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THE SIMILARITY PROBLEM FOR THE NONSELFADJOINT OPERATORS WITH ABSOLUTELY CONTINUOUS SPECTRUM: RESTRICTIONS TO THE SPECTRAL SUBSPACES

ALEXANDER V. KISELEV

ABSTRACT. The similarity problem for the restrictions of the nonselfadjoint operator possessing absolutely continuous spectrum only to its spectral subspaces corresponding to the Borel subsets δ of its spectrum (see [6]) is considered. Necessary and sufficient conditions of such similarity are obtained in the form of a pair of integral estimates on $\delta \subset \mathbb{R}$. The results are then applied to the analysis of one-dimensional nonselfadjoint Friedrichs model operator.

1. Preliminaries

The nonselfadjoint operator L acting in the Hilbert space H is called similar to a selfadjoint operator A if there exists a bounded, boundedly invertible operator X in H such, that $L = X^{-1}AX$.

In the present article we are going to consider a class of operators of the form [7]

$$L = A + iV,$$

where A is a selfadjoint operator in H defined on the domain $D(A)$ and the perturbation V admits the factorization $V = \frac{\alpha J \alpha}{2}$, where α is a nonnegative selfadjoint operator in H and J is a unitary operator in $E \equiv \overline{R(\alpha)}$. This factorization corresponds to the polar decomposition of the operator V. In order that the expression $A + iV$ be meaningful, we impose the condition that V be (A)-bounded with the relative bound less then 1, i.e. $D(A) \subset D(V)$ and for some a and b $(a < 1)$ the condition

$$\|Vu\| \le a\|Au\| + b\|u\|, \quad u \in D(A)$$

is satisfied, see [4]. Then the operator L is well-defined on the domain $D(L) = D(A)$.

Alongside with the operator L we are going to consider the maximal dissipative operator $L^{\|} = A + i\frac{\alpha^2}{2}$ and the one adjoint to it, $L^{-\|} \equiv L^{\|*} = A - i\frac{\alpha^2}{2}$. Since the functional model for the dissipative operator $L^{\|}$ will be used below, we require that $L^{\|}$ be completely nonselfadjoint, i.e. that it has no reducing selfadjoint parts. This requirement is not restrictive in our case due to the Proposition 1 in [7].

Now we are going to briefly describe the construction of the selfadjoit dilatation of the completely nonselfadjoint dissipative operator $L^{\|}$, following [1, 9], see also [7].

The characteristic function $S(\lambda)$ of the operator $L^{\|}$ is the contractive, analytic operator-valued function acting on the Hilbert space E, defined for $Im\lambda > 0$ by

$$S(\lambda) = I + i\alpha(L^{-\|} - \lambda)^{-1}\alpha, \quad Im\lambda > 0. \tag{1.1}$$

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In the case of unbounded α the characteristic function is first defined by the latter expression on the manifold $E \cap D(\alpha)$ and then extended by continuity to the whole space E.

Formula (1.1) makes it possible to consider $S(\lambda)$ for $Im\lambda < 0$ with $S(\overline{\lambda}) = (S^*(\lambda))^{-1}$. Finally, $S(\lambda)$ possesses boundary values on the real axis in the strong sense: $S(k) \equiv S(k + i0),\ k \in \mathbb{R}$ (see [1]).

Consider the model space $\mathcal{H} = L_2(\begin{smallmatrix} I & S^* \\ S & I \end{smallmatrix})$, which is defined in [9] as the Hilbert space of two-component vector-functions $(\tilde{g}, g)$ on the axis $(\tilde{g}(k), g(k) \in E, k \in \mathbb{R})$ with metric

$$\left(\begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right) = \int_{-\infty}^{\infty} \left(\begin{pmatrix} I & S^*(k) \\ S(k) & I \end{pmatrix} \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix}, \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix} \right)_{E \oplus E} dk.$$

It is assumed here that the set of two-component functions has been factored by the set of elements with the norm equal to zero.

Let's define the following orthogonal subspaces in $\mathcal{H}$:

$$D_- \equiv \begin{pmatrix} 0 \\ H_2^-(E) \end{pmatrix},\ D_+ \equiv \begin{pmatrix} H_2^+(E) \\ 0 \end{pmatrix},\ K \equiv \mathcal{H} \ominus (D_- \oplus D_+),$$

where $H_2^{+(-)}(E)$ denotes the Hardy class of analytic functions f in the upper (lower) half plane with the values in the Hilbert space E.

The subspace K can be described as $K = \{(\tilde{g}, g) \in \mathcal{H} :\ \tilde{g} + S^* g \in H_2^-(E), S\tilde{g} + g \in H_2^+(E)\}$. Let P_K be the orthogonal projection of $\mathcal{H}$ onto K:

$$P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+(\tilde{g} + S^* g) \\ g - P_-(S\tilde{g} + g) \end{pmatrix},$$

where $P_\pm$ are orthogonal projections of $L_2(E)$ onto $H_2^\pm(E)$.

The following theorem holds [1, 9]:

Theorem 1.1. *The operator $(L^\| - \lambda_0)^{-1}$ is unitary equivalent to the operator $P_K(k - \lambda_0)^{-1}|_K$ for all $\lambda_0, Im\lambda_0 < 0$.*

This means, that the operator of multiplication by k serves as the minimal $(clos_{Im\lambda \neq 0}(k - \lambda)^{-1}K = \mathcal{H})$ selfadjoint dilatation [1] of the operator $L^\|$.

Provided that the non-real spectrum of the operator L is countable, the characteristic function of the operator L is defined by the following expression:

$$\Theta(\lambda) \equiv I + iJ\alpha(L^* - \lambda)^{-1}\alpha, \qquad Im\lambda \neq 0,$$

and is a meromorphic, J-contractive $(\Theta^*(\lambda)J\Theta(\lambda) \leq J,\quad Im\lambda > 0)$ operator-function [2]. The characteristic function $\Theta(\lambda)$ admits a factorization in the form of the ratio of two bounded analytic operator-functions (in the corresponding half-planes $Im\lambda < 0,\ Im\lambda > 0$) triangular with respect to the decomposition of the space E into the orthogonal sum

$$E = (\mathcal{X}_+ E) \oplus (\mathcal{X}_- E), \quad \mathcal{X}_\pm \equiv \frac{I \pm J}{2} :$$

$$\Theta(\lambda) = \Theta_1'^*(\overline{\lambda})(\Theta_2'^*)^{-1}(\overline{\lambda}), \qquad Im\lambda > 0$$

$$\Theta(\lambda) = \Theta_2^*(\overline{\lambda})(\Theta_1^*)^{-1}(\overline{\lambda}), \qquad Im\lambda < 0,$$

where the following designations have been adopted [6]:

$$\begin{aligned}
\Theta_1(\lambda) &= \mathcal{X}_- + S(\lambda)\mathcal{X}_+, \\
\Theta_2(\lambda) &= \mathcal{X}_+ + S(\lambda)\mathcal{X}_-, \\
\Theta_1'(\lambda) &= \mathcal{X}_- + S^*(\overline{\lambda})\mathcal{X}_+, \\
\Theta_2'(\lambda) &= \mathcal{X}_+ + S^*(\overline{\lambda})\mathcal{X}_-,
\end{aligned}$$

and $S(\lambda)$ is defined by (1.1).

Following [6], we define the subspaces $N_\pm$ in $\mathcal{H}$ as follows:

$$\hat{N}_\pm \equiv \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} : \quad \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathcal{H}, \quad P_\pm \left(\Theta_1'^* \tilde{g} + \Theta_2^* g\right) = 0 \right\}$$

and introduce the following designation:

$$N_\pm = clos P_K \hat{N}_\pm.$$

Then, as it is shown in [7], one gets for $Im\lambda < 0 (Im\lambda > 0)$ and $(\tilde{g}, g) \in \hat{N}_{-(+)}$, respectively:

$$(L-\lambda)^{-1} P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{k-\lambda} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}.$$

The absolutely continuous and singular subspaces of the nonselfadjoint operator L were defined in [5]: let[1] $N \equiv \hat{N}_+ \cap \hat{N}_-$, $\tilde{N}_\pm \equiv P_K \hat{N}_\pm$, then

$$\begin{gathered}
N_e \equiv clos\left(\tilde{N}_+ \cap \tilde{N}_-\right) = clos P_K N \\
N_i \equiv K \ominus N_e(L^*).
\end{gathered} \tag{1.2}$$

We call operator L an "operator with absolutely continuous spectrum only" if $N_e = H$, i.e. $P_K N$ is dense in K.

The spectral projector $\mathcal{P}_\delta$ to the portion δ of the absolutely continuous spectrum was constructed in the model terms in [6]. Namely, the following theorem holds:

Theorem 1.2. *For any Borel set* $\delta \subset \mathbb{R}$

$$\mathcal{P}_\delta P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \mathcal{X}_\delta \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \tag{1.3}$$

where $\binom{\tilde{g}}{g} \in N$ *and* $\mathcal{X}_\delta$ *is the operator of componentwise multiplication by the characteristic function of the set* δ*. For the operator* $\mathcal{P}_\delta$ *defined by* (1.3) *on the linear set* $\tilde{N}_e \equiv \tilde{N}_- \cap \tilde{N}_+$ *the following assertions hold:*

(i) $\mathcal{P}_\delta \tilde{N}_e \subset \tilde{N}_e$;
(ii) $(L-\lambda_0)^{-1}\mathcal{P}_\delta = \mathcal{P}_\delta (L-\lambda_0)^{-1}, \quad Im\lambda_0 \neq 0$;
(iii) $\mathcal{P}_\delta \mathcal{P}_{\delta'} = \mathcal{P}_{\delta\cap\delta'}, \quad \delta, \delta' \subset \mathbb{R}$;
(iv) $\mathcal{P}_\delta u \longrightarrow u$ *as* $\delta \to (-\infty, \infty), \quad u \in \tilde{N}_e$;
(v) $\mathcal{P}_\delta u = \lim_{\varepsilon\to+0} \frac{1}{2\pi i} \int_\delta [(L-k-i\varepsilon)^{-1} - (L-k+i\varepsilon)^{-1}] u\,dk, \quad u \in \tilde{N}_e$.

[1]The linear set N is called a set of "smooth" vectors of the operator L (see [7])

In the second section of the present paper we are going to obtain the conditions, necesary and sufficient for the restrictions of the operator L posessing absolutely continuous spectrum only to its spectral subspaces to be similar to selfadjoint operators. Then, in the third section, we will apply these results to the analysis of the nonselfadjoint Friedrichs model operator in one simple case. See also [11] where analogous results are given for the similarity of the nonselfadjoint operator L as a whole to a selfadjoint one and [12], where the computations of the same nature are applied to the analysis of the nonselfadjoint extensions of symmetric operators with finite deficiency indices.

2. The restrictions of L to its spectral subspaces: functional model approach

We are going to rely on the following conditions considered for all $u \in H$:

$$\begin{aligned}
&\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L-k-i\varepsilon)^{-1}u\|^2 dk \leq C\|u\|^2 \\
&\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L^*-k-i\varepsilon)^{-1}u\|^2 dk \leq C\|u\|^2 \\
&\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L-k+i\varepsilon)^{-1}u\|^2 dk \leq C\|u\|^2 \\
&\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(L^*-k+i\varepsilon)^{-1}u\|^2 dk \leq C\|u\|^2
\end{aligned} \tag{2.1}$$

which if fulfilled for any $\varepsilon > 0$ are necessary and sufficient [8, 10] for a nonselfadjoint operator with the real spectrum to be similar to a selfadjoint one. Furthermore, notice that the first pair of estimates above is clearly equivalent to the second pair. This makes it possible to prove the following

Theorem 2.1. *Provided that the spectrum of L is absolutely continuous, the following assertions are equivalent:*

(a) *The restriction of L to its invariant subspace, corresponding to the "portion" of its spectrum contained in the Borel set $\delta \subset \mathbb{R}$, $\mathcal{P}_\delta H$, is similar to a selfadjoint operator;*
(b) *For any $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\int_\delta ((\Theta(k-i0)J\Theta^*(k-i0)-J)\mathcal{X}_+\alpha(L^{-\|}-k-i0)^{-1}u, \mathcal{X}_+\alpha(L^{-\|}-k-i0)^{-1}u)dk \leq C\|u\|^2$$

$$\int_\delta ((J-\Theta^*(k+i0)J\Theta(k+i0))\mathcal{X}_-\alpha(L^{-\|}-k-i0)^{-1}u, \mathcal{X}_-\alpha(L^{-\|}-k-i0)^{-1}u)dk \leq C\|u\|^2;$$

(c) *For any $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\int_\delta ((J - \Theta(k+i0)J\Theta^*(k+i0))\mathcal{X}_-\alpha(L^{\|} - k + i0)^{-1}u, \mathcal{X}_-\alpha(L^{\|} - k + i0)^{-1}u)dk$$
$$\leq C\|u\|^2$$
$$\int_\delta ((\Theta^*(k-i0)J\Theta(k-i0) - J)\mathcal{X}_+\alpha(L^{\|} - k + i0)^{-1}u, \mathcal{X}_+\alpha(L^{\|} - k + i0)^{-1}u)dk$$
$$\leq C\|u\|^2.$$

Proof. Our first goal is to rewrite the estimates (2.1) in the model terms. This will allow us to pass to limit as $\varepsilon \to +0$ in the corresponding estimates in the model representation for the operator L. To this end, we are first going to prove the following Lemma.

Lemma 2.2. *The estimates in* (2.1), *considered on the vectors $u \in \mathcal{P}_\delta H$, are one-to-one equivalent to the following ones:*

$$\begin{aligned}
\left\| P_+ \begin{pmatrix} \tilde{g} + S^* g \\ -(S\tilde{g} + g) \end{pmatrix} \right\|_{\mathcal{H}}^2 &\leq C \left\| P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2 \\
\left\| \begin{pmatrix} P_+(\tilde{g} + S^* g) - c_2(k) \\ -P_+(S\tilde{g} + g) + S(k)c_2(k) \end{pmatrix} \right\|_{\mathcal{H}}^2 &\leq C \left\| P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2 \\
\left\| P_- \begin{pmatrix} \tilde{g} + S^* g \\ -(S\tilde{g} + g) \end{pmatrix} \right\|_{\mathcal{H}}^2 &\leq C \left\| P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2 \\
\left\| \begin{pmatrix} -P_-(\tilde{g} + S^* g) + S^*(k)c_1(k) \\ P_-(S\tilde{g} + g) - c_1(k) \end{pmatrix} \right\|_{\mathcal{H}}^2 &\leq C \left\| P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2,
\end{aligned} \tag{2.2}$$

where $\binom{\tilde{g}}{g} \in \mathcal{X}_\delta N$ and

$$\begin{aligned}
T_1(\lambda) &\equiv [\mathcal{X}_- + S^*(\overline{\lambda})\mathcal{X}_+]^{-1} \\
T_2(\lambda) &\equiv [\mathcal{X}_+ + \mathcal{X}_- S(\lambda)]^{-1} \\
c_1(\lambda) &\equiv T_1(\lambda)(P_-(\tilde{g} + S^* g)(\lambda) + P_-(S\tilde{g} + g)(\lambda)) \\
c_2(\lambda) &\equiv T_2(\lambda)(P_+(\tilde{g} + S^* g)(\lambda) + P_+(S\tilde{g} + g)(\lambda)).
\end{aligned}$$

Proof. We will show that the first estimates in the statement of the Lemma 2.2 and in (2.1) are equivalent; the corresponding proof for the other three pairs of estimates is carried out in a similar manner.

Note, that clearly $\mathcal{X}_\delta N \subset N$, therefore one gets

$$(L - \lambda_0)^{-1} P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{k - \lambda_0} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}$$

for every $\binom{\tilde{g}}{g} \in \mathcal{X}_\delta N$. On the other hand, the straightforward computation shows that

$$(L-\lambda_0)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} = \begin{cases} \dfrac{1}{k-\lambda_0}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} + \dfrac{1}{k-\lambda_0}\begin{pmatrix}P_+(\tilde{g}+S^*g)(\lambda_0)\\ -P_+(S\tilde{g}+g)(\lambda_0)\end{pmatrix}, & Im\lambda_0>0\\ \dfrac{1}{k-\lambda_0}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} + \dfrac{1}{k-\lambda_0}\begin{pmatrix}-P_-(\tilde{g}+S^*g)(\lambda_0)\\ P_-(S\tilde{g}+g)(\lambda_0)\end{pmatrix}, & Im\lambda_0<0\end{cases}$$

Taking into account that the first estimate in (2.1), considered on all $u \in \mathcal{P}_\delta H$, in the model representation of the operator L can be written as

$$\sup_{\varepsilon>0}\varepsilon\int\left\|(L-k-i\varepsilon)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}\right\|^2 dk \le C\left\|P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}\right\|^2,$$

where $\binom{\tilde{g}}{g} \in \mathcal{X}_\delta N$, one can compute the left hand side in the latter estimate. Then

$$\varepsilon\int dx\left\|(L-x-i\varepsilon)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}\right\|^2 =$$

$$= \pi\int dx(\|P_+(\tilde{g}+S^*g)(x+i\varepsilon)\|^2 + \|P_+(S\tilde{g}+g)(x+i\varepsilon)\|^2) -$$

$$-2Re\int dx\int dk\frac{\varepsilon}{(x-k)^2+\varepsilon^2}(S(k)P_+(S\tilde{g}+g)(x+i\varepsilon), P_+(\tilde{g}+S^*g)(x+i\varepsilon)).$$

Having used the fact that (see [3])

$$\int dk\frac{\varepsilon}{(x-k)^2+\varepsilon^2}(S(k)P_+(S\tilde{g}+g)(x+i\varepsilon), P_+(\tilde{g}+S^*g)(x+i\varepsilon)) =$$
$$= \pi(S(x+i\varepsilon)P_+(S\tilde{g}+g)(x+i\varepsilon), P_+(\tilde{g}+S^*g)(x+i\varepsilon))$$

since $S(\lambda)$ is a bounded analytic operator-function in the upper semiplane of the complex plane, and then passing to the limit as $\varepsilon \to 0$, taking into account that $P_+(\tilde{g}+S^*g)(\lambda), P_+(S\tilde{g}+g)(\lambda) \in H_2^+(E)$ [3], we arrive to the following rezult:

$$\lim_{\varepsilon\to+0}\varepsilon\int dx\left\|(L-x-i\varepsilon)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}\right\|^2 = \pi\left\|P_+\begin{pmatrix}\tilde{g}+S^*g\\ -(S\tilde{g}+g)\end{pmatrix}\right\|^2_{\mathcal{H}}.$$

Therefore the equivalence claimed is proved.

When considering the estimates of (2.1) that involve the resolvent of the adjoint operator L^*, one has to use the following model representation for the action of the resolvent of the adjoint operator on the "smooth" vectors of the operator L (see [5, 7]):

$$(L^*-\lambda_0)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} =$$
$$= P_K\frac{1}{k-\lambda_0}\begin{pmatrix}\tilde{g}-P_+(\tilde{g}+S^*g)\\ g-P_-(S\tilde{g}+g)-\mathcal{X}_+[I+(S^*(\overline{\lambda_0})-I)\mathcal{X}_+]^{-1}*\\ *(P_-(\tilde{g}+S^*g)(\lambda_0)-S^*(\overline{\lambda_0})P_-(S\tilde{g}+g)(\lambda_0))\end{pmatrix}.$$

The rest of the proof in this case is essentially similar to the one carried out above. □

In order to complete the proof of Theorem 2.1, we need to rewrite the estimates obtained by virtue of Lemma 2.2 in terms of the initial Hilbert space H and of the operators in it. To this end, we will first rewrite our estimates in terms of $\mathfrak{H}$, the three-component representation of $\mathcal{H}$, see [9, 7]. The space $\mathfrak{H} \equiv D_- \oplus H \oplus D_+$ consists of three-component vector-functions $(\tilde{v}_-, u, \tilde{v}_+)$, where $\tilde{v}_- \in L_2(\mathbb{R}_-; E), \tilde{v}_+ \in L_2(\mathbb{R}_+; E)$ and $u \in H$. The unitary operator (see [7]) that maps $\mathfrak{H}$ onto $\mathcal{H}$ is given by the following formulas:

$$\tilde{g} + S^* g = -\frac{1}{\sqrt{2\pi}}\alpha(L^{\|} - k + i0)^{-1}u + S^*(k)v_-(k) + v_+(k)$$
$$S\tilde{g} + g = -\frac{1}{\sqrt{2\pi}}\alpha(L^{-\|} - k - i0)^{-1} + v_-(k) + S(k)v_+(k),$$

where[2]

$$v_\pm(k) \equiv \frac{1}{\sqrt{2\pi}}\int e^{ik\xi}\tilde{v}_\pm(\xi)d\xi \in H_2^\pm(E)$$

by the Paley–Wiener theorem [3].

We are going to use this mapping extensively. First of all, note that the fact that $\binom{\tilde{g}}{g} \in N$ in the model representation is equivalent to

$$\begin{cases} \mathcal{X}_-(\tilde{g} + S^* g) = 0 \\ \mathcal{X}_+(S\tilde{g} + g) = 0 \end{cases}, \tag{2.3}$$

which is of course also true for the subspace we are considering, $\mathcal{P}_\delta N$. Next, for the latter subspace we clearly have

$$\begin{cases} \mathcal{X}_\delta(\tilde{g} + S^* g) = \tilde{g} + S^* g \\ \mathcal{X}_\delta(S\tilde{g} + g) = S\tilde{g} + g \end{cases}, \tag{2.4}$$

and finally,

$$\begin{cases} \mathcal{X}_{+(-)}\alpha(L^{\|(-\|)} + (-)i0)^{-1}u \in H^2_{-(+)}(E) \\ \begin{cases} \mathcal{X}_+ v_-(k) = 0 \quad a.\ a.\ k \\ \mathcal{X}_- v_+(k) = 0 \quad a.\ a.\ k \end{cases} \\ \begin{cases} \mathcal{X}_- S^*(k)v_-(k) = \frac{1}{\sqrt{2\pi}}\mathcal{X}_-\alpha(L^{\|} - k + i0)^{-1}u \quad a.\ a.\ k \\ \mathcal{X}_+ S(k)v_+(k) = \frac{1}{\sqrt{2\pi}}\mathcal{X}_+\alpha(L^{-\|} - k - i0)^{-1}u \quad a.\ a.\ k \end{cases} \end{cases}, \tag{2.5}$$

where we have used (2.3) and the orthogonality of $H^2_+(E)$ and $H^2_-(E)$ in $L^2(E)$.

Let's prove now that the assertions (a) and (c) of Theorem 2.1 are equivalent. In order to do so we need to show, that the third and fourth estimates in the statement of Lemma 2.2 are respectively equivalent to the ones provided by the assertion (c) of Theorem 2.1.

[2]We assume here that the functions $\tilde{v}_-, \tilde{v}_+$ have been extended by zero to the complementary semiaxes.

We begin with the third estimate of Lemma. One immediately obtains:

$$\left\| \begin{pmatrix} P_-(\tilde g + S^* g) \\ -P_-(S\tilde g + g) \end{pmatrix} \right\|_{\mathcal{H}}^2 =$$
$$= \int\limits_\delta dk \left(\|P_-(\tilde g + S^* g)\|^2 + \|P_-(S\tilde g + g)\|^2 - 2Re(SP_-(\tilde g + S^* g), P_-(S\tilde g + g)) \right),$$

where we have used (2.3), (2.4) and the following simple observation:

$$\int\limits_{\mathbb{R}} (P_+ f_1(k), f_2(k)) dk = \int\limits_{\mathbb{R}} (P_+ f_1(k), P_+ f_2(k)) dk =$$
$$= \int\limits_\delta (f_1(k), f_2(k)) dk = \int\limits_\delta (P_+ f_1(k), f_2(k)) dk \quad \text{when } f_1(k) = \mathcal{X}_\delta f_1(k).$$

The direct computation now shows that the third estimate of Lemma is equivalent to the following one:

$$\int\limits_\delta dk \left(\|\mathcal{X}_- v_-\|^2 - \|\mathcal{X}_+ S^* v_-\|^2 \right) +$$
$$+ 2Re \int\limits_\delta dk \left(\frac{1}{\sqrt{2\pi}} \mathcal{X}_+ \alpha (L^\| - k + i0)^{-1} u, \left(S^*(k)\mathcal{X}_- - \mathcal{X}_+ S^*(k) \right) v_-(k) \right) \le$$
$$\le C\|u\|^2, \quad (2.6)$$

where we have taken into account that [7]

$$\int\limits_\delta \left\| \frac{1}{\sqrt{2\pi}} \mathcal{X}_+ \alpha (L^\| - k + i0)^{-1} u \right\|^2 dk \le C\|u\|^2.$$

The conditions (2.5) when applied to (2.6) show the equivalence of the latter estimate to

$$\int\limits_\delta dk \left((\mathcal{X}_- S \mathcal{X}_-)^{-1} (I - SS^*) (\mathcal{X}_- S^* \mathcal{X}_-)^{-1} \mathcal{X}_- \alpha (L^\| - k + i0)^{-1} u \right.,$$
$$\left. \mathcal{X}_- \alpha (L^\| - k + i0)^{-1} u \right) \le C\|u\|^2,$$

where $\mathcal{X}_- S \mathcal{X}_-$ is treated as a bounded linear operator on $\mathcal{X}_- E$ for a.a. k. It's not hard to show on the basis of Hilbert identity, that

$$(\mathcal{X}_- S(\lambda) \mathcal{X}_-)^{-1} = \mathcal{X}_- \Theta(\lambda) \mathcal{X}_-.$$

Then

$$((\mathcal{X}_- S(\lambda)\mathcal{X}_-)^{-1}(I - S(\lambda)S^*(\lambda))(\mathcal{X}_- S^*(\lambda)\mathcal{X}_-)^{-1}\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u) =$$
$$= (\mathcal{X}_-\Theta(\lambda)\mathcal{X}_-\Theta_2(\lambda)J(J - \Theta(\lambda)J\Theta^*(\lambda))J\Theta_2^*(\lambda)\mathcal{X}_-\Theta^*(\lambda)\mathcal{X}_-\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u) =$$
$$= (J(J - \Theta(\lambda)J\Theta^*(\lambda))J\mathcal{X}_-\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_-\alpha(L^{\parallel} - \overline{\lambda})^{-1}u),$$

since, clearly, $\Theta_2^*(\lambda)\mathcal{X}_-\Theta^*(\lambda)\mathcal{X}_- = \mathcal{X}_-$. The latter result leads to the desired estimate.

Analogous computations based on (2.3), (2.4) and (2.5) applied to the fourth estimate of the Lemma 2.2 show that the latter is equivalent to the following one:

$$\int_\delta dk((\mathcal{X}_- + S(k)\mathcal{X}_+)^{-1}(I - SS^*)(\mathcal{X}_- + \mathcal{X}_+S^*(k))^{-1}\mathcal{X}_+\alpha(L^{\parallel} - k + i0)^{-1}u,$$
$$\mathcal{X}_+\alpha(L^{\parallel} - k + i0)^{-1}u) \leq C\|u\|^2. \quad (2.7)$$

Taking into account that

$$\mathcal{X}_- + \mathcal{X}_+S^*(\lambda) = (\mathcal{X}_- + S(\lambda)\mathcal{X}_+)^* = \Theta_1^*(\lambda),$$

we have:

$$((\mathcal{X}_- + S(\lambda)\mathcal{X}_+)^{-1}(I - S(\lambda)S^*(\lambda))(\mathcal{X}_- + \mathcal{X}_+S^*(\lambda))^{-1}\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u) =$$
$$= (\Theta_1^{-1}(\lambda)\Theta_2(\lambda)J(J - \Theta(\lambda)J\Theta^*(\lambda))J\Theta_2^*(\lambda)\Theta_1^{*-1}(\lambda)\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u) =$$
$$= (\Theta^*(\overline{\lambda})J(J - \Theta(\lambda)J\Theta^*(\lambda))J\Theta(\overline{\lambda})\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u,$$
$$\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u) =$$
$$= ((\Theta^*(\overline{\lambda})J\Theta(\overline{\lambda}) - J)\mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u, \mathcal{X}_+\alpha(L^{\parallel} - \overline{\lambda})^{-1}u),$$

which finishes the proof of the equivalence of the assertions (a) and (c) of the Theorem 2.1.

The equivalence of assertions (a) and (b) is shown in analogous fashion, so we omit the corresponding calculations here. □

The results obtained are yet quite complicated since the integral estimates of Theorem 2.1 involve the boundary values of the resolvent of the dissipative operator $L^{\parallel}$ and its adjoint, rather then the boundary values of the operators L and L^* like the conditions (2.1). Therefore we are now going to prove a modification of the Theorem 2.1. Namely, the following result holds:

Theorem 2.3. *Provided that the spectrum of L is absolutely continuous, the following assertions are equivalent:*

(a) *The restriction of L to its invariant subspace, corresponding to the "portion" of its spectrum contained in the Borel set $\delta \subset \mathbb{R}$, $\mathcal{P}_\delta H$, is similar to a selfadjoint operator;*

(b) *For any $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\int\limits_\delta ((I - S^*(k)S(k))\mathcal{X}_+\alpha(L-k-i0)^{-1}u,$$
$$\mathcal{X}_+\alpha(L-k-i0)^{-1}u)dk \le C\|u\|^2$$
$$\int\limits_\delta ((I - S^*(k)S(k))\mathcal{X}_-\alpha(L^*-k-i0)^{-1}u,$$
$$\mathcal{X}_-\alpha(L^*-k-i0)^{-1}u)dk \le C\|u\|^2;$$

(c) *For any $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\int\limits_\delta ((I - S(k)S^*(k))\mathcal{X}_-\alpha(L-k+i0)^{-1}u,$$
$$\mathcal{X}_-\alpha(L-k+i0)^{-1}u)dk \le C\|u\|^2$$
$$\int\limits_\delta ((I - S(k)S^*(k))\mathcal{X}_+\alpha(L^*-k+i0)^{-1}u,$$
$$\mathcal{X}_+\alpha(L^*-k+i0)^{-1}u)dk \le C\|u\|^2.$$

Proof. This theorem is proved by direct computation. For example, for the first estimate of the assertion (b) of Theorem 2.1 one has:

$$((\Theta(\overline{\lambda})J\Theta^*(\overline{\lambda}) - J)\mathcal{X}_+\alpha(L^{-\|}-\lambda)^{-1}u, \mathcal{X}_+\alpha(L^{-\|}-\lambda)^{-1}u) =$$
$$= (\Theta_1^*(\overline{\lambda})\mathcal{X}_+(\Theta(\overline{\lambda})J\Theta^*(\overline{\lambda}) - J)\mathcal{X}_+\Theta_1(\lambda)\alpha(L-\lambda)^{-1}u, \alpha(L-\lambda)^{-1}u),$$

since

$$\alpha(L^{-\|}-\lambda)^{-1} = \Theta_1(\lambda)\alpha(L-\lambda)^{-1}.$$

Then

$$\Theta_1^*(\overline{\lambda})\mathcal{X}_+(\Theta(\overline{\lambda})J\Theta^*(\overline{\lambda}) - J)\mathcal{X}_+\Theta_1(\lambda) =$$
$$= (\mathcal{X}_+ - \mathcal{X}_+S^*(\lambda)\mathcal{X}_-)J(\mathcal{X}_+ - \mathcal{X}_-S(\lambda)\mathcal{X}_+) - \Theta_1^*(\lambda)\mathcal{X}_+J\mathcal{X}_+\Theta_1(\lambda) =$$
$$= (\mathcal{X}_+ - \mathcal{X}_+S^*(\lambda)\mathcal{X}_-)(\mathcal{X}_+ + \mathcal{X}_-S(\lambda)\mathcal{X}_+) -$$
$$- (\mathcal{X}_- + \mathcal{X}_+S^*(\lambda))\mathcal{X}_+(\mathcal{X}_- + S(\lambda)\mathcal{X}_+) =$$
$$= \mathcal{X}_+ - \mathcal{X}_+S^*(\lambda)\mathcal{X}_-S(\lambda)\mathcal{X}_+ - \mathcal{X}_+S^*(\lambda)\mathcal{X}_+S(\lambda)\mathcal{X}_+ =$$
$$= \mathcal{X}_+(I - S^*(\lambda)S(\lambda))\mathcal{X}_+,$$

where we have used the fact that by the Hilbert identity

$$\Theta^*(\overline{\lambda})\mathcal{X}_+\Theta_1(\lambda) = \mathcal{X}_+ - \mathcal{X}_-S(\lambda)\mathcal{X}_+.$$

In the case of the other respective pairs of estimates the proof is carried out similarly. $\square$

Corollary 2.4. *Provided that the spectrum of the operator L is absolutely continuous, the following conditions are sufficient for the restriction of L onto the subspace $\mathcal{P}_\delta H$ for any Borel set $\delta \subset \mathbb{R}$ to be similar to a selfadjoint operator:*

(a) *There exists a constant $C < \infty$ such that for all $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\begin{cases} \int_\delta \left\|\mathcal{X}_+\alpha(L-k-i0)^{-1}u\right\|^2 dk \le C\|u\|^2 \\ \int_\delta \left\|\mathcal{X}_-\alpha(L^*-k-i0)^{-1}u\right\|^2 dk \le C\|u\|^2 \end{cases} \tag{2.8}$$

(b) *There exists a constant $C < \infty$ such that for all $u \in \mathcal{P}_\delta H$ the following estimates hold:*

$$\begin{cases} \int_\delta \left\|\mathcal{X}_-\alpha(L-k+i0)^{-1}u\right\|^2 dk \le C\|u\|^2 \\ \int_\delta \left\|\mathcal{X}_+\alpha(L^*-k+i0)^{-1}u\right\|^2 dk \le C\|u\|^2 \quad . \end{cases} \tag{2.9}$$

In the last section of the present paper we are going to apply the results obtained above (more specifically, the result of the Corollary 2.4) to the analysis of the similarity problem for the operator of one-dimensional nonselfadjoint Friedrichs model.

3. Application: Friedrichs model operator

We consider the operator acting in the Hilbert space $L_2(\mathbb{R})$ defined by the formula

$$(Lu)(x) = xu(x) + \psi(x) \int u(t)\overline{\varphi(t)}\, dt, \qquad u, \varphi, \psi \in L_2(\mathbb{R}). \tag{3.1}$$

The determinant of perturbation $D(\lambda)$ in this case is given by the following expression: $D(\lambda) = 1 + \int \overline{\varphi(t)}\psi(t)(t-\lambda)^{-1}\, dt$. In order to simplify the calculation of the operators α and $\mathcal{X}_\pm$ let's restrict ourselves to the case of orthogonal functions φ, ψ: $(\varphi, \psi) = 0$.

Let's denote the class of the functions f analytic in the upper (lower) half-plane and satisfying the condition

$$\sup_{\varepsilon>0\,(\varepsilon<0)} \int |f(k+i\varepsilon)|^p \frac{dk}{1+k^2} < \infty$$

by $H_+^{p,\mathrm{loc}}$ ($H_-^{p,\mathrm{loc}}$).

The following lemma, characterizing the structure of the spectrum of the operator under investigation, holds:

Lemma 3.1. (i) *Let the spectrum of the operator* (3.1) *be absolutely continuous. Then* $(D(\lambda))^{-1} \in H_\pm^{2,loc}$, $(D(\lambda))^{-1}(\psi(t)(t-\lambda)^{-1}, \psi(t)) \in H_\pm^2$.

(ii) *Provided, that*

(a) $(D(\lambda))^{-1} \in H_\pm^{2+\delta,loc}$, $\delta > 0$,

(b) $\psi(t) \in L_\infty(\mathbb{R})$,

the spectrum of the operator (3.1) *is absolutely continuous.*

We finally note, that the condition (2.8) for the operator (3.1) can be reduced to the test of boundedness of the certain singular integral operators acting in $L_2(\mathbb{R})$. Namely, the following theorem can be proved:

Theorem 3.2. *Provided that the spectrum of the one-dimensional perturbation of the multiplication operator* (3.1) *is absolutely continuous and* $(\varphi, \psi) = 0$, *the boundedness of the* *singular integral operators with the kernels*

$$\begin{aligned} T_1(k,t) &= \frac{i\,\overline{\psi(t)}}{t-k-i0} + \frac{1 - i\,\|\varphi\|\|\psi\|^{-1}(\psi(x)(x-k-i0)^{-1},\psi(x))}{D(k+i0)}\,\frac{\overline{\varphi(t)}}{t-k-i0}, \\ T_2(k,t) &= \frac{i\,\overline{\varphi(t)}}{t-k-i0} + \frac{1 - i\,\|\psi\|\|\varphi\|^{-1}(\varphi(x)(x-k-i0)^{-1},\varphi(x))}{D_*(k+i0)}\,\frac{\overline{\psi(t)}}{t-k-i0}, \end{aligned} \tag{3.2}$$

where $D_*(\lambda) \equiv \overline{D(\overline{\lambda})}$, in the space $L_2(\delta)$ is sufficient for the similarity of the restriction of the operator L to a spectral set, corresponding to the Borel set $\delta \subset \mathbb{R}$, to a selfadjoint one.

The proof of this theorem is a straightforward application of the Corollary 2.4 to the operator under investigation.

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