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Multithreshold spectral phase transition examples in a class of unbounded Jacobi matrices. II

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Abstract

The paper deals with spectral analysis of a class of selfadjoint unbounded Jacobi matrices J with modulated entries. This means all their entries have the form of "smooth" increasing to infinity sequences multiplied by proper periodic sequences. For this class criteria of pure absolute continuity of the spectrum or its discretness and asymptotic of generalized eigenvectors of J are given. Some examples illustrating stability zones of spectrum structure are presented.

1 Introduction.

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical orthonormal basis in l^2. The paper deals with the class of tridiagonal matrices which induce densely defined operators J acting in l^2 by the formula

$$Je_n = \lambda_{n-1}e_{n-1} + \lambda_n e_{n+1} + q_n e_n, \quad n = 1, 2, \cdots,$$

where weights $\{\lambda_n\}_{n\in\mathbb{N}}$ and diagonal (potential) $\{q_n\}_{n\in\mathbb{N}}$ are sequences of real numbers and $\lambda_0 = 0$. Let S be the unilateral shift in l^2 and $De_n = \lambda_n e_n$, $Qe_n = q_n e_n$ are the diagonal operators. We always assume that $\sum \lambda_n^{-1} = +\infty$ (Carleman condition). Then $J = (SD + DS^* + Q)$. The main goal of the paper is spectral analysis of sufficiently rich class of $J's$ with modulated entries. Namely $\lambda_n = c_n\mu_n, q_n = b_n r_n$, where $\{c_n\}_{n\in\mathbb{N}}$ is periodic sequence non-zero numbers of the smallest period N, $\{b_n\}_{n\in\mathbb{N}}$ is also periodic sequence of real numbers of smallest period M, $\{\mu_n\}$ and $\{r_n\}$ are "smooth" sequences of real numbers with $\lim_n \mu_n = +\infty$, $\mu_n \neq 0$. The class of $J's$ induced by the above sequences $\{\lambda_n\}, \{q_n\}$ provides (as it will become clear below) many interesing and new examples of operators with complicated spectral behaviour with respect to perturbation of parameters $\{b_n\}, \{c_n\}$ and parities of M and N. Our interest in this class was partially inspired by noticing very particular but nice examples of Jacobi matrices with "linear" (in n) entries discussed from the point of view of group theory by Masson and Repka in [9] and Edward in [8]. Unbounded Jacobi matrices also appear in different fields: quantum group theory, birth and death Markoff processes etc. The paper [16] was devoted to the very special case when $M = 1$, $\mu_n = n^\alpha$, $\alpha \in (0, 1]$, $r_n = \delta\mu_n$. To some extent we continue here considerations started in [16] and for this reason the notations used below are close to those employed in [16]. In the present paper we try to understand the results of the above articles from our point of view i.e. asymptotic analysis of the transfer matrix. The main goal of the preceding paper was presentation of some explicit examples of spectral phase transition. In this paper we try to develop a sort of spectral analysis of sufficiently rich class of Jacobi matrices with modulated entries λ_n, q_n. Actually spectral theory of selfadjoint Jacobi matrices is a vast field which develops rapidly in recent years. Let us mention here only few works related to the above mentioned topics [1], [4], [6], [7], [13], [14], [15], [16], [17], [18], [19], [27]. In this paper we deal with spectral phase transition of the first type i.e. the space of parameters can be decomposed into separate regions in which spectrum of a given operator is either pure absolutely continuous or discrete. In turn according to physical terminology spectral phase transition of the second type refers to situations with mixed spectrum and existence of mobility edges and is even more interesting. Studies of spectral phase transition required applications of modern methods of spectral analysis of $J's$ and elaborating new ones. For example a sort of discrete version of semi-classical method (WKB asymptotics, Levinson-type theorem) was used. As it is well known spectral analysis of J due to Gilbert-Pearson theory [17] is strongly related to the study of asymptotic behaviour of solutions of the infinite system of equations

$$\lambda_{n-1}u_{n-1} + \lambda_n u_{n+1} + q_n u_n = \lambda u_n, \; n = 2, 3, \cdots \tag{1.1}$$

where $\lambda \in \mathbb{R}$. Let $\vec{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$ and

$$B_n := \begin{pmatrix} 0 & 1 \\ -\lambda_{n-1}\lambda_n^{-1} & (\lambda - q_n)\lambda_n^{-1} \end{pmatrix}. \tag{1.2}$$

Then (1.1) can be written as:

$$\vec{u}_{n+1} = B_n \vec{u}_n,\ n = 2, 3, \cdots.$$

The matrix B_n is called the transfer matrix of (1.1) The asymptotic analysis of the product $B_n B_{n-1} \cdots B_2$ was already essentially exploited in several paper e.g. [13], [14], [15], [16], [27], and others.

In what follows it will be always assumed that μ_n, r_n satisfy among others the following conditions $\lim_n \mu_{n+1}\mu_n^{-1} = 1$ and $\lim_n r_n \mu_n^{-1} = \delta$ is finite. Then for $n = lK + j$, where $K = M \vee N$ is the least common multiple of M and N, $1 \le j \le K$ is fixed, there exist $\lim_l B_{lK+j} := F_j$. The next necessary function we need in order to describe briefly the results of the paper is defined by $P_K(b, c; \delta) := Tr(F_{K+1} \cdots F_2)$, where $b := (b_1, \cdots, b_M)$, $c := (c_1, \cdots, c_N)$. Note that parameter δ is somehow redundant (being non-zero it could be absorbed by $b_n's$) but it is useful below for better understanding. Let $\{\tilde{b}_s\}$ (resp. $\{\tilde{c}_s\}$) be K-periodic extension of $\{b_s\}$ (resp. $\{c_s\}$). Denote by C_{per} (resp. B_{per}) the diagonal matrix defined by $\{\tilde{c}_s\}$ (resp. $\delta\tilde{b}_s\}$). Let us introduce periodic Jacobi matrix

$$J_{per} = SC_{per} + C_{per}S^* + B_{per}.$$

Observe that $P_K(b, c; \delta)$ defined above is strongly related to the characteristic polynomial $d_{J_{per}}(\lambda)$ of J_{per}. In fact, since $(c_0 := c_N)$

$$d_{J_{per}}(\lambda) := Tr \prod_{s=1}^{K} \begin{pmatrix} 0 & 1 \\ -c_{s-1}c_s^{-1} & (\lambda - b_s\delta)c_s^{-1} \end{pmatrix}$$

we have

$$d_{per}(0) = P_K(b, c; \delta).$$

First this relation was used in special case in [16]. It is well known that $\sigma_{ac}(J_{per})$ consists of exactly K intervals $[\alpha_j, \beta_j]$, $j = 1, \cdots, K$ which can have common end points (in degenerate case), see [22]. However this does not happen in generic situations [12]. Denoting by ' Int ' $\sigma_{ac}(J_{per}) = \bigcup_j (\alpha_j, \beta_j)$ the main result of §3 says (Th.3.1) : if $0 \in$ ' Int ' $\sigma_{ac}(J_{per})$ then J has pure absolutely continuous spectrum covering $\mathbb{R}$ under proper smoothness conditions (2.1). Moreover in Th.3.1 asymptotic formula for generalized eigenvectors (solutions $\vec{u}_n$ of (1.1)) is given. In turn in Th. 4.1 we show that $\sigma(J)$ is discrete provided 0 does not belong to $\sigma_{ac}(J_{per})$. Asymptotic of eigenvectors of J is also shown. Denote by $\{\rho_k\}_{k\in\mathbb{N}}$ the set of all eigenvalues of J ordered with respect to increasing of their modulus. In Th.4.2 estimations of Cesaro averages of $\{\rho_k\}$ are proved. In particular case $\mu_n \asymp n^\alpha$, $\alpha \in (0, 1)$ we have $\rho_n \asymp \mu_n$. This result was not found in [16] even in the case $\mu_n = n^\alpha$. Note that the point spectrum of J can be semibounded

or not semibounded depending mainly on b and c. Finally in Section 5 we give some explicit examples illustrating various types of spectral phase transition and stability of spectra under small perturbations of $b_n's$ or $c_n's$ and periods.
Concerning the methods we use in this work they are based on three main ideas.
i) Gilbert-Pearson theory reduces spectral analysis of J to the asymptotic behaviour of products $B_n \cdots B_2$, as $n \to \infty$.
ii) The asymptotic analysis is obtained by proper application of discrete version of WKB asymptotics for $B_n \cdots B_2$.
iii) In turn K-periodicuty forces us to employ WKB approach for K different subsequences $(n = lK + j,\ j = 1, \cdots, K)$ of the above products which appear to be smooth in l (not in n.) Actually this means that we collect the products into blocks of length K.

2 Preparatory facts and conditions.

In this section we discuss some conditions emposed on weights and diagonals. These conditions are expressed in an unified form giving us possibility to extend results of the present paper for a larger classes of Jacobi matrices. In what follows we shall use the class $\mathcal{D}^k$ of bounded sequences (or mxm matrices) with bounded varation introduced by G. Stolz in [27]. For a sequence $\{A(n)\}_{n\in\mathbb{N}} = A$ of mxm matrices, let $\Delta A(n) = A(n+1) - A(n)$, $\Delta^s A = \Delta(\Delta^{s-1}A)$, $s = 2, 3, \cdots$ and $\Delta^0 A = A$. We say that $A \in \mathcal{D}^k$ iff $\|\Delta^s A(\cdot)\| \in l^{k/j}$, $j = 1, \cdots, k$. Given a sequence $A(s)$ of mxm matrices the product $\prod_{s=n_1}^{n_2} A(s)$ is understood as: $A(n_2) \cdots A(n_1)$. This is so called chronological product.

Recall that $\{c_s\}$ be a periodic sequence of non-zero real numbers of minimal period N, and $\{b_s\}$ be also a periodic sequence of real numbers of minimal period M. The $K-$ periodic extension of sequence $\{b_s\}$ (resp. $\{c_s\}$) is denoted by $\{\tilde{b}_s\}$ (resp $\{\tilde{c}_s\}$). For any integer $1 \leq j \leq K$ the 2x2 matrix F_j is given by

$$F_j = \begin{pmatrix} 0 & 1 \\ -\tilde{c}_{j-1}\tilde{c}_j^{-1} & -\delta\,\tilde{b}_j\tilde{c}_j^{-1} \end{pmatrix}, \tag{2.3}$$

here $\tilde{c}_o := \tilde{c}_K$.
Let

$$G_s := \begin{pmatrix} 0 & 1 \\ -\tilde{c}_{s-1}\tilde{c}_s^{-1} & 0 \end{pmatrix}, E_s := \begin{pmatrix} 0 & 0 \\ 0 & -\delta\,\tilde{b}_s\tilde{c}_s^{-1} \end{pmatrix} \quad i.e\ F_j = G_j + E_j.$$

In what follows the principal role will play the matrix $C_\infty := \prod_{j=1}^{K} F_j$. The natural character of the notation will be clarified below.
In the next sections we shall need the following elementary

PROPOSITION **2.1** *(On the interplay between parities of M and N) Let $b = (b_1, \cdots, b_M)$ and $c = (c_1, \cdots, c_N)$. Under the above notations the function*

$$P_K(b, c; \delta) := TrC_\infty \tag{2.4}$$

can be written in the following form (depending on parities of M and N)
a) M- arbitrary, N- even

$$P_K(b,c;\delta) = P_o(c) + \delta^2 P_2(b,c) + \cdots + \delta^K P_K(b,c) \tag{2.5}$$

where $P_o(c) = (-1)^{K/2}(\alpha_N^{K/N} + \alpha_N^{-K/N})$, $\alpha_N := c_{N-1}\cdots c_3 c_1 (c_N \cdots c_4 c_2)^{-1}$ *and* $P_{2s}(b,c)$ *are homogeneous polynomials of degree* $2s$ *in* $b_1, \cdots, b_M$.
b) M- odd, N- odd

$$P_K(b,c;\delta) = \delta P_1(b,c) + \delta^3 P_3(b,c) + \cdots + \delta^K P_K(b,c), \tag{2.6}$$

where $P_{2s+1}(b,c)$ *are homogeneous polynomials of degree* $2s+1$ *in* $b_1, \cdots, b_M$.
c) M- even, N- odd

$$P_K(b,c;\delta) = (-1)^{K/2} 2 + \delta^2 P_2(b,c) + \cdots + \delta^K P_K(b,c), \tag{2.7}$$

where P_{2s} *have the same meaning as above.*

Proof. a) By definition

$$P_K(b,c;\delta) = Tr \prod_{j=1}^{K} (G_j + E_j). \tag{2.8}$$

The term $Tr \prod_{j=1}^{K} G_j$ can be computed explicitely and is equal to $P_o(c)$ as K is even. Observe that the product of odd number of anti-diagonal 2x2 matrices and any number of diagonal 2x2 matrices is again anti-diagonal. This can be checked easily because the products of anti-diagonal and diagonal matrices, is again anti-diagonal.
Hence

$$\delta^{2s+1} P_{2s+1}(b,c) = Tr \sum_{t_{2s+1} > \cdots > t_1} (X \cdots E_{t_{2s+1}} \cdots E_{t_1} \cdots Y)$$

where all matricess $X, \cdots, Y$ except $E_{t_{2s+1}}, \cdots, E_{t_1}$ are anti-diagonal , must be equal to zero. Indeed, since K is even so $K - 2s - 1$ is odd and in the above products we always multiply odd number of anti-diagonal matrices G_r by $(2s+1)$ diagonal matrices E_{t_i}
b) Since both M and N are odd so K is also odd. Therefore $K - 2s$ is odd.
Now

$$P_{2s}(b,c) = Tr \sum_{t_{2s} > \cdots > t_1} (X \cdots F_{t_{2s}} \cdots E_{t_1} \cdots Y),$$

where again all matrices except $E_{t_{2s}}, \cdots, E_{t_1}$ are anti-diagonal. Since we multiply $K - 2s$ matrices G_r by $2s$ diagonal matrices E_{t_i} the products in the above sum are again anti-diagonal and so $P_{2s}(b,c) = 0$.
c) By repeating the reasoning given in proof of a) we check that $P_{2s+1}(b,c) = 0$.
In turn

$$P_o(c) = Tr[\prod_{j=1}^{K} G_j] = (-1)^{K/2}[\alpha_K + \alpha_K^{-1}],$$

where $\alpha_K = \tilde{c}_{K-1}\tilde{c}_{K-3}\cdots\tilde{c}_N\tilde{c}_{N-2}\cdots c_3c_1(\tilde{c}_K\tilde{c}_{K-2}\cdots\tilde{c}_{N+1}\tilde{c}_{N-1}\cdots c_4c_2)^{-1}$.
Since K is even for each $\tilde{c}_j$ from the nominator of α_K one can find $\tilde{c}_{j+N}$ from the denominator of α_K (the number of factors in nominator and denominator is the same) and so $\alpha_K = 1$. ■

As it was mentioned in Introduction the class of Jacobi matrices considered in this paper is given by weights $\lambda_n = c_n\mu_n$, where $\{c_n\}$ is N-periodic sequence of non-zero, real numbers and diagonal $q_n = b_nr_n$, where $\{b_n\}$ is M- periodic sequence of real numbers. In what follows we always assume that μ_n and r_n satisfy the assumptions
i) $(\mu_{n+1}\mu_{n-1})^{1/2}\mu_n^{-1} - 1$ is in $\mathcal{D}^1$,
ii) μ_n^{-1} belongs to $\mathcal{D}^1$
iii) $r_n\mu_n^{-1} \in \mathcal{D}^1$.
iv)$\lim \mu_n = +\infty$ (2.7)
v) $\lim \mu_{n+1}\mu_n^{-1} = 1$
vi) $\sum_n \mu_n^{-1} = +\infty$ (the Carleman condition, see [3])
The above assumptions will play essential role in the analysis of spectral properties of J.

Remark **2.2** . We could not require iv) below. However then appears spectral parameter λ in the limit matrix F_j and so $P_K(b, c; \delta)$ would also depend on λ. We may (and will) always assume that all $c_k > 0$. This makes no loss of generality since this can be achieved by a suitable diagonal unitary equivalence of J with J' possesing positive c'_k.

Remark **2.3** . a) If $\varepsilon_n := \mu_{n-1}\mu_n^{-1} - 1$ is in $\mathcal{D}^1$ then $\{\mu_n\}$ satisfies i).
b) In turn if $\mu_n^{-1} \in l^2$ and $\varepsilon_n \in l^2$, then $\{\mu_n\}$ satisfies ii).
c) If μ_n is increasing then ii) holds.
Denote by i $'$) (the stronger version of i)) as: $(\mu_{n+1}\mu_{n-1})^{1/2}\mu_{n-1}^{-1} - 1$ is in l^1. In particular condition i $'$) implies that there exists a finite non-zero limit $d = \lim_n \mu_{n+1}\mu_n^{-1}$. Next iv) forces that $d \geq 1$, and using vi) one has $d = 1$. In other words v) is in this case derived from other conditions.

Remark **2.4** . Weights $\lambda_n = n^\alpha(1+\Delta_n)$, $\alpha > 0$, $\lim_n \Delta_n = 0$ considered in [12] satisfy i) iff

$$\sum_n |\Delta^2(\Delta_n) + \Delta_{n+1}\Delta_{n-1} - \Delta_n^2| < \infty$$

and ii) is equivalent to the convergence of $\sum_n |\Delta(\Delta_n)|n^{-\alpha}$, here $\Delta(\Delta_n) := \Delta_{n+1} - \Delta_n$ and $\Delta^2(\Delta_n)$ is the second difference

3 Absolutely continuous spectrum and asymptotics of generalized eigenvectors.

In this section we shall establish pure absolute continuity of J for the above defined class of weights and diagonals.The results will depend in essntial way on parities of periods M and N

as well as on regultarites of weights and diagonal expressed by assumptions i),ii), iii),iv), v), vi). The method of proof uses a reduction of the transfer matrix B_n for J to a matrix $\tilde{B}_n$ which resembles the transfer matrix of discrete Schrödinger operator. Advantage of this reduction is clear because it helps to extract the main part in the principal term of the asymptotic of the chronological product of transfer matrices. It can be applied in discrete and pure absolutely continuous cases as well.

The first step of our analysis of products $\prod_{k=1}^n B_k$ is based on the following procedure relating B_k with the matrix $\tilde{B}_k$ defined as

$$\tilde{B}_k = \begin{pmatrix} 0 & 1 \\ -\tilde{c}_{k-1}\tilde{c}_k^{-1}\mu_k^{-1}(\mu_{k+1}\mu_{k-1})^{1/2} & (\mu_{k+1}\mu_k^{-1})^{1/2}(\lambda - q_k)(\tilde{c}_k\mu_k)^{-1} \end{pmatrix}.$$

The matrix $\tilde{B}_k$ resembles the transfer matrix of discrete Schrödinger operator (which has the form $\begin{pmatrix} 0 & 1 \\ -1 & \lambda - q_n \end{pmatrix}$). This allows us (remind that $\lim_k(\mu_{k+1}\mu_k^{-1}) = 1$) to avoid condition $\mu_{n-1}\mu_n^{-1} \in \mathcal{D}^1$ for the left lower entry of B_n and replace it by less restrictive (i) $(\mu_{n+1}\mu_{n-1})^{1/2}\mu_n^{-1} \in \mathcal{D}^1$, see Remark 2.3. Let $V_k = diag(\mu_{k-1}^{1/2}, \mu_k^{1/2})$.
We have

$$B_k = V_{k+1}^{-1}\tilde{B}_k V_k. \tag{3.9}$$

Since V_k has obvious asymptotics $V_k \sim \sqrt{\mu_k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ it is clear that (3.1) reduces analysis of $\prod_{k=2}^n B_k$ to the corresponding $\prod_{k=2}^n \tilde{B}_k$. This fact will be exploited below in the proof of the following.

THEOREM **3.1** . *Let J be a Jacobi matrix with weights λ_n and diagonal q_n satisfying the above conditions i), ii), iii), iv), v), vi). If $\delta := \lim r_n\mu_n^{-1}$ then J has pure absolutely continuous spectrum and $\sigma_{ac}(J) = \mathbb{R}$ provided $|P_K(b, c; \delta)| < 2$, where $P_K(b, c; \delta) = Tr\prod_{j=1}^K F_j$, see (2.1). Moreover generalized eigenvectors $\vec{u}_n$ of (1.1) have the asymptotics*

$$\vec{u}_{n+1} = \mu_n^{-1/2}F_tF_{t-1}\cdots F_1 T diag(\prod_{k=1}^{l}\mu_+(k), \prod_{k=1}^{l}\mu_-(k))(I + o(1))\vec{e} \tag{3.10}$$

where $n = lK + t$, tends to infinity for fixed entire $t \in [1, K]$, $\mu_\pm(s)$ are eigenvalues of the matrix $\prod_{j=1}^K \tilde{B}_{sK+j}$,

$$\tilde{B}_r := diag(\mu_k^{1/2}, \mu_{k+1}^{1/2})B_k diag(\mu_{k-1}^{-1/2}, \mu_k^{-1/2}),$$

T is the invertible 2x2 matrix diagonalizing $C_\infty =: \prod_{j=1}^K F_j$, $\quad T^{-1}C_\infty T = diag(\mu_+(\infty), \mu_-(\infty))$ and $\vec{e}$ is a vector in $\mathbb{C}^2$.

Proof.
As usual the proof is based on Gilbert-Pearson theory of subordinacy [17]. Observe that $\tilde{B}_s$ can be written as the sum of K-periodic matrix F_s and the matrix H_s given by

$$\begin{pmatrix} 0 & 0 \\ \tilde{c}_{s-1}\tilde{c}_s^{-1}[1 - (\mu_{s+1}\mu_{s-1})^{1/2}\mu_s^{-1}] & (\lambda - \tilde{b}_s r_s)(\tilde{c}_s\mu_s)^{-1}(\mu_{s+1}\mu_s^{-1})^{1/2} + \delta\tilde{b}_s c_s^{-1} \end{pmatrix}$$

We claim that

$$C_l := \prod_{j=1}^{K} \tilde{B}_{(l-1)K+j}, \quad l = 1, 2, \cdots,$$

can be expressed in the form

$$C_l = \prod_{j=1}^{K} F_j + \tilde{H}_l, \tag{3.11}$$

where $\tilde{H}_l \in \mathcal{D}^1$. This can be checked by induction on K. Let us examine the case $K = 2$. Assume that n is odd number $n = 2(l-1)+1$ (the case of even n is similar). Then we have

$$C_l = \tilde{B}_{n+1}\tilde{B}_n = (F_2 + H_{n+1})(F_1 + H_n),\, C_{l+1} = (F_2 + H_{n+3})(F_1 + H_{n+2}).$$

By the definition of H_s it is easy to verify that $\|H_{n+2} - H_n\|$ belongs to l^1 (by using ii) and iii)). It follows that $\|C_{l+1} - C_l\|$ is also summable and this proves our claim. The induction step $K \to K+1$ is based on a similar argument.

Now for $n = (l-1)K + j$, $\; det\tilde{B}_n = \tilde{c}_{n-1}\tilde{c}_n^{-1}(1 + r_{jl})$, where $\{r_{jl}\} \in \mathcal{D}^1$ in l by i) and $\lim_{l\to\infty} r_{jl} = 0$ by v), $\quad j = 1, \cdots, K$.
Hence (by the K- periodicity of $\tilde{c}_n$)

$$detC_l = \prod_{j=1}^{K}(1 + r_{jl}) =: 1 + R_l, \quad \{R_l\} \in \mathcal{D}^1 \quad \text{and} \lim_l R_l = 0. \tag{3.12}$$

For the same reason $C_\infty := \prod_{j=1}^{K} F_{(l-1)K+j}$ does not depend on l.

Due to our assumption $|TrC_\infty| = |P_K(b, c; \delta)| < 2$ so eigenvalues $\mu_+(\infty)$ and $\mu_-(\infty)$ of C_∞ must be different because $detC_\infty = 1$. On the other hand $TrC_l = TrC_\infty + Tr\tilde{H}_l$, where $\Delta_l := Tr\tilde{H}_l \in \mathcal{D}^1$ and $\lim_{l\to\infty} \Delta_l = 0$.
Therefore the eigenvalues

$$\mu_\pm(l) = \frac{1}{2}(TrC_\infty + \Delta_l) \pm [\frac{(TrC_\infty + \Delta_l)^2}{4} - detC_l]^{1/2} \tag{3.13}$$

of C_l are also different and are complex conjugate for l sufficiently large ($\Delta_l \to 0$, as $l \to \infty$). By a proper choice of $\mu_+(l), \mu_-(l)$ and of the corresponding eigenvectors $\vec{e}_1(l), \vec{e}_2(l)$ of C_l one can check that $\mu_\pm(l) \in \mathcal{D}^1$ and $\vec{e}_s(l) \in \mathcal{D}^1$ due to the following .

Lemma **3.2** . *Let $M_l \in \mathcal{D}^k$ be a sequence of mxm matrices. Suppose that there exists $\lim_{l\to\infty} M_l = M$, where the matrix M is invertible and has simple spectrum. Then for a proper choice of eigenvalues $\mu_1(l), \cdots, \mu_m(l)$ and the corresponding eigenvectors $\vec{e}_1(l), \cdots, \vec{e}_m(l)$ of M_l, for $l \gg 1$ we have,*
a) $\mu_s(\cdot) \in \mathcal{D}^k$, $\quad s = 1, \cdots, m$.
b) $\lim_{l\to\infty} \vec{e}_s(l) = \vec{e}_s$, where $M\vec{e}_s = \mu_s\vec{e}_s$ and $\mu_s = \lim_{l\to\infty} \mu_s(l)$.
c) $\vec{e}_s(\cdot) \in \mathcal{D}^k$ coordinatewise.

Proof. We use the following fact (see [16]): if $\{g_s(l)\}_{l=1}^{\infty}$, $s = 1, \cdots, m$ belong to $\mathcal{D}^k$ and the function $f(x_1, \cdots, x_m)$ belongs to C^k in a neighbourhood of the point $(g_s, \cdots, g_m)$, where $g_s = \lim_{l\to\infty} g_s(l)$ (provided these limits exist) then the sequence $f(g_1(l), \cdots, g_m(l))$ also belongs to the scalar class $\mathcal{D}^k$.
Due to the formula for the eigenvalues of M_l it is clear that they are C^∞ functions in entries of M_l in suitable domains by the simplicity of $\sigma(M)$. From this follows immediately the statement a), since all entries of M_l are in $\mathcal{D}^k$. Because $\sigma(M)$ is simple we can suppose that M is a diagonal matrix without loss of generality. Now choose $\vec{e}_s(l)$ (for l suff. large) such that its s-th coordinate equals identically to 1. Then the spectral equation $M_l\vec{e}_s(l) = \mu_s(l)\vec{e}_s(l)$ after substituting $(\vec{e}_s(l))_s = 1$ can be reduced to $(m-1)x(m-1)$ system of linear equations for the rest of $m-1$ coordinates, with determinant different from zero, for $l \gg 1$. This gurantee the unique solution for the vector $\vec{e}_s(l)$, $l \gg 1$. The procedure can be performed separately for $s = 1, \cdots, m$. In turn, Cramers formulae for the solutions and elementary facts: $\mathcal{D}^k\mathcal{D}^k \subset \mathcal{D}^k$ (put above $f(x_1, x_2) = x_1x_2$) and $\mathcal{D}^k/\mathcal{D}^k \subset \mathcal{D}^k$ for non-zero scalar limit of the denominator (put above $f(x_1, x_2) = x_1\ x_2$, $|x_2| \geq \varepsilon > 0$) prove c). Finally the construction of $\vec{e}_s(\cdot)$ gives statement b). Special case of Lemma 3.2 can be found in our preprint [16]. ■

Diagonalization of $C_\infty + \tilde{H}_l$ gives

$$C_\infty + \tilde{H}_l = T_l diag(\mu_+(l), \mu_-(l))T_l^{-1}, \tag{3.14}$$

here $T_l := (\vec{e}_1(l), \vec{e}_2(l))$ and the eigenvectors are understood as columns, and we choose l_o so large that $\mu_+(l) \neq \mu_-(l)$ for $l \geq l_o$. For any $L = l \cdot K$ and $n_0 = l_0K$
we have

$$\prod_{n=n_o}^{L} \tilde{B}_n = \prod_{k=l_o}^{l} C_k = \prod_{k=l_o}^{l} (C_\infty + \tilde{H}_k) \tag{3.15}$$

$$= (T_l\{\prod_{k=l_o+1}^{l} [diag(\mu_+(k), \mu_-(k))T_k^{-1}T_{k-1}]\}diag(\mu_+(l_o), \mu_-(l_o))T_{l_o}^{-1}.$$

Now

$$T_k^{-1}T_{k-1} = (T_{k-1} + \Delta T_{k-1})^{-1}T_{k-1} = I + \Gamma_k, \quad \text{and}\, \{\|\Gamma_k\|\} \in l^1$$

because $\{T_k\} \in \mathcal{D}^1$. Therefore the last product in (3.7) can be written as:

$$T_l\{\prod_{k=l_o+1}^{l} [diag(\mu_+(k), \mu_-(k))(I + \Gamma_k)]\}diag(\mu_+(l_o), \mu_-(l_o))T_{l_o}^{-1} \tag{3.16}$$

$$= (\prod_{s=l_o}^{l} |\mu_+(s)|)T_l[\prod_{k=l_o}^{l} (U_k + \tilde{\Gamma}_k)]T_{l_o}^{-1},$$

where $U_k = diag(\mu_+(k)|\mu_+(k)|^{-1}, \mu_-(k)|\mu_-(k)|^{-1})$ (as $\mu_+(k) = \overline{\mu_-(k)}$) and $\{\|\tilde{\Gamma}_k\|\} \in l^1$.
By easy version of Levinson theorem [5], [16] we have identically

$$\prod_{k=l_o}^{l} (U_k + \tilde{\Gamma}_k) = (\prod_{i=l_o}^{l} U_i)[\prod_{k=l_o}^{l} \{(\prod_{s=l_o}^{l} U_s)^{-1}(U_k + \tilde{\Gamma}_k)(\prod_{m=l_o}^{k-1} U_m)\}] = (\prod_{i=l_o}^{l} U_i)[\prod_{k=l_o}^{l} (I + \Gamma_k^{(1)})],$$

where

$$\|\Gamma_k^{(1)}\| = \|\tilde{\Gamma}_k\|.$$

Hence moving in (3.8) the factor $\prod_{s=l_o}^{l} |\mu_+(s)|$ back to $\prod_{k=l_o}^{l}(U_k + \tilde{\Gamma}_k)$ we have

$$\prod_{k=l_o}^{l} (C_\infty + \tilde{H}_k) = T_l diag(\prod_{k=l_o}^{l} \mu_+(k), \prod_{k=l_o}^{l} \mu_-(k))[\prod_{k=l_o}^{l} (I + \Gamma_k^{(1)})]. \tag{3.17}$$

Finally writing $T_l = T_\infty + o(1)$ the asymptotics of $\prod_{k=l_o}^{l}(C_\infty + \tilde{H}_k)$ is given by

$$T_\infty^{-1}\{diag(\prod_{k=l_o}^{l} \mu_+(k), \prod_{k=l_o}^{l} \mu_-(k))\}C(I + o(1)), \tag{3.18}$$

where C is a fixed invertible matrix (depending on l_o). ¿From formulae (3.7), (3.1) and definition of B_n it is clear that $\mu_\pm(l) \neq 0$ for any $l = 1, 2, \cdots$. Therefore we may replace l_0 in (3.10) by 1. The above asymptotics provides us formula (3.2) only for special sequence of natural numbers ($L = l \cdot K$) but due to uniform boundness of $\|B_k^{-1}\|$ it can be extended to arbitrary natural numbers (we leave the details to the reader). Finally (3.2) obviously implies that (1.1) has no subordinated solutions because $|\mu_+(k)| = |\mu_-(k)|$, $k \gg 1$. This completes the proof. ∎

4 Discrete Spectrum, asymptotics of principal and supplementary solutions.

It turns out that asymptotics of solutions to (1.1) described by formula (3.2) (see Th.3.1) can be extended also to the case $|P_K(b, c; \delta)| > 2$. As the consequence of this asymptotics it will be shown discreteness of $\sigma(J)$ provided $|TrC_\infty| > 2$. Below we shall need the following discrete version of the Levinson theorem [5], [16].

Proposition **4.1** *(N. Levinson). Let A, A_n, V_n and R_n be sequences of 2* x *2 matrices such that*
1) A is constant, invertible matrix with two different eigenvalues $\lambda_1 \neq \lambda_2$ and
$A\vec{e}_i = \lambda_i \vec{e}_i \neq 0$ $i = 1, 2$.
2) $\{\|V_{n+1} - V_n\|\} \in l^1$ and $\|V_n\| \to 0$, as $n \to \infty$,
3) $\{\|R_n\|\} \in l^1$.
4) In the special case $Re\lambda_1 = Re\lambda_2$, we need extra condition:
Let $d(n) := Re[\lambda_1(n) - \lambda_2(n)]$. Suppose that either $\sum_{k=1}^{n} d(k)$ tends to $+\infty$, as $n \to \infty$ and $\sum_{k=n_1}^{n_2} d(k)$ is uniformly bounded from below ($n_1, n_2 > 0$) or $\sum_{n_1}^{n_2} d(k)$ is uniformly bounded fromm above. Consider the infinite system of reccurence equations

$$\vec{x}_{n+1} = (A + V_n + R_n)\vec{x}_n. \tag{4.19}$$

Then there exist two non-zero solutions $\vec{x}^{(1)}(n)$, $\vec{x}^{(2)}(n)$ of (4.1) such that

$$\vec{x}^{(i)}(n) = [\prod_{k=p}^{n} \lambda_i(k)](\vec{e}_i + o(1)), \text{ as } n \to \infty, \quad i = 1, 2, \tag{4.20}$$

for some p sufficiently large. Here $\lambda_i(n)$ are eigenvalues of $A+V_n$ chosen such that $\lim_{n\to\infty}\lambda_i(n)=\lambda_i,\ i=1,2.$

Actually Proposition 4.1 is only a special case of more general result when condition 2) of $\mathcal{D}^1$ type is replaced by more general condition of $\mathcal{D}^k$ type. It will be considered in another paper. Here we only mention that this generalization has one essential difference in comparison with Prop.4.1. The asymptotics in (4.2) will contain beside the spectrum of RHS of (4.1) information on the eigenvectors of $(A+V_n)$.

In what follows we will apply Prop.4.1 to the matrices $C_l=C_\infty+\tilde{H}_l$. More precisely, we apply it in the case $A=C_\infty, V_k=\tilde{H}_k,\ R_k\equiv 0$ because $\tilde{H}_k\in\mathcal{D}^1$ (see §3).Since in our case $det\, C_l\neq 0$ for all l we can replace p in formula (4.2) by 1. Recall that eigenvalues of $(C_\infty+\tilde{H}_l)$ are $\mu_\pm(l)$. They are given by formula (3.5) in nonelliptic case too. According to standard terminolgy [11] $\mu_+(\infty)=\overline{\mu_-(\infty)}$ in elliptic case and $\mu_\pm(\infty)\in\mathbb{R}$ in nonelliptic (hyperbolic) situation. But now $\mu_+(l)>1$ and $\mu_-(l)<1$ for l sufficiently large, because here $|TrC_\infty|>2$ and $detC_\infty=1$. Let $C_\infty\vec{e}_1=\mu_-(\infty)\vec{e}_1$, $C_\infty\vec{e}_2=\mu_+(\infty)\vec{e}_2$. Using (4.2) we can write for $n=(l-1)N+j$ the asymptotic formula for two linearly independent solutions $\vec{u}^{(i)}_{n+1}$ of (1.1) as:

$$\vec{u}^{(i)}_{n+1}=V^{-1}_{n+1}F_j\cdots F_1[\prod_{k=1}^{l}\mu_\pm(k)](\vec{e}_i+o(1)), \tag{4.21}$$

here F_s are given by (2.1). We choose in (4.3) $\mu_-(k)$ for $i=1$ and $\mu_+(k)$ for $i=2$. Evoking definitions of V_n and F_j we can rewrite (4.3) as follows

$$\vec{u}^{(i)}_{n+1}=\mu_n^{-\frac{1}{2}}\prod_{k=1}^{l}\mu_\pm(k)\begin{pmatrix}0 & 1\\ -\tilde{c}_{j-1}\tilde{c}_j^{-1} & -\delta\tilde{b}_{j-1}\tilde{c}_j^{-1}\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ -\tilde{c}_0\tilde{c}_1^{-1} & -\delta b_1\tilde{c}_1^{-1}\end{pmatrix}(\vec{e}_i+o(1)). \tag{4.22}$$

Asymptotics of solutions given by (4.4) allows to obtain estimations of the Green matrix $G(k,n;\lambda_0)=((J-\lambda_0)^{-1}e_k,e_n)$, where $\lambda_0\notin\mathbb{R}$ and $k,\ n\in\mathbb{N}$. These estimates permit to show compactness of $(J-\lambda_0)^{-1}$.

Let $u^D(\cdot,\lambda_0)$, $u^N(\cdot,\lambda_0)$ be two Weyl solutions to (1.1) with $\lambda=\lambda_0$ and boundary conditions given by $u^D(0)=0$, $u^D(1)=1$ respectively $u^N(0)=1$, $u^N(1)=0$. Denote by $\varphi^N_+(n,\lambda_0)$ the unique (by Carleman condition) Jost [11] solution which belongs to l^2,

$$\varphi^N_+(n,\lambda)=u^D(n,\lambda_0)+m(\lambda_0)u^N(n,\lambda_0),$$

where $m(\lambda)$ is the well known Weyl function [2]. The Green matrix $G(k,n,\lambda_0)$ can be written as usual

$$C(\lambda_0)u^D(k_<,\lambda_0)\varphi^N_+(k_>,\lambda_0), \tag{4.23}$$

where $k_<=min(k,n), k_>=max(k,n)$, see [2].
Using (4.4) we know that

$$\varphi^N_+(n,\lambda_0)=O(\mu_n^{-1/2}\prod_{k=1}^{n}\mu_-(k)),\ n\to\infty. \tag{4.24}$$

Surely $u^D(n,\lambda_0)$ must grow ($\lambda_0 \notin \sigma(J)$) and therefore again by 4.4

$$u^D(n,\lambda_0) = O(\mu_n^{-1/2}\prod_{k=1}^{n}\mu_+(k)),\ n\to\infty. \tag{4.25}$$

Indeed, evoking definition of C_l and using K periodicity of $\tilde{c}_j$ we have

$$\prod_{l=2}^{m}\mu_+(l)\mu_-(l) = \prod_{l=2}^{m} detC_l = \prod_{l=2}^{m}(\prod_{j=1}^{K} det\tilde{B}_{(l-1)K+j}) \tag{4.26}$$

$$= \prod_{l=2}^{m}[\prod_{j=1}^{K}\tilde{c}_{j-1}\tilde{c}_j^{-1}(\mu_{(l-1)K+j+1}\mu_{(l-1)K+j-1})^{1/2}\mu_{(l-1)K+j}^{-1}] = \prod_{s=K+1}^{mK}[(\mu_{s+1}\mu_{s-1})^{1/2}\mu_s^{-1}]$$

$$= (\mu_{mK+1}\mu_{mK}^{-1}\mu_k\mu_{K+1}^{-1}.$$

By (v) the above equalities prove the desired convergence of the left hand side of (4.8). Using (4.5)-(4.8) we have

$$|G(k,n;\lambda_0)| \le C(\mu_n\mu_k)^{-1/2}\prod_{s=k+1}^{n}\mu_-(s), \tag{4.27}$$

where $C = C(\lambda_0) > 0$ and $n \ge k$ but k is sufficiently large (here for $k = n$ the product $\prod_{s=k+1}^{n}\mu_-(s)$ is understood to be equal to 1.)
Since $\mu_-(s) \le r < 1$, for $s \gg 1$ (4.9) implies that

$$|G(k,n;\lambda_0)| \le C_1(\mu_n\mu_k)^{-1/2}r^{|k-n|} \tag{4.28}$$

for $n \ge k$ and k sufficiently large, say $k \ge k_0$. Surely estimation (4.10) also holds for $k < k_0$ and $k < n$.
The case $n < k$ follows by symmetry in (4.5) and so (4.10) holds for all k,n. Estimation (4.10) implies that $G(k,n;\lambda_0)$ can be written in the form

$$G(k,n;\lambda_0) = \mu_k^{-1/2}[F(k,n;\lambda_0)r^{|n-k|}]\mu_n^{-1/2}, \tag{4.29}$$

where $|F(k,n;\lambda_0)| \le M$, for some $M > 0$ and all $k,n \in \mathbb{N}$.
It follows that the operator G induced by the matrix $\{G(k,n;\lambda_0)\}_{k,n\in\mathbb{N}}$ can be expressed as the product

$$G = ABA, \tag{4.30}$$

where A is the diagonal operator with the diagonal $\{\mu_n^{-1/2}\}_{n\in\mathbb{N}}$ and B is the operator defined in l^2 by the matrix $\{F(k,n;\lambda_0)r^{|n-k|}\}$.
Note that

$$\sum_k |F(k,n;\lambda_0)|r^{|n-k|} \le 2M(1-r)^{-1},\ n\in\mathbb{N}$$

and

$$\sum_n |F(k,n;\lambda_0)|r^{|n-k|} \le 2M(1-r)^{-1},\ k\in\mathbb{N}.$$

Therefore B is bounded in l^2 (and $\|B\| \leq 2M(1-r)^{-1}$), [10].
Since A is compact ($\mu_n \to +\infty$, as $n \to \infty$) G must be also compact and the spectrum of J is discrete.
Suming up the above considerations we have proved

THEOREM **4.2** . *If weights λ_n and diagonal q_n satisfy assumptions (2.7) and $|P_K(b,c;\delta)| > 2$ then*
a) the spectrum of J is discrete,
b) there exist two non-zero solutions $\vec{x}^{(1)}(n), \vec{x}^{(2)}(n)$ of the system (1.1) such that
$\vec{x}^{(i)}(n) = \mu_n^{-1/2} F_j \cdots F_1 [\prod_{s=1}^{l} \mu_\pm(s)](\vec{e}_i + o(1))$,
where $n = lK+j$, $1 \leq j \leq K$, $\mu_\pm(s)$ are eigenvalues of $C_s := \prod_{j=1}^{K} \tilde{B}_{sK+j}$, and $\vec{e}_i$ are eigenvectors of $C_\infty := \lim_{l\to\infty} C_l$.

Let $\{\rho_k\}_{k\in\mathbb{N}}$ be the sequence of all eigenvalues of J enumerated according to increasing order of their modulus. It will be shown in Th.4.3 that $|\rho_k|$ can be estimated from below and the Cesaro average of the sequence $\{|\rho_k|^{-1}\}$, $k \in \mathbb{N}$ is proportional to the Cesaro average of $\{\hat{\mu}_k\}_{k\in\mathbb{N}}$, here $\{\hat{\mu}_k\}_{k\in\mathbb{N}}$ stands for the increasing rearrangement of $\{\mu_k\}_{k\in\mathbb{N}}$.

THEOREM **4.3** . *Let J be the Jacobi matrix with weights and diagonal satisfying (2.7).*
Suppose that $|P_K(b,c;\delta)| > 2$. Let $\sigma(J) = \{\rho_k\}_{k\in\mathbb{N}}$, $|\rho_k| \leq |\rho_{k+1}|$ then
a) there exists a constant $c > 0$ such that $|\rho_k| \geq c\hat{\mu}_{[\frac{k}{2}]-1}$, $k = 3, \cdots$
b) there are positive constans c_1, c_2 such that

$$c_1 \sum_{k=1}^{n} \hat{\mu}_k^{-1} \leq \sum_{k=2}^{n} |\rho_k|^{-1} \leq c_2 \sum_{k=1}^{n} \hat{\mu}_k^{-1},\ n = 2, 3. \cdots \tag{4.31}$$

Proof. Fix $\lambda_0 \notin \mathbb{R}$. Employing the notation given above in Th.4.2 we know that $G = (J-\lambda_0 I)^{-1} = ABA$, see (4.12). Let $\{s_k(A)\}_{k\in\mathbb{N}}$, $\{s_k(J)\}_{k\in\mathbb{N}}$ be the sequences of singular numbers of A and J. Elementary properties of singular numbers of operators imply that for $m > 1$ ($\rho_2 \neq 0$ since J has simple spectrum)

$$\begin{aligned} |\rho_m|^{-1} &= s_m(J)^{-1} \leq C s_m((J-\lambda_0 I)^{-1}) \\ &= C s_m(ABA) \leq C\|B\| s_{[\frac{m}{2}]-1}(A^2) = C\|B\| \hat{\mu}_{[\frac{m}{2}]-1}^{-1}, \end{aligned}$$

here $[r]$ stands for the entire part of $r \in \mathbb{R}$. This proves a).
The proof of b) is more complicated. First observe that for any compact operator A_1 and a bounded operator B_1 we have

$$\sum_{k=1}^{m} s_k(A_1 B_1 A_1) \leq \|B\| \sum_{k=1}^{m} s_k(A_1)^2. \tag{4.32}$$

Inequality (4.13) is a simple consequence of Horn inequality for product of two operators and general inequalities for cross-norms [9]. Applying (4.14) to $A_1 = A$ and $B_1 = B$ we get

$$\sum_{k=2}^{n} |\rho_k|^{-1} \leq \|B\| \sum_{k=1}^{n} \hat{\mu}_k^{-1}. \tag{4.33}$$

Formally speaking in most cases Cesaro type estimation (4.14) is worse than inequality a). Let P_n be the orthogonal projection on the subspace spanned by eigenvectors $\{e_{t_s}\}_{s=1}^n$ (chosen from the canonical basis in l^2) corresponding to the first singular numbers $s_1(A), \cdots, s_n(A)$. Particular choice of indices t_s corresponds to the monotonic rearrangment of $\{\mu_n\}$. Then applying Th.5.1, Chapt. 2 in [9] we have

$$C\sum_{k=2}^{n} |\rho_k|^{-1} \geq \sum_{k=1}^{n} s_k((J-\lambda_0 I)^{-1}) \geq Tr[P_n(J-\lambda_0 I)^{-1}P_n] = \sum_{k=1}^{n} G(t_k, t_k; \lambda_0) \tag{4.34}$$

The last sum can be estimated from below by const. $\sum_{k=1}^{n} \mu_{t_k}^{-1}$. Indeed, using (4.4), (4.5), (4.8) and invertibility of F_j, $j = 1, \cdots, K$. We can write for $t \gg 1$

$$G(t, t; \lambda_0) \asymp \mu_t^{-1},$$

because $D_t := \prod_{l=1}^{t} \mu_+(l)\mu_-(l)$ is convergent. Combining (4.15) and (4.16) we obtain the desired estimation ($\sum_{k=1}^{n} \mu_{t_k}^{-1} = \sum_{k=1}^{n} \hat{\mu}_k^{-1}$ by definition). The proof is complete. ■

As a consequence of Th.4.3 one can show using elementary estimations.

Corollary **4.4** . *If* $\mu_k \asymp k^\alpha$, $\alpha \in (0,1)$ *then* $|\rho_k| \asymp k^\alpha$.

Proof. It is easy to check that $\hat{\mu}_k \asymp k^\alpha$. Th.4.2 a) implies the lower bound estimate $|\rho_k| \geq \tilde{d}_1\hat{\mu}_k = d_1 k^\alpha, d_1, \tilde{d}_1 > 0$. Using (4.13) for any $\Theta \in (0,1)$ we have

$$\sum_{k=2}^{n} |\rho_k|^{-1} = \sum_{k=2}^{[n\Theta]-1} |\rho_k|^{-1} + \sum_{k=[n\Theta]}^{n} |\rho_k|^{-1} \leq (n - [n\Theta] + 1)|\rho_{[\Theta n]}|^{-1} + c_2 \sum_{k=2}^{[n\Theta]-1} \hat{\mu}_k^{-1}$$

Applying again (4.13) and $\hat{\mu}_k \asymp k^\alpha$ we get (with the same c_1, c_2) :

$$(n - [n\Theta] + 1)|\rho_{[\Theta n]}|^{-1} \geq c_1 \sum_{k=1}^{n} \mu_k^{-1} - c_2 \sum_{k=2}^{[n\Theta]-1} \hat{\mu}_k^{-1} \geq c_3 n^{1-\alpha} - c_4([n\Theta])^{1-\alpha} \tag{4.35}$$

for sufficiently small $c_3 > 0$ and large c_4 independent on Θ. Choosing $\Theta \ll 1$ it is clear that the LHS of (4.17) is greater or equal $c_5 n^{1-\alpha}$, $c_5 = c_3 - c_4\Theta^{1-\alpha} > 0$
Therefore

$$|\rho_{[\Theta n]}| \leq c_5^{-1}(n+1)n^{\alpha-1}.$$

Since Θ has been chosen independently on n we obtain the desired estimation. ■

¿From the above considerations it is actually possible to obtain asymptotics of ρ_n. We hope to do it later.

Note that we could not remove modulus sign from $|\rho_k|$ in all the above estimations because the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ can have accumulation points at $+\infty$ and $\{-\infty\}$ simultanously. This can be illustrated by the following .

EXAMPLE **4.5** . a) (non semiboundedness) Let $\mu_n = r_n = n^\alpha$, $\alpha \in (0,1)$, $b_n = (-1)^n$, $c_n \equiv 1$. Recall that $J = B+Q$, where $B = S\Lambda + \Lambda S^*$ and $Q = diag(b_n n^\alpha)$. Choose the sequence $f_N = (1,\cdots,1,0,0,\cdots)^T$, where all coordinates with indices greater than N are equal to zero. Easy computations shows that $(Bf_N, f_N)\|f_N\|^{-2} \sim 2N^\alpha(\alpha+1)^{-1}$ and $(Qf_N, f_N)\|f_N\|^{-2} = O(N^{\alpha-1})$, and so $(Jf_N, f_N)\|f_N\|^{-2} \to +\infty$, as $n \to \infty$.

In turn for $\tilde{f}_N = (1,-1,1,-1,\cdots,1,-1,0,0,\cdots)^T$ (zero coordinates for $k > N$) we have $(B\tilde{f}_N, \tilde{f}_N)\|\tilde{f}_N\|^{-2} \sim -2N^\alpha(\alpha+1), (Q\tilde{f}_N, \tilde{f}_N)\|\tilde{f}_N\|^{-2} = O(N^{\alpha-1})$.
Therefore $(J\tilde{f}_N, \tilde{f}_N)\|\tilde{f}_N\|^{-2} \to -\infty$, as $n \to \infty$.

b) (semiboundedness). Choosing the same $\mu_n, r_n\, c_n$ and 2-periodic sequence b_n with $b_1 b_2 > 4$ one can prove applying Cauchy inequality that J is semibounded from below provided $b_1, b_2 > 0$. Actually using subordination of operators in Kato sense (like in [16], Sec.4) it is easy (again only by Cauchy inequality) to check that $\sigma(J)$ is discrete. Therefore in the case of discrete spectrum J can be either semibounded or not semibounded. Remind that in the case of absolutely continuous spectrum (see Th.3.1) J is always not semibounded. Concernig the querstion of semiboundedness of J there is an explicit answer (in generic case) in terms of characteristic polynomial $d_{J_{per}}(\lambda)$. We hope to come back to this topic in the future paper.

REMARK **4.6** . (Case $\delta = 0$.) Note for any weights satisfying (2.7) Prop. 2.1, Th.3.1, Th.4.1 imply that the spectrum of J is pure absolutely continuous and covers $\mathbb{R}$ provided K is odd (hence $M = 1$ and N odd). Really $P_K(b,c;0) \equiv 0$. In turn if K is even and N is also even (the additional condition depending on M) $\sigma(J)$ is discrete under extra (generic type) condition: $c_1,\cdots c_{N-1} \neq c_2 \cdots c_N$. Actually $|P_k(b,c;0)| = \alpha_N^{K/N} + \alpha_N^{-K/N} > 2$, see (2.3). Despite the absence of influence of particular values of $b_n's$ in generic case (modulation of diagonal) its trace remains in the possible eveness of the common period K and so it changes spectra character for arbitrary small perturbation of δ keeping fixed b.

EXAMPLE **4.7** . Take $\mu_n = n^\alpha, r_n = n^\beta, 0 < \beta < \alpha \leq 1$. Then $\delta = 0$ and surely (2.7) is satisfied and we can easily apply Remark 4.6 by choosing various periods of M and N.

5 Examples of curves and surfaces of spectral phase transition and stability of their topology

In this section we present some examples illustrating geometric complexity of domains (zones of stable spectral structure in the space of parameters of modulation) with fixed phase states : $\sigma(J)$ discrete or pure absolutely continuous in our case. We introduce this space because the spectral structure of J depends only on parameters of modulations (see (2.6)) a.e. except the phase transition points. The boundaries of these domains correspond to the first type spectral phase transition (in suitable regions) under small perturbation of $b_n's$ or $c_n's$. Spectral situation on the boundaries of the domains (phase transition points in the space of parameters $(b,c,\delta) \in \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R}$) is rather complicated and very intresting. Its study requires finding new tools and we hope to consider this in a future paper. In what follows we always assume (except the last example) that the numbers M and N are fixed. The next general assumption is that the diagonal $\{q_n\}$ does not dominate the weights λ_n (we say that $\{q_n\}$ dominates $\{\lambda_n\}$ if $\lim inf\, q_n^2(\lambda_n^2+\lambda_{n-1}^2)^{-1} > 2$). Otherwise as it is easy to prove $\sigma(J)$ is discrete, see [16, Th 4.1]. In Prop. 5.1 we fix all parameters of J except $b_n{}'s$ which are supposed to be small. Using Prop. 2.1, Th. 3.1 and Th.4.2 one easily prove.

Proposition **5.1** *Under our assumptions (2.7) we have the following possibilities for types of spectrum of J.*
a) If M is arbitrary and N is even then $\sigma(J)$ is discrete for $b_n{}'s$ sufficiently small (as the term $P_0(c)$ dominates the remaining ones in formula (2.3) for $P_K(b,c;\delta)$) if $c_1c_3\cdots c_{N-1} \neq c_2c_4\cdots c_N$,
b) if both M and N are odd then $\sigma(J)$ is pure absolutely continuous for sufficiently small $b_n{}'s$.
c) if M is even, N is odd and $\delta \neq 0$ then $\sigma(J)$ can be either discrete pure absolutely continuous or depending whether the value of

$$(-1)^{K/2}P_2(b,c) = (-1)\sum_{i<j}\tilde{b}_i(\tilde{b}_j)(\tilde{c}_i\tilde{c}_j)^{-1} \tag{5.36}$$

$$[(\tilde{c}_{j-2}\cdots\tilde{c}_{i+3}\tilde{c}_{i+1})(\tilde{c}_{j-1}\cdots\tilde{c}_{i+4}\tilde{c}_{i+2})^{-1}(\tilde{c}_{i-2}\cdots\tilde{c}_{j+1})(\tilde{c}_{i-1}\cdots\tilde{c}_{j+2})^{-1}]$$

is positive or negative, respectively, provided b_n's are sufficiently small. Again the term $P_2(b,c)$ being nonzero dominates the remaining ones in formula (2.5) for $P_K(b,c;\delta)$,

The above Proposition displays the interplay between parities of M and N. Below we describe the table which explains the spectral structure of J in generic case, i.e. for almost all values of parameters $b_n{}'s, c_n{}'s, \delta$ provided N and M are fixed. It happens that the answer does not depend on μ_n, r_n under assumptions (2.7) ouside the spectral phase transition points. Remind that the modulation of the diagonal $b_n{}'s$ are supposed to be sufficiently small. The last condition is necessary to dominate the higher terms in the polynomials in δ in $P(b,c;\delta)$.

M- period of diagonal

		even	**odd**
N- period of weights	**even**	discrete (in generic case)	discrete (in generic case)
N- period of weights	**odd**	either discrete or abs. cont. (dependence on sign of the expression in (5.1))	pure absolutely continuous

In order to illustrate geometry of regions of "stable" spectrum of J with fixed phase state we give below a few simple examples.

EXAMPLE **5.2** . Let $M = N = 2,\ \lambda_n = c_n n^\alpha,\ \alpha \in (0,1]$ and $q_n = b_n n^\alpha$. Then $K = 2, \delta = 1$ and $P_2(b,c;1) = -[c_1c_2^{-1} + c_2c_1^{-1}] + b_1b_2(c_1c_2)^{-1}$. For symmetry reasons we assume below that sign of c_1 can be arbitrary and $c_2 = 1$. Denote $c_1 = t,\ b_1b_2 = v$. Then

$$P_2(b,c;1) = \frac{v}{t} - (t + \frac{1}{t}).$$

We have in coordinates (t,v) two different unbounded simply connected domains Ω_1 and Ω_2 of discrete spectrum of J corresponding to the inequality $|\frac{v}{t} - (t + \frac{1}{t})| > 2$ and two unbounded simply connected domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ of pure absolutely continuous spectrum in the case $|\frac{v}{t} - (t + \frac{1}{t})| < 2$. Note that all Ω_k are symmetric with respect to v-axis ($t \to -t$) and $\partial\Omega_k$ are picewise parabolic curves.

$$\Omega_1 = \{(t,v) : v \in (1,+\infty),\ |t| < (\sqrt{v} - 1)\},$$

so it starts from the point (0,1) and extends to infinity becoming wider with respect to the growing parameter v. Ω_2 contains the whole open lower half plane $\prod_-$ and the parts of $\prod_+$ around (0,0), given by $\{(t,v),\ v \in [0,1),\ |t| < 1 - \sqrt{v}\}$, and two unbounded pieces $\{(t,v),\ v \in [0,+\infty),\ |t| > \sqrt{v} + 1\}$

In turn $\tilde{\Omega}_1 \cup \tilde{\Omega}_2 = \mathbb{R}^2 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ has the form of two symmetric pipes in $\prod_+$ extended to infinity. Additionally vertical axis $t = 0$ which was formaly excluded from considerations ($c_1 = 0$) reduced J to the trivial case of infinite orthogonal sum of 2x2 matrices $\begin{pmatrix} b_1r_{2n-1} & \mu_{2n} \\ \mu_{2n} & b_2r_{2n} \end{pmatrix}$, and therefore $\sigma(J)$ is pure point and actually discrete. From this picture we see that for any horizontal line $\{(t,v_0),\ t \in \mathbb{R}\},\ v_0 > 0$ we have the following pattern: for any $v_0 \neq 1 \quad \sigma(J)$ is discrete for $|t|$ sufficiently large or small and pure absolutely continuous in the intermediate case. This is clear because for small $|t| \quad J$ becomes close to the above mentioned orthogonal sum of 2x2 matrices with discrete spectrum. The same holds true for $|t|$ sufficiently large because $|t|^{-1}J$ has similar behaviour only the numeration of 2x2 matrices is changed. For $v_0 = 1$ we have an interesting phenomenon : $\sigma(J)$ is never discrete for $|t|$ arbitrary small (except $t = 0$) in contradistinction to the cases where $v_0 \in \mathbb{R}_+ \setminus \{1\}$. Remind that for $v_0 < 0$ the spectrum of J is always discrete. In other words if the signs of the diagonal modulation b_1, b_2 are different $\sigma(J)$ should be discrete independently on the modulation of weights. This can be explained in a way: due eveness of N for $t \neq 1$ and zero diagonal we alawys have discrete spectrum, Remark 4.6. Therefore it is not supprising that the spectrum of J remains discrete in most cases (except two pipes) also in presence of the diagonal. More unasual seems to be its discreetness for $t = 1$, when $\sigma(J)$ is pure absolutely conttinuous for zero diagonal, for arbitrary modulation of the diagonal with different signs and $M = 2$.

EXAMPLE **5.3** . Even more interesting picture of spectral phase transition appears in higher dimension parameter space (b,c). Let $M = 3, N = 2$ and $b = (v,w,0), c = (1,t)$. Suppose that $\mu_n = r_n = n^\alpha,\ \alpha \in (0,1)$. Direct computation gives $P_6(b,c) := Tr(F_6F_5\cdots F_1) = -t^3 + tvw + (w^2 + v^2)t^{-1} + (vw - 1)t^{-3}$. The above formula shows that $P_6(b,c) = P_6(v,w,t)$ is odd function of t and so in analysis of domains where $|P_6(v,w,t)| < 2$ (or > 2) we can assume

that $t > 0$.
a) Take $w = \alpha v,\ \alpha > 0$. Strightforward computation shows that
$|P_6(v, \alpha v, t)| < 2$ if and only if

$$|t^3 - 1|(\alpha t^4 + t^2 + \alpha^2 t^2 + \alpha)^{-1/2} < |v| < (t^3 + 1)(\alpha t^4 + t^2 + \alpha^2 t^2 + \alpha)^{-1/2}. \qquad (5.37)$$

Deonoting the function on the right (resp. left) hand side of (5.2) by $v_2(t)$ (resp. $v_1(t)$) one can check that

$$\lim_{t\to 0_+} v_2(t) = \lim_{t\to 0_+} v_1(t) = \alpha^{-1/2}, \qquad (5.38)$$

$$\lim_{t\to\infty} [v_2(t) - v_1(t)] = 0 \qquad (5.39)$$

but $\lim_{t\to\infty} v_2(t) = \lim_{t\to\infty} v_1(t) = \infty$
Since $P_6(-v, -w, t) = P_6(v, w, t)$ we may assume that $v \geq 0$, and so varying α we see that in the octant $v \geq 0,\ w \geq 0,\ t > 0$ one has one unbounded domain of parameters (v, w, t), where $\sigma(J)$ is pure absolutely continuous. Due to (5.3) and (5.4) this domain shrinks as $t \to 0_+$ or as $t \to \infty$. Since for the point $(0, 0, 1)$ $\sigma(J)$ is pure absolutely continuous (use [16, Cor.3.3] or [7]) the plane $t = 1$ is special because for (v, w) arbitrary but $|v|$ and $|w|$ sufficiently small $(0 < (v + w)^2 < 4)$ $\sigma(J)$ is always pure absolutely continuous. This does not contradict Prop.5.1 a) because $c_1 = c_2 = 1$. Observe that the curves $(v_1(t), \alpha v_1(t), t)$ and $(v_2(t), \alpha v_2(t), t)$ never interesect $(v_1(t) < v_2(t)\,,\ t > 0)$ and functions $v_1(t), v_2(t)$ are monotonic in suitable two intervals.
b) If $w = \alpha v$ but $\alpha < 0$ the picture looks different. Assume that $-1 < \alpha < 0$ (the case $\alpha < -1$ can be treated similarly). Then $|P_6(v, \alpha v, t)| < 2$ iff

$$(t^3 - 1)^2 < p_\alpha(t) v^2 < (t^3 + 1)^2 \qquad (5.40)$$

and $p_\alpha(t) := \alpha t^4 + (1+\alpha^2)t^2 + \alpha > 0$. One can easily check that $p_\alpha(t) > 0$ iff $t \in ((-\alpha)^{1/2}, (-\alpha)^{-1/2})$ Therefore for $t \in (0, (-\alpha)^{1/2}]$ or $t \in [(-\alpha)^{-1/2}, \infty)$, $\sigma(J)$ is always discrete for $w = \alpha v$. In other words the situation in the second octant $\{(v, w, t),\ v > 0,\ w < 0,\ t > 0\}$ looks different than in the first one. This is not so surprising because $P_6(v, w, t)$ is neither even nor odd function of w. Write (5.5) in the form

$$|t^3 - 1| p_\alpha(t)^{-1/2} < |v| < (t^3 + 1) p_\alpha(t)^{-1/2}\ t \in ((-\alpha)^{1/2}, (-\alpha)^{-1/2}). \qquad (5.41)$$

Denote

$$f_\alpha(t) := |t^3 - 1| p_\alpha(t)^{-1/2},\ g_\alpha(t) := (t^3 + 1) p_\alpha(t)^{-1/2}.$$

We have

$$\lim_{t\to(-\alpha)^{1/2}_+} [g_\alpha(t) - f_\alpha(t)] = +\infty \qquad (5.42)$$

and

$$\lim_{t\to(-\alpha)^{-1/2}} [g_\alpha(t) - f_\alpha(t)] = +\infty. \qquad (5.43)$$

Since $f_\alpha(t) < g_\alpha(t),\ t \in ((-\alpha)^{1/2}, (-\alpha)^{-1/2})$ it is clear that (5.6) desribes in the second octant one unbounded domain which is contained in the layer between two planes $t = (-\alpha)^{1/2}$, $t =$

$(-\alpha)^{-1/2}$. Moreover, this domain approaches the set given by $|t^2 - \frac{1}{t}| < |v| < t^2 + \frac{1}{t}$, as $\alpha \to 0_-$. Again functions $f_\alpha(t), g_\alpha(t)$ are monotonic in suitable intervals and the above domain shrinks when t tends to critical values $(-\alpha)^{1/2}, (\alpha)^{-1/2}$. Observe that for $\alpha = -1$ and $t \neq 1$, $P_6(v, -v, t) < -2$ and $\sigma(J)$ is always discrete. Note also that $((-\alpha)^{1/2}, (-\alpha)^{-1/2}) \to \{1\}$, as $\alpha \to -1_+$.
Finally, in the case $\alpha = -1$, $t = 1$ we do not the answer because $P_6(v, -v, 1) = -2$ (the border situation). As it was mentioned above this situation requires a new approach. Sumning up we have for $t \in \mathbb{R}$ four simply connected zones of parameters (v, w, t) which give $\sigma(J)$ pure absolutely continuous and four zones where $\sigma(J)$ is discrete.
The following comments seem to clarify a litlle geometry of domains from perturbation theory point of view. First, if $v^2 + w^2$ tends to $\{\infty\}$ then $\sigma(J)$ should be discrete since diagonal dominates weights. Second, if $t \to 0$ $(or\, t \to \{\infty\})$ $\sigma(J)$ also becomes discrete because J is close to infinite orthogonal sum of diagonal matrices (see Ex.5.2). In turn in a neighborhood of $(t, v, w) = (t, 0, 0)$ absolutely spectrum can appear only for $t = \pm 1$, because of dominating weights and evenness of $\mathbb{N} = 2$. Therefore appearance of σ_{ac} near $(\pm 1, 0, 0)$ has a resonance character. This seems to be even more reasonable for appearance of σ_{ac} near $(t, v, w) = (0, \alpha^{-1/2}, \alpha^{1/2})$. Actually for small t and $v = \alpha^{-1/2}$ the modulation of diagonal with $w = -\alpha^{1/2} - (\alpha^{-1/2} + \alpha^{3/2})t^2 + O(t^3)$ (see formula for $P_6(b, c)$) gives σ_{ac} due to resonance interplay between modulations of weights and diagonal. Surely the above reasoning gives only rough explanation of the whole picture, pricese answer requires detailed but elementary analysis of the characteristic polynomials.

Remark **5.4** . It is clear that 2M is a new period (not the smallest) along with M and by a slight change of $b_n's$ it becomes the smallest one. Hence the odd case for the period is not stable in contrast with the even case. The following example shows also instability of the spectral structure of J regarding to changes of the parity of M. Actually instability due to changing of periods is a well-known fact leading to the appearance of Cantor-like spectra. It was a basis for construction of almost periodic potential theory [4], [22].

Example **5.5** . Let $M = 3, N = 1$, $\mu_n = r_n = n^\alpha$, $0 < \alpha \leq 1$.Take $c_k \equiv 1$ and $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1\varepsilon_2 < 0$. Define $b = (1, -1, 0)$. Then $Tr(F_3F_2F_1) = 0$ and so by Th4.2 the spectrum of J (defined for the above weights and diagonal) is absolutely continuous. Now change $M = 3$ by $M = 6$, and define the new

$$b' = (1, -1, 0, 1 + \varepsilon_1, -1 + \varepsilon_2, 0),\ c'_k \equiv 1,\ \mu'_n \equiv r'_n \equiv \mu_n.$$

We have

$$Tr(F'_6F'_5 \cdots F'_1) = -2 + \varepsilon_1\varepsilon_2 < -2,$$

and applying Th.3.1 to the matrix J' corresponding to the above $\{b'_k\}, \{c'_k\}, \{\mu'_k\}, \{r'_k\}$ we know that $\sigma(J')$ is discrete despite small perturbation of $\{b_k\}$ and $\{c_k\}$. Surely we can choose here ε_1 and ε_2 arbitrary small but with different signs.

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References

[1] W. Van Assche, " Asymptotics for orthogonal Polynomials", Lect. Notes in Math. 1265 Springer-Verlag, Berlin, 1987.

[2] F.V. Atkinson, " Discrete and Continuous Boundary Value Problems " Academic Press, 1964.

[3] Yu M. Berezanskii, Expansions in Eigenfunctions of Selfadjoint Operators, Naukova Dumka, Kiev (1965) (in Russian).

[4] R. Carmona and J. Lacroix, " Spectral Theory of Random Schrödinger Operators ", Birkhäuser, Boston, 1990.

[5] E.A. Coddington, N. Levinson, "Theory of Ordinary Differential Equations", Mc Graw - Hill (1955).

[6] J.M. Combes, P.D. Hislop and E. Mourre, Spectral averaging, perturbation of singular spectrum, and loealization, Trans. Amer. Math. Soc. 348 (1996), 4883-4894.

[7] J. Dombrowski, Cyclic Operators, Commutators, and Absolutely Continuous Measures, Proc. Amer. Math. Soc., Vol 100, No 3, (1987) 457-462.

[8] J. Edward Spectra of Jacobi matrices differential equations on the circle, and the $su(1,1)$ Lie algebra SIAM J. Math. Anal. 34:3 (1993) 824-831.

[9] I. Gohberg, M.G. Krein, "Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert space", Amer. Math. Soc., Providence, R. I. (1969).

[10] P. Halmos, "A Hilbert space problem book", Van Nostrand, Princeton (1967).

[11] P. Hartman, "Ordinary differential equations", John Wiley (1964).

[12] H. Heinig, Inversion of periodic Jacobi matrices, Math. Islled., Chisinau 1:27 (1973) 180-200.

[13] J. Janas, S. Naboko, Jacobi matrices with power like weights-grouping in blocks aproach, J. Func. Anal. 166 (2) (1999) 218-243.

[14] J. Janas and S. Naboko, Jacobi Matrices with Absolutely Continuous Spectrum, Proc. Amer. Math. Soc. 127 (3) (1999) 791-800.

[15] J. Janas and S. Naboko, Asymptotics of generalized eigenvectors for unbounded Jacobi matrices with power-like weights, Pauli matrices commutation relations and Cesaro averaging (to appear in Operator Theory: Advances and Applications, volume dedicated to M.G. Krein).

[16] J. Janas and S. Naboko, Multithreshold spectral phase transition examples in a class of unbounded Jacobi matrices, Res. Reports in Math. , No 7 Stockholm Univ. (1999)

[17] S. Khan and D.B. Pearson, Subordinacy and Spectral Theory for Infinite Matrices, Helv. Phys. Acta, 65 (1992) 505-527.

[18] A. Kiselev, Y. Last and B. Simon, Modified Prüfer and EFGP Transforms and the Spectral Analysis of One-Dimensional Schrödinger Operators, Commun. Math. Phys. 194 (1997) 1-45.

[19] A.G. Kostuchenko and K.A. Mirzoev, Generalized Jacobi matrices and deficiency indices of differential operators with polynomial coefficients, Funct. Anal. Appl. 33 (1) (1999) 30-45.

[20] D. Masson , J. Repka Spectral theory of Jacobi matrices in $l^2(\mathbb{Z})$ and the $su(1,1)$ Lie algebra SIAM J. Math. Anal. 22 (1991) 1131-1146.

[21] S. Molchanov, The local structure of the spectrum of the one dimensional Schrödinger operator, Comm. Math. Phys. 78 (1981) 429-446.

[22] J. Moser, An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum, Comm. Math. Helvetici. 56 (1981), 198-224.

[23] P.B. Najman, On the theory of periodic and limit-periodic Jacobi matrices, Doklady ANUSSR 143 (1962), 277-280.

[24] F.W. Olver, "Asymptotic and special functions", New York, London, A.P. (1974)

[25] B. Simon, The classical moment problem as a selfadjoint finite difference operator (preprint) (1997).

[26] P. Stollmann, Localization for random perturbation of anisotropic periodic media, Isreal J. Math. 107 (1998) 125-139.

[27] G. Stolz, Spectral Theory for Slowly Oscillating Potentials I. Jacobi Matrices, Manuscripta Math. 84 (1994) 245-260.

Jacobi Matrices - Periodically Modulated Entries

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