. Consider the monomial semigroups S of the type (1) in Theorem 3.11, that is $S = S_{m,s,b}$ by Remark 3.12. We get, by deleting a large generator, seven different classes of semigroups with $\mathrm{crit}(S) = 1$:

- (IA) $S = \{0, m, 2m, \ldots, sm, sm + b, sm + b + 2, \longrightarrow\}$, with $s > 1$, $m \geq 4$ and $1 \leq b < m$.
- (IB) $S = \{0, m, m + 2, \ldots, m + 2 + x - 1, m + 2 + x + 1, \longrightarrow\}$, with $1 \leq x < m - 2$ and $m \geq 4$.
- (IC) $S = \{0, m, m + b, m + b + 2, \longrightarrow\}$, with $m \geq 4$ and $1 \leq b < m$.
- (ID) $S = \{0, m, 2m, \ldots, sm, (s + 1)m - 1, (s + 1)m, (s + 1)m + 2, \longrightarrow\}$, with $s > 1$ and $m \geq 3$.
- (IE) $S = \{0, m, 2m - 1, 2m, 2m + 2, \longrightarrow\}$, with $m \geq 4$.
- (IF) $S = \{0, m, m + 3, \ldots, 2m + 1, 2m + 3, \longrightarrow\}$, with $m \geq 3$.

- (IG) $S = \{0, m, 2m, \ldots, sm, (s+1)m, (s+1)m+1, (s+1)m+3, \longrightarrow\}$, with $m \geq 3$ if $s \geq 1$ and $m \geq 4$ if $s = 0$.

If we delete a large generator from (IB) we have two new classes of semigroups with $\mathrm{crit}(S) = 1$

- (IBa) $S = \{0, m, m+2, \ldots, m+1+x, m+4+x, \longrightarrow\}$, with $x > 1$and $m \geq 4$

- (IBb) $S = \{0, m, m+2, m+4, \ldots, 2m-1, 2m, 2m+1, 2m+2, 2m+4, \longrightarrow\}$, with $m \geq 4$.

If we delete a large generator from (ID), we have a new semigroup with $\mathrm{crit}(S) = 1$:

- (IDa) $S = \langle 3, 8 \rangle$

If we delete a large generator from the (IE), we have a new semigroup with $\mathrm{crit}(S) = 1$

- (IEa) $S = \langle 4, 7, 13 \rangle$,

If we delete 11 from (IF) with $m = 3$, we have a new semigroup with $\mathrm{crit}(S) = 1$:

- (IFa) $S = \langle 3, 7 \rangle$

If we delete any generator from (IA), (IBa), (IBb), (IC), (IDa), (IEa), (IFa) or (IG), we have no new semigroup with $\mathrm{crit}(S) = 1$.

Consider now the monomial semigroup of the type (2) on the Theorem 3.11, that is $S = S_{m,r}$ by Remark 3.12. We get, by deleting a large generator, two different classes of semigroups with $\mathrm{crit}(S) = 1$:

- (IIA) $S = \{0, m, m+1, m+4, \longrightarrow\}$, with $m \geq 4$

- (IIB) $S = \{0, m, m+1, m+3, m+5, \longrightarrow\}$, with $m \geq 5$

If we delete a large generator from (IIA), we have a new class of semigroups and a new semigroup with $\mathrm{crit}(S) = 1$:

- (IIAa) $S = \{0, m, m+1, m+4, \ldots, 2m-1, 2m, 2m+1, 2m+2, 2m+4, \longrightarrow\}$, with $m \geq 5$

- (IIAb) $S = \langle 4, 5 \rangle$

If we delete any large generator from (IIAa), (IIAb) or (IIB) we have no new semigroup with $\mathrm{crit}(S) = 1$.

Consider the monomial semigroup of the type (3) of the Theorem 3.11, that is $S = S_m$ by Remark 3.12.

We can not delete any large generator because all generators are below the conductor.

By Corollary 3.18 we have $P(S) = 1$ if $\mathrm{crit}(S) = 1$. We want to show that a semigroup with $P(S) = 1$ and $\mathrm{crit}(S) > 1$ is $S = \langle 4, 6, 11\rangle$.

Example 4.4. Let $S = \langle 4, 6, 11\rangle$. We have that $\mathrm{crit}(S) = 2$ and 1 and 3 are the only m-critical numbers with $m > 1$ (in this case $m = 2$). We know By Remark 3.3 that R has a canonical rapresentation of this form:

$R = [[f_1 = t^4 + at^5 + bt^7 + ct^9 + dt^{13}, f_2 = t^6 + et^7 + ft^9 + gt^{13}, f_3 = t^{11} + ht^{13}, t^r \mid r \geq 14]]$. Since $v(R) = S$ and $f_1^3 - f_2^2 \in R$, we have that $e = (3/2)a$. By $t = x(1-(a/4)x)$ we have $R = [[f_1' = x^4 + ax^7 + bx^9 + cx^{13}, f_2' = x^6 + dx^9 + ex^{13}, f_3' = x^{11} + fx^{13}, x^r \mid r \geq 14]]$. Since S is not monomial and using the same argument as in the proof of the Theorem 3.17, we have that $P(S) = 1$.

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