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On the number of ideals of finite colength

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1 Introduction

If I is an ideal of a (commutative) ring R and $l_R(R/I) = h$, we say that I has colength h. Maximal ideals have colength 1, and there may be many other ideals of finite colength even in non-Noetherian rings. If R is a one-dimensional Noetherian domain, every non-zero ideal has finite colength.

We are interested in the class of rings in which there is a finite number of ideals for each finite colength, and how this number grows as a function of h. In Artinian rings there are no ideals of colength h, if $h >> 0$ and we show that, in a Noetherian ring of dimension at least two, the number of ideals of colength h grows exponentially with h (for a precise statement, cf. Theorem 2.8). If R has a finite number of ideals for each finite colength, then R is semilocal and each localization at a maximal ideal has the same property. Thus the one-dimensional (Noetherian) local rings are a natural class of rings to investigate. For a large natural subclass of those, we give a precise measure of the growth: if (R, m) is a one-dimensional analytically unramified residually rational local ring with finite residue field and the integral closure $\bar{R}$ has d maximal ideals, with $|R/m| \geq d$, we prove that the number of ideals of colength h is a polynomial of degree $d-1$ in h, if $h \geq l_R(\bar{R}/R : \bar{R})$ (cf. Theorem 3.7). In particular, when $d = 1$, i.e. when R is analytically irreducible, the number of ideals of colength h is, for large h, a constant that of course depends on the cardinality of the residue field of R.

All the information on the number of ideals of finite colength of a ring R can be collected in a generating function, the colength series of R, which in the case of our subclass of rings has a nice form.

To prove the mentioned results, we use the value semigroup associated to a one-dimensional analytically unramified ring and we refer for that to [1].

2 Generalities

Let R be a (not necessarily Noetherian) ring. We consider ideals J of R of finite colength (i.e. $l_R(R/J) = h < \infty$). This is equivalent to say that R/J is an Artinian ring.

Notice that many ideals of finite colength may exist also in non-Noetherian rings. $R = \mathbb{Z} + X\mathbb{Q}[[X]]$ is an example of a non-Noetherian ring with ideals of any colength $h \in \mathbb{N}$. As a matter of fact $R/X\mathbb{Q}[[X]] \cong \mathbb{Z}$ and so, if p is a prime in $\mathbb{Z}$, $p^h R = p^h\mathbb{Z} + X\mathbb{Q}[[X]]$ is an ideal of R of colength h.

Lemma 2.1 *Suppose that $J \subseteq I$ are ideals of a quasilocal ring (R, m) with $l_R(I/J) = 1$. Then $Im \subseteq J$.*

Proof. If I is finitely generated, this follows from Nakayama's lemma, but the statement is always true. Let $t \in I \setminus J$. Then $t \notin J + tm$, since otherwise $t = j + tm_1, j \in J, m_1 \in m$, so $t(1 - m_1) = j \in J$, and $t \in J$ since $1 - m_1$ is invertible. Since $J \subseteq J + tm \subset I$ and the last inclusion is proper, the first inclusion cannot be proper, and we get $J + tm = J$ for all $t \in I$, so $Im \subseteq J$.

Proposition 2.2 *Let (R, m) be a quasilocal ring. If J is an ideal of colength h, then $m^h \subseteq J$. In particular J is m-primary.*

Proof. We use induction on h. If $h = 1$, then $J = m$. Suppose $l_R(R/J) = h$ and that the statement is proved for ideals of colength $h - 1$. Choose an ideal $I \supseteq J$ of colength $h - 1$. Then $m^{h-1} \subseteq I$, so $m^h \subseteq mI \subseteq J$ by Lemma 2.1.

Corollary 2.3 *Let R be a ring with a finite number of ideals for each colength $h \in \mathbb{N}$ and let m be a maximal ideal, then the localization R_m has also a finite number of ideals for each colength $h \in \mathbb{N}$.*

Proof. By Proposition 2.2, ideals in R_m of finite colength are mR_m-primary, and there is a 1-1 correspondence between mR_m-primary ideals Q_m in R_m and m-primary ideals Q in R, and $l_{R_m}(R_m/Q_m) = l_R(R/Q)$.

If the ring (R, m) is Noetherian, then each m-primary ideal is of finite colength, but in general this is not true. By Proposition 2.2, if the maximal ideal of R is idempotent, i.e. $m = m^2$ (this happens for example in a one-dimensional non-discrete valuation domain), then the only ideal of finite colength is the maximal ideal, but each non-zero ideal is m-primary. However, if we restrict to Noetherian rings, we get:

Proposition 2.4 *Let (R, m) be a local (i.e. quasilocal and Noetherian) ring. Then there exists, for each $h \in \mathbb{N}$, an ideal of colength h.*

Proof. By induction on h. Let I be an ideal of colength $h - 1$. Any ideal J which is maximal in the set of proper subideals of I is of colength h.

In the sequel we will restrict to (Noetherian) local rings. There is no restriction to assume that R is complete:

Proposition 2.5 *If (R, m) is local with (m-adic) completion $(\hat{R}, \hat{m})$, there are just as many ideals of colength h in R as in $\hat{R}$.*

Proof. By Proposition 2.2, $m^h \subseteq I$ if I is of colength h (and correspondingly for ideals in $\hat{R}$), and $R/m^h \cong \hat{R}/\hat{m}^h$.

Notice however that even such a simple ring as $R = \mathbb{C}[X, Y]/(X, Y)^2 = \mathbb{C}[x, y]$ has infinitely many ideals of colength 2. Any maximal chain of ideals in R looks like this:

$$R \supset (x, y) \supset (ax + by) \supset (0)$$

and there are infinitely many choices for $(a, b) \neq (0, 0)$ giving different ideals.

The following proposition gives the class of rings we will study.

Proposition 2.6 *Let (R, m) be a local ring. Then, for each $h \in \mathbb{N}$, there is a finite number of ideals of colength h if and only if R is a DVR, an Artinian principal ideal ring, or if R/m is finite.*

Proof. Suppose that the number of ideals of colength 2 is finite. The ideals of colength 2 are in 1-1 correspondence to R/m-subspaces of m/m^2 of codimension 1. Then either m/m^2 is one-dimensional or R/m is a finite field. In the first case $m = (x)$ is a principal ideal and, since by Krull intersection theorem $\bigcap_{i \geq 0} m^i = 0$, we get that every element of R is of the form ex^i, for some $i \geq 0$ and some unit e. It follows that, if $m^i \neq 0$ for each i, then R is a DVR and, if $m^i = 0$ for some i, R is an Artinian principal ideal ring.

If R is a DVR or an Artinian principal ideal ring, the number of ideals of each colength is at most one, so we assume that R/m is a finite field. By induction we can assume that there are finitely many ideals J_i of colength $h - 1$. The ideals of colength h corresponds to R/m-subspaces of J_i/mJ_i of codimension 1, which are finitely many.

We are interested in the growth of the number of ideals of colength h in a local ring R as a function of h. We denote the number of ideals in R of colength h by $\Omega_R(h)$, or just $\Omega(h)$ if the ring R is understood from the context. If R is Artinian, then $\Omega(h) = 0$, if $h >> 0$. We will first see that, if $\dim R \geq 2$, then $\Omega(h)$ cannot be bounded by a function which grows less than exponentially, thus the following theorem shows that it is natural to restrict to one-dimensional rings. We will use the following, certainly well known, lemma.

Lemma 2.7 *Let V be a vector space of dimension n over a field with q elements. Then the number of subspaces of dimension (or codimension) $[n/2]$ is at least $q^{[n/2]^2}$.*

Proof. The number of subspaces of dimension $[n/2]$ is $(q^n - 1)(q^n - q) \cdots (q^n - q^{[n/2]-1})/((q^{[n/2]} - 1)(q^{[n/2]} - q) \cdots (q^{[n/2]} - q^{[n/2]-1}))$. If $n \geq 4$ (and so $[n/2] - 1 \geq 1$), since $(q^n - q^i)/(q^{[n/2]} - q^i) \geq q^{[n/2]}$, and we have at least two factors, the statement follows. If $n < 4$, the result is still true since $[n/2]^2 = [n/2]$.

Theorem 2.8 *Let* (R, m) *be a local ring. If* $\dim R \geq 2$*, there is a positive rational number* F *such that, for each* N *there is an* $M \geq N$ *with* $\Omega(M) > q^{FM}$.

Proof. Suppose that $\dim R = 2$. Then we have $l_R(R/m^n) = an^2 + bn + c$ and $l_R(m^n/m^{n+1}) = 2an + a + b$, for some $a > 0$, if $n >> 0$. Let $M_n = [(2an + a + b)/2]$ (integer part). There are, by Lemma 2.7, at least $q^{(M_n)^2}$ subspaces of codimension M_n in m^n/m^{n+1}, so there are at least $q^{(M_n)^2}$ ideals of colength $an^2 + bn + c + M_n$ in R. If we let $M'_n = an^2 + bn + c + M_n$, we will show that $(M_n)^2 > FM'_n$ for some $F > 0$, if $n >> 0$. Since $(M_n)^2 > (an)^2/2$ if $n >> 0$ and $M'_n < 2an^2$ if $n >> 0$, it suffices to show that $(an)^2/2 = F \cdot 2an^2$, for some $F > 0$, and so $F = a/4$ will do. If $\dim R > 2$, let I be an ideal in R such that $\dim R/I = 2$. Obviously $\Omega_R(n) \geq \Omega_{R/I}(n)$.

In the next section we will see that we can get good control over the growth of $\Omega(h)$ for a large class of one-dimensional rings.

3 Analytically unramified one-dimensional rings

In this section we consider a particular class of one-dimensional rings. In all this section R will be an analytically unramified one-dimensional local ring, i.e. a one-dimensional reduced Noetherian local ring, such that the integral closure $\bar{R}$ is finite over R. An important class of examples of such rings are the local rings of an algebraic curve.

As we noticed in the previous section, it is not restrictive to suppose that R is complete. So we can suppose that, if $P_1, \ldots, P_d$ are the minimal primes of R, each R/P_i is analyticlly irreducible, with integral closure V_i, a DVR. Thus we have $R \subseteq R/P_1 \times \cdots \times R/P_d$ and $\bar{R} = V_1 \times \cdots \times V_d$.

We also suppose that R is residually rational (i.e. that all localizations at maximal ideals of $\bar{R}$ have the same residue field as R) and that the cardinality of the residue field of R is at least equal to the number d of minimal primes.

Since $R \subseteq R/P_1 \times \cdots \times R/P_d \subseteq V_1 \times \cdots \times V_d$, each element $x = (x_1, \ldots, x_d) \in R$ has a value $v(x) = (v_1(x_1), \ldots, v_d(x_d))$, where, for $i = 1, \ldots, d$, v_i is the valuation of the DVR V_i (it is convenient to assume $v_i(0) = \infty$).

The value semigroup of R is $S = v(R) = \{v(x); x \in R\} \subseteq (\mathbb{N} \cup \{\infty\})^d$ and each ideal $I \subseteq R$ has its value set $v(I) = \{v(x); x \in I\} \subseteq S$. On S there is a natural partial ordering, $(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d)$, if $\alpha_i \leq \beta_i$ for all i. For other properties of S, we refer to [1].

If $C = (R : \bar{R})$ is the conductor, then C is an ideal (of R and) of $\bar{R}$, so $C = {t_1}^{\delta_1} V_1 \times \cdots \times {t_d}^{\delta_d} V_d$, where t_i is the uniformizing parameter of V_i. Thus $v(C) = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{\infty\})^d \mid \alpha_i \geq \delta_i, \text{ for } i = 1, \ldots, d\}$. We will always denote $\min v(C)$ by $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)$ in the sequel. Notice in particular that each element $x \in \bar{R} = V_1 \times \cdots \times V_d$, with $v(x) \geq \boldsymbol{\delta}$ (i.e. $v(x) \in v(C)$) is in R, because it is in C.

If $J \subseteq I$ are ideals of R, it is possible to compute $l_R(I/J)$ looking at $v(I)$ and $v(J)$ (cf. [1, Section 2.1]).

Finally, in order to study how the number of ideals of colength h grows with h in R, we have to suppose that the residue field of R is finite, cf. Proposition 2.6.

In this setting, that is fixed for all Section 3, and with the notation introduced above, if I, J are ideals of R, we define $I \sim J$ if there exists an element x in the quotient ring of R such that $v(I) = v(xJ)$. This is an equivalence relation and we call a *shape for the ideals of* R an equivalence class. If I is an ideal in the equivalence class $\mathcal{I}$, we say that $\mathcal{I}$ is the shape of I or I is of shape $\mathcal{I}$.

Notice that the shapes are finitely many for a ring R.

Example 1. For the ring $R = k[[(t,u),(t^3,u^2)]] = k[[x,y]]/(x^3 - y) \cap (x^2 - y)$ that has the following value semigroup

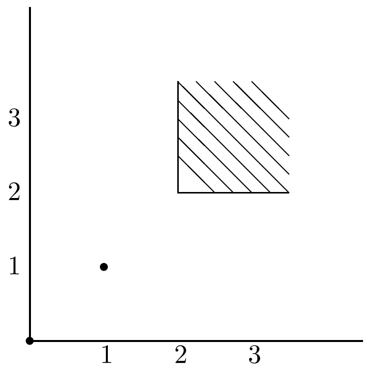
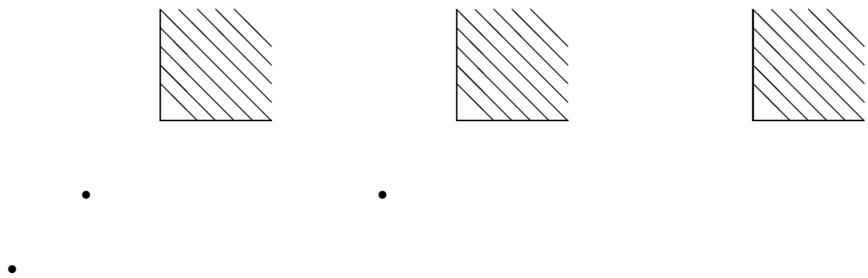


Fig. 1. The value semigroup of R

we have the following shapes:



$\mathcal{I}_1$=the principal shape. $\mathcal{I}_2$=shape$(((t,u),(t^3,u^2))R)$. $\mathcal{I}_3$=shape(C).

Definition. Given a shape $\mathcal{I}$ for the ideals of R, we define the function $\Omega_{\mathcal{I}}(h)$ as the number of ideals of R of shape $\mathcal{I}$ and colength h. Of course we have $\Omega(h) = \sum_{\mathcal{I}} \Omega_{\mathcal{I}}(h)$.

3.1 The analytically irreducible case

We first consider the analytically irreducible case, i.e., we assume that the integral closure of the ring R is a DVR which we denote by (V, t). We denote the conductor $C = R : V$ by $t^\delta V$.

Lemma 3.1 *The map* $\phi_i(I) = t^i I, i \geq 0$*, from ideals* I *with* $\min v(I) = \delta$ *to ideals* J *with* $\min v(J) = i + \delta$ *is a bijection which preserves the shape of the ideal.*

Proof. Since $t^i I$ is a fractional ideal and $t^i I \subseteq C \subseteq R$, we get that $\phi_i(I)$ is an ideal of R. The map is bijective with $\phi_i^{-1}(J) = t^{-i}J$ as inverse. The shape is preserved by the definition of shape.

Lemma 3.2 $\Omega_{\mathcal{I}}(h)$ *is constant, if* $h \geq \delta$*.*

Proof. If $\min v(I) < \delta$, then $l_R(R/I) = \#(v(R) \setminus v(I)) = \#((v(R) \setminus v(I)) \cap [1, \delta)) + \#((v(R) \setminus v(I)) \cap [\delta, \infty)) < l_R(R/C) + l_R(V/R) = l_R(V/C) = \delta$. Thus $I \subseteq C$, if $l_R(R/I) \geq \delta$. According to Lemma 3.1, the number of ideals of shape $\mathcal{I}$ is constant (i.e. independent of $\min v(I)$) for all ideals inside the conductor.

We now state the main result for analytically irreducible rings.

Proposition 3.3 *If* R *is analytically irreducible, then* $\Omega(h)$ *is constant, if* $h \geq l_R(V/C)$*.*

Proof. We have $\Omega(h) = \sum_{\mathcal{I}} \Omega_{\mathcal{I}}(h)$. The sum is finite and each summand is a constant, by Lemma 3.2, if $h \geq \delta = l_R(V/C)$.

As usual it is convenient to collect the information in a generating function. We define the *colength series* of R to be $\mathrm{CL}_R(Z) = \sum_{h=0}^{\infty} \Omega(h) Z^h$. For an analytically irreducible ring R we get $\mathrm{CL}_R(Z) = p(Z)/(1 - Z)$. Then $p(Z) \in \mathbb{Z}[Z]$, and $p(1) = \Omega(h)$, for $h \geq l_R(V/C)$.

The constant $\Omega(h), h >> 0$, of course depends on $q = |R/m|$. We will determine this dependence in an example.

Example 2. Let $R = k[[t^3, t^4, t^5]]$. There are the following shapes of ideals: $\mathcal{I}_1 = \mathrm{shape}(R)$, $\mathcal{I}_2 = \mathrm{shape}((t^3, t^5, t^7))$, $\mathcal{I}_3 = \mathrm{shape}((t^3, t^4))$, and finally $\mathcal{I}_4 = \mathrm{shape}((t^3, t^4, t^5))$. We get, if $q = |k|$,
$\Omega(0) = \Omega_{\mathcal{I}_1}(0) = \Omega(1) = \Omega_{\mathcal{I}_4}(0) = 1$,
$\Omega(2) = \Omega_{\mathcal{I}_2}(2) + \Omega_{\mathcal{I}_3}(2) + \Omega_{\mathcal{I}_4}(2) = q + q^2 + 1$,
$\Omega(h) = \Omega_{\mathcal{I}_1}(h) + \Omega_{\mathcal{I}_2}(h) + \Omega_{\mathcal{I}_3}(h) + \Omega_{\mathcal{I}_4}(h) = q^2 + q + q^2 + 1$, if $h \geq 3$.
$\mathrm{CL}_R(Z) = (1 + (q + q^2)Z^2 + q^2 Z^3)/(1 - Z)$.

We could generalize the example and show that, if $v(R) = \langle \delta, \delta + 1, \ldots, 2\delta - 1 \rangle$, then the constant $\Omega(h)$ is a polynomial of degree $[\delta/2]^2$ in $q = |R/m|$, if $h \geq \delta$. In general the dependence of $\Omega(h)$ of q is more complicated. We can, however, show that $\Omega(h)$ is always *bounded* by a polynomial of degree $[\delta/2]^2$ in q.

3.2 The non-analytically irreducible case

We consider now the analytically unramified case with $d > 1$. Recall that, as above, $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d)$ is $\min v(C)$. By the properties (1) and (2) of [1, Proposition 2.1], the semigroup S is given by the union of a finite number of sets that, modulo a reordering of the coordinates, are of the following form:

$$T = \{(\alpha_1, \dots, \alpha_u, \delta_{u+1}, \dots, \delta_s, \beta_{s+1}, \dots, \beta_d)\} \quad (1)$$

where $\delta_i \leq \alpha_i \in \mathbb{N}$, for $i \leq u$, and β_i are fixed integers, $0 < \beta_i < \delta_i$, for $s + 1 \leq i \leq d$.

If $s = 0$, T is just a single element of S, and $s = d$, if $T = v(C)$. Any subset T of S of this form has a minimum element, $\min T = (\delta_1, \dots, \delta_s, \beta_{s+1}, \dots, \beta_d)$. Denote by $T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ the union of the previous subsets with the same minimum $(\delta_1, \dots, \delta_s, \beta_{s+1}, \dots, \beta_d)$. We have $v(C) = T_{\delta_1,\dots,\delta_d}$ and, for a point $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) < \boldsymbol{\delta}$, we have $\boldsymbol{\beta} = T_{\beta_1,\dots,\beta_d}$.

Lemma 3.4 *Let $x = (x_1, \dots, x_d) \in R$, with $v(x) = (a_1, \dots, a_s, \beta_{s+1}, \dots, \beta_d) \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$. If $\delta_i - a_i \leq n_i \leq \infty$, for $i = 1, \dots, s$, then*

$$(x_1, \dots, x_d)(t_1^{n_1}, \dots, t_s^{n_s}, 1, \dots, 1) \in R.$$

Proof. Every element $x' = (x'_1, \dots, x'_s, 0, \dots, 0)$ of $\bar{R}$, with $v_i(x'_i) \geq \delta_i$, for $i = 1, \dots, s$ has a value in $v(C)$, so $x' \in C \subseteq R$. In particular $(x_1, \dots, x_s, 0, \dots, 0) \in R$ and so $(x_1, \dots, x_d) - (x_1, \dots, x_s, 0, \dots, 0) = (0, \dots, 0, x_{s+1}, \dots, x_d) \in R$, and thus every element $(x'_1, \dots, x'_s, x_{s+1}, \dots, x_d)$ with $v_i(x'_i) \geq \delta_i$, for $i = 1, \dots, s$ belongs to R. The element $(x_1, \dots, x_d)(t_1^{n_1}, \dots, t_s^{n_s}, 1, \dots, 1)$ in the statement of the lemma is such an element.

Lemma 3.5 *a) Let I be an ideal of R with $\min v(I) = \min T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$. Then the map $\phi(I) = (t_1^{n_1}, \dots, t_s^{n_s}, 1, \dots, 1)I$, from ideals I with $\min v(I) = (\delta_1, \dots, \delta_s, \beta_{s+1}, \dots, \beta_d)$ to ideals J with $\min v(J) = (\delta_1 + n_1, \dots, \delta_s + n_s, \beta_{s+1}, \dots, \beta_d)$ is a bijection which preserves the shape of the ideal.*
b) The number of ideals of R of shape $\mathcal{I}$ with $\min v(I) \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ is a constant.

Proof. a) By Lemma 3.4, if $x = (x_1, \dots, x_d) \in I$, then $(t_1^{n_1}, \dots, t_s^{n_s}, 1, \dots, 1)x \in R$. So the fractional ideal $(t_1^{n_1}, \dots, t_s^{n_s}, 1, \dots, 1)I$ is contained in R and is an ideal of R. The map is bijective with $\phi^{-1}(J) = (t_1^{-n_1}, \dots, t_s^{-n_s}, 1, \dots, 1)J$ as inverse. The shape is preserved by the definition of shape.
b) By a) the number of ideals of R of shape $\mathcal{I}$ with $\min v(I) \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ does not depend on the element $\boldsymbol{\alpha} = \min v(I)$.

Denote the constant in Lemma 3.5b) by $f_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$.

For an analytically unramified ring with $d > 1$ minimal primes, we need to make the definition of the function $\Omega_{\mathcal{I}}(h)$ finer.

Definition. Given a shape $\mathcal{I}$ for the ideals of R, we define the function $\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ as the number of elements $\boldsymbol{\alpha} \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ such that there exists an ideal I in R of shape $\mathcal{I}$ and colength h, with $\min v(I) = \boldsymbol{\alpha}$.

The computation of the function $\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ gives an answer to our problem at semigroup level and is the first step in the computation of the growth (with h) of $\Omega(h)$, the number of ideals of colength h. It is convenient to introduce before next lemma another notation. If I is an ideal of shape $\mathcal{I}$, with $\min v(I) = \min T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$, set $b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}) = l_R(R/I)$.

Lemma 3.6 *$\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ is a polynomial of degree at most $s-1$, if $h \geq b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$.*

Proof. We know that $T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ is a finite union of subsets T of type (1) described in the beginning of this section. We will first count the number of elements $\boldsymbol{\alpha} \in T$ (where T is of type (1)), such that there exists an ideal I of R of shape $\mathcal{I}$ and colength h, with $\min v(I) = \boldsymbol{\alpha}$. If such an ideal I with $\min v(I) \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ exists, we have to count the number of ways to write $h - b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ as a sum of u non-negative summands $h_i = \alpha_i - \delta_i$, where $u \leq s$. This is given by

$$\binom{h - b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}) + u - 1}{u-1}$$

which is a polynomial in h of degree $u - 1 \leq s - 1$.

Since $T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ is a finite union of subsets of type (1), then, by the principle of inclusion-exclusion, $\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ is an alternating sum of polynomials of degree $\leq s-1$, thus a polynomial of degree at most $s-1$.

Theorem 3.7 *If R is analytically unramified, with d minimal primes, then $\Omega(h)$ is a polynomial in h of degree $d-1$, if $h \geq l_R(\bar{R}/C)$.*

Proof. We have $\Omega(h) = \sum_{\mathcal{I}} \Omega_{\mathcal{I}}(h)$ and the sum is finite. Thus it is enough to prove the theorem for a fixed shape $\mathcal{I}$.

By Lemma 3.6, $\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$, i.e. the number of elements $\boldsymbol{\alpha} \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ such that there exists an ideal I in R of shape $\mathcal{I}$ and colength h, with $\min v(I) = \boldsymbol{\alpha}$, is a polynomial in h of degree at most $s-1$, if $h \geq b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$. All the numbers $b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ are bounded by $b_{\mathcal{I}}(T_{\delta_1,\dots,\delta_d}) = b_{\mathcal{I}}(v(C))$. Moreover, if $\mathcal{R}$ is the shape of a principal ideal, all the numbers $b_{\mathcal{I}}(v(C))$ are bounded by $b_{\mathcal{R}}(v(C))$. It follows that, for $h \geq b_{\mathcal{R}}(v(C)) = l_R(R/C) + l_R(\bar{R}/R) = l_R((\bar{R}/C)$ and for each shape $\mathcal{I}$, $\omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$ is a polynomial in h of degree at most $s-1$. It is actually a polynomial of degree $d-1$, when $T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d} = v(C)$, because an ideal I of shape $\mathcal{I}$ with $\min v(I) \in v(C)$ certainly exists.

On the other hand we have to count, fixed a certain $T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$, for each $\boldsymbol{\alpha} \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$, how many ideals I of R of shape $\mathcal{I}$ and colength h, with $\min v(I) = \boldsymbol{\alpha}$ exist. By Lemma 3.5, this number, $f_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$, does not depend on the element $\boldsymbol{\alpha} \in T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}$ chosen. So, for $h \geq l_R(\bar{R}/C)$, we get that

$$\Omega_{\mathcal{I}}(h) = \sum_{T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}} \omega_{\mathcal{I}}(h, T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d}) \cdot f_{\mathcal{I}}(T_{\delta_1,\dots,\delta_s,\beta_{s+1},\dots,\beta_d})$$

is a polynomial in h of degree $d-1$.

Example 3. We use the same ring R as in Example 1 from the beginning of this section. We get $f_{\mathcal{I}_1}(T_{0,0}) = 1, f_{\mathcal{I}_1}(T_{1,1}) = q, f_{\mathcal{I}_1}(T_{2,2}) = q(q-1), f_{\mathcal{I}_2}(T_{0,0}) = 0, f_{\mathcal{I}_2}(T_{1,1}) = 1, f_{\mathcal{I}_2}(T_{2,2}) = q-1, f_{\mathcal{I}_3}(T_{0,0}) = 0, f_{\mathcal{I}_3}(T_{1,1}) = 0, f_{\mathcal{I}_3}(T_{2,2}) = 1$. Furthermore $\omega_{\mathcal{I}_1}(h, T_{0,0}) = 1$ for $h = 0$ and $= 0$ otherwise, $\omega_{\mathcal{I}_1}(h, T_{1,1}) = 1$ for $h = 2$ and $= 0$ otherwise, $\omega_{\mathcal{I}_1}(h, T_{2,2}) = h-3$ for $h \geq 4$ and $= 0$ otherwise, $\omega_{\mathcal{I}_2}(h, T_{0,0}) = 0$ for each h, $\omega_{\mathcal{I}_2}(h, T_{1,1}) = 1$ for $h = 1$ and $= 0$ otherwise, $\omega_{\mathcal{I}_2}(h, T_{2,2}) = h-2$ for $h \geq 3$ and $= 0$ otherwise, $\omega_{\mathcal{I}_3}(h, T_{2,2}) = h-1$ for $h \geq 2$ and $= 0$ otherwise. Thus $\Omega_R(0) = 1$, $\Omega_R(1) = 1$, $\Omega_R(2) = 1+q$, $\Omega_R(3) = 1+q$, and $\Omega_R(h) = 1+q+(h-3)q^2$, if $h \geq 4 = l_R(\bar{R}/C)$.

For the generating function $\mathrm{CL}_R(Z)$ of $\Omega(h)$ we get the following result.

Corollary 3.8 $\mathrm{CL}_R(Z) = p(Z)/(1-Z)^d$, *where* $p(Z) \in \mathbb{Z}[Z]$, $\deg p(Z) = l_R(\bar{R}/C)$, $p(1)$ *equals the number of ideals* I *with* $\min v(I) = \boldsymbol{\alpha}$ *for any* $\boldsymbol{\alpha} \geq \boldsymbol{\delta}$, *and* d *equals the number of maximal ideals in* $\bar{R}$.

Example 4. The generating function from Example 1 (and Example 3) becomes

$$\mathrm{CL}_R(Z) = (1 - Z + qZ^2 - qZ^3 + q^2Z^4)/(1-Z)^2.$$

References

[1] V. Barucci - M. D'Anna - R. Fröberg, Analytically unramified one-dimensional semilocal rings and their value semigroups, *J. Pure Appl. Algebra* **147** (2000), 215-254.