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GEOMETRY OF THE INNER MAXIMAL FUNCTION II

Mats Erik Andersson

Abstract. A maximal function with restrictions on the test sets is studied. As compared to the Hardy–Littlewood maximal function, only affine copies contained in the region are allowed as test sets. The question is whether this maximal function must be positive everywhere in the region. A complete characterization is achieved, stating that this happens exactly when the region is a bounded or unbounded copy of the fixed convex contour.

For a long time the Hardy–Littlewood maximal function has been of exceptional value in analysis. There are variations on its standard definition:

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f|\,dx, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),\ Q \subseteq \mathbb{R}^n \text{ is a cube.}$$

One common idea is to make a local maximal function by the additional demand $|Q| < t$, for a given $t > 0$. Another alteration is the theme of this short note.

Let us consider a fixed compact and convex set $K \subseteq \mathbb{R}^2$ with interior, as well as a region $\Omega \subseteq \mathbb{R}^2$. The collection $\mathcal{K} = \mathcal{K}_\Omega$ consists of all affine copies V of K such that $V \subseteq \Omega$. Observe that V is constructed by dilation and translation of K; no rotation is involved. We define the inner maximal function M_Ω with respect to Ω and $\mathcal{K}$ as

$$M_\Omega f(x) = M_{\Omega,K} f(x) = \sup_{x\in V\in\mathcal{K}} \frac{1}{|V|}\int_V |f|\,dx, \quad f \in L^1_{\text{loc}}(\Omega).$$

The word inner is chosen for its suggestive geometrical meaning, instead of the more prosaic word local. With this notation the Hardy-Littlewood function could be written $M_{\mathbb{R}^2} = M_{\mathbb{R}^2,Q}$, where Q would be a square in the plane.

It is a basic fact that for each $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, not almost everywhere zero, one has $Mf > 0$ everywhere: simply choose a sufficiently large square. It would be interesting to know under what circumstances M_Ω shares with $M_{\mathbb{R}^2}$ the property of being positive everywhere in Ω. Since the equality $M_\Omega f(x) = 0$ means that it is possible to "hide" the point $x \in \Omega$ from $\operatorname{supp} f$ when covering sets are from $\mathcal{K}$, it seems natural to introduce a notion.

This material is the refined development of an earlier more specialized subject. The original ideas formed during an extended visit in the Autumn 1998 to the Institute of Mathematics at the Czech Academy of Sciences, Prague. Its hospitality is thankfully acknowledged. The visit was initiated through an exchange program administered by the Royal Academy of Sciences, Stockholm.

Definition. The region Ω is said to be well-covered by $\mathcal{K}$ in case for every f in $L^1_{\mathrm{loc}}(\Omega) \setminus \{0\}$ and $x \in \Omega$, we have $M_\Omega f(x) = M_{\Omega,K} f(x) > 0$.

A simple argument shows that it suffices to consider instead of $f \in L^1_{\mathrm{loc}}(\Omega)$ point masses δ_y, $y \in \Omega$, and to demand $M_\Omega \delta_y(x) > 0$ for all $x, y \in \Omega$, in the obvious sense. Equivalently, there is to every pair $x, y \in \Omega$ a set $V \in \mathcal{K}$ with $\{x, y\} \subset V$.

Lemma 1. *Any region Ω well-covered by $\mathcal{K}$ has to be convex.*

Proof. Take $x, y \in \Omega$ arbitrary. Then we have $M_\Omega \delta_y(x) > 0$, which produces an affine copy K^* of K with $x, y \in K^*$, $K^* \subseteq \Omega$. By convexity of K the full segment $[x, y] = \{t + (1-t)y \mid 0 \leq t \leq 1\}$ is contained in K^* and hence in Ω. It follows that also Ω is convex.

For points $x, y \in \mathbb{R}^2$ the notation $[x, y] = \{t + (1-t)y \mid 0 \leq t \leq 1\}$ for segments will be used in this paper. There is no danger of confusing this with closed intervals on the real line.

Remark. Of course the convexity of Ω is not sufficient for well-coveredness. Letting K be the unit square we can consider $\Omega = \,]-2, 2[\, \times \,]0, 1[$. If f is supported in $]1, 2[\, \times \,]0, 1[$, then $M_\Omega f(x) = 0$ for all $x \in \Omega \cap \{(x_1, x_2) \mid x_1 \leq 0\}$.

Definition. The set K, being convex and bounded, defines for each $\kappa \in \partial K$ a unique closed cone $V(\kappa)$, at most a half plane and with vertex at the origin, giving $K \subseteq \kappa + V(\kappa)$. These latter sets, as $\kappa \in \partial K$, will be referred to as the closed wedge domains generated by K. The interior of such a wedge domain is naturally said to be an open wedge domain.

An open half plane L is said to be a resting half plane for K if $\overline{L} \supseteq K$ and there is $\kappa \in \partial L \cap \partial K$ such that the left or right tangent to K at κ is contained in ∂L.

These two notions allow us to formulate the main result of this document.

Theorem. *Let Ω be well-covered by $\mathcal{K}$. Then Ω is either the full plane, an affine copy of K°, a translation of an open wedge domain generated by K, or a translation of a resting half plane for K.*

It is trivial to see that all listed options are actually well-covered by $\mathcal{K}$.

As the case of bounded regions Ω turns out to be more streamlined, the next two sections deal in turn with the bounded and unbounded cases. First some geometric objects must be introduced.

Definition. Consider for any $\theta \in [0, 2\pi[$ the orthonormal vectors $A_\theta = (\cos\theta, \sin\theta)$ and $B_\theta = (\sin\theta, -\cos\theta)$. A closed band of direction θ is for real parameters $s \leqslant t$ the set

$$R(\theta, s, t) = \big\{\xi A_\theta + \zeta B_\theta \,;\, \xi \in \mathbb{R},\, s \leqslant \zeta \leqslant t\big\}.$$

For a plane set U we define the θ-width of U according to

$$\theta\text{-width}(U) = \inf \big\{t - s \,;\, U \subseteq R(\theta, s, t),\, s < t\big\}.$$

Finally, a particular quantity relating Ω to K will be necessary to tell the size of Ω in relation to that of K.

Definition. $r(\Omega, K) = \sup \big\{a > 0 \,;\, aK + b \subseteq \Omega \text{ for some } b \in \mathbb{R}^2\big\}$.

THE CASE OF A BOUNDED REGION.

One needs first to understand how boundary points of Ω can be sensed by the inner maximal function, that is, in what respect the covering property with affine copies of K can include boundary points. Of the next two lemmata, the latter improves on the former.

Lemma 2. *Assume Ω is bounded and well-covered by $\mathcal{K}$. To any $\alpha \in \Omega$ and $\beta \in \overline{\Omega}$ there exist $a > 0$ and $b \in \mathbb{R}^2$ with $\{\alpha, \beta\} \subset aK + b \subseteq \overline{\Omega}$.*

Proof. One may assume that $\alpha \neq \beta$ and $\beta \in \partial\Omega$. Choose $\beta_n \in \Omega$ such that $\beta_n \to \beta$. By well-coveredness there are $a_n > 0$ and $b_n \in \mathbb{R}^2$ with $\{\alpha, \beta_n\} \subset a_n K + b_n \subseteq \Omega$.

Since Ω is bounded, the sets $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded, so increasing sub-indices $n(j)$ exist with $a_{n(j)} \to a$ and $b_{n(j)} \to b$ as $j \to \infty$. Due to $\alpha \neq \beta$ and $\alpha, \beta_{n(j)} \in a_{n(j)} K + b_{n(j)}$, by necessity $a > 0$ follows.

Put $\|K\| = \sup\{|k|\,;\, k \in K\}$, a finite quantity. For any $x = a_n y + b_n \in a_n K + b_n$ an estimate obtains;

$$\operatorname{dist}(x, aK + b) \leqslant |a_n y + b_n - ay - b| \leqslant |a_n - a|\,\|K\| + |b_n - b|.$$

Writing $\varepsilon_N = \sup_{j \geqslant N} |a_{n(j)} - a|\,\|K\| + |b_{n(j)} - b|$, one has $\varepsilon_N \to 0$ and

$$x \in \bigcup_{j=N}^{\infty} \big(a_{n(j)} K + b_{n(j)}\big) \quad \text{implies} \quad \operatorname{dist}(x, aK + b) \leqslant \varepsilon_N.$$

Consequently, ($\overline{B}(0, r)$ is here the closed disk at the origin of radius r)

$$\{\alpha\} \cup \{\beta_{n(N)}, \beta_{n(N+1)}, \dots\} \subset \bigcup_{j=N}^{\infty} \big(a_{n(j)} K + b_{n(j)}\big) \subseteq aK + b + \overline{B}(0, \varepsilon_N).$$

Letting N tend to infinity, it follows that $\alpha, \beta \in aK + b$ as well as $aK + b = \limsup_{j \to \infty} a_{n(j)} K + b_{n(j)} \subseteq \overline{\Omega}$. The latter limes superior being understood in the sense of sets. These last two observations complete the desired proof.

Lemma 3. *Assume Ω is bounded and well-covered by $\mathcal{K}$. To any $\alpha, \beta \in \overline{\Omega}$ and $\beta \in \overline{\Omega}$, there exist $a > 0$ and $b \in \mathbb{R}^2$ with the property $\{\alpha, \beta\} \subset aK + b \subseteq \overline{\Omega}$.*

Proof. According to Lemma 2, only the case $\alpha \in \overline{\Omega}$ and $\beta \in \partial\Omega$ remains. Essentially the same argument as in the preceding proof goes through. The major change being made in the stage of choosing parameters a_n and b_n. Instead of referring to well-coveredness, the statement of Lemma 2 itself insures the existence of these parameters. The rest of the proof goes through with the minimal change of inserting the closure $\overline{\Omega}$ instead of Ω where applicable for inclusion relations.

Now the characterization of well-coveredness for bounded regions can be resolved.

Proposition 4. *Assume Ω is bounded and well-covered by $\mathcal{K}$. Then parameters $a > 0$ and $b \in \mathbb{R}^2$ exist, such that $\Omega = aK^\circ + b$.*

Proof. Choose $\alpha, \beta \in \partial\Omega$ with $|\alpha - \beta| = \operatorname{diam}\Omega$. According to Lemma 3 there are $a_0 > 0$ and $b_0 \in \mathbb{R}^2$ with $\{\alpha, \beta\} \subseteq a_0 K + b_0 \subseteq \overline{\Omega}$.

It is now claimed that $r(\Omega, K) = a_0$; already by definition $a_0 \leqslant r(\Omega, K)$ holds. Define temporarily $x, y \in K$ by $\alpha = a_0 x + b_0$ and $\beta = a_0 y + b_0$. The contrapositive to the claim is now $a_0 < r(\Omega, K)$, which would produce $a > a_0$ and $b \in \mathbb{R}^2$ with $\{ax + b, ay + b\} \subset aK + b \subset \Omega$. Then a contradiction obtains:

$$\operatorname{diam} \Omega \geqslant a|x - y| > a_0|x - y| = \operatorname{diam} \Omega.$$

Thus the claimed value $r(\Omega, K) = a_0$ has been certified. An affine transformation applied to K next allows $a_0 = 1$, $b = 0$, and the simplified relations

$$(*) \qquad K \subseteq \overline{\Omega}; \qquad aK + b \subseteq \overline{\Omega} \text{ implies } a \leqslant 1.$$

Suppose finally that $K \subsetneq \overline{\Omega}$. The rest of the proof aims at contradicting $(*)$ under this assumption. Since K is strictly contained in $\overline{\Omega}$, there is a band $R(\theta, s, t)$ containing $\overline{\Omega}$ in such a way that both components of $\partial R(\theta, s, t)$ intersects $\partial\Omega$, whereas at least one of them does not meet ∂K.

Study for this direction θ the quantities $\delta = \theta\text{-width}(K)$ and $\varepsilon = \theta\text{-width}(\Omega)$. Clearly $\varepsilon > \delta > 0$ obtains. Choose now $x, y \in \partial\Omega \cap R(\theta, s, t)$, one point in each component of $\partial R(\theta, s, t)$. Lemma 3 supplies $a > 0$ and $b \in \mathbb{R}^2$ with $x, y \in \partial(aK + b)$ and $aK + b \subseteq \overline{\Omega}$. This means

$$\varepsilon = \theta\text{-width}(\{x, y\}) \leqslant \theta\text{-width}(aK + b) = a \cdot \theta\text{-width}(K) = a\delta,$$

whence $a \geqslant \varepsilon/\delta > 1$, in spite of $aK + b \subseteq \overline{\Omega}$. Thus we have, as intended, arrived at a contradiction, which in turn forces upon us the conclusion $K = \overline{\Omega}$. By convexity we deduce $K^\circ = \Omega$, the desired statement.

Corollary 5. *Suppose Ω is well-covered by $\mathcal{K}$ and that θ-width$(\Omega) < \infty$ for some direction θ. Then $\Omega = aK^\circ + b$ for some $a > 0$ and $b \in \mathbb{R}^2$.*

Proof. The assumption on θ-width implies $r(\Omega, K) < \infty$. Every choice of $\alpha, \beta \in \Omega$ thus gives $|\alpha - \beta| \leqslant r(\Omega, K) \operatorname{diam}(K) < \infty$. Hence Ω has bounded diameter and so is bounded. The claim is now resolved by a reference to Proposition 4.

Unbounded regions

To begin with, it must be stressed that Lemma 2 need not be true for well-covered, but unbounded Ω. Thus the method of proof for bounded regions cannot be carried over without major modification.

Examples 6. Consider the convex sets $K_1 = \{(x, y)\,;\, 0 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant 1\}$ and $K_2 = \{(x, y)\,; y \geqslant 0, x^4 \leqslant y^2 \leqslant x\}$ together with the regions $U_1 = \{(x, y)\,;\, x, y > 0\}$ and $U_2 = \{(x, y)\,;\, x \in \mathbb{R}, y > 0\}$. It is perfectly straightforward to see that each U_j is well-covered by the set families generated by each K_i.

Furthermore, the conclusion in Lemma 2 holds with $K = K_1$ and $\Omega = U_1$ precisely for $\beta \in U_1 \cup \{(0, y)\,;\, y \geqslant 0\}$, whereas $\Omega = U_2$ gives validity only for $\beta \in U_2$. On the other hand, the choice $K = K_2$ and $\Omega = U_1$ demands $\beta \in \{(0, 0)\} \cup U_1$ for Lemma 2 to hold, whereas $\Omega = U_2$ again reduces validity to $\beta \in U_2$.

The completion of our Theorem, as stated earlier, demands that unbounded, well-covered regions $\overline{\Omega}$ be shown to coincide with $\alpha + V(\kappa)$, where $\alpha \in \partial\Omega$ and $\kappa + V(\kappa)$ arises as a closed wedge domain relative to the given contour K, or that Ω coincide with a translated resting half plane for K.

Lemma 7. *Let Ω be well-covered and such that $\partial\Omega$ contains a straight segment. Then this segment is parallel to some right or left tangent line of ∂K. When affinely mapping the segment into the tangent, Ω and K will be part - that is after a translation - of the same half plane spanned by the tangents.*

Proof. Suppose $[\alpha, \beta] \subset \partial\Omega$. Let L be the closed half plane such that $0 \in \partial L = \ell$, $[\alpha, \beta] \subset \alpha + \ell$, and $\Omega \subseteq L$. Take $\nu \in L$, $|\nu| = 1$, $\nu \perp \ell$, and consider the strip $L(\rho) = \ell + \{t\nu\,;\, 0 \leqslant t \leqslant \rho\}$.

It is possible to select $\alpha_n, \beta_n \in \Omega$ with $\alpha_n \to \alpha$ and $\beta_n \to \beta$; hence also $a_n > 0$, $b_n \in \mathbb{R}^2$ such that $\{\alpha_n, \beta_n\} \subset a_n K + b_n \subseteq \Omega$. The convexity of K provides $\kappa \in \partial K$ and $\kappa + L \supseteq K$. Consider now the closed wedge domain $\kappa + V(\kappa)$.

For each $\rho > 0$, the inclusion $a_n K + b_n \subseteq V(\kappa) + a_n\kappa + b_n$ delivers an estimate

$$\begin{aligned}\operatorname{diam}\big([a_n K + b_n] \cap [\alpha + L(\rho)]\big) &\leqslant \operatorname{diam}\big([V(\kappa) + a_n\kappa + b_n] \cap [\alpha + L(\rho)]\big)\\ &\leqslant \operatorname{diam}\big(V(\kappa) \cap L(\rho)\big).\end{aligned}$$

Consequently, for all $\rho > 0$,

$$0 < |\alpha - \beta| = \limsup_{n\to\infty} |\alpha_n - \beta_n| \leqslant \operatorname{diam}\big(V(\kappa) \cap L(\rho)\big).$$

Letting $\rho \to 0+$, we conclude that at least one of the rays in $\partial V(\kappa)$ must be contained in $\ell = \partial L$. This is precisely the claim.

Parts of the classification can now be completed. Let Ω be well-covered and unbounded. The full plane $\Omega = \mathbb{R}^2$ is an obvious possibility and any smaller region Ω must, by convexity, be contained in a half plane. Should Ω be exactly a half plane, Lemma 7 says that it must in fact be a resting half plane for K. Thus only the cases of Ω being strictly smaller than any supporting half plane still remain to be studied. The next standard argument, for completeness sketched here, provides the means to locate any corner. Observe that any still unclassified, but well-covered region now satisfies the hypothesis of the following statement.

Lemma 8. *Let Ω be a convex set with interior and strictly contained in any supporting half plane. Then there exists a closed half plane L such that $L^\circ \cap \Omega \neq \emptyset$ and also $0 < \operatorname{diam}(L^c \cap \Omega) < \infty$.*

Proof. Translation, reflection, and rotation allow the simplifications $(0,0) \in \partial\Omega$, $\Omega \subset \mathbb{R} \times]0, \infty[$, and that the ray $\mathbb{R}^+ \times \{0\}$ be a right-hand tangent to $\partial\Omega$ at $(0,0)$, as well as the existence of a point $\sigma = (s,t) \in \Omega^c$ such that $s < 0$, $t > 0$.

Furthermore, there is close to the origin, a point $\rho = (r,q) \in \partial\Omega$ with $r > 0$. Let ℓ be the unique straight line passing through σ and ρ; take L to be the half-plane with boundary ℓ and not containing the origin. Clearly $L^\circ \cap \Omega \neq \emptyset$ is achieved. In addition, $\ell \cap \Omega \subsetneq [\sigma, \rho]$ and hence $L^c \cap \Omega \subseteq Q$, where Q is a quadrilateral with corners at the points σ, ρ, $(x,0)$, and $(r,0)$. Consequently, $0 < \operatorname{diam}(L^c \cap \Omega) \leqslant \operatorname{diam}(Q)$, which completes the claim.

Lemma 9. *Let Ω be well-covered by $\mathcal{K}$ and such that Lemma 8 is applicable. Then there exist $\alpha \in \partial\Omega$ and $\kappa \in \partial K$ such that the closed tangent cone of α with respect to Ω, here written $\alpha + U(\alpha)$, has the property $U(\alpha) \subseteq V(\kappa)$.*

Proof. By slightly tilting the half plane L from Lemma 8, we may assume there are unique $\alpha \in \partial\Omega$ and $\kappa \in \partial K$ that have translations of L as supporting half planes. This follows from the convexity of Ω and K.

Let U_n be open cones that exhaust $U(\alpha)$, that is $U(\alpha)^\circ = \cup_n U_n$. There are hence quantities $\delta_n \to 0$ such that $(\alpha + U_n) \cap \overline{B}(\alpha, \delta_n) \subseteq \Omega$. The set on the left-hand side is henceforth denoted $\alpha + U_n^*$.

Take any $\beta \in \alpha + U_n^*$ with $|\beta - \alpha| = \delta_n$, and additionally locate $\{\alpha_j\}_{j=1}^\infty$ such that $\alpha_j \to \alpha$ and $\alpha_j \in \alpha + U_n^*$. Suitable parameters a_n and b_n achieve $\{\beta, \alpha_j\} \subset a_j K + b_j \subseteq \Omega$. Write also $S(\rho) = \{x \in \alpha + L\,;\, \mathrm{dist}(x, \alpha + \ell) \leqslant \rho\}$, where $\ell = \partial L$.

Putting $\varepsilon_j = \mathrm{dist}(\alpha_j, \alpha + \ell)$ we have the useful facts

$$\mathrm{diam}\,\big[\Omega \cap S(\varepsilon_j)\big] \to 0 \quad \text{and} \quad a_j\kappa + b_j \in \Omega \cap S(\varepsilon_j).$$

It follows that $a_j\kappa + b_j \to \alpha$ as $j \to \infty$. Hence the inclusion $[\beta, \alpha_j] \subset a_j\kappa + b_j + V(\kappa)$ delivers, as $j \to \infty$, the relation $[\beta, \alpha] \subseteq \alpha + V(\kappa)$. The freedom in choosing β thus gives $\alpha + U_n^* \subseteq \alpha + V(\kappa)$ for any n, which when using the exhaustion $\{U_n\}$ simplifies to $U(\alpha) \subseteq V(\kappa)$; this was the claim.

Remark. Observe that the preceding lemma is applicable also for bounded, well-covered regions, so there is no hope of identifying Ω from the above statement alone.

The final step for our classification resembles the preceding proof very much but of course uses the unboundedness as an essential ingredient.

Proposition 10. *Let Ω be an unbounded region such that Lemma 9 is applicable. With the same α and κ it follows that $\Omega = \alpha + V(\kappa)$.*

Proof. It suffices, by Lemma 9, to prove $\alpha + V(\kappa) \subseteq \overline{\Omega}$, which more or less will be achieved by reversing the exhaustion used in the previous argument. Notation and choices made during the proof of Lemma 9 remain in effect.

Let $\{V_n\}_{n=1}^\infty$ be open cones exhausting $V(\kappa)$, so $V(\kappa)^\circ = \cup_n V_n$. Take $\delta_n \to 0$ with $V_n^* = V_n \cap \overline{B}(0, \delta_n)$ such that $\kappa + V_n^* \subseteq K$. Fix a $\rho > 0$ and choose $\alpha_\rho \in \Omega \cap S(\rho)$. From the unboundedness of Ω there are $\beta_j \in \Omega$ such that $|\beta_j| \to \infty$. Thus $\{\alpha_\rho, \beta_j\} \subset a_j K + b_j \subseteq \Omega$ for suitable $a_j > 0$ and $b_j \in \mathbb{R}^2$. It follows that $\gamma_j = a_j\kappa + b_j \in \Omega \cap S(\rho)$ and $\gamma_j + a_j V_n^* \subseteq \Omega$ for every $n \geqslant 1$. Also $a_j \to \infty$, due to $|a_\rho - \beta_j| \to \infty$.

Since $\{\gamma_j\}_1^\infty \subseteq \Omega \cap S(\rho)$ is bounded, there is a subsequence $\gamma_{j(m)} \to \gamma_\delta \in \overline{\Omega} \cap S(\rho)$. From $\gamma_{j(m)} + a_{j(m)} V_n^* \subseteq \Omega$ and $V_n = \cup_m\, a_{j(m)} V_n^*$, it follows that $\gamma_\rho + V_n \subseteq \overline{\Omega}$. This holds for any $n \geqslant 1$, so also $\gamma_\rho + V(\kappa) \subseteq \overline{\Omega}$ follows.

Finally, $\gamma_\rho \in \overline{\Omega} \cap S(\rho)$ forces $\gamma \to \alpha$ as $\rho \to 0+$. Consequently $\alpha + V(\kappa) \subseteq \overline{\Omega}$ obtains from $\gamma_\rho + V(\kappa) \subseteq \overline{\Omega}$. This inclusion thus finishes the proof.

The proposition says that any well-bounded region, which is not the full plane, a half plane, or a bounded region, must in fact be the translation of an open wedge domain generated by K. Therefore the claimed classification has now been completed and our Theorem has been fully verified.

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